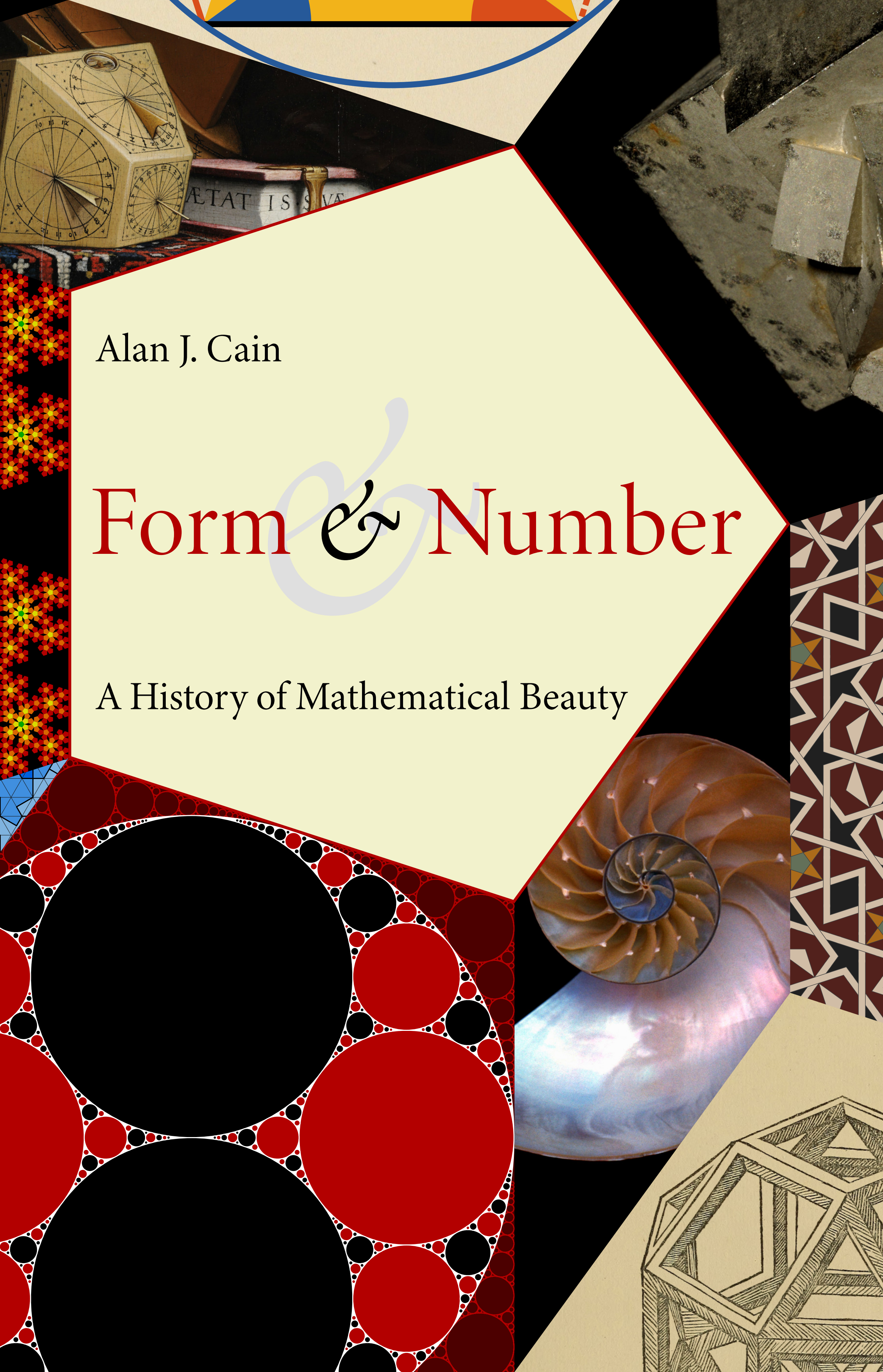

Alan J. Cain

Form & Number

A History of Mathematical Beauty

In the millennia since humans started to do mathematics, they have admired, sought, studied, and argued over its beauty.

¶ In ancient Egypt and Mesopotamia can be discerned the first hints of an aesthetic appreciation of mathematics. Greek philosophers from Plato to Proclus recognized and examined mathematical beauty. Mathematicians from Apollonius onward acknowledged its influence. The heritance of antiquity lasted throughout the Middle Ages in both the Christian and Islamic worlds, and into the European Renaissance.

¶ Mathematical beauty played a role in the work of Copernicus, Galileo, and Kepler. The giants of the seventeenth century — Fermat, Descartes, Pascal, Huygens, Newton, Leibniz, the Bernoullis — evaluated their own and others' work in aesthetic terms. When modern aesthetics emerged as a discipline during the eighteenth century, scholars like Shaftesbury and Francis Hutcheson thought it natural to look at mathematical beauty; later, there would be philosophical reaction against the very concept. Outside mathematics proper, artists such as Albrecht Dürer and M. C. Escher were influenced by its beauty.

¶ Starting in the nineteenth century, there began a practice of justifying the pursuit of mathematics using its beauty, a tendency that grew out of the increasing abstraction and autonomy of mathematics. While the bloodiest war in history raged, G. H. Hardy defended the pursuit of mathematics on aesthetic grounds.

¶ Mathematicians such as Henri Poincaré and Jacques Hadamard thought beauty a guide to discovery. Mathematical beauty also came to be seen as an indicator of the validity of physical theories. Technological advance led to computer proof and to computer-generated images of mathematical objects, both of which were the subject of aesthetic debate.

¶ This book offers the first in-depth general history of beauty in mathematics and of the study of beauty in mathematics. Its intention is to examine the historical development of the experience of beauty in mathematics, of the influence of such experience both inside and outside mathematics, and of the scrutiny to which this experience has been subjected by mathematicians and philosophers.

Form & Number

Alan J. Cain

Form & Number

A History of Mathematical Beauty

Lisbon
2024

Caveat lector:

This is a ‘beta version’ of *Form & Number*. The author welcomes comments, corrections, or constructive criticisms; please send them to the email address below.

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To the extraordinary A. B.,
who was my companion through adventure and hardship,
and who never had the chance to read it.

‘[illegible]
[illegible]
[illegible]
[illegible]’

— [illegible] /
[illegible] ⟦VIII.46–9⟧.

IN PRAETERITŌ
◆
IN PRAESENTI

To my best friend,
in whom, as I go to wander far on roads that I will not tell,
I have unwavering faith.

‘黄葉之
落去奈倍尒
玉梓之
使乎見者
相日所念’

— 柿本 人麻呂: 『萬葉集』, 巻第二, 歌第二百九.

‘ And at first all things
were without reason and measure.
But when the world began to get into order, […]
God fashioned them by form and number. ’

— Plato, *Timaeus*, STE 53a–b
(trans. B. Jowett).

‘ All these have form because they have number:
take away their form and number and they will be nothing.
[…] Look at artists skilled in the production of all
corporeal forms: even their art consists of numbers. ’

— Augustine, *On Free Choice of the Will*, § 2.16.42
(trans. O. V. Bychkov).

‘ for intense artistic feelings of pleasure,
in form, colour, and composition —
and for those abstract notions of form and number
which render geometry and arithmetic possible. ’

— Alfred Russel Wallace,
Contributions to the Theory of Natural Selection, p. 351.

‘ For the harmony of the world is made manifest in
Form and Number, and the heart and soul
and all the poetry of Natural Philosophy
are embodied in the concept of mathematical beauty. ’

— D’Arcy Wentworth Thompson,
On Growth and Form, pp. 1096–7.

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Preface

‘A preface, being the entrance to a book,
should invite by its beauty.’

— Isaac Disraeli,
Curiosities of Literature, vol. i, § ‘Prefaces’, p. 71.

‘Except for this preface, this study is completely self-contained.’

— Raymond Smullyan, *First-Order Logic*, p. vii.

This work offers a history of beauty in mathematics and of the study of beauty in mathematics. Its intention is to examine the historical development of the experience of beauty in doing or contemplating mathematics, of the scrutiny to which such experience has been subjected by mathematicians and philosophers, and of the influence of beauty on the development of mathematics and the lives of its practitioners. To the best of the author’s knowledge, this book is the first in-depth general history of the subject. Previous scholarly attention has either examined the attitudes of individual thinkers,[1] or has produced short summarizing works.[2]

And yet on both philosophical and historiographical grounds, the development of mathematical beauty merits examination. It has sometimes been thought that the history of mathematics as a discipline and the history of mathematical ideas walk hand-in-hand, possibly due to the platonism that most mathematicians accept for practical purposes: mathematical ideas are considered not to be *formulated*, but rather to be *perceived* in a platonic mathematical reality. Thus the actual situation of mathematicians has little bearing on their ideas: ‘great men have played decisive roles in determining the course of mathematics. But it is their ideas that have been featured; biography is entirely subordinate’.[3] Yet even the most ardent mathematical platonist, who fully approves of this historiographical approach, must surely concede that aesthetic and other forms of value — or, at least, our *opinions* on the degree to which those values apply to some piece of mathematics — do not subsist in a platonic heaven. Thus the role of aesthetic concerns in the history of mathematics, in influencing the direction of research or the motivation of its practitioners, cannot be reduced to the history of mathematical ideas.

The use of the term ‘mathematical beauty’ is not a strict limit on the book’s subject. Other aesthetic categories arise from time to time. In particular, mathematical elegance seems closely linked to, and is sometimes only very finely distinguished from, mathematical beauty.

[1] For example, on Leibniz: Breger, ‘Die mathematisch-physikalische Schönheit bei Leibniz’; or on Kant: Wenzel, ‘Beauty, Genius, and Mathematics’.

[2] For example, Sinclair & Pimm, ‘A Historical Gaze’; Jullien, *Esthétique et Mathématiques*, chs i–iii.

[3] Kline, *Mathematical Thought*, vol. 1, p. ix; cf. Heard, ‘The Evolution of the Pure Mathematician’, p. 31.

The application of mathematics to aesthetics in general, with the aim of creating measures or models of aesthetic value,[1] falls outside the purview of this book, except when it relates to aesthetic value seen *in* mathematics.

So there are three main strands in this book: the evolution of the taste for beauty in mathematics; the development of philosophical attitudes to beauty in mathematics; and the influences of these on the development of mathematics and on the mathematicians who experience it. Certainly the first two strands are intertwined. Just as it would be impossible to study the history of philosophy of art without considering the development of art, one cannot understand philosophical scrutiny of mathematical beauty without considering beautiful mathematics. This is an instance of one of the dualities of aesthetics: the establishing of the facts of beauty and the explanation of them; in short, aesthetic facts and aesthetic explanation.[2]

SCOPE

> ‘ it will perhaps be expected that I should explain,
> in a few words, the nature and limits of my general plan. ’
>
> — Edward Gibbon,
> *The Decline and Fall of the Roman Empire*, Preface.

The scope of this book is limited: save for the occasional excursion, it considers only the western tradition of mathematics and philosophy. Mediaeval Islamicate thinkers, the heritors and supersessors of the Greeks, are included. Those of China, India, Japan, …, are not. The author is aware of the great achievements of mathematics in these cultures, but is too conscious of his unfamiliarity with their intellectual — and especially philosophical — traditions to imagine he could treat them adequately, let alone competently, at present. But within the scope of occidental thought, numerous subjects turn out to be relevant to mathematical beauty: art, architecture, epistemology, ethics, literature, music, mysticism, ontology, poetry, physics, psychology.

One pitfall the author has attempted to avoid is an ‘aesthetics of the gaps’, whereby an unknown reason for a mathematical choice is ascribed to aesthetic value, or where an aesthetic theory that *could be* applied to mathematics is inferred to *have been* so applied. In situations such as these, at least some positive evidence must be adduced. For example, much of Alexander Gerard’s (1728–95) theory of beauty could be applied to beauty in mathematical proofs and theorems. But Gerard only discussed mathematics in the context of novelty,[3] which he considered to be a distinct aesthetic category, and so he must be held not to have thought of mathematics as falling within the compass

[1] King, *The Art of Mathematics*, pp. 145–8.

[2] Tatarkiewicz, *Hist. Aesthetics*, vol. 1, p. 3.

[3] Gerard, *An Essay on Taste*, p. 10.

of his theory of beauty. What authors do not speak about we must pass over in silence.

When treating of other thinkers’ attitudes to the aesthetics of mathematics, the author tries to describe fairly their views, but considers himself free to engage critically. Furthermore, an author cannot be expected to remain entirely neutral in choosing examples of beauty, elegance, or other aesthetic categories, and the present author openly confesses the influence of his own personal tastes.

IMPETUS

> ‘ But as I went deeper into the subject […]
> I found that much that had previously been written about it
> was not to be depended upon;
> that frequently one author had copied another,
> and that errors and opinions which had once gained admission
> remained embedded in the literary tradition. ’
>
> — Fridtjof Nansen, *In Northern Mists*, vol. 1, p. vii
> (trans. A. G. Chater).

Part of the present author’s motivation is that those scholars who examine beauty in mathematics often seem unaware of its history. There might be a remark about G. H. Hardy’s (1877–1947) *A Mathematician’s Apology*[*] or a quotation of Bertrand Russell’s (1872–1970) dictum about ‘a beauty cold and austere’[1][†] or a reference to Pythagoras[‡] (c. 570–c. 490 BCE) or Plato[§] (c. 427–348/347 BCE), but they stand in isolation; there is no discussion — and presumably no awareness — of the intellectual tradition in which they sit and which traces back from each thinker to the last. Consider something the highly-regarded historian of mathematics Morris Kline (1908–92) wrote in 1972:

> ‘For about a hundred years now mathematicians have come to recognize what was felt and asserted by the Greeks but had been lost sight of in the intervening centuries: mathematics is an art and mathematical work must satisfy aesthetic requirements.’[2]

This statement stands at odds with the recognition of mathematical beauty throughout the Middle Ages, Renaissance, Enlightenment, and early modern period, by such mathematicians as Ibn al-Haytham[¶] (‘Alhazen’; 965–c. 1040), Leonardo Pisano[♠] (‘Fibonacci’; c. 1170–after 1240), Girolamo Cardano[♥] (1501–76), Johannes Kepler[♦] (1571–1630), Pierre de Fermat[♣] (1607–65), René Descartes[A] (1596–1650), Leonhard Euler[B] (1707–83), and Joseph-Louis Lagrange[C] (1736–1813).

Yet it is difficult to condemn such nescience. There are scattered references to the aesthetics of mathematics in the modern literature, but the distribution is too sparse to have reached that critical mass

[*] ▸ pp. 498–509

[1] Russell, ‘The Study of Mathematics’, p. 60.

[†] ▸ pp. 479–85

[‡] ▸ p. 44

[§] ▸ pp. 58–81

[2] Kline, *Mathematics in Western Culture*, p. 466.

[¶] ▸ pp. 207–9

[♠] ▸ pp. 179–80

[♥] ▸ pp. 264–8

[♦] ▸ pp. 279–94

[♣] ▸ pp. 297–303

[A] ▸ pp. 303–5

[B] ▸ pp. 334–41

[C] ▸ pp. 393–401

that forms a canon. Chains of references — when they exist at all — frequently peter out all too quickly. Consequently, excepting the works of one or two very famous authors, discussion of previous writers is rare and engagement with their views still rarer. Aesthetics perhaps lies closer to the personal than the technical side of mathematics, and so is treated by professional mathematicians with less care than 'proper' mathematical writing. [Such weaker scholarly rigour outside the technical core of mathematics has been identified as one of the causes behind the formation of mathematical legends.[1]]

Perhaps also as a consequence of this lack of a canon, the interplay of aesthetics and mathematics seems to be the subject of a disproportionate share of exaggeration, ambiguity, equivocation, and downright nonsense. At one, more innocent, end of the spectrum is the remark of Kline discussed above. At the other, more culpable, end are the golden numberists' claims that the extreme and mean (or 'golden') ratio is embodied in the Egyptian pyramids, the Parthenon, the *Aeneid*, and other masterpieces of western art; such claims have long since been refuted.[2] The extreme and mean ratio *does* feature in the history of mathematical beauty, but its true role is very unlike what the mythical accretions suggest.[*] Undoubtedly the most malignly influential focus of confusion has stemmed from the misunderstanding and misquotation of Hardy's *Apology*, especially concerning his views on the relationship between beauty and utility in mathematics.[†]

[The general history of mathematics has not been immune to myth presented as fact. The mathematician and historian of mathematics Bartel van der Waerden (1903–96) complained that 'books on the history of mathematics copy their assertions uncritically from other books, without consulting the sources! How many fairy tales circulate as "universally known truths"!'[3] And L. E. Sigler (1928–97) wrote that 'with mathematical history, these lamentable errors or fantasies catch on, persist, and seem never to be correctable'.[4] As a concrete example, a particular proof of the pythagorean theorem, which seems to date from the late eighteenth century, was mistakenly attributed to Leonardo da Vinci (1452–1519) in 1906. This error continued for over a century and was widely disseminated,[5] even appearing in a caveated form in the work of such a meticulous scholar as T. L. Heath (1861–1940).[6]]

If, in the end, this book serves no purpose other than to inform its readers of the rich heritage of scholarship relevant to the aesthetics of mathematics, and to clear away some of the accumulated detritus; even if every other argument of the author's is rejected, then he shall rest content. 'If this book draws more attention to a most important subject and stimulates others to labour in this field it will have done its work.'[7]

[1] Cooke, 'Life on the Mathematical Frontier', esp. p. 466.

[2] Markowsky, 'Misconceptions about the Golden Ratio'.

[*] ▸ pp. 552–73

[†] ▸ pp. 512–16

[3] van der Waerden, *Science Awakening*, p. 6.

[4] Sigler, 'Leonardo Pisano', p. xvi.

[5] Lemmermeyer, 'Leonardo da Vinci's Proof of the Pythagorean Theorem'.

[6] Heath, notes to Euclid, *Elements*, Prop. I.47, p. 265.

[7] Yates, *Giordano Bruno*, p. xi.

SOURCES

‘ no Abbé Vallet had published books on the abbey’s presses
(for that matter, nonexistent).
French scholars are notoriously careless
about furnishing reliable bibliographical information,
but this case went beyond all reasonable pessimism. ’

— Umberto Eco, *The Name of the Rose*, preface
(trans. W. Weaver).

The lack of careful scholarly practice in many works that discuss mathematical beauty, together with the inattention to mathematical (or, more generally, intellectual) beauty in modern philosophical aesthetics, led the author to adopt a policy of seeking primary sources, both mathematical and philosophical, whenever possible. Indeed, this book is largely based upon consulting the primary sources for the mathematicians and philosophers who created and analysed mathematical beauty. In particular, the author chose *never* to use a quotation without tracking it down either to a primary source or, when it has been impossible to consult the relevant primary source, to a scholarly work that cites *precisely* that primary source.

Fortunately, many of the relevant primary sources, particularly older books and journals, are available on or via sites such as the Internet Archive, the Biodiversity Heritage Library, the Deutsche Digitale Bibliothek, `e-rara.ch`, the European Digital Mathematics Library, and Gallica.[1] Being able to search rapidly for and through such texts has been very helpful. However, general online searches are often swamped by pages of quotations that lack proper sourcing. Even when a source is given, it may be unreliable: as in other fields, a mathematical quotation can become detached from its origin, perhaps be distorted in the re-telling, and attach itself to a famous name.[2] In seeking the true genesis of quotations, the author has met many dead ends.

Take a specific example: François Le Lionnais (1901–84) wrote that Carl Friedrich Gauss (1777–1855) described the law of quadratic reciprocity as the ‘*jewel of arithmetic* [*joyau de l’arithmétique*]’.[3] Gauss’s biographer G. Waldo Dunnington (1906–74) said that Gauss called it ‘the “gem” of the higher arithmetic’.[4] Gauss was certainly captivated by the result, offering multiple proofs and praising it in aesthetic terms.* But did he actually use an expression like ‘jewel’ or ‘gem’ — or rather, a Latin or German equivalent — in this context? Neither Le Lionnais nor (more disappointingly) Dunnington gave a source; neither did most other authors. A translation of Gauss’s first memoir on biquadratic residues says:

> ‘The theory of quadratic residues can be reduced to the most beautiful jewel among the fundamental theorems of higher arithmetic’,[5]

but the original Latin is:

[1] `archive.org`
`biodiversitylibrary.org`
`ddb.de`
`e-rara.ch`
`eudml.org`
`gallica.bnf.fr`

[2] Cooke, ‘Life on the Mathematical Frontier’, pp. 467 sqq.

[3] Le Lionnais, ‘La Beauté en Mathématiques’, p. 442; trans. C., emphasis in orig.

[4] Dunnington, *Carl Friedrich Gauss*, p. 46.

* ▸ pp. 405–7

[5] Knoebel et al., *Mathematical Masterpieces*, p. 244.

> ‘Theoria residuorum quadraticorum ad pauca theoremata fundamentalia reducitur, pulcherrimis Arithmeticae Sublimioris cimeliis adnumeranda.’[1]

which translates to:

> ‘The theory of quadratic residues reduces to a few fundamental theorems, among the most beautiful treasures of the higher arithmetic’.

The Latin ‘cimeliis’ is more accurately translated as ‘treasures’, and in any case is certainly plural: Gauss did not single out any one theorem here.

Carl Jacobi (1804–51) wrote that Gauss called the law of quadratic reciprocity ‘the gem of the higher arithmetic [Kleinod der höhern Arithmetik]’,[2] but did not give a source; Gauss was alive when Jacobi wrote this, and may have used the expression in personal communication. In the manuscript of the first part of his ‘Report on the Theory of Numbers’, presented to the British Association for the Advancement of Science in 1859, H. J. S. Smith (1826–83) cited this report by Jacobi to support Gauss's use of the expression.[3] One might speculate that the use of this report as a secondary source led to the entry of the claim into the English-language literature.

Thus the author is left with a puzzle. Even searching Gauss's *Werke*, he has been unable to find any instance of Gauss describing the law of quadratic reciprocity as ‘the jewel’ or ‘the gem’ of arithmetic. The author has of course not *read* Gauss's entire corpus, and perhaps the idea can be found there, but in different words. The most the author can say with certainty is that *Jacobi said* that Gauss called the theorem ‘the gem of the higher arithmetic’.

And lack of citation is not an especially recent phenomenon. There is an oft-quoted statement by G. N. Watson (1886–1965):[4]

> ‘The study of Ramanujan's work and of the problems to which it gives rise inevitably recalls to mind Lamé's remark that, when reading Hermite's papers on modular functions, “on a la chair de poule”[5].’[6]

Watson, writing in 1936, did not supply a reference for Gabriel Lamé's (1795–1870) remark. In a review of Charles Hermite's (1822–1901) collected works, W. J. Greenstreet (1861–1930) wrote that: ‘C. Jordan used to say that he once heard Lamé remark of Hermite's memoirs on the theory of modular functions that in reading them “*on a la chair de poule*”’.[7] Like Watson, Greenstreet supplied no source, but may have personally heard Jordan say this. Searching Camille Jordan's (1838–1922) collected papers led the author to the original obituary of Hermite, which makes it clear that Greenstreet was correct: Jordan *heard* Lamé make the remark (and agreed with him that Hermite's work produced that effect).[8] This time, despite the twofold absence of citation, the ensuing search had a happy ending.

[1] Gauss, ‘Theoria Residuorum Biquadraticorum’, p. 67.

[2] Jacobi, ‘Ueber die complexen Primzahlen’, p. 314; trans. C.

[3] Smith, ‘Report on the Theory of Numbers (pt 1)’, p. 56, n. ∗.

[4] See, for example, Chandrasekhar, ‘Beauty and the quest for beauty in science’, p. 26.

[5] ‘one gets goosebumps’

[6] Watson, ‘The final problem’, p. 80.

[7] Greenstreet, Review of Hermite, *Œuvres* vol. 3, emphasis in orig.

[8] Jordan, ‘Charles Hermite’, p. 130.

Having been forced to embark on numerous such quests in the course of writing this book, the author deliberately chose to err on the side of pedantry in citation. The reader should not find themself starting from here on a hunt for sources.

Caveat lector:

This is a 'beta version' of *Form & Number*. The author welcomes comments, corrections, or constructive criticisms; please send them to the email address on the copyright page.

Notes to the reader

> ‘ The first thing he did was to sit down and read all the notes ’
>
> — Milorad Pavić, *Dictionary of the Khazars*, p. 167
> (trans. C. Pribićević-Zorić).

> ‘ The general reader is advised not to bother about the Notes ’
>
> — Arthur Koestler, *The Sleepwalkers*, p. 16.

JARGON

> ‘ The power of terminology,
> because of its *structural implications*,
> is well known in science ’
>
> — Alfred Korzybski, *Science and Sanity*, p. lxiii.

In this book, an eponymous adjective or noun — *Pythagorean*, *Cartesianism*, *Hardian* — is capitalized if it denotes a direct relationship to the person concerned; it is uncapitalized if the relationship is indirect. [This policy is inspired by, but quite distinct from, a practice of Ian Hacking (1936–2023).[1]] Thus, ‘the Platonic theory of Forms’, since this was Plato’s own theory, but ‘mathematical platonism’, since this is a modern idea that only descends from Plato’s ontology.[2] This rule does not apply in quotation, where capitalization and orthography are always per the original.

Names of persons are linked by an ampersand when they are discussed as the collective author of a work. Thus the phrase ‘Wolfgang Haken and Kenneth Appel’ denotes two individuals, while the phrase ‘Haken & Appel’ denotes the collective author of one or more of their various joint publications.

Cross-references are marked by the symbol ▸ and the relevant page or pages in a sidenote marked by a symbol or letter. For instance, here is a cross-reference to the discussion of Albrecht Dürer in Chapter 8.*

CITATION

> ‘ “Ibid is a well-known authority on everything,” said Xeno. ’
>
> — Terry Pratchett, *Pyramids*, pt III.

The citation system is ‘author/editor and short title’ with a few bibliographic abbreviations (see page 1034 for the key),

[1] See Hacking, *Why Is There Philosophy of Mathematics at All?*, pp. 21–2.

[2] Cf. Shapiro, *Thinking about Mathematics*, p. 28; see also Bernays, ‘On platonism in mathematics’, p. 258.

* ▸ pp. 252–8

supplemented by the use of ‘*ibid.*’, but only if the previous citation is on the same page, and ‘*loc. cit.*’, but only to repeat the immediately preceding citation, and only if that citation is on the same page. This policy saves space while obviating the usual objection to these abbreviations: *viz.*, that they compel the reader to search backwards through the book to find the unabbreviated citation.

In citations and in the bibliography, the spelling of each name is always per the publication. This policy creates some inconsistencies in spelling between different citations, different bibliography items, and the main text.

Some texts are cited using standard systems of pagination, rather than the physical pagination of the cited edition: Stephanus pagination for Plato and Plutarch; Bekker pagination for Aristotle; Meibomius pagination for Diogenes Laertius; *und so weiter*. Citations using such systems use explanatory prefixes such as ‘MEI’ for ‘Meibomius’ (see pages 1034–5 for the key).

Square brackets [■] in a citation enclose supplementary location information, such as a section or letter number. Double square brackets ⟦■⟧ in a citation enclose supplementary location information relevant only to the particular edition of a work that is cited using a standard system of pagination or sectioning.

The abbreviation ‘trans. C.’ indicates that a quotation has been translated from a non-English language source by the present author.

TRANSLITERATION

> ‘ Arabic names are spelt anyhow,
> to prevent my appearing an adherent of
> one of the existing “systems of transliteration”. ’
>
> — T. E. Lawrence, *Seven Pillars of Wisdom*, app. II, p. 664.

Transliteration of non-Latin scripts generally follows the United States Library of Congress systems, although for Arabic script the ‘half-circle’ modifiers ʿ and ʾ are preferred to transliterate ع [ʿ*ayn*] and ء [*hamzah*] respectively.

Acknowledgements

> ‘A book and its author profit from blame and not from approval.’
>
> — E. J. Bickerman, *Chronology of the Ancient World*, p. 7.

> ‘Valéry acknowledges that he hardly reads at all
> and yet doesn’t hesitate to offer his opinion’
>
> — Pierre Bayard,
> *How to Talk About Books You Haven’t Read*, p. 25
> (trans. J. Mehlman).

It was at St Andrews, in December 2006, as he sat musing in the University Library, while the students were anticipating the Christmas break, that the idea of writing a history of mathematical beauty first started to the author’s mind. But other commitments intruded and only in 2016 did he start research and writing in earnest: when the world went mad, this work proved ‘an incomparable anodyne’.[1]

Yumi Murayama carefully read large parts of the text and made extensive and incisive comments, which led to countless improvements. Murayama had earlier critiqued *An Annotated Mathematician’s Apology*; the author’s three essays in that work were partially incorporated into Chapters 12 and 13.

Karl Berry meticulously proof-read the entire text, including the bibliography and indexes, pointing out a multitude of minor errors and making copious suggestions for improving the text and typography. He also ran his own suite of automated checks on the LuaLaTeX source files, uncovering various small flaws.

Frank Plastria indicated numerous obscurities and minor errors: grammatical, mathematical, orthographical, typographical.

Werner Lemberg noted various typographical infelicities and defects in diagrams, and offered advice on German capitalization.

Other minor errors and inconsistencies were pointed out by Erkko Lehtonen, Richard Lucas, and Alexander Willand.

Tung Ken Lam pointed to an online edition of a book which the author had missed. Luís Trabucho de Campos supplied a copy of his book *O Número*.

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[1] Cf. Hardy, *Apology*, § 28.

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Acknowledgements

Almada,
St John’s Eve 2024

A. J. C.

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The Problem of Beauty in Mathematics

A prolegomenon

> ‘ BURNS: […] I hear you say the words “mathematical beauty”
> and I’m sure you mean something by it,
> but really I have no idea what. *Audience laughter.*
> PLATO: Nor should you. ’
>
> — Rebecca Newberger Goldstein,
> *Plato at the Googleplex*, ch. δ.

> ‘ We know on excellent authority that beauty is truth,
> that it is the expression of the ideal,
> the symbol of divine perfection,
> and the sensible manifestation of the good.
> […] Such phrases stimulate thought and give us a momentary
> pleasure, but they hardly bring any permanent enlightenment. ’
>
> — George Santayana,
> *The Sense of Beauty*, § 1.

Mathematics is a human art. Even if mathematics as a body of knowledge is independent of us, or at least about things that are independent of us, mathematics as a process that creates this knowledge is a human activity. This process cannot be entirely uninfluenced by those parts of human nature that are not wholly rational — or at least not obviously so. This book is about the interplay between mathematics and one such aspect of human thought: the appreciation of beauty.

THE ROLE OF AESTHETICS

Aesthetic concerns play a crucial role in mathematical practice. Aesthetic terms (*beautiful*, *elegant*, *cute*, *neat*; *ugly*, *inelegant*, *tedious*, *messy*) are used by mathematicians in judging mathematical work and in describing the appeal of and motivation for mathematics. In their answer to the question ‘What is mathematics?’ in their celebrated book of the same name, Richard Courant (1888–1972) &

◀ (*Key on* page 1024)

The role of aesthetics

Herbert Robbins (1915–2001) turned to the aesthetic motivation in the very first sentence:

> 'Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection.'[1]

The aesthetic aspect of mathematical practice demands scholarly attention, for it raises many important questions: How do aesthetic considerations influence the way mathematicians work? How have they influenced the historical development of mathematics? Do people of different cultural backgrounds make different aesthetic judgements of the same mathematics? How has mathematics influenced aesthetics as a philosophical discipline? Is there a connection between aesthetics and epistemology in mathematics? (Does the poet's '"Beauty is truth, truth beauty"'[2] hold in mathematics?) Despite their importance, the literature examining such questions is sparse, and is more often written by mathematicians than by philosophers of aesthetics or of mathematics.

When aesthetics emerged (at least in hindsight) as a recognizable discipline in the eighteenth century, writers on the subject such as Jean-Pierre de Crousaz (1663–1750), Francis Hutcheson (1694–1746), and Yves-Marie André (1675–1764) considered beauty in mathematics a natural topic of study. But this did not last: today, studies by philosophers of aesthetics often do not consider mathematics at all, whether from lack of interest, lack of knowledge, or lack of confidence.[3] In modern reference books such as *The Oxford Handbook of Aesthetics*, the Blackwell *Companion to Aesthetics*, and *The Routledge Companion to Aesthetics*, beauty in mathematics rates only the occasional passing mention.

Rarely too have philosophers of mathematics engaged, but this is perhaps symptomatic of a larger phenomenon. Especially during the twentieth century, 'philosophy of mathematics' became focused on foundational questions of ontology and epistemology, to the detriment of examining issues that concern the way in which mathematicians actually *do* mathematics, such as conjecture formation, analogy, plausible reasoning, visualization, and diagrammatic reasoning.[4] In particular, certain philosophers of mathematics seemed to believe that if mathematics could reasonably be said to be represented in terms of some formalism like logic or set theory, then it did not matter that mathematicians generally do not work using such a formalism: the philosophical study of mathematics could be restricted, without (significant?) loss, to the philosophical study of this formalism. This mindset has been acutely criticized: 'if we find we are not required to look at the details of mathematical thinking, we are not posing the right questions'.[5] Philosophy of mathematics should be descriptive and should not proceed from normative assumptions.[6] And, in particular, value judgements, such as aesthetic judgements, should be confronted by philosophers.[7] The difference between the two outlooks

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[1] Courant & Robbins, *What is Mathematics?*, Start of § 'What is mathematics?', n.p.

[2] Keats, 'Ode on a Grecian Urn', ll. 49–50.

[3] Cf. King, *The Art of Mathematics*, pp. 145, 147.

[4] Corfield, *Towards a Philosophy of Real Mathematics*; Mancosu, 'Introduction'.

[5] Corfield, 'Categorification as a heuristic device', p. 71.

[6] Cf. Rota, 'Phenomenology of Mathematical Truth', pp. 135–6; see also Cellucci, 'Indiscrete Variations', pp. 225–6.

[7] Corfield, *Towards a Philosophy of Real Mathematics*, pp. 38–9.

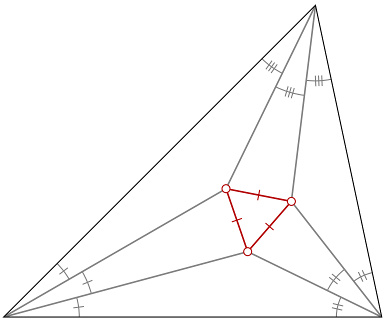
is so profound that a working mathematician once semi-seriously proposed that the field should be divided into two: the foundational 'philosophy of Mathematics' (majuscule 'M') and the practice-oriented 'philosophy of mathematics' (minuscule 'm').[1] The weight once assigned to foundational studies was such as to distort the surrounding philosophical landscape. More recently, philosophy of mathematics has rebalanced towards study of the thought and practice of actual thinking and practising mathematicians.[2]

And let it be emphasized that none of the aesthetic concepts and practices of working mathematicians are self-explaining. Two proofs, equally rigorous, are offered of the same theorem. One is celebrated because it is beautiful and the other is not, although epistemologically both proofs secure the theorem. Unless such judgements are to be dismissed as arbitrary or meaningless or irrational, philosophy must give some account of them. And the building-blocks of such an account will not be found in the foundational studies, but in examining the practice of real mathematics.[3]

This is why it seems to be a good moment to examine the history of beauty in mathematics, from the perspectives of both the mathematician who seeks it and the philosopher who studies it. But in preparation for this task, it is worthwhile to make first a survey of the scope of the subject and the main questions that arise.

QUESTIONING BEAUTY

What is mathematical beauty? 'I don't know. I can't define it in abstract terms — yet. But I think I can recognise it when I see it; and I am looking at it now':[4]



This is Morley's trisector theorem: *In any triangle, the points of intersection of the trisectors of adjacent angles are the vertices of an equilateral triangle.* To the author, it is a theorem both beautiful and surprising. It seems like the kind of result in which ancient Greek geometers would have delighted, but it was first published in 1900 by Frank Morley (1860–1937).[5]

[1] Harris, *Mathematics without Apologies*, Preface.

[2] Aspray & Kitcher, 'An Opinionated Introduction', pp. 17–19; Mancosu, 'Introduction', pp. 3 sqq.

[3] Cf. Larvor, 'What can the philosophy of mathematics learn [...]?', pp. 405–6.

[4] The reply by Clark, *Civilisation*, p. 1 to the question 'What is civilisation?'.

[5] Morley, 'On the metric geometry of the plane *n*-line'; see also Oakley & Baker, 'The Morley trisector theorem'.

Questioning beauty

The noun *beauty*, the adjective *beautiful*, the adverb *beautifully*: each applies to a great variety of objects, concrete and abstract, that *prima facie* have little in common. A beautiful person, a beautiful baby, a beautiful sonnet, a beautiful painting, a beautiful flower, a beautiful symphony, a beautiful death, a beautiful lesson, a beautiful moral, a beautiful landscape, a beautiful game, a beautiful proof. The word *beautiful* is clearly a term of praise and admiration, but beyond that one wonders why is it applied in so many situations.[1] Subtract the word *beautiful*: does anything connect the bare objects? Is there actually some commonality in the responses stimulated in each situation where beauty is recognized? Other possibilities present themselves: the word *beautiful* might be applied to, say, a sculpture and a poem in the same way that a word like *cold* describes a physical state and an emotional state. Or the word *beauty* might simply be applied metaphorically to some concepts.

There seems at first to be a division between sensory beauty and intellectual beauty: the former being beauty that is perceived through the senses (beauty in paintings, music, scenery), the latter being beauty that is not so perceived (beauty in mathematics, chess, morals). One must of course become familiar with a thing in which one perceives intellectual beauty, but it does not matter how one learns of it: a mathematical proof or a chess problem is beautiful regardless of whether one looks at it on a board, sees it on a screen, reads it in a book, or hears another person describe it. In contrast, sensory beauty is dependent on its presentation to the senses: a painting must be seen, a symphony must be heard. A written or spoken description may convey the judgement that it is beautiful, and perhaps suggest some inkling of the beauty itself, but it cannot convey the *experience* of the beauty.

But the dividing line is unclear. A beautiful mathematical proof surely has intellectual beauty: but if it uses diagrams and visual thinking, is its beauty related to beauty in vision? There may be some commonality in visualizing a beautiful geometric transformation and thinking of a beautiful picture. Is there some parallel between the beauty experienced in reading a piece of literature and the beauty experienced in reading the text of a mathematical proof? Further, a diagram illustrating some piece of mathematics may be beautiful in itself: Figure 0.1 shows a tiling of the hyperbolic plane by regular octagons. The diagram is beautiful, but is there some connection to the theorem that the hyperbolic plane can be tiled by any regular polygon? (The only regular polygons that can tile the euclidean plane are the equilateral triangle, the square, and the regular hexagon.)

The testimonies of working mathematicians suggest that beauty in mathematics is just the same as beauty in art. The 1998 Kyoto laureate Kiyosi Itō (1915–2008) wrote: 'In precisely built mathematical structures, mathematicians find *the same sort of beauty* others find in enchanting pieces of music, or in magnificent architecture.'[2]

[1] Cf. Scruton, 'In Search of the Aesthetic', p. 236.

[2] Itô, 'My Sixty Years', Abstract, emphasis added.

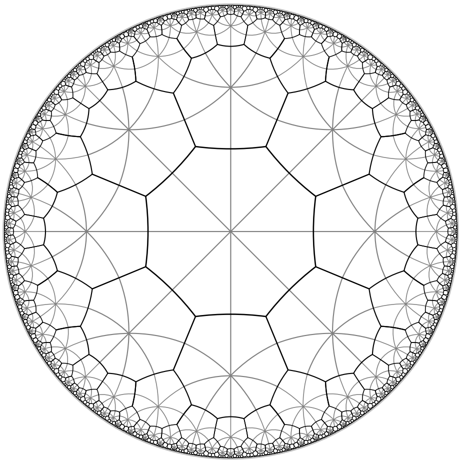


FIGURE 0.1
Tiling of the hyperbolic plane (represented as a Poincaré disc) by regular octagons.

Consider also the reaction of the analyst G. N. Watson (1886–1965) to the following formula, discovered by Srinivasa Ramanujan[*] (1887–1920):

$$\int_0^\infty e^{-3\pi x^2}\frac{\sinh \pi x}{\sinh 3\pi x}\,dx = \frac{1}{e^{2\pi/3}\sqrt{3}}\sum_{n=0}^{\infty}\frac{e^{-2n(n+1)\pi}}{(1+e^{-\pi})^2(1+e^{-3\pi})^2\cdots(1+e^{-(2n+1)\pi})^2}.$$

Watson wrote of this formula:

> ‘[It] gives me a thrill which is *indistinguishable* from the thrill which I feel when I enter the Sagrestia Nuova of the Capelle Medicee and see before me the austere beauty of the four statues representing “Day”, “Night”, “Evening”, and “Dawn” which Michelangelo has set over the tombs of Giuliano de’ Medici and Lorenzo de’ Medici.’[1]

(See Figure 0.2.) The biologist and mathematician J. B. S. Haldane (1892–1964) drew a parallel between the responses to music and to mathematics while reflecting on G. H. Hardy’s (1877–1947) attitude towards mathematics[†] and aesthetic value, saying that Hardy’s ‘writings gave me some of the emotions which others derive from classical music’.[2]

The strongest evidence in favour of a commonality between beauty in mathematics and visual or musical beauty may be a 2014 study in which sixteen volunteers, all postgraduate students or postdoctoral researchers in mathematics, made an evaluation of the beauty of sixty mathematical formulae, and then repeated the process while their brain activity was scanned using functional magnetic resonance imaging.[3][‡] When the volunteers viewed formulae that they had previously evaluated as beautiful, there was a spike in brain activity in a certain portion of the orbito-frontal cortex. This volume is precisely

[*] ▸ pp. 667–72

[1] Watson, ‘The final problem’, p. 80, emphasis added.

[†] ▸ pp. 498–509

[2] Haldane, *Everything has a History*, p. 242.

[3] Zeki, Romaya et al., ‘The experience of mathematical beauty’, pp. 1–2.

[‡] ▸ pp. 760–2, 837–8

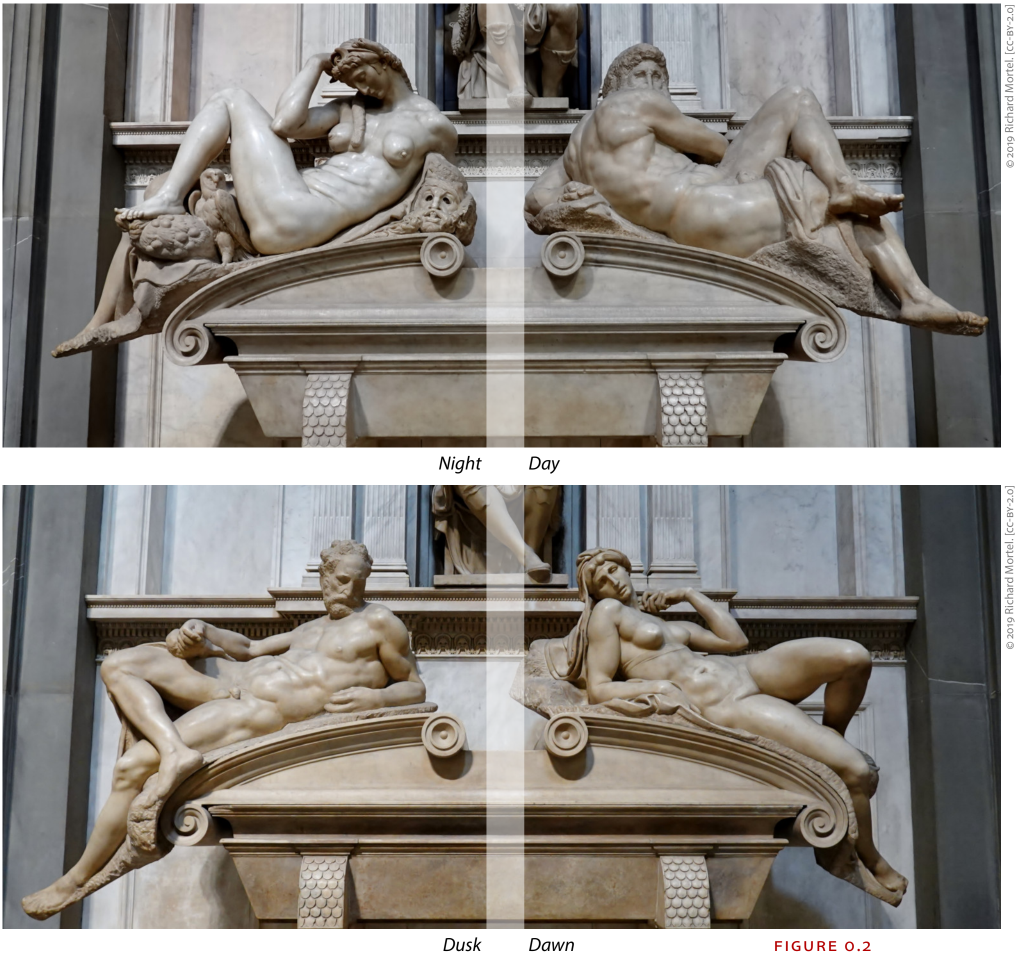

© 2019 Richard Mortel. [CC-BY-2.0]
Night
Day
© 2019 Richard Mortel. [CC-BY-2.0]
Dusk
Dawn

FIGURE 0.2 Michelangelo's *Night*, *Day*, *Dusk*, and *Dawn*. The first two adorn the tomb of Giuliano de' Medici; the latter two that of Lorenzo de' Medici.

where brain activity spikes during exposure to beautiful visual or musical stimuli. Other pleasurable experiences, such as the enjoyment of food or erotic pleasure results in brain activity increasing in different (but neighbouring) volumes.[1]

LOCI OF MATHEMATICAL BEAUTY

> ‘ we shall attend also to some of the things which the great mathematicians have considered worthy of loving understanding for their intrinsic beauty. ’
>
> — E. T. Bell, *Men of Mathematics*, ch. 1.

Although recognizing mathematical beauty seems easy, defining it is a difficult problem. A perhaps easier one is to identify *where* one can find it, by uncritically listing those aspects of mathematics in which various authors have seen beauty.

[1] Zeki, Romaya et al., 'The experience of mathematical beauty', p. 10.

Results. The notion that there can be beauty in theorems, propositions, or lemmata is a popular one. Morley's trisector theorem[*] is an example from geometry; elsewhere, Fermat's two-square theorem was thought by both G. H. Hardy[1] and E. T. Bell[2] (1883–1960) to be one of the most beautiful results in number theory, and was rated highly in a survey of beautiful theorems:[†]

FERMAT'S TWO-SQUARE THEOREM. *Every prime number of the form* $4k+1$ *is a sum of two square numbers.* □

Equations/formulae. An equation or formula is arguably a type of result, but there may be a fine distinction between beauty in an equation or formula and in a result. For example, consider the Rogers–Ramanujan identities:

$$\begin{aligned}1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \cdots \\ = \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})(1-q^{14})\cdots},\end{aligned}$$

$$\begin{aligned}1 + \frac{q^2}{1-q} + \frac{q^6}{(1-q)(1-q^2)} + \frac{q^{12}}{(1-q)(1-q^2)(1-q^3)} + \cdots \\ = \frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})(1-q^{13})\cdots}.\end{aligned}$$

G. H. Hardy praised them highly, always phrasing his approbation of them as *formulae*, not theorems. They were of 'obvious interest and beauty', were 'very beautiful',[3] and were such that 'it would be difficult to find more beautiful formulæ'.[4]

Terence Tao (1975–) considered 'the amazing identities of Ramanujan' to be exemplars of beautiful mathematics.[5] Watson's reaction to one of Ramanujan's equations has already been mentioned,[‡] but there are beautiful equations unconnected with Ramanujan. For his film *Rites of Love and Math*, the mathematician Edward Frenkel (1968–) had to choose a formula to play the role of a 'formula of love'. The search for a suitable candidate was guided by aesthetic principles: 'It had to be sufficiently complicated […] but also aesthetically pleasing. We wanted to convey that a mathematical formula could be beautiful in content as well as form.'[6] Frenkel eventually chose an example from his own work:

$$\begin{aligned}&\int_{\mathbb{CP}^1} \omega F(qz,\bar{q}\bar{z}) \\ &= \sum_{m,\bar{m}=0}^{\infty} \fint_{|z|<\varepsilon^{-1}} \omega_{z\bar{z}} z^m \bar{z}^{\bar{m}}\, dz\, d\bar{z} \cdot \frac{q^m \bar{q}^{\bar{m}}}{m!\bar{m}!}\, \partial_z^m\, \partial_{\bar{z}}^{\bar{m}}\, F\Big|_{z=0} \\ &\quad + q\bar{q} \sum_{m,\bar{m}=0}^{\infty} \frac{q^m \bar{q}^{\bar{m}}}{m!\bar{m}!}\, \partial_w^m\, \partial_{\bar{w}}^{\bar{m}}\, \omega_{w\bar{w}}\Big|_{w=0} \cdot \fint_{|w|<q^{-1}\varepsilon^{-1}} F w^m \bar{w}^{\bar{m}}\, dw\, d\bar{w}\,.\end{aligned}$$

[*] ▸ p. 3

[1] Hardy, *Apology*, § 12.

[2] Bell, *Men of Mathematics*, ch. 4.

[†] ▸ pp. 834–7

[3] Hardy, preface to Rogers & Ramanujan, 'Proof of certain identities', pp. 211, 212.

[4] Hardy, 'Srinivasa Ramanujan', p. xxxiv.

[5] Tao, 'What is good mathematics?', p. 624.

[‡] ▸ p. 5

[6] Frenkel, *Love and Math*, ch. 18.

The script required the formula to be tattooed onto a character's skin, and the special effects artist judged Frenkel's choice to be too complicated. Frenkel therefore simplified the formula to the following:[1]

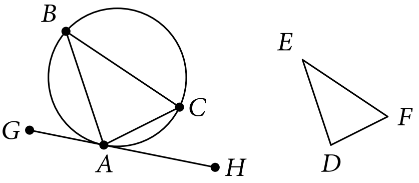
$$\begin{aligned}\int_{\mathbb{P}^1} &\omega(z,\overline{z})F(qz,\overline{qz})\,\mathrm{d}z\,\mathrm{d}\overline{z}\\ &=\sum \fint \omega z^m \overline{z}^n\,\mathrm{d}z\,\mathrm{d}\overline{z}\times\frac{q^m\overline{q}^n}{m!n!}F\\ &\quad+q\overline{q}\sum\frac{q^m\overline{q}^n}{m!n!}\,\partial^m\,\partial^n\,\omega\times\fint F w^m\overline{w}^n\,\mathrm{d}w\,\mathrm{d}\overline{w}\,.\end{aligned}$$

This is therefore an example of a professional mathematician choosing and simplifying an equation on an entirely aesthetic basis, *with no care for semantics*. In some sense, the choices here thus illustrate 'raw' aesthetic judgement of equations as purely syntactic forms.

Proofs. Beauty in proofs is possibly the most commonly recognized form of mathematical beauty. As a single example, Benno Artmann (1933–2010) considered beautiful[2] the proof of Proposition IV.2 in Euclid's (*fl.* c. 300 BCE) *Elements*:

PROPOSITION C. *'In a given circle to inscribe a triangular equiangular with a given triangle.'*[3]

Proof of C. Let $\mathcal{K}$ be the given circle and let $\triangle DEF$ be the given triangle.



Select any point A on $\mathcal{K}$ and draw the tangent GH to $\mathcal{K}$ at A. Construct the angle $\angle HAC$ equal to $\angle DEF$ and the angle $\angle GAB$ equal to $\angle DFE$. Then, by Proposition III.32 of the *Elements* the angle $\angle HAC = \angle DEF$ is equal to the angle $\angle ABC$ in the alternate segment of the circle. Similarly, the angle $\angle GAB = \angle DFE$ is equal to the angle $\angle ACB$. Therefore the remaining angle $\angle BAC$ is equal to the remaining angle $\angle EDF$. Therefore the triangle $\triangle ABC$ is equiangular to the triangle $\triangle DEF$. $\boxed{\text{C}}$

Proof steps. The combinatorialist Gian-Carlo Rota (1932–99) thought that individual steps in a proof can also be beautiful: 'The most common instance of beauty in mathematics is a brilliant step in an otherwise undistinguished proof.'[4] Rota gave no examples, but he may have been thinking of steps such as the following example from Archimedes' (c. 287–212 BCE) first book *On the Sphere*

[1] Frenkel, *Love and Math*, ch. 18, third image.

[2] Artmann, *Euclid*, pp. 219–21.

[3] Euclid, *Elements*, Prop. IV.2.

[4] Rota, 'Phenomenology of Mathematical Beauty', p. 172.

and the Cylinder. Archimedes announced his intention of proving three-dimensional results: that the surface of a sphere is four times its great circle, and that the volume of a sphere is two-thirds of a cylinder that exactly encloses it.[1] Yet he started by proving a series of two-dimensional theorems on circles and polygons.[2] Later, without warning, comes the key step: Archimedes imagined rotating polygons about an axis:[3] all the two-dimensional results acquire three-dimensional analogues and the desired results follow in short order.[4][*]

Conjectures. Beauty is seen in certain conjectures. For instance, Tao described as beautiful the conjecture that any set of positive integers A such that $\sum_{n\in A}(1/n) = \infty$ must contain arithmetic progressions of any finite length.[5] Petr Hliněný (1971–) considered beautiful Seiya Negami's (1957–) plane cover conjecture, which asserts that any graph G has a finite planar cover (a surjective homomorphism from a finite planar graph onto G that is bijective on vertex neighbourhoods) if and only if G can be embedded into the projective plane.[6]

A beautiful conjecture, if proved, may become a beautiful result (depending perhaps on any shadow cast by an ugly proof). But it is not clear whether the beauty of a conjecture is contingent on potential proof. If a conjecture is disproved, must it cease to be beautiful, or can it retain a wistful beauty? (*If only...*)

Questions. A mathematical problem or question can have aesthetic value. A question, posed by Esther Klein (1910–2005), that Béla Bollobás (1943–) found beautiful[7] is: *Does there exist, for each n, a number $N(n)$ such that any set of $N(n)$ points in the plane, no three collinear, contains n points that form the vertices of a convex polygon?*

For example, the minimum value for $N(4)$ is 5. To see that $N(4) > 4$, note that the configuration of 4 points in Figure 0.3(a) does not contain points that form the vertices of a convex quadrilateral. To see that $N(4) \leqslant 5$, note that with 5 points, there are two possibilities: first, if the convex hull has 4 or 5 vertices, any four vertices in the convex hull form a convex quadrilateral (Figure 0.3(b)); second, if the convex hull has 3 vertices, the two surrounded points and the two vertices on one side of the line through these points form a convex quadrilaterial (Figure 0.3(c)).

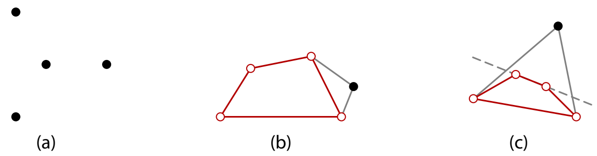


FIGURE 0.3
Proving that $N(4) = 5$ in Klein's question: (a) 4 points are not sufficient; (b, c) the two cases showing that 5 points suffice.

It is known that $N(n)$ always exists; the Erdős–Szekeres conjecture is that the minimum value for $N(n)$ is $2^{n-2} + 1$, though the best known bound far exceeds this value.[8]

[1] Archimedes, *Sphere and Cylinder*, bk 1, Introduction.

[2] *Ibid.*, bk 1, Props 2–6, 21–2.

[3] *Ibid.*, bk 1, Prop. 23.

[4] See Netz, 'The Aesthetics of Mathematics', pp. 256–7.

[*] ▸ pp. 107–8

[5] Tao, 'What is good mathematics?', p. 627.

[6] Hliněný, '20 Years of Negami's Planar Cover Conjecture', Abstract.

[7] Bollobás, 'Paul Erdős', p. 22.

[8] Erdős & Szekeres, 'A Combinatorial Problem in Geometry', p. 464.

Solutions. The motion of a vibrating string is described by what is today called the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Jean le Rond d'Alembert's (1717–83) solution to this partial differential equation was called 'very beautiful'[1] by Leonhard Euler (1707–83).

Objects. Individual mathematical objects may be beautiful. For instance, the *Klein quartic* or *Klein curve*, a compact Riemann surface of genus 3 that possesses a high degree of symmetry, and which can be thought of as a hyperbolic analogue of a regular solid,[*] has been described as

> 'a beautiful mathematical construction that has been studied by mathematicians for more than a century, from many points of view: geometry, symmetry, group theory, algebraic geometry, topology, number theory, complex analysis.'[2]

[Note that the term 'construction' was used here in the sense of a constructed object rather than a *process* of construction, which is discussed below.]

Another example might be sets, which Paul Benacerraf (1931–2025) said were 'beautiful (at least to some)'.[3]

Constructions. Michail Monastyrsky (1945–) remarked on the singularity of John Milnor's (1931–) work on topology in the following terms: 'It is very difficult to find an analogous invention in the past to his beautiful construction of the different differential structures on the seven-dimensional sphere.'[4] Monastyrsky seems not to have been assigning beauty to the differential structures as mathematical objects: he saw the beauty in the process, not in the result. Similarly, both Artmann[5] and Robin Hartshorne[6] (1938–) thought Euclid's construction of the regular pentagon[7] was beautiful.

Definitions. Rota thought that a definition could be beautiful:

> 'The axiomatization of the notion of category, carried out by Eilenberg and MacLane in the forties, is an example of beauty in a definition, though a controversial one.'[8]

Witness also the mathematician Lloyd Trefethen's (1955–) recollection of his first encounter with abstract algebra: 'The definition of a group was so beautiful!'[9]

Examples. An example may possess beauty, which may be linked to the beauty of the definitions or results that it exemplifies. However, not all examples of the same definition or result are equal:

> 'a *good* example is a thing of beauty. It shines and convinces. It gives insight and understanding. It provides the bedrock of belief.'[10]

[1] Euler, 'Sur la vibration des Cordes', § IV; trans. C.

[*] ▸ pp. 26–8

[2] Thurston, '*The Eightfold Way*', p. 1.

[3] Benacerraf & Putnam, 'Introduction', p. 31.

[4] Monastyrsky, 'Some trends', p. 4.

[5] Artmann, *Euclid*, p. 99.

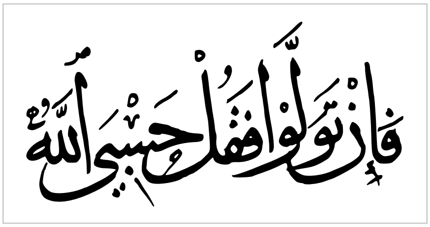
[6] Hartshorne, *Geometry*, p. 45.

[7] Euclid, *Elements*, Prop. IV.11.

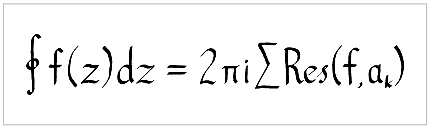
[8] Rota, 'Phenomenology of Mathematical Beauty', p. 173.

[9] Trefethen, *An Applied Mathematician's Apology*, p. 5.

[10] Atiyah, 'Advice to a Young Mathematician', p. 1007, emphasis added.

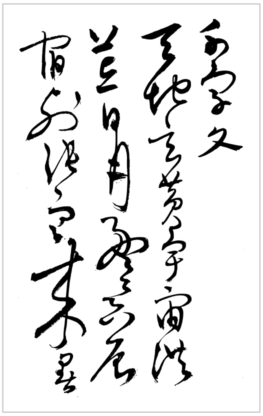

Public domain

Calligraphy from a 15th century CE Qur'an manuscript (9:129)



$$\oint f(z)\,dz = 2\pi i \sum \mathrm{Res}(f, a_k)$$

Cauchy's residue theorem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Calligraphy from a manuscript of the Thousand Word Classic, after Zhang Bi (1425–87)

FIGURE 0.4 Illustration of Bell's comparison of mathematical symbolism and Chinese and Arabic calligraphy.

Theories. A theory, in the sense of a body of definitions, concepts, and results dealing with some subject, can be thought of as beautiful. Gerhard Domanski said that '[t]he ordering of a whole field, like the work of Bourbaki ... is of great beauty';[1] for William Dunham (1947–), the theory of the Lebesgue integral is beautiful;[2] for József Beck (1952–), the theory of Nim-like games (in which a player unable to move loses the game) is beautiful.[3]

All the loci discussed above are purely mathematical, in the sense that they can in principle be detached from how they are communicated. But there are further potential loci of beauty that are more closely linked to the medium of communication.

Texts. A mathematical text might be beautiful independently of the mathematics it describes. However, such evaluation of a text is perhaps unfamiliar to the working mathematician, who is usually accustomed to think separately of the meaning of a piece of mathematics and of its form of expression. A philologist, by contrast, would place great emphasis on the particular textual form of the mathematics.[4] For this reason, a note of caution is in order: though mathematicians may write of 'a beautiful paper', this may actually be — or in all probability *is* — a judgement of the mathematics in the paper, not the text itself.

Symbols. The symbols of mathematics may possess beauty, independent of content. While a mathematician can admire both the symbols and the mathematical concepts expressed, J. L. Bell (1945–) thought that the non-mathematician could still find the symbols beautiful while being unaware of the concepts, just as someone can see beauty in Chinese or Arabic calligraphy while being unable to read the language (see Figure 0.4).[5] The astrophysicist Trịnh Xuân Thuận (1948–) said that he obtained 'a certain sense of abstract

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[1] Domanski, quot. in Wells, 'Are these the most beautiful?', p. 38.

[2] Dunham, *The Calculus Gallery*, p. 220.

[3] Beck, 'Games, Randomness and Algorithms', p. 311.

[4] Artmann, *Euclid*, p. 147.

[5] Bell, 'Reflections on Mathematics and Aesthetics', p. 172; cf. Leaman, *Islamic Aesthetics*, p. 70.

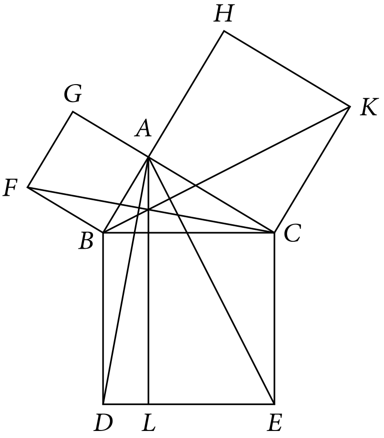

FIGURE 0.5
Diagram of the proof of the pythagorean theorem in Euclid, *Elements*, Prop. I.47.

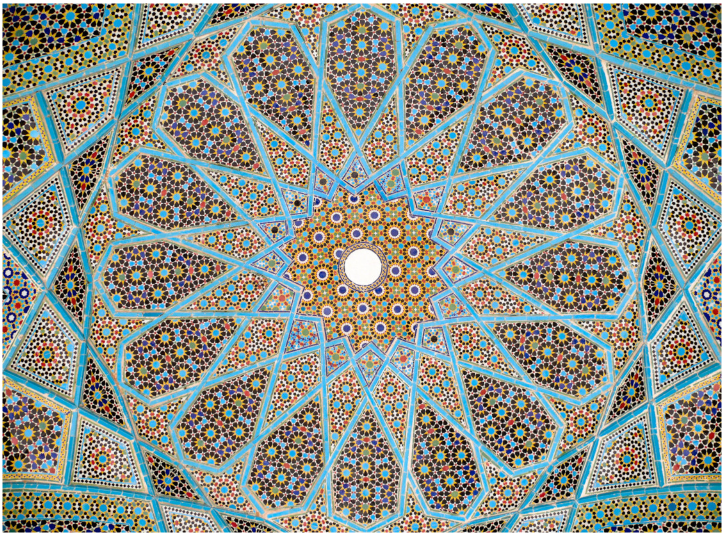
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FIGURE 0.6
Ceiling of the Tomb of Hafez in Shiraz.

beauty' from the visual form of equations that he understood, resembling his response to Chinese calligraphy.[1]

Diagrams. Mathematical diagrams can also be considered beautiful, and this beauty may be distinct, but not necessarily separate, from whatever beauty attaches to the mathematics they describe. The historian of mathematics Reviel Netz (1968–) observed that 'ancient diagrams possess considerable aesthetic value in their austere systems of interconnected, labelled lines'[2] (see, for instance, Figure 0.5).

Mathematical structure of artistic compositions. Certain works of art have a mathematical structure, and possibly there is some connection between the beauty of the mathematics and the beauty of the work of art. Examples include the tiling patterns found in Islamic art and architecture* (see Figure 0.6), the labyrinths found some

[1] Trinh, *Chaos and Harmony*, p. 7.

[2] Netz, *The Shaping of Deduction*, p. xvii.

* ▸ pp. 217–32

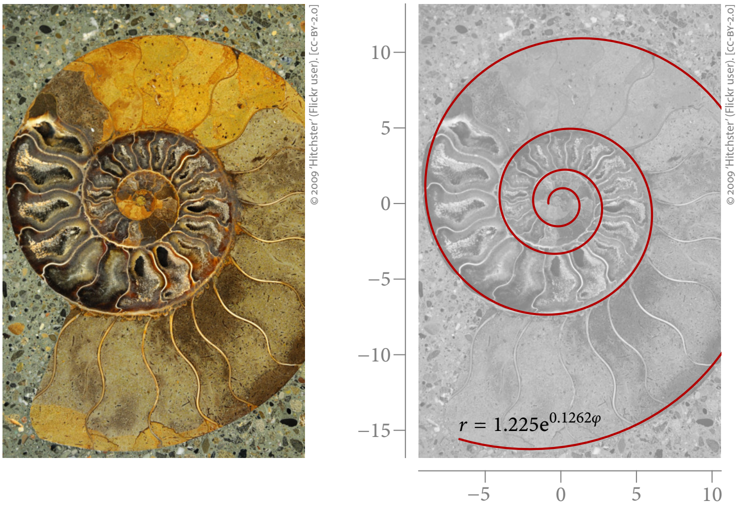

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FIGURE 0.7
A fossil nautilus and a logarithmic spiral that its shape approximates.

mediaeval cathedrals, and the symmetries and tilings in various works of M. C. Escher* (1898–1972).

Mathematical aspects of natural objects. The aesthetic role of mathematical structure in works of nature is perhaps similar to its role in works of art. The beauty of the shell of a nautilus may be connected to the logarithmic spiral that its shape approximates (see Figure 0.7): D'Arcy Wentworth Thompson (1860–1948) described the shell (among others) as 'exquisitely beautiful'; the spiral is 'simple and very beautiful'.[1]

AESTHETIC JUDGEMENTS

One might say of a sculpture that it is small, or is carved from wood, or depicts a certain mythical character, or is ugly. Of these four statements, the first three express non-aesthetic judgements and the last an aesthetic judgement. The first three somehow arise from a notion of what counts as small, from a perception of a wood grain or other evidence of the sculpture's material, from knowledge of the sculpture's origin and the relevant mythology. The last is an aesthetic judgement and its origin is at least not so easily described. Similarly, one might say of a piece of mathematics — a theorem and its proof, for instance — that it is correct, or concerns such-and-such a field, or proves so-and-so's conjecture, or is elegant. Again, the first three statements arise from one's ability to read and understand mathematics and evaluate its soundness, from one's background in mathematics, from one's knowledge of the current state of a field. Again, the last is an aesthetic judgement whose basis is not so easy to explain.

* ▸ pp. 698–700

[1] Thompson, *On Growth and Form*, pp. 748, 751; cf. Judson, *The Search for Solutions*, p. 37.

In mathematics, *beautiful* and *elegant* are probably the most common aesthetic descriptors. But one encounters a variety of apparently aesthetic terms in the literature: *awkward*, *charming*, *crude*, *cute*, *delicate*, *dull*, *lucid*, *messy*, *neat*, *polished*, *striking*, *sublime*, *tedious*, *ugly*. But what distinguishes such terms as aesthetic? Why are they different from non-aesthetic terms such as *brief*, *deep*, *ingenious*, *intuitive*, *long*, *rigorous*, *trivial*? Are terms like *careful*, *meticulous*, *simple*, or *technical* aesthetic or non-aesthetic?

Answering such questions is one of the fundamental problems of aesthetics. Although this book's main focus is beauty, the role of other aesthetic categories and judgements in mathematics cannot be ignored. For instance, in mathematics, elegance and beauty are often held to be closely related and are sometimes barely distinguished.[1] Or beauty is bound up with elegance (aesthetic), brevity (non-aesthetic), and simplicity (who can say?):

> 'Mathematicians have customarily regarded a proof as beautiful if it conformed to the classical ideals of brevity and simplicity. The most important determinant of a proof's perceived beauty is thus the degree to which it lends itself to being grasped in a single act of mental apprehension.'[2]

Beginning a discussion of aesthetics with judgement seems reasonable, since other aesthetic notions can be at least roughly understood in terms of their relationship to it: one makes an aesthetic judgement when an aesthetic experience occurs; in making such a judgement one perceives aesthetic values; one ascribes an aesthetic property to the object of attention or contemplation; one uses aesthetic concepts; one expresses all of this in aesthetic terms.[3] But among the fundamental problems of aesthetics are the questions of whether and how one can distinguish aesthetic and non-aesthetic judgements.

Immanuel Kant[*] (1724–1804) identified two criteria that distinguish judgements of beauty: subjectivity and universality.[4] Subjectivity means that one's judgement is based on an inner response. Universality means that one expects others to agree with one's judgement: hence one naturally says that '*it is* beautiful', as if the beauty is a property of the object. These criteria seem to characterize aesthetic judgements. Subjectivity is shared by aesthetic judgements and judgements of agreeableness (such as liking a particular food), but not by empirical judgements. Universality is shared by aesthetic judgements and empirical judgements, but not by judgements of agreeableness. These criteria are related to the different ways in which one would aim to persuade someone of the correctness of one's judgement. For an empirical judgement, one can present evidence and explain how the evidence supports the judgement. For an aesthetic judgement, one can present evidence, as an art critic might, in pointing out features of the object that may not have been noticed, or perhaps cause one's interlocutor to look upon it in a new way, but this falls short of explaining how these features support the judgement: one thinks that

[1] See, for example, Davis, *The Nature and Power of Mathematics*, pp. 187–8, 240.

[2] McAllister, 'Mathematical Beauty', p. 22.

[3] Zangwill, 'Aesthetic Judgment', § 4.1.

[*] ▸ pp. 374–82

[4] Kant, *Critique of the Power of Judgment*, KGS vol. 5, pp. 203–4, 211–16 [§§ 1, 6–8].

if only the other party had a fuller or clearer perception of the object, it would trigger the same inner response.

Using the Kantian criteria, judgements of elegance and neatness are (*prima facie*) aesthetic judgements, while judgements of rigour or triviality are not. Judgements of simplicity or meticulousness are harder to categorize.

But part of the historical debate has concerned whether the application of aesthetic terms such as *beauty* or *elegance* in mathematics actually reflects genuinely aesthetic judgement. Rota, for instance, argued that mathematicians apply the term *beauty* when they feel that a piece of mathematics is enlightening, thus re-interpreting mathematical beauty as mathematical enlightenment.[1]* Others, such as the philosopher Nick Zangwill (1957–), held that aesthetic properties can be possessed only by objects perceivable through the senses, and thus that mathematical objects cannot possess them, and thus that aesthetic terms are applied only metaphorically in mathematics.[2]†

Another fundamental question in aesthetics is whether an aesthetic value is (in some sense) ‘in’ the object to which it is attributed. An object is judged beautiful: is beauty a property of the object? If one accepts that objects have aesthetic properties, how do they correspond to the non-aesthetic ones? It seems reasonable that aesthetic properties do not ‘float free’: the aesthetic and non-aesthetic properties of an object are somehow bound. The usual view in philosophical aesthetics is that the aesthetic properties of an object *supervene* upon its non-aesthetic properties. The thesis of aesthetic supervenience is that two objects differ *aesthetically* only if they differ *non-aesthetically*. That is, it is impossible for two objects to be aesthetically different but non-aesthetically identical.[3] Applying the supervenience thesis to mathematics prompts questions of ontology in mathematics. Mathematical platonism holds that the objects of mathematics subsist independently of us. For the platonist, the relationship between their aesthetic and non-aesthetic properties may closely parallel the correspondence between the aesthetic and non-aesthetic properties of a physical object. But there is then the question of how one can perceive these properties, which is called the *problem of access* to the platonic objects.[4] Further, if one holds that aesthetic properties supervene on non-aesthetic properties of a mathematical proof, one is faced with the question of what it means for two proofs to be identical. Presumably trivial changes in notation maintain the identity of a proof, for in some sense the text is a representation of an abstract proof. But is a proof published in a research journal identical to its exposition at a slower pace (perhaps with examples) in a textbook? Thus the role of aesthetic judgement in mathematics is interwoven with broader philosophical questions about mathematics.

[1] Rota, ‘Phenomenology of Mathematical Beauty’.

* ▸ pp. 742–5

[2] Zangwill, *The Metaphysics of Beauty*, esp. pp. 140–3.

† ▸ pp. 749–54

[3] Levinson, ‘Aesthetic Supervenience’, p. 93.

[4] Brown, *Philosophy of Mathematics*, pp. 16 sqq.

AN ART OF MATHEMATICS

An art of mathematics

Mathematics has always been art in the sense of craft, practice, *technē* [τέχνη]. In some ways it has come to resemble an art in the sense of a humanity, a fine art, a pursuit of aesthetic ends:

> 'mathematics has developed so many humanistic aspects that for its devotees it may be said to play the role of a humanity. […] I am using the term "humanity" to indicate an aesthetic pursuit, usually employing the tools of art, literature, or music — in a nutshell, the use of symbols of some sort to achieve beauty, simplicity, harmony, or other types of what are generally considered to be aesthetically satisfying qualities.'[1]

But even if the practice of and motivation for mathematics suggest that it can be classed as an art-humanity, it may be maintained that its products do not qualify as art. For example, the artist and writer Robert Emmett Mueller (1925–2017) held that art consists in 'replaying' sensory experience in what he termed a 'mnemesthetic reaction' (referring to the memory of the sensory events, from Mnemosyne, the Greek goddess of memory). Thus he argued that, although aesthetic experiences can be found in mathematics or chess, neither field is art: they are aesthetic but not mnemesthetic.[2] Another way of drawing the distinction is to hold that beauty in mathematics is *natural* beauty, like that of a mountain or waterfall, so that the pursuit of mathematics is no more an art than exploration of a landscape is.[3]

Further, taste in art is at least to some extent culturally shaped. In one respect, taste in mathematics exhibits an extreme form of cultural determination, for whatever aesthetic value exists in (for example) a mathematical proof is *inaccessible* to those who have not undergone the cultural influence of mathematical training.[4]* While mathematical beauty can be in some degree accessible to the layperson through diagrams or other media, there is a boundary beyond which only the initiated can have taste. But in other respects cultural shaping of mathematical taste seems to have received scant attention.

The debate on whether mathematics is an art is clearly connected with the history of beauty in mathematics, which is almost paradoxical: by the twentieth century, it had become uncontroversial that 'beauty was no part of the concept of art, that beauty could be present or not, and something still be art'.[5] But there is nevertheless a relationship between the philosophy of art on one side and aesthetics *qua* study of aesthetic judgements, properties, and so forth on the other. In contrast, there seems to be only limited dialogue between aesthetics and the philosophy of mathematics. Establishing such a relationship might be worthwhile, but it requires care because it may be incorrect to apply to mathematics concepts in aesthetics that have developed with reference to art.[6] Further, such a relationship will only be possible if philosophy of mathematics continues its shift away from a focus on foundational matters.[7]

[1] Wilder, *Evolution of Mathematical Concepts*, p. 6.

[2] Mueller, 'Mnemesthetics', p. 191, n. 6.

[3] Rieger, 'The Beautiful Art of Mathematics', p. 246.

[4] Cf. Zeki, 'Neural Correlates of the Experience of Beauty', 32:20 sqq., 41:42 sqq.

* ▸ pp. 574–600

[5] Danto, *The Abuse of Beauty*, p. xv.

[6] Reiner, 'Mathematics and the Arts', p. 10.

[7] Cf. *loc. cit.*

THE PURSUIT OF BEAUTY

Thus closes this brief introductory chapter: the problem has been set out and the scope of its subject roughly sketched. Now begins the main body of the book: HISTORIA, THEORIA, and PHYSIKOS. All three parts, of a broadly historical character, examine the development of mathematical beauty: whether and in what way standards of beauty changed; the evolving role of beauty in the practice of mathematics; philosophical accounts of mathematical beauty; and influences of mathematical beauty beyond mathematics itself. HISTORIA deals with concerns that are almost entirely historical; THEORIA deals with matters that directly affect current attitudes towards the practice and evaluation of mathematics; PHYSIKOS treats of the role of mathematical beauty in modern natural science. The shorter last part, SYNTHESIS, serves as a conclusion and meditation on the history of mathematical beauty.

EX HIS CREATOREM
Preclarissimus liber elementorum Euclidis perspicacissimi: in artem Geometrie incipit quam foelicissime:
Punctus
Superficies plana
Circulus
Diameter
μούς· τίνας
ας καλῶ, εἰδήσεις
ου. Τὸ δὲ τρίτον πολλὰ
ρήματα χρήσιμα πρός τε τ
ερεῶν τόπων καὶ τοὺς διορισ
εῖστα καὶ κάλλιστα ξένα, ἃ
νείδομεν μὴ συντιθέμενον
τρεῖς καὶ τέσσαρας γραμμὰ
ριον τὸ τυχὸν αὐτοῦ καὶ τοῦ
ο ἦν δυνατὸν ἄνευ τῶν προσ
λειωθῆναι τὴν σύνθεσιν. τὸ

I

Historia

> ‘ ἱστορ-ία, […]
> *inquiry* […]; *systematic or scientific observation* […].
> 2. *knowledge so obtained* […].
> II. *written account of one’s inquiries, narrative, history* […]. ’
>
> — Henry Liddell & Robert Scott,
> *A Greek–English Lexicon*, ‘ἱστορ-ία’.

Faint glimmerings of an aesthetic appreciation of mathematics can be seen in the surviving mathematics of ancient Mesopotamia and Egypt, which are the temporal limits of mathematical history. Ethnomathematics indicates a preference for pattern and order that renders vague the intellectual boundaries of mathematics itself and of the kinds of aesthetic contemplation that could count as ‘mathematical’.

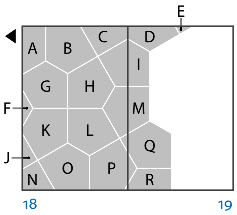
These are the starting-points for the investigation of the history of mathematical beauty, and thus the concern of the first chapter of this study. Subsequent chapters consider Classical Greek and Hellenistic times, later antiquity, the Christian and Islamic Middle Ages, the European Renaissance, the emergence of modern mathematics and of aesthetics as a philosophical discipline, and beyond; during the journey, mathematical, philosophical,

Partie Synth. Pl. 5.
Fig. 67.
Fig. 68.
Fig. 69.
Fig. 70.
Fig. 71.
Fig. 72.
Fig. 73.
Fig. 74.
Fig. 75.

scientific, political, literary, and artistic concerns will be seen to have been connected to beauty in mathematics.

Chronologically, this part reaches into the twentieth century. Thematically, it concerns matters that are purely historical: though they describe thought from which descends the present-day contemplation and analysis of beauty in mathematics, these are the chapters in the history of mathematical beauty that are *closed*. In the second part, THEORIA, are the chapters that are still being written. The third part, PHYSIKOS, concerns the modern interaction of mathematical beauty and physical science, and thus confronts the interaction of beauty and empirical adequacy.

> ‘The difference between *poiêtikê* and *historia*
> can thus be expressed precisely as follows:
> the latter is faithful to experience in its factuality’
>
> — Silvia Carli, ‘Poetry is More Philosophical than History’, p. 325.



A Detail from Paolo Barbotti, *Cicero discovering the tomb of Archimedes*. B William Blake, *The Ancient of Days*. C Construction of ‘C’ following Jaugeon’s specification in what became the ‘Romain du Roi’ (Jaugeon, ‘Description et perfection des arts et métiers’, § ‘Construction de la Lettre capitale droite c’). D Geometrical design from the Hall of the Bark in the Alhambra. E Rhind mathematical papyrus. F A honeycomb. G Diagram of the Copernican system by Johann Gabriel Doppelmayr. H Leonardo da Vinci, ‘Virtruvian Man’. I Identification of the five regular solids with the four Empedoclean elements and the cosmos, based on illustrations in Kepler, *The Harmony of the World*, bk II, § XX, p. 111. J Spiral ornament from the Book of Durrow (Allen, *Celtic Art in Pagan and Christian Times*, p. 169). K Illustration of the Ptolemaic system from Cellarius, *Harmonia Macrocosmica*, pl. between pp. 12, 13. L Opening page of the first printed edition of Euclid’s *Elements*, published in Venice by Erhard Ratdolt in 1482 (Euclid, *Preclarissimus liber elementorum*, fol. 2r). M Stefano della Bella, ‘Galileo and personifications of Astronomy, Perspective and Mathematics’, frontispiece for Galileo Galilei, *Opere* (1656), the first collected edition of his works. N Detail showing Euclid from Raphael, *School of Athens*. O Detail from the frontispiece of Johann I Bernoulli, *Opera Omnia*, vol. 1, showing Bernoulli pointing to a cycloid. P Passage from Apollonius’ *Conics* with the earliest description of theorems as ‘most beautiful’ (*Coniques*, vol. 1.2, p. 4, ll. 1–12; bk 1, preface). Q Babylonian mathematical tablet (Yale Babylonian Collection YPM BC 21354; formerly YBC 7289) showing an approximation of $\sqrt{2}$. R Crystallographic diagrams from Haüy, *Traité de Cristallographie*, Atlas, pl. 5.

F © 2014 ‘prilfish’. [CC-BY-2.0]. A, B, D, E, G–O, Q, R Public domain.

Beginnings and Endings

1

Aesthetics on the thresholds of mathematics

> ‘The great glory of mathematics is its durative nature; that it is one of humankind’s longest conversations’
>
> — Barry Mazur, *Imagining Numbers*, § 65.

> ‘Man wondered at the heavens above him;
> he wondered at life
> and he wondered even more at death;
> all was a mystery. […]
> The mystery of form and the mystery of number
> he connected with the mystery of the universe about him’
>
> — David Eugene Smith, *History of Mathematics*, vol. i, p. 16.

When did mathematics begin? In one sense, this question can be answered by listing the times and places in which mathematics, as a set of activities dealing with quantity and space — and recognizable by analogy to modern mathematics — emerged without historical precursors: in the river valley civilizations of the Nile, the Tigris and Euphrates, the Indus, the Yellow River, and also in Mesoamerica. But in another sense, it depends on the answer to another question: Where does mathematics *end*? That is, where are the borders of mathematics? What topics of study and what forms of activity count as mathematics? Even if one considers only mathematics that is taught and researched today in academic institutions of various levels, formulating a definition or description can be difficult: standard dictionaries vary widely in the emphasis they place on the scope, structure, subject-matter, and standards of mathematics.[1]

The term *mathematics* derives from the Greek word *mathēma* [μάθημα], meaning *that which is learnt*, *lesson*, *learning*, *knowledge*, and only in particular to what today would be called *the mathematical sciences*.[2] In ancient Greece, the narrow sense would have included arithmetic, geometry, and astronomy, and, depending on the author, musical theory. Greek mathematics owes *something* to Mesopotamia and to Egypt (though there is no Sumerian or Akkadian equivalent of the word ‘mathematics’[3]). The flow of mathematics from the other main well-springs ultimately became — for reasons more political than intellectual — tributary to the main stream that descends from *mathēma* to today’s mathematics.

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[1] Hacking, *Why Is There Philosophy of Mathematics at All?*, § 2.3 sqq.; cf. Thurston, ‘On proof and progress in mathematics’, p. 162.

[2] LSJ, ‘μάθ-ημα’.

[3] Robson, *Mathematics in Ancient Iraq*, p. 289.

In its course of its descent, the discipline has gained some fields and lost others and altered its notions of proof and rigour. Thus to project the modern conception of mathematics onto the past can entail the exclusion of a wide range of intellectual endeavour. As the mathematician and historian of ancient science Otto Neugebauer (1899–1990) put it, the 'mathematical' tools required to manage the affairs of state and commerce in antiquity are what '*no mathematician* would call mathematics'.[1] That is to say, a professional mathematician today, equipped with a system of numeration and calculation that has had millennia further to develop, might have a tendency to dismiss the systems of Egyptian or Mesopotamian antiquity as primitive, uninteresting, *trivial*.

To use a distinction introduced by the historian of mathematics Jens Høyrup (1943–), the main stream of mathematics is *scientific*, but the mathematics that was in use in ancient Mesopotamia and Egypt was *sub-scientific*:

> '*Scientific knowledge* is knowledge which is pursued *systematically* and for its own sake (or at least without any intentions of application) *beyond the level of everyday knowledge*. [...]
>
> *Sub-scientific knowledge*, on the other hand, is *specialists' knowledge* which (at least as a corpus) is acquired and transmitted *in view of its applicability*.'[2]

This scientific/sub-scientific division is distinct from the modern pure/applied division. Roughly, in the modern division, *pure* refers to the pursuit of fundamental knowledge and *applied* to its remoulding and synthesis for practical ends: different levels or aspects of the same research. 'Scientific' and 'sub-scientific' refer to autonomous *types* of mathematics.[3]

The adjective *sub-scientific* should not be interpreted as a judgement of the value or sophistication of the mathematics. The value of sub-scientific mathematics is evident in the important role it played in society, and its practitioners developed it to a high degree of intricacy. Nor does applying the term entail denying aesthetic appreciation. But as a matter of historiographical fact, it is difficult to find direct evidence of a role for aesthetic appreciation in sub-scientific mathematics. Roughly speaking, all the surviving aesthetic discussions of mathematics are either by mathematicians who pursued scientific mathematics or by philosophers who examined scientific mathematics.

But there is indirect evidence of a role for aesthetic appreciation, if one draws on the insights of *ethnomathematics*.[4] The field of ethnomathematics arose from another effect of delineating mathematics using the modern conception: exclusion of other activities and ideas, particularly in small-scale or traditional cultures, that never joined the main mathematical stream but which are nevertheless mathematical.[5] Ethnomathematics encompasses the study of these otherwise-excluded arts, although the term's range of applicability is debated, both in term of cultures and of activities and knowledge. In particular,

[1] Neugebauer, *The Exact Sciences in Antiquity*, p. 72, emphasis added.

[2] Høyrup, 'Sub-scientific Mathematics', p. 64, emphases in orig.

[3] *Ibid.*, pp. 64–5.

[4] Ascher, *Ethnomathematics*; Powell & Frankenstein, *Ethnomathematics*; Eglash, 'Anthropological Perspectives'.

[5] Ascher, *Mathematics Elsewhere*, pp. 1–3.

it was originally applied specifically to non-literate cultures,[1] but the ethnomathematical approach, situating the practice of mathematics in the broader culture, can 'identify the external constraints on mathematical thinking, or […] reveal the conceptualisation of number, space, shape, symmetry'.[2] The term encompasses activities that employ or develop concepts that can be classed as geometrical, combinatorial, numerical, algebraical, or logical, but which are not located within a formal structure of mathematical study, but rather within areas that might be categorized as 'navigation, calendrics, divination, religion, social relations, or decoration'.[3] To take a single field of endeavour, techniques of divination such as described in the Chinese *I Ching* [易經 *yìjīng*][4] or practised in the Malagasy *sikidy*[5] involve combinatorial selections and manipulations.

This chapter considers the role of aesthetic appreciation in early mathematics and in ethnomathematics; that is, at the beginnings and the endings of mathematics. However, most of what can be said is either tentatively speculative or negative, in the senses of emphasizing what is *not* known and what is known *not* to be the case. The earliest explicit mentions of beauty in mathematics — and the earliest philosophical engagement with it — are from the ancient Greek world, and will be the focus of the next chapter.

ART, GEOMETRY, SYMMETRY, RITUAL

There is *something* mathematical in the geometric patterns found engraved in 70 000-year old pieces of ochre in the Blombos Cave in South Africa.[6] Whatever reason the human who carved these pieces had for their design, he or she must surely have recognized and extended the repeating pattern. One cannot say whether this early engraver found the pattern pleasing or whether there was some aesthetic end to the work. And yet in them one sees the first glimmerings of the investigation of the nature of space.

There are many other practices that indicate an informal exploration of space, driven at least partly by an aesthetic concern, and based on unarticulated but often sophisticated mathematics. The most obvious example is how the decorative arts of many cultures have thoroughly explored the symmetries of the plane. A few examples will suffice here.

Frieze patterns in cultures such as ancient Egypt or the Inca Empire are evidence of a non-abstract exploration of symmetries. [For this reason, the visual art of ancient Egypt has been said — albeit with hyperbole — to be the 'prehistory of group theory'.[7]] Each of the seven possible frieze symmetries (see Figure 1.1) can be found among, for example, Roman mosaic or Hungarian needlework patterns.[8] That is, artists lacking any formal mathematical notion of 'symmetry'

[1] Ascher & Ascher, 'Ethnomathematics', p. 26.

[2] Robson, *Mathematics in Ancient Iraq*, p. 25.

[3] Ascher, *Mathematics Elsewhere*, p. 3.

[4] Needham, *Science and Civilisation in China*, vol. 2, § 13(g).

[5] Ascher, *Mathematics Elsewhere*, §§ 1.5–7.

[6] Henshilwood et al., 'Emergence of Modern Human Behavior'.

[7] Speiser, *Die Theoerie der Gruppen*, p. 1.

[8] See, respectively, Radaelli, *Symmetry in Crystallography*, Figs 2.4–8; Hargittai & Hargittai, *Symmetry through the Eyes of a Chemist*, p. 377.

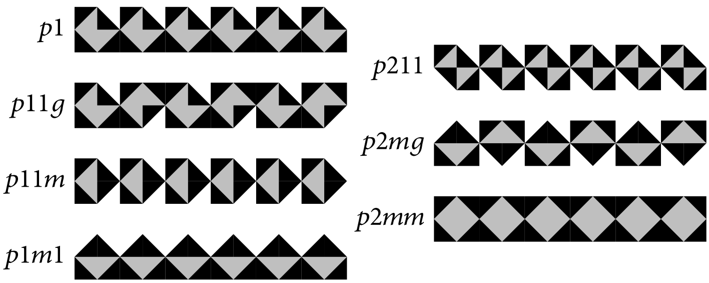


FIGURE 1.1
Patterns exhibiting the seven possible frieze symmetries.

nevertheless found through experiment all possible symmetries of friezes.

Symmetry is evident in many of the Japanese family emblems known as *mon* [紋] or *kamon* [家紋]. Most designs fit into a roundel, and among the countless designs one can find examples exhibiting both cyclic symmetry (that is, rotational but no reflectional symmetry) and dihedral symmetry (rotational and reflectional symmetry). It is straightforward to find examples whose symmetry groups are the cyclic groups C_n and the dihedral groups D_n for $n = 2, \ldots, 9$ (see Figure 1.2). None of this is to suggest that the designers of these *kamon* were consciously seeking such examples, but they must have perceived the symmetries and had probably at least some notion that cyclic and dihedral symmetries, and simple reflectional symmetry are the possible symmetries for a shape that fits within a roundel.

It is often noted that humans have a preference for symmetry.[1] There are cultures where deliberate asymmetries are introduced into otherwise symmetric designs. But this practice is not because the symmetry itself is displeasing, but because of some other factor. For example, in traditional Japanese art and architecture, precise symmetry and uniformity was seen as clashing with the asymmetry of nature[2] and destroying ‘freshness of imagination’.[3] In Navajo weaving, a pathway [*ch’ihónít’i*] leading from the inner design to the edge prevented aspects of the weaver’s self from remaining entangled in the textile.[4]

But there is more to the exploration of space than symmetry. In southern India and Sri Lanka there exists a tradition in which women create intricate designs on the threshold of the home by carefully trickling rice flour onto the floor. The figures, known as *kolam* [கோலம்; कोलम], are tied to ritual and philosophy, but even to an untrained observer, they are intriguing for their complexity and beauty.[5] Women learn how to create the *kolam* from their mothers, and attaining this skill forms an important part of their education since it indicates grace, dexterity, mental discipline, inner harmony. The term *kolam* in Tamil refers to more than just these figures: the meanings include ‘*beauty*, *gracefulness*, *form*, *shape*, and *appropriate dress*’,[6] which points to their intertwining with aesthetic and moral concerns. There are many different designs, some quotidian, some for particular occasions or rituals. Most *kolam* have at least reflectional symmetry; others have

[1] See, for example, Eisenman, ‘Complexity-simplicity I’; Rhodes et al., ‘Facial symmetry’; Osborne, ‘Symmetry as an aesthetic factor’.

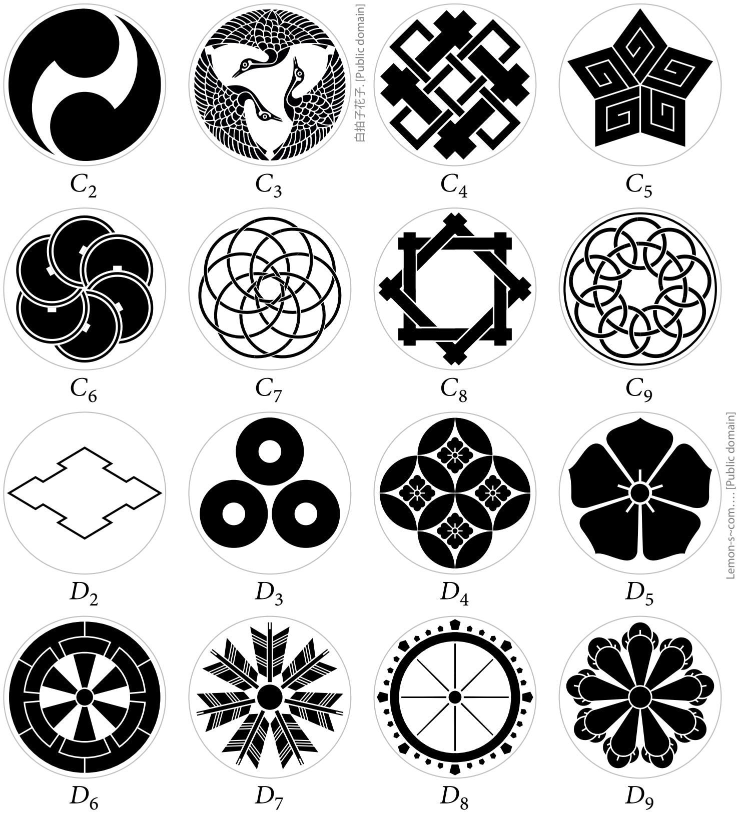
[2] Ramberg, ‘Some Aspects of Japanese Architecture’, esp. pp. 40–1.

[3] Okakura, *The Book of Tea*, p. 96.

[4] Ahlberg Yohe, ‘Situated Flow’, esp. pp. 26–7.

[5] Ascher, *Mathematics Elsewhere*, ch. 6; Siromoney, ‘South Indian *kolam* patterns’.

[6] Ascher, *Mathematics Elsewhere*, p. 163.



白拍子花子. [Public domain]

Lemon-s~com.... [Public domain]

FIGURE 1.2
Kamon with cyclic and dihedral symmetries C_n and D_n for $2 \leqslant n \leqslant 9$.

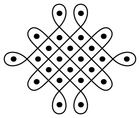


FIGURE 1.3
The *Ring*, a *kolam*.

dihedral symmetry, such as the one known as the *Ring* (see Figure 1.3). Others can be seen as being created through the transformation and combination, often iteratively, of certain subunits;[1] this perspective has attracted the attention of computer scientists who have shown how to place such *kolam* into a framework of generative grammars.[2] For instance, each member of the family of *kolam* known as *Anklets of Krishna* (see Figure 1.4) is made up of a single continuous curve surrounding dots known as *pulli*. The curve can be drawn by following a sequence of 'moves' such as a straight segment, a curve to the right or left, or a loop. The next member of the family is formed by replacing all 'moves' of a certain type with a sequence of moves. The next member is formed by repeating this replacement, and so on.

Tracing elaborate geometrical paths, often highly symmetrical, using a single continuous curve is an activity found in other cultures too. These activities should not be collapsed together, for their cultural and social contexts vary from children's games to aspects of storytelling, and the styles of the paths vary widely.[3]

The mathematician or computer scientist who studies *kolam* or other path-tracing arts, or the symmetries of artworks such as *kamon* or friezes is not necessarily replicating or even making an analogue of the techniques or thought processes used by those who created

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[1] Ascher, 'The Kolam Tradition'.

[2] Siromoney, Siromoney & Krithivasan, 'Array Grammars and Kolam'.

[3] Ascher, *Ethnomathematics*, ch. 2.

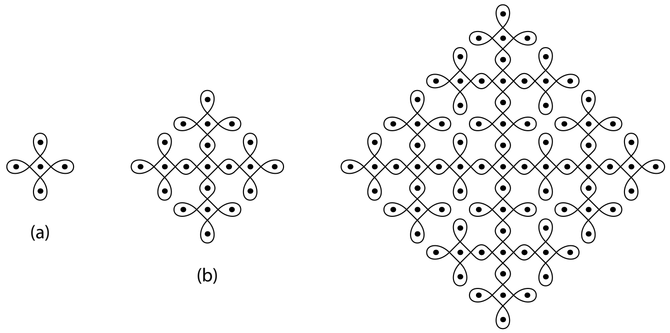
(a)
(b)
(c)

FIGURE 1.4
The *Anklets of Krishna*, a family of *kolam*.

and still create them.[1] A perspective from within ‘core’ mathematics can yield increased understanding of these activities, but it cannot be assumed to translate into how the practitioners conceived of what they did.[2] Nevertheless, there is an indication that there is no fundamental incompatibility between the exploration of space outside of formal mathematics, at least partly guided by aesthetic concerns, and the systematic concepts that are developed within the main stream of mathematics, at least partly guided by the aesthetic goals of *its* practitioners.

REGULAR SOLIDS

> ‘ certainly one of the most beautiful and singular discoveries made in the whole history of mathematics. ’
>
> — Hermann Weyl, *Symmetry*, p. 74.

A polygon is regular if all its sides are the same length and all its angles are equal. A polyhedron is *convex* if it contains every line joining two points inside it. It is *regular* if its faces are equal (or, to be mathematically precise, congruent) regular polygons and the configurations of faces around each vertex are all the same. For a polyhedron with equal regular faces, the following three conditions are equivalent: i) the polyhedron is regular; ii) for each pair of vertices, there is a symmetry of the polyhedron carrying one vertex to the other; iii) the polyhedron is inscribable in a sphere, in the sense that there is a sphere on which lie all of its vertices.

There are exactly five convex regular polyhedra, known collectively as the platonic solids because of their role in Plato’s *Timaeus*,[*] though the term *regular solids* will be preferred here. The five regular solids are: the tetrahedron, or (triangular-based) pyramid; the cube; the octahedron; the dodecahedron; and the icosahedron (see Figure 1.5). This classification is one of the landmark achievements of the pre-Euclidean period of Greek mathematics.[†]

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[1] Ascher, *Mathematics Elsewhere*, p. 168.

[2] Ascher, *Ethnomathematics*, p. 54.

[*] ▸ pp. 67–70

[†] ▸ pp. 68–70

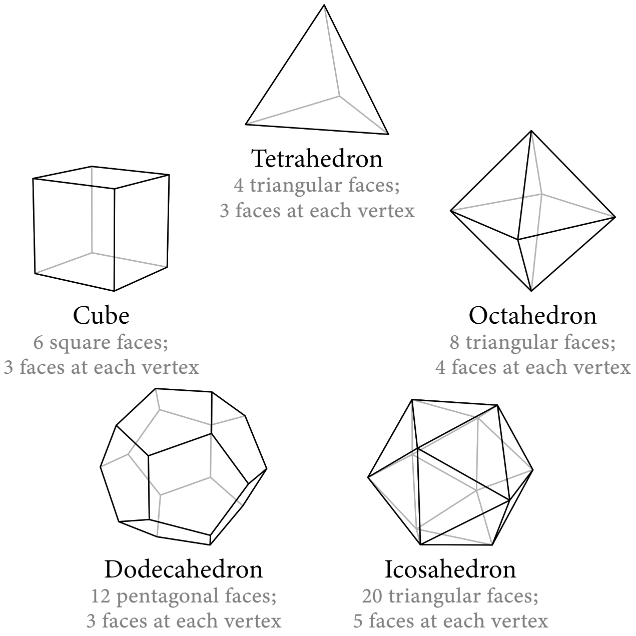
Tetrahedron
4 triangular faces;
3 faces at each vertex
Cube
6 square faces;
3 faces at each vertex
Octahedron
8 triangular faces;
4 faces at each vertex
Dodecahedron
12 pentagonal faces;
3 faces at each vertex
Icosahedron
20 triangular faces;
5 faces at each vertex

FIGURE 1.5
The five regular solids.

Some have argued for an even earlier knowledge of the regular solids. A particular claim, which seems to have appeared first in 1979, in Keith Critchlow's (1933–2020) book *Time Stands Still*,[1] is that all five of the regular solids are found among Neolithic 'carved stone balls' found in Scotland.[2] These stone balls have protrusions or knobs that can number up to 160; an example is shown in Figure 1.6.[3] On some of these stone balls, the arrangement of the knobs is supposed to mirror either the faces or the vertices of the five regular solids. This face/vertex ambiguity is a feature of an oft-reproduced photograph from *Time Stands Still* that illustrates the claim.[4] In this photograph, ribbons are wrapped around the five stones to illustrate the edges of the polyhedra. These ribbons influence the viewer towards perceiving the knobs as vertices or as faces — that is, towards seeing a particular polyhedron or its dual. A critical inspection of the photograph shows that the apparently dodecahedral and icosahedral stones actually appear to have the same configuration of twelve knobs. If knobs correspond to vertices, both are supposed to be icosahedra. A dodecahedron would have twenty knobs, and although two carved stone balls with twenty knobs have been found, neither is close to having a dodecahedral configuration.[5] [None of this is to dismiss the stone balls as beautiful works of art in their own right.]

The cube is such a basic shape, and so commonly found in nature and artifice, that it must have been recognized and named very early in any culture's history. Likewise the tetrahedron can be assigned an early date. An Old Babylonian (early 2nd millennium BCE) mathematical text concerning twenty equilateral triangles can be interpreted

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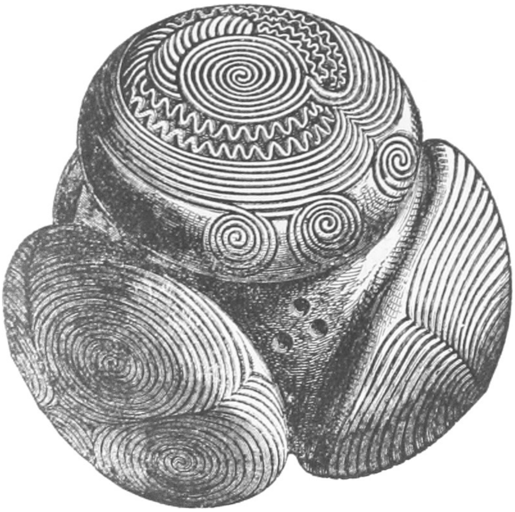
[1] See Critchlow, *Time Stands Still*, ch. 7.

[2] For the history of this claim, see Lloyd, 'How old are the Platonic Solids?', pp. 131–2.

[3] Marshall, 'Carved stone balls'.

[4] See Lloyd, 'How old are the Platonic Solids?', Fig. 1.

[5] *Ibid.*, p. 136.



Public domain

FIGURE 1.6
A Neolithic carved stone ball with exceptionally intricate carvings, found in Towie, Aberdeenshire, Scotland (National Museum of Scotland X.AS 10; Evans, *Ancient Stone Implements*, p. 421; Marshall, 'Carved stone balls', Fig. 1(4)).

as dealing with the figure formed by unfolding the icosahedron.[1] It is difficult to imagine the icosahedron being found before the octahedron, so presumably it was known by the Old Babylonians too. However, there is no evidence for the *concept* of a regular solid before the Greeks. Even if the Old Babylonians also knew the dodecahedron, there is nothing to suggest they would have grouped together the five regular solids in a category of their own, excluding, for instance, solids all of whose faces are congruent regular polygons, but where there are different configurations of faces at different vertices. (For an example of such a solid, consider the triangular dipyramid, which has six equilateral triangular faces, with three faces meeting at some vertices and four faces meet at others; see Figure 1.7.)

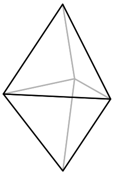


FIGURE 1.7
The triangular dipyramid.

EGYPTIAN MATHEMATICS

According to Herodotus (d. c. 425 BCE), the Greeks, who certainly did appreciate mathematics aesthetically, received geometry from Egypt, where it had been developed for measuring land.[2] Aristotle (384–322 BCE) said that

> 'the sciences which do not aim at giving pleasure or at the necessities of life were discovered, and first in the places where men first began to have leisure. This is why the mathematical arts were founded in Egypt; for there the priestly caste was allowed to be at leisure.'[3]

Unfortunately for Aristotle's claim, which implies that mathematics would have been pursued for its own sake, there was no separate 'priestly caste' in Egypt until after the Middle Kingdom (c. 2055–

[1] Friberg, *A Remarkable Collection*, § 11.3.

[2] Herodotus, *The Histories*, § II.109.

[3] Aristotle, *Metaphysics*, BEK A.1.981b21–3.

c. 1650 BCE).[1] Although no mathematical texts survive from the Old Kingdom (c. 2686–c. 2160 BCE), there is indirect evidence that there was a sophisticated system of numeration in place, which was necessary for the proper administration of the state.[2] Furthermore, the mathematics in those texts which survive from the Middle Kingdom period is generally applied to practical problems.[3] Aristotle's mistake was to project onto Egypt his own intellectual environment of fourth-century Athens, in which mathematics was a part of a liberal education, not a practical discipline.[4] The account of Herodotus is nearer the mark, at least insofar as it describes a practical origin for mathematics.[5]

Yet the assessments of scholars vary about whether mathematics in Egypt was pursued for its own sake. The mathematician and historian of mathematics C. S. Roero (1952–) noted two diametrically opposed views of Egyptian mathematics.[6] Arnold Buffum Chace (1845–1932), who was trained as a mathematician and continued to do mathematics when he left academia to manage his family's business, maintained that:

> 'the Rhind Papyrus [...] is not a mere selection of practical problems especially useful to determine land values [...]. Rather I believe that they studied mathematics and other subjects for their own sakes [...]. [T]he Rhind Papyrus, while very useful to the Egyptian, was also "an example of the cultivation of mathematics as a pure science, even in its first beginnings."'[7]

Gerald Toomer (1934–), a historian of mathematics and astronomy, took a contrary view:

> 'The Rhind and Moscow Papyri are handbooks for the scribe, giving model examples of how to do things which were a part of his everyday tasks. [...] The truth is that Egyptian mathematics remained at much too low a level to be able to contribute anything of value. The sheer difficulties of calculation with such a crude numeral system and primitive methods effectively prevented any advance or interest in developing the science for its own sake. It served the needs of everyday life [...] and that was enough.'[8]

It is important to remember, however, that three millennia separate the earliest number-representations in the pre-dynastic and early dynastic periods from the incorporation of Egypt into the Hellenistic world when it was conquered by Alexander in 332 BCE. From these ages, very few mathematical texts have survived: one leather roll, two wooden boards, two major papyri, and a scattering of papyrus fragments and ostraca. Most of these texts are from the Middle Kingdom period. From such brief glimpses of Egyptian mathematics, any reconstruction of its full scope can only be partial.[9]

Literacy and numeracy were necessary to work as a government official, and, starting in the Old Kingdom period, high-ranking offi-

[1] All Egyptian dating is from Shaw, *The Oxford History of Ancient Egypt*.

[2] Imhausen, *Mathematics in Ancient Egypt*, p. 35.

[3] van der Waerden, *Science Awakening*, p. 16.

[4] Guthrie, *Earlier Presocratics*, p. 35.

[5] Cf. Burnet, *Early Greek Philosophy*, p. 19.

[6] Roero, 'Egyptian mathematics', p. 43.

[7] Chace & Manning, *Rhind Mathematical Papyrus,* vol. 1, pp. 42–3.

[8] Toomer, 'Mathematics and Astronomy', pp. 37, 45.

[9] Imhausen, *Mathematics in Ancient Egypt*, pp. 63–9, 131–2.

cials had themselves depicted in art as reading or writing.[1] Taking this practice together with the apparent fact that numeracy was an essential part of the scribe's work from the earliest pre-dynastic (c. 3200–c. 2686 BCE) times, and with the placement of mathematical objects in tombs in the later New Kingdom (c. 1550–1069 BCE) period,[2] it is not much of a stretch to suggest that a high social value was placed on mathematics throughout Egyptian history.

There is no positive evidence that any part of the development of Egyptian mathematics was driven by aesthetic appreciation. In particular, despite modern claims to the contrary, the extreme and mean ratio or 'golden section' was neither known nor incorporated into architecture.[*] But parts of Egyptian mathematics were praised for their aesthetic value in later times. For instance, a certain problem-solving method in the Rhind Papyrus was considered elegant by the historian of mathematics R. J. Gillings.[3] The method of 'unit fractions' found admirers in the nineteenth century.[†] And there is a hint — nothing more — of a contemporary aesthetic choice in this latter method.

The 'unit fraction' system denoted the n-th part of the unit by placing a special symbol above the usual symbols denoting the number n. Neugebauer introduced the modern notation $\overline{n}$, which is not unlike the original Egyptian one. Since $\overline{n}$ denotes the fraction $1/n$, it is customary to call the Egyptian system 'unit fractions', but it is important to note that there was no concept of a 'non-unit fraction', although there was a special symbol for $2/3$. A quantity that today could be written as $3/8$ would have been denoted $\overline{4}\ \overline{8}$, since $3/8 = 1/4 + 1/8$.

There can be many different ways of expressing the same quantity using unit fractions. For example,

$$\frac{8}{13} = \overline{2}\ \overline{10}\ \overline{65} = \overline{2}\ \overline{12}\ \overline{52}\ \overline{78}.$$

There seems to have been a rule against repeated terms in unit fraction representations, which meant, in particular, that the result of $2 \div n$ could not be expressed as $\overline{n}\ \overline{n}$. There are two extant tables listing the results of $2 \div n$ for various n, which would have been an aid to reckoning used in standard computations. In the earlier, from one of the Lahun Mathematical Fragments,[4] n ranges through the odd numbers up to 21; in the later, from the Rhind Papyrus,[5] it ranges through the odd numbers up to 101. Where they overlap, the two tables give identical representations, suggesting that there was some standard: this raises the question of how such a standard was formulated.

Gillings proposed some precepts governing the choice of representation.[6] Of the possible terms, it seems that those with a smaller denominator (to use an anachronistic term) were preferred; they were set down in increasing order of denominator; fewer terms were preferred; a smaller first denominator was preferred, but a larger one was accepted if the last denominator was consequently much smaller; even denominators were preferred to odd denominators, regardless

[1] Imhausen, *Mathematics in Ancient Egypt*, p. 31.

[2] *Ibid.*, pp. 27, 165–6, 169.

[*] ▸ pp. 556–7

[3] Gillings, *Mathematics in the Time of the Pharaohs*, p. 136.

[†] ▸ p. 439

[4] University College, London (UC) 32159; see Imhausen, 'Egyptian Mathematics', pp. 18–19.

[5] *Ibid.*, pp. 19–20.

[6] Gillings, *Mathematics in the Time of the Pharaohs*, pp. 49–80.

of any increase in the denominators or the number of terms. There are many possible objections to Gillings's precepts, both in terms of their accuracy and in terms of his reconstruction of how the search for each representation would have proceeded.[1] And though Gillings' precepts and later refinements may *explain* certain choices, they are not fine-grained enough to *predict* them. Perhaps the search for such precepts is in fact futile: 'it is not so surprising that [...] we should be able to make up, post facto, some standards by which we come to the conclusion that the scribe chose the simplest and best possible'.[2]

But the very difficulty of setting up a system of conditions or rules that explains and predicts the representation of $2 \div n$ perhaps hints that there was some kind of aesthetic judgement involved, for aesthetic concepts are often thought of as not governed by rules or conditions.[3] It is unjustified to project modern aesthetic concepts back into ancient Egypt, but it is at least conceivable that something akin to aesthetic choice explains the selected representations.

MESOPOTAMIAN MATHEMATICS

Starting around the middle of the fourth millennium BCE, a sophisticated body and practice of mathematics developed in the polities that existed in Mesopotamia. Like Egyptian mathematics, it was connected to the administration of the state,[4] and was sub-scientific in Høyrup's sense.* While their Egypt counterparts wrote on papyrus, Mesopotamian scribes used cuneiform script impressed into clay tablets, which are more likely to survive the ages. Consequently, the corpus of extant mathematical texts is vastly greater for Mesopotamia.[5]

Non-utilitarian mathematics

Sub-scientific mathematics can be divided into the mathematics that was actually applied and into non-applied mathematics. In the case of Mesopotamia, little of the former mathematics survives, though the resulting accounts and tax documents and inventories do. The latter, non-applied, mathematics ranged from training problems to more purely 'recreational' mathematics, and used the same methods as the applied mathematics but with a deviation from reality such as the use of exaggerated magnitudes.[6] This non-applied, non-utilitarian, kind of mathematics was fundamentally determined by *methods*, for which new problems were created to illustrate its power and the skill of those who could apply it. [In contrast, Greek mathematics was motivated by the problem, which drove the development of new methods.]

[1] See the debate in Bruckheimer & Salomon, 'Some Comments'; Gillings, 'Response to "Some Comments [...]"'

[2] Bruckheimer & Salomon, 'Some Comments', p. 446.

[3] See, for example Sibley, 'Aesthetic Concepts'.

[4] Høyrup, 'Mesopotamian Mathematics', p. 58.

* ▸ p. 22

[5] Robson, *Mathematics in Ancient Iraq*, Tbl. B.22.

[6] Høyrup, 'Sub-scientific Mathematics', pp. 66 sqq.

In the Early Dynastic period (c. 3000–c. 2350 BCE) lie the faintly discernible beginnings of non-utilitarian mathematics. Rather than sticking to those calculations that were paedagogically or practically necessary, mathematically-trained scribes systematically investigated all possibilities. An example is a problem involving an allocation of 7 *sila* of grain to each worker. In reality, workers always received a multiple of 5 *sila*: the intent of the problem probably revolves around the necessary division by 7, which in the sexagesimal number system of the time is much harder than division by 1, 2, 3, 4, 5, or 6.[1]

In the later Old Babylonian period (c. 1850–c. 1600 BCE) the study of non-utilitarian mathematical problems becomes more evident.[2] Although these explorations did bring out deeper insights, it does not follow that there was an enjoyment of mathematics or even an interest in mathematics for its own sake. Old Babylonian non-utilitarian mathematics remained sub-scientific. It was never completely abstracted away from applications to the world. It was always concerned with real-world things, not mathematical entities themselves. But although the *things* it concerned were real-world, non-utilitarian mathematics dealt with problems and tasks that were unreal. To take a single example, one collection of problems dealing with quantity surveying includes questions where only sums or differences of parameters are given, while real calculations would use the actual measured parameters.[3] In Høyrup's phrase, such problems are '"pure" *in substance*, […] *applied in form*'.[4] What then was their purpose? Non-utilitarian mathematics served to demonstrate the ingenuity and virtuosity of the mathematically-trained scribe: challenges would be issued and solved or not.[5] Thus, this development of mathematics was driven by a desire to obtain or maintain social status. However, the need to set challenging problems that could nevertheless be solved would have driven the development of mathematical insight as a byproduct, regardless of any intrinsic curiosity.[6]

But this motivation was not a constant feature as the polities of Mesopotamia rose and fell. Even the permissibility of non-utilitarian mathematics may have been at times restricted. Robert Englund argued that the Third Dynasty of Ur (c. 2113–c. 2004 BCE; abbreviated to Ur III) ruled a bureaucratic, heavily centralized and controlled, slave society.[7] Building on Englund's view, Høyrup argued that Ur III mathematics teaching reflected this control: its intent was 'deliberately eliminating even the slightest appeal to independent thought on the part of the students'.[8] There appears not to have been any culture of non-utilitarian challenge problems such as existed in the later Old Babylonian period.[9] Speculatively, perhaps it would lessen any pleasure the student or scribe derived from mathematical work.

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[1] Robson, *Mathematics in Ancient Iraq*, p. 42.

[2] Robson, *Mathematics in Ancient Iraq*, ch. 4; Høyrup, 'Varieties of Mathematical Discourse', p. 12.

[3] Yale Babylonian Collection (YBC) 4657; Robson, 'Mesopotamian Mathematics', pp. 116–18.

[4] Høyrup, 'Varieties of Mathematical Discourse', p. 15, emphases in orig.

[5] Høyrup, 'Was Babylonian Mathematics Created by 'Babylonian Mathematicians'?', pp. 115–16.

[6] *Loc. cit.*

[7] Englund, *Organisation und Verwaltung*.

[8] Høyrup, 'How to educate a Kapo', p. 28.

[9] *Ibid.*, esp. p. 27.

Geometry and symmetry

Geometric decoration was evident even in early pottery of the Samarra culture (c. 5500–c. 4800 BCE), involving such elements as 'alternation of direction of movement and/or symmetrical motion of contiguous bands [...] emphasis of fourfold rotation or of quatrofoldradial symmetry of finite designs [...] uniformity and precision of draftsmanship'.[1] Fivefold and sixfold symmetry is also encountered.[2] As an artistic style, it could be described as 'neat, busy, and abstract'.[3]

Geometry and symmetry seem to have continued to play a role in the visual culture of the emerging polities of Mesopotamia. Abstract geometric decorations were used on the board for 'The Royal Game of Ur',[4] which dates to the Early Dynastic Period.[5] There are geometrical doodles[6] on non-mathematical documents and drawings of labyrinths[7] from the Old Babylonian period. There is of course much debate about the degree of mathematical intentionality that such artefacts exhibit. However, one can compare the non-mathematical designs with the evidence of actual mathematical texts. In particular, there seems to be a commonality in the symmetries of the decorations and labyrinths and in the symmetries of diagrams used in an Old Babylonian collection of forty plane geometry problems (see Figure 1.8).[8] Every diagram exhibits symmetry: of the twenty-nine diagrams that have survived largely intact, twenty possess the same symmetries as the square (reflection horizontally, vertically, and in both diagonals, and quadrifold rotation), six have rectangular symmetry (reflection in horizontal and vertical axes, rotation through a half-turn); and the remaining three have bilateral symmetry (reflection in a horizontal or a vertical axis, but not both).[9] The labyrinths have fourfold rotational symmetry, the decorations of the gaming board have the symmetries of the square or rectangle, bilateral symmetry, or eightfold rotational symmetry. There seems to be no instance of twofold rotational symmetry without any reflective symmetry, which can be found in other cultures.[10] A near-miss is a larger labyrinth with two paths from opposite entrances leading to the centre, but even if drawn on an idealized grid, it does not have perfect rotational symmetry.[11] [Before the tablet was baked and cleaned, it was mistakenly thought that in this labyrinth only one of the entrances led to the centre and the other to a blind path.[12]] This points to some kind of aesthetic preference common to the game-board decorator, the labyrinth artist, and the creator or compiler of the collection of problems.

According to the surviving texts, the problems in the collection all involved finding the areas of some or all of the shapes in the diagram, starting from the side-length of the outer square.[13] The forty exercises are loosely organized in increasing order of complexity. No solutions survive for these problems. However, the symmetry of the diagrams must have been involved at some level in their solutions. First, the

[1] Leslie, 'Style Tradition and Change', p. 60.

[2] Robson, 'The Uses of Mathematics', p. 97.

[3] Leslie, 'Style Tradition and Change', p. 60.

[4] British Museum (BM) 120834.

[5] Robson, *Mathematics in Ancient Iraq*, pp. 45–6.

[6] *Ibid.*, p. 47.

[7] Schøyen Collection (MS) 4516; Friberg, *A Remarkable Collection*, p. 228.

[8] British Museum (BM) 15285; Robson, 'Mesopotamian Mathematics', pp. 93–9.

[9] Robson, *Mathematics in Ancient Iraq*, p. 50.

[10] *Ibid.*, pp. 46–51.

[11] Schøyen Collection (MS) 3194; Friberg & Phillips, 'Fragments of Three Tablets', § 14.1.

[12] Friberg, *A Remarkable Collection*, pp. 224–7.

[13] Robson, 'Mesopotamian Mathematics', pp. 93–9.

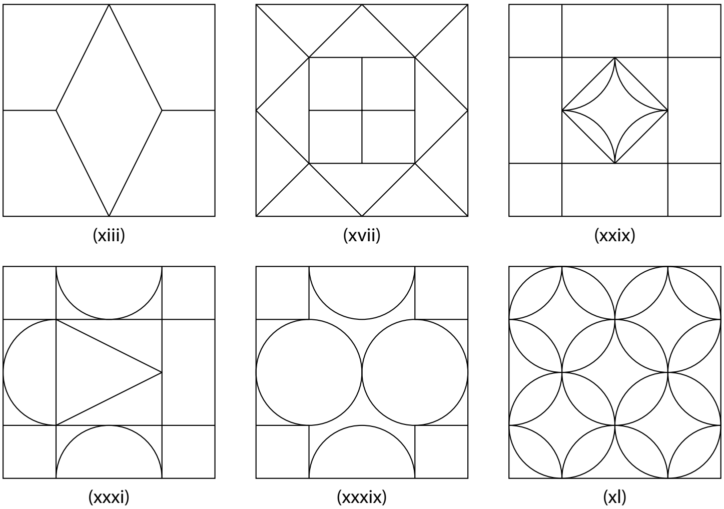

FIGURE 1.8 Selected diagrams from BM 15285, a collection of Old Babylonian plane geometry problems (Robson, 'Mesopotamian Mathematics', pp. 93–9; numbering per original).

question text alone is inadequate to understand the problem. Consider, for example, problem (xvii), whose text is:

> 'The square-side is 1 cable. ⟨Inside it⟩ I drew 12 wedges and 4 squares. What are their areas?'[1]

[The modern equivalent of 1 cable is approximately 360 metres.[2] The text in angle brackets ⟨ ⟩ is a restoration of an accidental elision from the original tablet.] The term *wedge* translates the Akkadian *santakkum* [𒆕], which names any figure with three (possibly non-straight) sides.[3] Thus without the diagram there would be no relationship between the wedges and squares and the specified length, and so the question would be impossible to answer. Furthermore, with the diagram but without the assumption of symmetry, the problem would still be insoluble; since a diagram is never completely precise, the true configuration could be closer to that shown in Figure 1.9, where the wedges have different sizes. In fact, the surviving part of the diagram on the original tablet is very neat, but the point remains that the equality of the sizes of the wedges is not explicitly stated, but comes from an implicit assumption that the approximate visual symmetry of the diagram expresses the exact symmetry of the mathematical configuration. This is not to say that this assumption was articulated by the scribes of Old Babylonia; rather, it was part of the intellectual foundation of their mathematics. By contrast, a modern mathematician, steeped in the distrust of diagrams, would not take the symmetry for granted, but might investigate the possibility of it being a consequence of the supplied conditions, or alternatively the possibility that the areas of the triangles and squares are independent of the symmetry of the diagram. Neither possibility in fact holds. However, with the assumption of symmetry, it is straightforward to

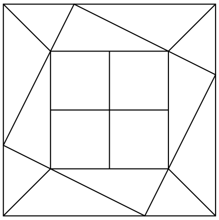

FIGURE 1.9 Modified diagram from problem (xvii) of BM 15285.

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[1] Robson, 'Mesopotamian Mathematics', p. 95.

[2] See Robson, *Mathematics in Ancient Iraq*, app. A.

[3] Robson, 'Mesopotamian Mathematics', p. 68.

see that each wedge and small square has an area 1/16 of the original square.

Similar arguments apply to the other surviving problems. The assumption of symmetry is vital to their solution. But it seems likely that in many cases the diagram was chosen for aesthetic reasons and the problem was devised to fit the diagram. Certainly these geometric problems seem to have no direct application. But the choice of diagram was driven not simply by aesthetic concerns but by the need to create problems soluble by available techniques. Thus there is a tension between the aesthetic and mathematical requirements.

The end of the beginning

THE END OF THE BEGINNING

Even if mathematics was pursued for its own sake in ancient Egypt or Mesopotamia, it does not follow that 'its own sake' had any aesthetic component. However, 'its own sake' does leave *space* for aesthetic concerns to play a role. The symmetry preferences in the configuration of Mesopotamian geometric problems and the Egyptian preferences for certain numbers in unit fraction representations hint at an aesthetic faculty at work.

There is no evidence that mathematics was a leisure pursuit in either Egypt or Mesopotamia. In both contexts, mathematics was a vital part of the functioning of a well-organized state. In the Old Babylonian period, it was part of the training of scribes, who could set challenge problems to show their virtuosity, but it does not follow that its practitioners had high social standing.[1]

What, then, of the role of aesthetic concerns? In the ethnomathematical investigation of space, and in the geometry and geometrical designs used in Mesopotamia, there appears to be some kind of attraction, which may fit into the category of the aesthetic, for kinds of structure that can — from the perspective of the main stream — be called mathematical. But this attraction is entangled in an interplay of cultural, mathematical, and near-mathematical ideas from which it is difficult to extract a purely aesthetic strand.[2]

In the cases of ancient Egypt and Mesopotamia, there is no evidence for the kind of reflection on mathematics that the Greeks undertook. It is known that mathematics was important to the state, and at least in the Old Babylonian period it was connected to the idea of a well-run and just state.[3] But it had not attained autonomy: it remained bound to its applicability. And one might speculate that only autonomous mathematics can stimulate curiosity about mathematics. To make a gross oversimplification: the concepts of three oxen or three loaves or three soldiers do not provoke intellectual inquiry into their natures, but the concept of *three* might possibly. As a corollary of this lack of reflection, there are no writings describing the practitioners

[1] Robson, *Mathematics in Ancient Iraq*, p. 286.

[2] Cf. Ascher, *Ethnomathematics*, p. 186.

[3] Robson, *Mathematics in Ancient Iraq*, p. 266.

of mathematics gaining any aesthetic pleasure from their work. It is impossible to rule out them having experienced such pleasure, but it would not have been something that they thought to record.

2

Beauty without Extravagance

Mathematical beauty in Greece to the time of Aristotle

> ‘We cultivate beauty without extravagance,
> and intellect without loss of vigour’
>
> — Pericles,
> as related by Thucydides, *The Peloponnesian War*, § ii.40
> (trans. M. Hammond).

> ‘they had a subtle appreciation of beauty,
> with an inexhaustible delight in the free play of intellectual energy’
>
> — Werner Jaeger, *Paideia*, vol. i, p. 337
> (trans. G. Highet).

Studying Greek thought, whether in philosophy, science, art, or elsewhere, is different from studying that of Mesopotamia or Egypt. There is a higher degree of intellectual continuity between the Greek world and today, a continuity that is perhaps best exemplified by the descent from Euclid to modern mathematics. The historical road of mathematics of course leads further back, but while it exists and is traceable, it is less clear. Again, it was in Greece that philosophy in the western tradition emerged as a recognizable discipline. This is not to say that philosophy began when Thales of Miletus (c. 625–c. 547 BCE) ‘suddenly dropped from the sky’, which the classicist F. M. Cornford (1874–1943) once acerbically suggested was an axiom in the history of philosophy,[1] but simply that there developed a discipline that can be in retrospect be recognized as philosophical, even if it is the heir both of the natural science of older civilizations and of the earlier world-accounts of myth, religion, and song.[2]

One of the features of this emerging study was that it involved considering problems that were not of direct practical import. The astronomy of Mesopotamia, for instance, was not primarily a disinterested investigation of the world, but was rather bound up with what was then the practical matter of celestial divination and astrology. Indeed, the very difference between the astronomy and the astrology of this period is a distinction made by later scholars.[3]

This feature is part of the answer to the question, asked by the classical scholar W. K. C. Guthrie (1906–81), of why it feels natural to

[1] Cornford, *Thucydides Mythistoricus*, p. x.

[2] Cornford, *From Religion to Philosophy*.

[3] Rochberg, *The Heavenly Writing*, pp. x sqq.

speak of Egyptian and Mesopotamian *science*, but of Greek *philosophy*.[1] But science in its 'pure' form and philosophy both require one to step away from immediate experience: a degree of abstraction and generalization to make them possible, and a degree of wonder or curiosity about the world to make them worthwhile. Utilitarian ends may ultimately be served by such free intellectual endeavour, but they cannot and do not drive it. [Guthrie noted that Plato and Aristotle agreed on this.[2]] Curiosity may motivate disinterested intellectualizing, but alone cannot direct it.

Aesthetic value seems to have been *one* of the driving and directing forces, and it reappears in different aspects of the ongoing philosophical enterprise.[3] In mathematics, the first hint of it is in Anaximander's[*] (c. 610–c. 546 BCE) system of the world. His cosmology attempted to impose a rational structure on the universe, and this structure seems to have been shaped, in a limited way, by an aesthetic concern manifested in its geometric and numerical structure.

Nothing in the corpus of Mesopotamian astronomical texts appeals to a spatial depiction of the universe; only in mythological texts are references to some 'celestial circles' found.[4] The world-systems of Anaximander and those who came later seem to be something new in thought: an attempt to assemble a rational cosmology.[5] It is perhaps unsurprising to find aesthetic value playing a role in devising such a structure: as the twentieth-century philosopher C. D. Broad (1887–1971) noted, introducing 'unity and tidiness into the world' is an act 'which appeals very strongly to our æsthetic interests'.[6] And another philosopher, Heinrich Gomperz (1873–1942), explicitly identified a kind of artistic creativity in his analysis of explanation of phenomena in early Greek science. He suggested that such an explanation is a kind of analogy that relates those phenomena to other, more familiar ones. These analogies he called *thought-patterns*: for example, one such thought-pattern is biological/anthropological/theological, relating the phenomena to processes such as growth and decay, human thought and action, or divine power. Another is artistic:

> 'The scientist takes the attitude of an architect about to found a city or to lay out a temple. He *feels* that certain forms, measures and proportions are called for under the circumstances, and thence he immediately concludes that the facts actually conform to them.'[7]

Mathematics too began to involve an explicit investigation that went beyond practical requirements. What is traditionally called Greek mathematics — that of Euclid, Archimedes, Apollonius — is *scientific* in Høyrup's sense.[†] This mathematics was separate from sub-scientific Greek mathematics, which included both professional calculators and geometricians who calculated for traders or engineers.[8] In Greek, *arithmetic* [*arithmetikē* ἀριθμητική] denoted theoretical study, *logistic* [*logistikē* λογιστική] practical calculation. Like the Mesopotamian and Egyptian praxes, the Greek sub-scientific mathematics

[1] Guthrie, *Earlier Presocratics*, p. 34.

[2] Guthrie, *Earlier Presocratics*, p. 30; cf. Plato, *Theaetetus*, STE 155d sqq.; Aristotle, *Metaphysics*, BEK A.2.982a30 sqq.

[3] Gilbert & Kuhn, *A History of Esthetics*, p. 3.

[*] ▸ pp. 42–3

[4] Rochberg, *The Heavenly Writing*, pp. 126–7.

[5] Guthrie, *Earlier Presocratics*, p. 38.

[6] Broad, *The Mind and Its Place in Nature*, p. 76.

[7] Gomperz, 'Problems and Methods', p. 166, emphasis added.

[†] ▸ p. 22

[8] Asper, 'The two cultures of mathematics', pp. 109–11.

was centred on methods for solving problems. With the emergence of scientific Greek mathematics, '[t]he problem — something to be *done* — gave way to the theorem — something to be *contemplated*.'[1]

What caused this emergence of scientific mathematics? It was doubtless related to the Greek philosophical enterprise, which was not concerned to work with the world, but rather to develop theoretical *understanding* of it.[2] Both mathematics and philosophy — at least from Socrates' (c. 469–399 BCE) time — were based upon a non-hierarchical kind of discourse. In neither was there a one-way master–student dynamic, but instead open scrutiny and argument over the grounds of an assertion, with the aim of seeing if it was sound.[3] Both may ultimately come from debate in the assembly of the democratic city-state.[4]

As mathematics developed as a system, it became autonomous and detached from the empirical world. Socrates' older contemporary Protagoras (c. 490–c. 420 BCE), either being unaware of this detachment or not realizing its significance, complained that geometers had a tangent touching a circle only at a point, not along a length.[5] Mathematics had become a special kind of knowledge. This specialization was part of a tendency that began to elevate pure intellectualizing over knowledge obtained through the senses. Parmenides (*fl.* early 5th century BCE) seems to have been the first to prefer the judgement of reason over that of sight or hearing.[6] Democritus (c. 460–c. 370 BCE) held that there were two kinds of knowledge or judgement: 'genuine' and 'bastard'.[7] In the latter class was that provided by the senses such as sight and hearing. 'Genuine' knowledge was supplied only by the intellect. And although Plato reputedly despised the work of Democritus,[8] there is a glimmering here of Plato's distinction between knowledge of the world and of the Forms.* This division between purely intellectual and sense-derived knowledge is not specifically an aesthetic one, but it is compatible with a special application of aesthetic value to intellect and its products.

This chapter focuses on the period from the start of the sixth century BCE up to the death of Aristotle in 322 BCE. It was during this period that Greek philosophy and mathematics — or, perhaps better, philosophers and mathematicians — emerged, and the end of Aristotle's life, which was shortly before Euclid was active in Alexandria, is a good point to take stock. Note, however, that although only a small proportion of *any* literature survives from this time, there is much more extant in philosophy than in mathematics. There is vastly more to say about the attitude to mathematical beauty of Plato alone — a philosopher who was interested in mathematics — than there is to say about the attitudes of all those known primarily as mathematicians. Although there has been excellent and successful work done on reconstructing the development of mathematics during this period,[9] this does not reveal the opinions, aesthetic or otherwise, of the mathematicians involved.

[1] Cornford, *Before and After Socrates*, p. 6, emphases added.

[2] Høyrup, 'Varieties of Mathematical Discourse', pp. 13–15.

[3] *Ibid.*, p. 23.

[4] *Ibid.*, p. 24.

[5] Aristotle, *Metaphysics*, BEK B.998a1–4.

[6] DK, 28B7; orig. Plato, *Sophist*, STE 242a; Sextus Empiricus, *Adversus Mathematicos*, § VII.114; trans. KRS, p. 248 [§ 294].

[7] DK, 68B11; orig. Sextus Empiricus, *Adversus Mathematicos*, §§ VII.138–9; trans. KRS, p. 412 [§ 554].

[8] Diogenes Laertius, 'Democritus', p. 40.

* ▸ pp. 71–4

[9] For example, Knorr, *The Evolution of the Euclidean Elements*; Szabó, *The Beginnings of Greek Mathematics*; Fowler, *The Mathematics of Plato's Academy*.

LANGUAGE MATTERS

> ‘ We should never underestimate what a word can tell us,
> for language represents the previous accomplishment of thought. ’
>
> — Hans-Georg Gadamer, ‘The relevance of the beautiful’, p. 12
> (trans. N. Walker).

Before delving into Greek considerations of beauty, a linguistic caveat is necessary: no single ancient Greek word coincided semantically with *beautiful*.[1] The two words that most often signify beauty are the adjective *kalos* [καλός] and the noun *kallos* [κάλλος]. The two terms are etymologically related but have different usage.[2] A preliminary discussion of the usage of these and certain other terms will prepare the way ahead.

Ancient Greek is a highly inflected language. In particular, nouns and adjectives take different forms depending on gender, number, and case. The details are not important in this history, but for reference the forms that will be encountered are listed with the discussions below.

kalos. The adjective *kalos* [καλός] is usually translated *beautiful*, but reaches beyond today’s category of the aesthetic towards *good*, *useful*, *fine*, *auspicious*, *noble*, *honourable*.[3] In the sense of *beautiful*, it was applied to outward forms, especially of persons, but also to deeds and reputation. This latter sense — which has been called *moral beauty* — easily enters into Greek ethical discourse where today it might seem odd — although certainly still defensible[4] — to use the word ‘beautiful’. To borrow another author’s example,[5] Aristotle wrote that ‘it is for a noble end [*tō telei kalou* τῷ τέλει· καλοῦ] that the brave man endures and acts as courage directs’.[6]

kalos	*kalē*	*kalon*	*kala*	*kalou*	*kalō*	*kalēn*
καλός	καλή	καλόν	καλά	καλοῦ	καλῷ	καλήν

kalliōn. The adjective *kalliōn* [καλλίων] is the comparative of *kalos*; indicating ‘more *kalos*’.[7]

kallion
κάλλιον

kallistos. The adjective *kallistos* [κάλλιστος] is the superlative of *kalos*; indicating ‘most *kalos*’. It is not derived from *kallos*.[8]

kallistos	*kallista*	*kallistē*	*kalliston*	*kallistōn*
κάλλιστος	κάλλιστα	καλλίστη	κάλλιστον	καλλίστων

kallos. The noun *kallos* [κάλλος] is more limited than *kalos* and is usually rendered as *beauty*.[9] It is narrower in its scope than the English term, denoting primarily visual or physical beauty. [For comparison, the entry for καλός in Liddell & Scott’s *A Greek–English Lexicon* distinguishes five main divisions of meaning, with many sub- and sub-sub-divisions; the entry for κάλλος has only one main division with three sub-divisions.[10]]

kallos	*kallei*	*kallē*	*kallous*
κάλλος	κάλλει	κάλλη	κάλλους

[1] See Guthrie, *Plato: The Man and His Dialogues*, pp. 176–7.

[2] Konstan, *Beauty*, pp. 35, 61.

[3] LSJ, ‘καλός’.

[4] See Riegel, ‘Beauty, το καλον, and its Relation to the Good’, esp. ch. 1.

[5] Kosman, ‘Beauty and the Good’, p. 342.

[6] Aristotle, *Nicomachean Ethics*, BEK 1115b23.

[7] LSJ, ‘κάλλιων’.

[8] LSJ, ‘κάλλιστος’; see also Konstan, *Beauty*, pp. 37–8.

[9] LSJ, ‘κάλλος’.

[10] LSJ, ‘καλός’, ‘κάλλος’.

agathos. The adjective *agathos* [ἀγαθός] is usually translated *good*. Unlike *kalos*, it lacks any aesthetic content.[1] It signifies an object of desire, something worth having, or something fit for its purpose.[2]

agathos *agathon* *agatha*
ἀγαθός ἀγαθόν ἀγαθὰ

glaphyros. The adjective *glaphyros* [γλαφυρός] can be translated as *polished*, *neat*, *delicate*, *subtle*, *exact*, *skilful*, *refined*, *elegant*.[3]

glaphyros *glaphyra* *glaphyron*
γλαφυρός γλαφυρὰ γλαφυρόν

charieis. The adjective *charieis* [χαρίεις] can be translated as *graceful*, *beautiful*, *elegant*, *clever*.[4]

charien
χαρίεν

symmetria. The noun *symmetria* [συμμετρία] can be translated *symmetry*, but is broader than the modern sense of a transformation, such as a reflection or rotation, that preserves an object, or the concept of being so preserved. It includes the notions of *commensurability*, *due proportion*, *harmony*.[5]

symmetria *symmetrois* *symmetrias*
συμμετρία συμμέτροις συμμετρίας

Whether and how to resolve the issues of translation is beyond the scope of the present discussion, but the reader must be conscious of the issue.[6] When appropriate, the original Greek will be inserted into the text.

THE MILESIANS

Aristotle honoured Thales of Miletus (c. 624–c. 546 BCE) as the first thinker to attempt to understand the world in natural terms, without reference to myth and the supernatural.[7] What distinguished the speculations of the Milesians from earlier thinkers was this ‘discovery of nature’: phenomena in the world are neither random nor the will of capricious deities, but are regular and governed by cause and effect. There had not emerged a coherent scientific methodology or scope of inquiry, but the Milesians aimed to explain something about the natural world *in terms of* the natural world, *not* in terms of divine or supernatural forces.[8] With the Milesians there also began rational debate about world-systems. A storyteller recounting a myth and the world-system it presented did not need to engage with or even consider competing myths and their world-systems. But from the Milesians onwards, Greek philosophers put forward theories that were in competition with each other. They were thus required to present the strengths of their own theories and the weaknesses of their opponents’: put simply, the aim was to find the *best* theory.[9]

According to Proclus (410/12–485 CE), who wrote a thousand years later, Thales travelled to Egypt, whence he introduced geometry to

[1] LSJ, ‘ἀγαθός’.

[2] Nettleship, *Lectures on the Republic*, pp. 218 sqq.

[3] LSJ, ‘γλαφυρ-ός’.

[4] LSJ, ‘χαρίεις’.

[5] LSJ, ‘συμμετρία’.

[6] For a discussion, see Kosman, ‘Beauty and the Good’; Lear, ‘Response to Kosman’.

[7] Aristotle, *Metaphysics*, BEK A.3.983b20.

[8] Lloyd, *Early Greek Science*, pp. 8–9.

[9] *Ibid.*, pp. 11–12.

Greece, and his method was a mixture of theoretical and empirical reasoning.[1] [If Herodotus[2] was correct — which is doubtful[3] — that Thales predicted a solar eclipse (most often considered to be the one of 28 May 585 BCE), then it is likely he was familiar also with the more advanced Mesopotamian astronomy.] Proclus credited Thales with a number of elementary theorems and proofs, in particular that a circle is bisected by its diameter;[4] however, Proclus relied on sources that are no longer extant and such demonstrations as Thales discovered would have been proofs that were judged according to the standards of the early Greek mathematics.[5] Assuming that the accounts given by Proclus are accurate (and here it should be re-emphasized that he wrote about a millennium after Thales lived), Thales thought that mathematics was important enough to pass on what he had learned, and recognized that certain 'obvious facts', such as the circle-bisection result, needed proof. Beyond this, it seems impossible to discern what he thought *about* mathematics and whether he perceived aesthetic value in it.

The successors of Thales at Miletus, Anaximander (c. 610–c. 546 BCE) and Anaximenes (c. 585–c. 525 BCE), constructed non-mythological cosmologies, and Anaximander's is notable for its peculiarly mathematical character.[6] For Anaximander, the Earth was shaped like a cylinder, three times as deep as it is wide.[7] The Earth was located at the centre of the universe and remained there since it had the same distance from all points;[8] thus the position and stability of the Earth depended on symmetry and presage a pre-Euclidean definition of a circle or sphere.[9] The introduction of this geometric idea thus dispensed with the requirement to posit some support to keep the Earth in its place: here, Anaximander was ahead of his time, for his predecessor Thales and his successors Anaximenes, Anaxagoras (c. 500–c. 428 BCE), and Democritus (c. 460–c. 370 BCE) all felt the need of some underpinning to secure the Earth.[10]

Around the Earth were fire-filled circular rings with holes into their interiors. From these holes streamed jets of fire, which were seen as the various heavenly bodies. The outermost ring had a diameter either 27 or 28 times that of the Earth, and the fire emerging from it was perceived as the Sun. The middle ring's diameter was 19 times that of the Earth, and its dimmer fire was perceived as the Moon. The inner ring or rings presumably had many openings with jets of fire emerging, which were perceived as the stars. These rings revolved around the Earth, which explained the daily solar motion, the monthly lunar motion, and the movement of the stars.[11] (See Figure 2.1.)

It has been suggested that the different values for the diameter of the Sun-ring are due to measuring variously to the inner side, middle, or outer side of the rings,[12] which leads to the plausible idea that the diameters of the inner, middle, and outer rings formed the arithmetic progression 9, 18, 27. This scheme is almost difficult to resist,[13] especially in light of the numerical ratio of 3 : 1 in the dimensions of the

[1] Proclus, *Commentary on Euclid*, FRI 65.

[2] Herodotus, *The Histories*, § 1.74.

[3] Neugebauer, *The Exact Sciences in Antiquity*, pp. 142–3.

[4] Proclus, *Commentary on Euclid*, FRI 157, 250, 299, 352.

[5] Guthrie, *Earlier Presocratics*, p. 53.

[6] For a general account of Anaximander's cosmology, see Kahn, *Anaximander*.

[7] *Ibid.*, p. 81.

[8] *Ibid.*, p. 76.

[9] Cf. Plato, 'Epistle 7', STE 342b.

[10] Aristotle, *On the Heavens*, BEK 294a29 sqq., 294b13 sqq.; cf. Guthrie, *Earlier Presocratics*, p. 162.

[11] Kahn, *Anaximander*, p. 86; see also Guthrie, *Earlier Presocratics*, p. 94.

[12] Guthrie, *Earlier Presocratics*, p. 94.

[13] See, for example, Kahn, *Anaximander*, p. 88; Guthrie, *Earlier Presocratics*, pp. 95–6.

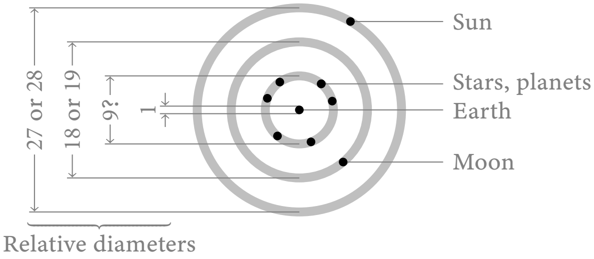


FIGURE 2.1
A schematic outline of Anaximander’s cosmology.

Earth. But it is dangerous to draw any conclusion about a possible aesthetic preference for such arithmetic progressions, for it is the very neatness of the sequence that encourages this reconstruction.

Anaximander’s choice of rings rather than isolated bodies moving in circular paths tied in with his metaphysics and seems to be irrelevant to any aesthetic concerns. But the role of circularity in his cosmology is germane, for it marked the beginning of the prominence of the circle in cosmology, a prominence that lasted until the work of Johannes Kepler[*] (1571–1630). At one level, observation shows that the Sun, Moon, and planets move cyclically. A closed shape *such as* the circle can be seen as a natural figure to embody cyclic movement. But the choice of the circle was not determined solely by the phenomena. Rather, the circle was somehow the most natural choice. The naturalness of the choice may be due to the circle having been conceptualized by the Greeks as a figure drawn by a compass, by rotating a radius about a centre. By way of contrast, Old Babylonian mathematics conceived of a circle as a figure surrounded by a circumference: all diagrams of circles from this period give the length of the circumference, the square of this length, and the area of the circle, but never the radius or diameter.[1]

Anaximander’s Earth is circular in plan as well, but its circularity was not required to explain the shadow of the Earth on the Moon during lunar eclipses, which were caused by the closing of the orifice in the middle ring rather than by the interposition of the Earth between the Moon and the Sun. At least among the surviving accounts of his cosmology, there seems to be no empirical justification for the circular plan of the Earth. Perhaps, therefore, having posited circles elsewhere, he simply stuck with this choice to give a unity to his cosmology.

But the arithmetic progression and the circles, taken together, point to a concern with symmetry.[2] And if Anaximander was concerned with seeking the best theory, this in turn suggests that symmetry had positive value. This is at least compatible with the beauty explicitly assigned to the circle by Pythagoras and later Greek thinkers.[†]

As the classicist John Burnet pointed out, there is ‘an audacity almost amounting to ὕβρις [*hubris*]’[3] in moving from the observation that the heavens exhibit cyclical behaviour to constructing a universal order for nature. It would require no extra audacity to suppose (or impose) an aesthetic unity.

[*] ▸ pp. 279–94

[1] Robson, *Mathematics in Ancient Iraq*, pp. 65–6.

[2] Cf. Lloyd, *Early Greek Science*, p. 80.

[†] ▸ p. 44

[3] Burnet, *Early Greek Philosophy*, p. 25.

PYTHAGORAS

The fundamental difficulty in studying pre-Socratic philosophy is that its reconstruction depends on seeing it through the veil of second-hand accounts, aided by the few direct glimpses afforded by the fragments that survive in quotation. With Pythagoras (c. 570–c. 490 BCE), the difficulty is redoubled: he was already cloaked in legend during his own lifetime. From later antiquity onwards, he was seen as a promethean genius who founded philosophy, mathematics, musical theory, and cosmology; perhaps the single greatest exemplar of the Greek habit of seeking a 'first inventor' [*prōtos heuretēs* πρῶτος εὑρετής].[1]

It is very difficult to discern anything of pythagorean thought from before Plato's time. No texts written by Pythagoras survive; indeed, even in antiquity it was debated whether he had written anything at all.[2] Walter Burkert's (1931–2015) fundamental work *Lore and Science in Ancient Pythagoreanism* (1972; originally published in German in 1962) showed that much of the traditional account of Pythagoras was invented or changed by Plato's successors. Burkert's own conclusion was that the most ancient testimony, uninfluenced by Plato, showed that Pythagoras was a spiritual leader, a shamanistic figure, who did not contribute to, let alone found, the Greek philosophical and mathematical enterprises.[3] On the other hand, the account by Guthrie[4] (which also appeared in 1962) and the later studies by Leonid Žmud'[5] (1956–) and Charles H. Kahn[6] (1928–2023) paint a picture closer to the traditional view of Pythagoras as an innovator in philosophy and mathematics. Modern scholarship has certainly clarified the difficulties in achieving an accurate portrayal of the historical Pythagoras, and perhaps it is time to admit that he is simply inaccessible.[7]

Pythagoras may have written nothing; certainly nothing he wrote survives. Only one of the *akousmata* [ἀκούσματα; singular *akousma* ἄκουσμα], which are orally-transmitted sayings attributed to Pythagoras, bears on mathematical beauty: 'He held that the most beautiful figure [*tōn schēmaton to kalliston* τῶν σχημάτων τὸ κάλλιστον] is the sphere among solids, and the circle among plane figures.'[8] Although Anaximander's cosmology* may suggest that value was placed on circles before Pythagoras, this *akousma* is the first explicit statement that assigns *aesthetic* value to circles and spheres. Thus begins a recurring theme of aesthetic preference for circles and spheres in Greek thought. It should be emphasized, however, that there is no hint of a circle- or sphere-based cosmology like Anaximander's among the *akousmata*. Rather, the cosmology that the *akousmata* describe was primarily mythological, with no overarching natural order or process, and no mathematical structure.[9]

[1] Kleingünther, 'Πρωτος Ευρετης'.

[2] Burkert, *Lore and Science*, p. 218.

[3] *Ibid.*, p. 165.

[4] Guthrie, *Earlier Presocratics*, ch. IV.

[5] Zhmud, *Wissenschaft, Philosophie und Religion*; Zhmud, *Pythagoras and the Early Pythagoreans*.

[6] Kahn, *Pythagoras and the Pythagoreans*.

[7] Lloyd, 'Pythagoras'.

[8] Diogenes Laertius, 'Pythagoras', MEI 8.35; on its authenticity, see Burkert, *Lore and Science*, p. 168, n. 18.

* ▸ pp. 42–3

[9] Huffman, 'Reason and Myth', pp. 60–8.

EARLY PYTHAGOREANISM

Regardless of the historical Pythagoras' contribution, mathematical and scientific study seems to be present in the pythagorean tradition by the late fifth century BCE when Philolaus of Croton (c. 470–c. 385 BCE) wrote, and even more so in the time of Archytas of Tarentum (c. 425–c. 355 BCE), who is the first and only pythagorean who can with certainty be recognized as an accomplished mathematician.[1] However, it can be debated how accurate it is to label Philolaus and Archytas as pythagoreans. They were certainly *called* pythagoreans, and in particular Plato quoted words of Archytas but cited 'the Pythagoreans'.[2] But Aristotle hedged by identifying Philolaus and his contemporaries as 'so-called [*kaloumenoi* καλούμενοι]' pythagoreans.[3] Perhaps Philolaus and Archytas inherited the religious and moral aspects of pythagoreanism, but developed the rational investigation of the world in new ways.

Early pythagoreans understood the world in terms of number. This understanding was not, as with modern science's mathematical theories, based on number as measuring and quantifying natural phenomena, but rather on number as ordering and corresponding with such phenomena. The sources of pythagorean understanding of the world in terms of number are twofold. First, pebble arithmetic, wherein a number was represented as a group of pebbles [*psēphoi* ψῆφοι] arranged in some way. Second, the production of consonant musical intervals by plucking strings whose lengths are in small-number ratios, a discovery that can still cause delight.[4] [An appealing but physically impossible ancient tradition had Pythagoras passing a smithy, hearing harmonious intervals produced by some of the hammers, and, upon investigating, finding that the weights of the hammers were in the ratio of small integers.[5] Other tales tell of Pythagoras carrying out experiments: hanging weights from strings that he plucked or striking jars filled with varying amounts of water and finding the numerical relation in the weights or the amounts. These experiments do not produce the result the tales claim.[6]]

According to Aristotle, the pythagoreans — not just the 'so-called' pythagoreans — thought that things 'are' or 'consist of' numbers.[7] However, this may just be Aristotle's overly succinct formulation of the view that the world can be understood through numbers, *amounting to* the idea that all things are numbers.[8] An extant fragment of Philolaus says that 'all the things that are known have number. For it is not possible that anything whatsoever be understood or known without this',[9] which points to an epistemological rather than an ontological role for number. Perhaps Aristotle was concerned to contrast the pythagorean and platonist traditions, and in opposition to the platonist ontology of numbers, which elevated them above the world, gave an unintentionally distorted account of the pythagorean view.[10]

One can discard the idea that the pythagoreans held everything to be number but retain the notion that they thought of numbers as

[1] Huffman, 'Reason and Myth', p. 60; Schofield, 'Archytas', p. 69.

[2] Plato, *Republic*, STE VII.530d; see DK, 47B1, ll. 7–8; Huffman, *Archytas*, p. 109.

[3] For instance, Aristotle, *Metaphysics*, BEK A.5.985b22, 989b30; see also Huffman, 'Reason and Myth', p. 60; Huffman, *Philolaus*, p. 32.

[4] Cornford, 'The Harmony of the Spheres', p. 19.

[5] Burkert, *Lore and Science*, pp. 375–6.

[6] Lloyd, 'Pythagoras', pp. 37–8.

[7] Aristotle, *Metaphysics*, BEK A.5.986a3, 986a19, 6.987b28, M.8.1083b17, 3.1090a22; Aristotle, *On the Heavens*, BEK 300a15; Burkert, *Lore and Science*, p. 31.

[8] Burkert, *Lore and Science*, p. 48; Huffman, *Philolaus*, p. 59.

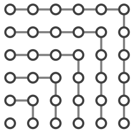
[9] DK, 44B4; orig. Stobaeus, *Anthologium*, § I.xxi.7b ⟦vol. 1, p. 188⟧; trans. Huffman, *Philolaus*, p. 172.

[10] Huffman, *Philolaus*, pp. 63–4.

bound to the sensible world: numbers did not, as Plato thought, exist or subsist in a higher realm only accessible to the intellect. A number was a concrete thing, such as an arrangement of pebbles: they did not think of an arbitrary number, they grasped a fixed number of pebbles.[1] The term for 'number' here is *arithmos* [ἀριθμός], which always refers to a whole number, a number that is counted, and 'is closely connected with things, and in fact is itself a thing, or at least an ordering of things'.[2] In the examples given by Aristotle, the identification of numbers and things never connects an individual thing with an individual number, but rather many things with a system of numbers.[3]

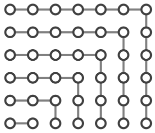
Pebble arithmetic

Pebble arithmetic led to a pythagorean preference for odd numbers over even. Odd numbers were associated with 'limit', even with the 'unlimited'. Aristotle explained this in terms of pebble arithmetic.[4] Gnomons formed of odd numbers, when stacked together, form a series of squares: the same shape regardless of how many gnomons are stacked together:



(G)

In contrast, gnomons of even numbers form a series of rectangles, each of different proportions:



These figures immediately illuminate three of the entries in the 'table of ten opposites' reported by Aristotle, though only ascribed to some members of the pythagorean school, in which ten principles are arranged in columns of 'cognates':[5]

[πέρας]	limit	:	unlimited	[ἄπειρον]
[περιττόν]	odd	:	even	[ἄρτιον]
[ἕν]	one	:	plurality	[πλῆθος]
[δεξιόν]	right	:	left	[ἀριστερόν]
[ἄρρεν]	male	:	female	[θῆλυ]
[ἠρεμοῦν]	resting	:	moving	[κινούμενον]
[εὐθύ]	straight	:	curved	[καμπύλον]
[φῶς]	light	:	darkness	[σκότος]
[ἀγαθόν]	good	:	bad	[κακόν]
[τετράγωνον]	square	:	oblong	[ἑτερόμηκες]

[1] Burkert, *Lore and Science*, p. 435.

[2] *Ibid.*, p. 265.

[3] *Ibid.*, p. 48.

[4] Aristotle, *Physics*, BEK 203a13 sqq.

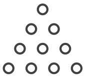
[5] Aristotle, *Metaphysics*, BEK A.5.986a22 sqq.; Burkert, *Lore and Science*, pp. 51–2.

From Aristotle's explanation, gnomon diagrams are instances of limit–odd–square and unlimited–even–oblong. There does seem to be something aesthetically pleasing in how the stacking of gnomons of odd numbers always forms squares, which at least fits with the placement of odd numbers in the 'good' column. Whether this pleasing combination *contributed* to it being so placed is unknown.

Today, diagram (G) would be seen as illustrating the theorem *An ascending series of consecutive odd numbers, starting from* 1, *sums to a square number*, and the diagram remains an excellent proof of this result: although the figure only shows $1+3+5+7+9+11 = 36 = 6^2$, one sees immediately that the result is true for all sums of odd numbers $1 + 3 + \ldots + (2k - 1)$. Two and a half millennia later, this proof is still considered an exemplar of beauty.[1] However, early pythagoreans, though clearly aware of this rule, probably did not conceive of it as a theorem and proof. The result is immediately revealed by the pebbles, rather than being the product of a chain of deductions. Such goodness as they perceived (regardless of whether they felt it as beauty) would doubtless have been assigned to the numbers — the physical numbers embodied in the pebbles — that fitted the rule, rather than to the result. Indeed, Burkert argued that a proof could be viewed negatively, as robbing the rule of its mystery and symbolism.[2]

Perfect 10

In spite of the preference for odd numbers, Aristotle wrote that the 'so-called Pythagoreans' considered the number 10 'to be perfect [*teleion* τέλειον] and to comprise the whole nature of numbers'.[3] Several related factors seem to have contributed to this unique status accorded to 10. The number 10 had a special symmetrical representation as a triangular arrangement of pebbles, that the pythagoreans called the tetractys [τετρακτύς *tetraktys*]:[4]



As aleady mentioned, Aristotle's phrase 'so-called Pythagoreans' refers to Philolaus and his contemporaries. But the tetractys seems to be an older concept, with an origin connected to music: one of the *akousmata* says: '"What is the oracle of Delphi?" "The tetractys; that is, the harmony in which the Sirens sing"'.[5] Later tradition gives some reasons for the status of 10: it is the sum of the first four counting numbers $1 + 2 + 3 + 4 = 10$, and these four numbers contain the most important musical intervals (2 : 1, the octave; 3 : 2, the fifth; 4 : 3, the fourth; 4 : 1, the double octave) as well as the later platonic point–line–plane–solid pattern, in which one point exists alone, two points define a line, three a plane, and four a solid.[6] However, while the unique status of 10 and the general importance of musical intervals to

[1] Polster, *Q.E.D.*, pp. 32–3.

[2] Burkert, *Lore and Science*, p. 433.

[3] Aristotle, *Metaphysics*, BEK A.5.986a8.

[4] Burkert, *Lore and Science*, p. 72.

[5] DK, 58C4; Iamblichus, *De Vita Pythagorica Liber*, p. 47, ll. 15–16 [para. 82]; trans. Burkert, *Lore and Science*, p. 187.

[6] Sextus Empiricus, *Against the Logicians*, §§ 1.94–100; Burkert, *Lore and Science*, p. 72.

the pythagoreans is attested by Aristotle,[1] the point–line–plane–solid pattern is not.[2] Notice that if the later tradition is correct that the first four counting numbers were seen as containing the most important intervals, then the perfection of 10 arises from both pebble arithmetic and musical intervals.

According to Aristotle, the perceived perfection of 10 was the reason Philolaus posited an unseen 'counter-earth' [*antichthon* ἀντίχθων] in his cosmology: with the counter-earth, the number of heavenly bodies would be 10.[3] It is perhaps more likely that the counter-earth was suggested to explain why lunar eclipses are more frequent than solar eclipses,[4] especially since 'the' comet, which, according to Aristotle, was thought of as a planet by Philolaus ('so-called Pythagoreans'), could otherwise have filled out the numbers in place of the counter-earth.[5] It has also been suggested that Philolaus was actually hypothesizing that later study would reveal a counter-earth and so bring the number of heavenly bodies to the perfect 10, rather than inferring the existence of the counter-earth from the number 10.[6] Inference or hypothesis: either serves to reinforce the value placed on 10.

[Note that the sense in which 10 is 'perfect' is completely distinct from Euclid's notion of a *perfect number* as a number equal to the sum of its proper divisors, like $6 = 1+2+3$ or $28 = 1+2+4+7+14$. [The word 'perfect' translates the adjective *teleios* [τέλειος] used by both Aristotle and Euclid.] There is no evidence of this notion of a perfect number prior to Euclid.[7] This is not to imply that Euclid defined it *ex nihilo*, just that its history cannot be traced further back. However, in any case the concept would be too abstract for the pythagoreans, with their concrete view of numbers.[8] The same applies to *amicable numbers* (sometimes called 'friendly numbers'), which are a pair of numbers each of which is equal to the sum of the other's proper divisors, such as $220 = 1 + 2 + 4 + 71 + 142$ and $284 = 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110$. The earliest mention of amicable numbers is by Iamblichus* (c. 245–c. 325 CE), who did not attribute the mathematical concept to Pythagoras, merely the analogous characterization of a friend as 'another I'.[9]]

Thus far, there is little that can be said about any early pythagorean aesthetic appreciation of mathematics, beyond a plausible locus of mathematical beauty in numbers as pebble diagrams or more generally as concrete things.

* ▸ pp. 145–8

[1] Aristotle, *Metaphysics*, BEK A.4.985b31 sqq.

[2] Burkert, *Lore and Science*, p. 72.

[3] Aristotle, *Metaphysics*, BEK A.5.986a8 sqq.

[4] Graham, 'Philolaus', pp. 57–9.

[5] Aristotle, *Meteorology*, BEK 342b30 sqq.

[6] Huffman, *Philolaus*, p. 75.

[7] Heath, *Hist. Greek Math.*, vol. 1, p. 74; cf. Heath, notes to Euclid, *Elements*, Definition VII.22.

[8] Burkert, *Lore and Science*, p. 431.

[9] Iamblichus, *In Nicomachi*, p. 35, ll. 1–7; trans. Dickson, *Hist. Theory of Numbers*, vol. 1, p. 38.

PHILOLAUS

Philolaus of Croton (c. 470–c. 385 BCE) may in fact have been instrumental in adapting the early pythagorean number symbolism to the rational study of the world. Philolaus is the earliest author of whom there are extant fragments that explicitly invoke ‘number’ as a way to explain the world.[1] Shades of number-based explanation can be found earlier in the thought of Anaximenes, who maintained that change and difference in quality and kind can be explained as change or difference in quantity and distribution of a single *archē* [ἀρχή], meaning ‘origin’ or ‘first principle’,[2] and in that of Empedocles (c. 495–435 BCE), who described the numerical combination of elements that formed a substance.[3] Philolaus may have wanted to bring the reliability of mathematical reasoning to bear upon the problem of understanding the world, and this wish may in turn point to proof as being essential for Philolaus’ approach.[4] However, if Philolaus was so motived, then the ‘proof’ may simply have been the certainty in the immediate recognition of results that pebble arithmetic yielded. Or it may point to an appreciation of deductive, non-pebble, proofs. Either position is compatible with the fragment of Aristotle preserved by Iamblichus[5] that reported: ‘The Pythagoreans spent their time on mathematics espoused the exactitude of its arguments because it alone among human undertakings contained demonstrations’.[6]

Philolaus’ cosmogony demonstrates that he posited a spherical universe,[7] and it is tempting to see an influence of the Pythagorean *akousma* on the beauty of the sphere. But other authors who are not recorded as being members of the pythagorean tradition also described a spherical universe; the idea may have been ‘in the air’. The pantheist Xenophanes (c. 570–c. 470 BCE) saw God as spherical according to most ancient authorities.[8] Parmenides (*fl.* early 5th century BCE), often considered the greatest of the pre-Socratic philosophers, argued for a spherical universe by implicitly appealing to the principle of sufficient reason: there is no reason for the universe to be bigger in one direction than another.[9] In the cosmogony described by Empedocles, the sphere also featured as a stage in the formation of the cosmos when Love (as opposed to Strife) was totally dominant and unified all things.[10] Nothing contradicts the idea that aesthetic value played some role for Xenophanes or Parmenides or Empedocles when they formulated their cosmogonies, but there is no positive evidence. For Philolaus himself, another fragment suggests he saw the symmetry of the sphere as important, but there is no hint of an aesthetic appreciation of this symmetry.[11]

But there is certainly an aesthetic element in Philolaus’ cosmogony. The world-order [*kosmos* κόσμος] arose from limited things and unlimited things.[12] But these are unlike and so must be bound together by a third principle: they are ‘bonded together by harmony [*harmonia* ἁρμονία]’. The origin of this harmony is not addressed; it is simply

[1] Huffman, *Philolaus*, p. 71.

[2] Gottlieb, *The Dream of Reason*, ch. 1; Guthrie, *Earlier Presocratics*, § III.D.(4).

[3] DK, 31B96; orig. Simplicius, *In Physicorum*, p. 300, ll. 21–4; trans. KRS, p. 302 [§ 374].

[4] Huffman, *Philolaus*, p. 72.

[5] Burkert, *Lore and Science*, pp. 49–50.

[6] Iamblichus, *On the General Science of Mathematics*, FES 78.8–11; for the passage being derived from Aristotle, see Burkert, *Lore and Science*, pp. 49–50 & n. 112.

[7] DK, 44B7; orig. Stobaeus, *Anthologium*, § I.xxi.8 ⟦vol. 1, p. 189⟧; trans. Huffman, *Philolaus*, p. 227.

[8] Guthrie, *Earlier Presocratics*, pp. 376–7.

[9] DK, 28B8, ll. 42–5; orig. Simplicius, *In Physicorum*, p. 146, ll. 15–18; trans. KRS, p. 252 [§ 299, ll. 42–5].

[10] DK, 31B27; orig. Simplicius, *In Physicorum*, p. 1138, ll. 28–31; trans. KRS, p. 295 [§ 358].

[11] DK, 44B17; orig. Stobaeus, *Anthologium*, § I.xv.7 ⟦vol. 1, p. 148⟧; trans. Huffman, *Philolaus*, p. 215.

[12] DK, 44B1; orig. Diogenes Laertius, ‘Philolaus’, MEI VIII.85; trans. Huffman, *Philolaus*, p. 93.

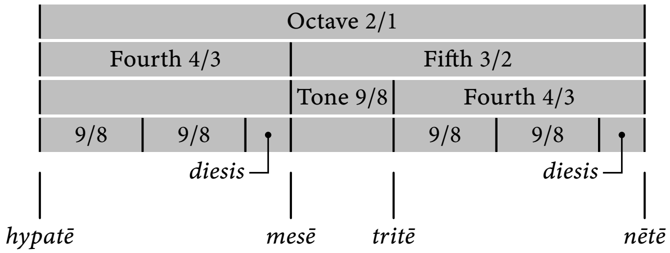


FIGURE 2.2 Illustration of the structure of the octave in Philolaus' harmonic theory. (Lower notes are to the left; higher notes are to the right.)

held to be necessary.[1] This mention of harmony is the introduction into cosmology of an *explicitly* aesthetic element. And the harmony goes beyond order itself. As Cornford put it, '[t]he very essence of order is a measure or limit imposed upon the infinite or unlimited.'[2] But order need not be harmonious, so the harmony is an independent quality. This harmony that governs the cosmos has a diatonic scale, which Philolaus described in detail.[3] The octave has ratio 2/1 and is divided into a fourth [*syllaba* συλλαβά] with ratio 4/3, which spans from the lowest note *hypatē* [ὑπάτη] to the note *mesē* [μέση], and a fifth [*di oxeian* δι᾽ ὀξειᾶν] with ratio 3/2, which spans from *mesē* to the highest note *nētē* [νητη]. The fifth is made up of a ratio 9/8, which spans from *mesē* to *tritē* [τρίτη], and a fourth, which spans from *tritē* to *nētē*. (The note names are given in Attic rather than Doric forms, to ease comparison with later systems. The note *tritē* is more usually called *paramesē* [παραμέση] and '*tritē*' names a different note.[4]) The fourth comprises three 9/8 ratios and a *diesis* [δίεσις] which has ratio 256/243, although this value does not appear in the surviving fragments.[5] (Figure 2.2 illustrates this structure.) While Philolaus was describing a system that would have been familiar to practising musicians, his account evokes especially the symmetry of its mathematical structure.[6]

Philolaus was clearly familiar with the use of ratios in musical analysis, but in the surviving fragments of his harmonic theory, he did not actually make any mathematical use of ratios. His discussion only requires recognition that one can sensibly discuss the 'size' of intervals, that certain intervals are equal, and that smaller intervals can be added together to make larger ones. Although Philolaus gave the ratios corresponding to some of the intervals he considers, a delicate grammatical point suggests that he only thought of adding (musical) *intervals*, not calculating with (mathematical) ratios. This difference indicates that his harmonic theory was a hybrid of empirical and mathematical perspectives that would later separate and be treated as incompatible.[7]

Note that each of these ratios Philolaus gave (that is, those mentioned above except for the *diesis*), is of the form $(n + 1)/n$. Such ratios are called *superparticular* or *epimoric* [*epimorios* ἐπιμόριος]. According to Ptolemy (c. 100–c. 170 CE), the pythagoreans held that each concord must correspond to a ratio where either the lower term divides the larger or the difference between the two terms measures both: this condition is equivalent to saying that the ratio, in lowest terms, must be either of the form $n/1$ (which is called a *multiple ratio* in the *Division of the Canon*[*]) or of the form $(n+1)/n$.[1] This requirement fits with Philolaus' discussion, because the ratio 256/243 for the *diesis* could only be calculated, not found empirically; if Philolaus did not calculate with ratios, he would not have known it.

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[1] DK, 44B6; orig. Stobaeus, *Anthologium*, § I.xxi.7d ⟦vol. 1, pp. 188–9⟧; trans. Huffman, *Philolaus*, pp. 123–4; see also Huffman, *Philolaus*, pp. 128–9; Burkert, *Lore and Science*, p. 257.

[2] Cornford, 'The Harmony of the Spheres', p. 20.

[3] DK, 44B6(a); orig. Stobaeus, *Anthologium*, § I.xxi.7d ⟦vol. 1, p. 129⟧; trans. Huffman, *Philolaus*, pp. 146–7.

[4] Barker, *The Science of Harmonics*, pp. 265–6.

[5] *Ibid.*, p. 269.

[6] Barker, 'Pythagorean harmonics', p. 191.

[7] Barker, *The Science of Harmonics*, pp. 270–2.

The important innovation in Philolaus' cosmogony is the use of an aesthetic–mathematical concept to bind together fundamental substances.[2] Given Aristotle's statement that the 'so-called Pythagoreans', meaning Philolaus, thought of the whole heaven as a *harmonia* and a number and tried to link to cosmology every possible aspect of number and harmony,[3] and given the fundamental role of the ratios 2/1, 3/2, and 4/3 in Philolaus' harmonic theory, there is a strong indication that the tetractys played a role in Philolaus' thought.[4] Further, these ratios, like all superparticular ratios, connect an even and an odd number and so form a harmony of limited and unlimited.[5]

Philolaus' use of *harmonia* in the structure of the universe seems to have been an influence on Plato, who used a similar device in the *Timaeus*,[†] though not on Archytas, who seems to have avoided any such cosmogonical role for *harmonia*.[‡]

POLYKLEITOS

The sculptor Polykleitos (*fl.* c. 500 BCE) wrote a treatise, known as the *Canon*, that gave a mathematical prescription to obtain beauty in art, and created a statue, also called the *Canon*, exemplifying the tenets of the treatise. His writings are lost, but two fragments were preserved by Philo of Byzantium (*fl.* c. 250 BCE) and Plutarch (before 50–after 120 CE), and two summaries were written by Galen (129–c. 210/217 CE) and possibly a third by Plutarch.[6]

Philo quoted Polykleitos thus: 'beauty, he said, comes about *para mikron* [παρὰ μικρὸν] through many numbers'.[7] Different meanings for *para mikron* have been proposed; the likely possibilities are 'from minute calculation' or 'little by little'.[8] Galen said that the *Canon* statue 'received this name from its having a precise commensurability of all its parts to one another',[9] and that Polykleitos thought that beauty is located in

> 'the proper proportion of the parts, such as for example that of finger to finger and all these to the palm and base of the hand, of those to the forearm, of the forearm to the upper arm and of everything to everything else, just as described in the Canon of Polykleitos. [...] So then, all philosophers and doctors accept that beauty resides in the due proportion of the parts of the body.'[10]

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[*] ▸ pp. 103–4

[1] Ptolemy, *Harmonics*, § I.5, DÜR 11.

[2] For the connection between DK, 44B6 and DK, 44B6(a), see Huffman, *Philolaus*, pp. 158–60.

[3] Aristotle, *Metaphysics*, BEK A.5.986a1 sqq.

[4] Barker, *The Science of Harmonics*, pp. 282–3.

[5] Burkert, *Lore and Science*, p. 383.

[†] ▸ pp. 62–5

[‡] ▸ p. 56

[6] Collected in Stewart, 'The Canon of Polykleitos', pp. 124–5.

[7] Philo, *Belopoiika*, ch. 2 ⟦pp. 7–8⟧; trans. Stewart, 'The Canon of Polykleitos', p. 124, [§ 1] n. 19.

[8] Stewart, 'The Canon of Polykleitos', p. 126.

[9] Galen, *De Temperamentis*, KÜH I.566; trans. Stewart, 'The Canon of Polykleitos', p. 125, [§ 3] n. 22.

[10] Galen, *De Placitis Hippocratis et Platonis*, KÜH V.448–9; trans. Stewart, 'The Canon of Polykleitos', pp. 125, 131 [§ 4] n. 23 & postscript.

It is possible that there was some influence of the 'everything is number' aspects of pythagorean thought on the *Canon*.[1] In particular, the pythagorean musical theory, with its focus on small-number ratios, may have led Polykleitos to think of number in terms of proportion and harmony.

DEMOCRITUS

With the work of Democritus (460/57–c. 370 BCE), aesthetics began to emerge as a subject of study, at least in terms of the philosophy of art.[2] None of his books survive, save in fragments, but the catalogue by Thrasyllus of Mendes (d. 36 CE) lists six pertaining to art and beauty.[3]

Democritus was a polymath. In particular, he was an accomplished mathematician,[4] and was reportedly proud of his accomplishments in the field, allegedly having boasted: 'no one has ever surpassed me in the composition of treatises with proofs'[5] (though the authenticity of this quotation is disputed[6]). Thrasyllus' catalogue shows that Democritus wrote six books on mathematics.[7]

He *may* have been the first to discover, but not prove, that the volume of a cone is one-third that of the cylinder with the same height and base, and that of a pyramid is one-third that of a prism with the same height and base. In the twentieth century, the mathematician Max Dehn (1878–1952) held up the latter result as one of the most beautiful results of Greek mathematics up to the time of Archimedes.[8] But the authority for assigning the discovery to Democritus is Archimedes,[9] who elsewhere wrote that both the results and proofs were due to Eudoxus.[10]*

There is nothing to suggest that this philosopher-mathematician explicitly considered or even perceived beauty in mathematics. But in the book titled *Sayings of Democrates* [Δημοκρατους Γνωμαι], whose title may or may not be a corruption of 'Democritus', and, even if it is, may represent a later stage in the transmission of his thought rather than be based on his own writings,[11] there are tantalizing fragments that hint at an appreciation of intellectual beauty: 'Beauty [*kallos*] of body is animal-like, if there is no intelligence [*nous* νοῦς] behind it'; 'It is a mark of the divine mind always to be thinking of something fine [*kalon*].'[12]

ARCHYTAS

Philosopher, mathematician, and statesman: Archytas (c. 425–c. 355 BCE) seems to fit simultaneously the pythagorean and

[1] Stewart, 'The Canon of Polykleitos', p. 131.

[2] Tatarkiewicz, *Hist. Aesthetics*, vol. 1, p. 89.

[3] Diogenes Laertius, 'Democritus', p. 48.

[4] Heath, *Hist. Greek Math.*, vol. 1, pp. 176–81.

[5] DK, 68B299; orig. Clemens, *Stromata*, § 1.16.69; trans. Freeman, *Ancilla*, p. 119.

[6] Taylor, *The Atomists*, p. 158, n. 3.

[7] DK, 68A33, §§ VII–VIII; trans. Diogenes Laertius, 'Democritus', pp. 47–8.

[8] Dehn, 'The Mentality of the Mathematician', p. 21.

[9] Archimedes, *The Method*, p. 13.

[10] Archimedes, *Sphere and Cylinder*, bk 1, Introduction.

* ▸ pp. 84–6

[11] Taylor, *The Atomists*, pp. 223–7.

[12] DK, 68B105, 68B112; trans. Taylor, *The Atomists*, pp. 237–8 [§§ 105, 112].

the Platonic paradigms of the scientist-philosopher and the philosopher-king. He was credited with an argument for the infinitude of the universe,[1] found a method to duplicate the cube, and was an undefeated general [*stratēgos* στρατηγός], elected seven times to that post by the citizens of Tarentum.[2] Archytas was arguably the key figure who led to the engagement of three groups of thinkers: the Italian pythagorean school, the school of philosophers around Plato, and the 'first network' of pre-Euclidean Greek mathematicians.[3]

Later tradition consistently labels Archytas a pythagorean, although Aristotle wrote now-lost books on Archytas' thought which are separate from his writings on the pythagoreans. He certainly descended in some respect from the pythagorean tradition, but before the fourth century CE, Archytas was never linked to pythagorean religion or mysticism;[4] he may simply have followed a pythagorean way of life.[5] Aristotle wrote books *On the Archytan Philosophy* and *A Summary of the Timaeus and the Works of Archytas*.[6] These writings were double the attention he devoted to fifth-century pythagoreanism, suggesting that, for Aristotle, Archytas was a far more important philosopher than Pythagoras or Philolaus.[7]

As a mathematician, Archytas' supreme achievement was his solution to the problem of duplicating the cube: how to find the side length of a cube whose volume is twice that of a cube with a given side length. In modern terms, this requires multiplying the side length by $\sqrt[3]{2}$. According to Theon of Smyrna (*fl.* 127–32 CE[8]), who cited Eratosthenes' (c. 276–c. 195 BCE) lost work *Platonicus*, the question arose when an oracle required the Delians, if they wished to be rid of a plague, to construct a cubic altar twice the size of the existing one. Hippocrates of Chios (*fl.* late 5th century BCE) proved that the problem of duplicating the cube can be reduced to finding two geometric means in continued proportion. Thus, if the original side length is ℓ, it is sufficient to find x and y such that $\ell/x = x/y = y/(2\ell)$; then $x = \ell\sqrt[3]{2}$. But a method for finding two geometric means in continued proportion can also be applied to scale an object to any multiple of its original volume; such a method is thus of direct practical importance. Hippocrates' result served as the basis for all later attempts to duplicate the cube, including that of Archytas,[9] and the name 'duplicating the cube' also became attached to the more general abstract problem of finding two geometric means in continued proportion.[10] Archytas' own solution is a sophisticated three-dimensional argument that finds the required geometric means using the point where three surfaces intersect (see Figure 2.3).[11]

The three means

One of the surviving fragments of Archytas' work discusses the 'three means in music', which came to play a significant role in later harmonics, mathematics, and cosmogony:

[1] DK, 47A24; orig. Simplicius, *In Physicorum*, pp. 467–8; trans. Huffman, *Archytas*, p. 541.

[2] DK, 47A1; orig. Diogenes Laertius, 'Archytas', MEI 79; trans. Huffman, *Archytas*, p. 256.

[3] Netz, 'The problem of Pythagorean mathematics', pp. 167–71, 181–2.

[4] Huffman, *Archytas*, p. 25.

[5] *Ibid.*, p. 8.

[6] DK, 47A13; orig. Diogenes Laertius, 'Aristotle', MEI 5.25; trans. Huffman, *Archytas*, p. 580.

[7] Huffman, *Archytas*, p. 45.

[8] Heath, *Hist. Greek Math.*, vol. 2, p. 239.

[9] *Ibid.*, vol. 1, p. 246.

[10] Netz, *Ludic Proof*, p. 160.

[11] Heath, *Hist. Greek Math.*, vol. 1, pp. 246–8; DK, 47A14; orig. Eutocius, *Comm. Sphaera et Cylndro*, pp. 84–8; trans. Huffman, *Archytas*, pp. 343–4.

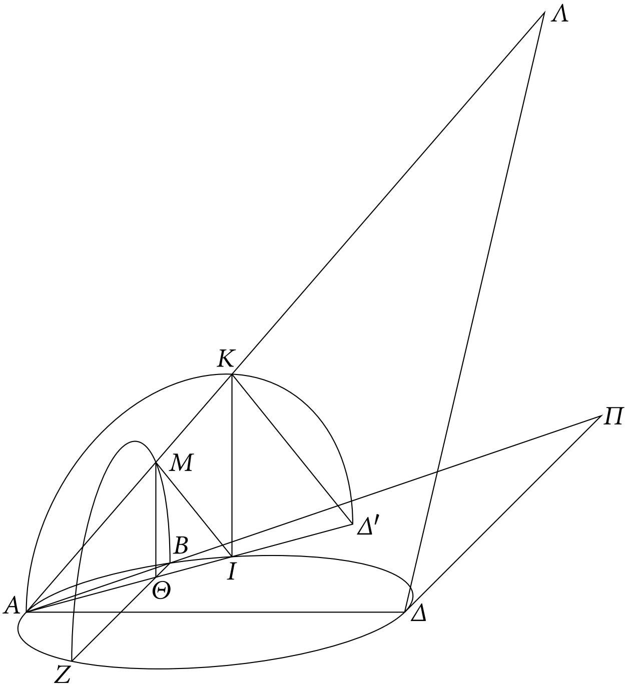


FIGURE 2.3
Archytas’ solution to finding two geometric means in continued proportion between the segments AB and $A\Delta$. A circle is drawn with diameter $A\Delta$, and AB is moved to become a chord of this circle. The point Π is the intersection of the extension of AB and the tangent to this circle at Δ. The point K is defined as the intersection of three surfaces: (1) that traced by a semicircle with diameter $A\Delta$, perpendicular to the plane $AB\Delta$, rotated about the axis through A perpendicular to this plane; (2) the semicylinder perpendicular to the semicircle $AB\Delta$; (3) the cone formed by rotating the triangle $A\Delta\Pi$ about $A\Delta$. The result is that $AB : AI :: AI : AK :: AK : A\Delta$.

Arithmetic mean. ‘The mean is arithmetic, whenever three terms are in proportion by exceeding one another in the following way: by that which the first exceeds the second, by this the second exceeds the third.’ (In modern terminology, the three terms p, q, r are such that $p = q + c$ and $q = r + c$ for some c, or, equivalently, $q = (p + r)/2$.)

Geometric mean. ‘The mean is geometric, whenever they [the terms] are such that as the first is to the second so the second is to the third.’ (That is, the three terms p, q, r are such that $p/q = q/r$, or, equivalently $q = \sqrt{pr}$.)

Harmonic mean. ‘The mean is subcontrary, which we call harmonic, whenever they [the terms] are such that, by which part of itself the first term exceeds the second, by this part of the third the middle exceeds the third.’[1] (That is, the three terms p, q, r are such that $p = q + p/n$ and $q = r + r/n$ for some n, or, equivalently, $q = 2pr/(p + r)$, or, again equivalently $(p - q)/(q - r) = p/r = n$. The last formulation is perhaps the simplest: the two differences have the same ratio as the two outer terms.)

These three means were known in the time of Pythagoras and his followers. The third was originally called ‘subcontrary’, but Archytas and Hippasus (*fl.* 5th century BCE) and their followers called it ‘harmonic’ because it ‘manifestly embraced the ratios of what is harmonic and melodic’.[2] For instance, 2, 4/3, 1 form a harmonic mean, and, as was noted in Philolaus’ harmonic theory,* the fourth 4/3 and fifth 3/2

[1] All quotations from DK, 47B2; orig. Porphyry, *Kommentar zur Harmonielehre des Ptolemaios*, pp. 92–4; trans. Huffman, *Archytas*, pp. 163–4, insertion in trans. See also Heath, *Hist. Greek Math.*, vol. 1, p. 85.

[2] DK, 47B2; trans. Huffman, *Archytas*, p. 164.

* ▸ p. 50

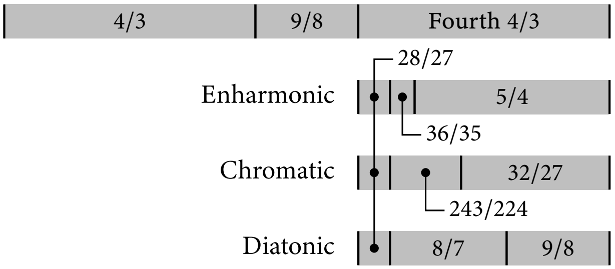
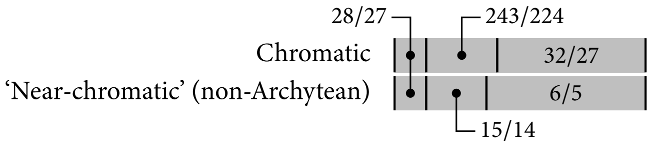

4/3
9/8
Fourth 4/3
28/27
Enharmonic
5/4
36/35
Chromatic
32/27
243/224
Diatonic
8/7
9/8

FIGURE 2.4
Illustration of the enharmonic, diatonic, and chromatic divisions of the fourth in Archytas' harmonic theory (as related by Ptolemy, *Harmonics*, § I.13, DÜR 30-1). (Lower notes are to the left; higher notes are to the right.)

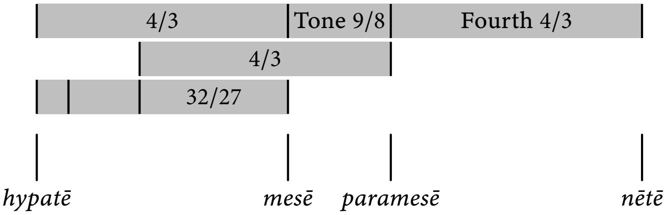
(which is composed of a fourth and a tone 9/8) make up the octave 2/1. [The new name *may* have actually been coined by Philolaus, but this attribution is questionable.[1]]

Harmonic theory

Ptolemy described Archytas' division of the fourth in his *Harmonics*;[2] this testimony is generally regarded as authentic.[3] According to this account, Archytas proposed three ways of dividing the fourth: the *enharmonic*, with intervals whose ratios are 28/27, 36/35, 5/4, listed from lowest to highest; the *chromatic*, with ratios 28/27, 243/224, 32/27; and *diatonic*, with ratios 28/27, 8/7, 9/8 (see Figure 2.4). Notice that each ratio in Archytas' enharmonic and diatonic genera, and the lowest ratio in the chromatic genus, is a superparticular ratio. However, the other two ratios in the chromatic genus are not superparticular. Recall that Philolaus seems not to have conceived of studying harmonics using mathematical manipulation of ratios, and so the non-superparticular ratio corresponding to his *diesis* would not be a concern. However, Archytas *did* use such calculation and yet consciously chose non-superparticular ratios for the chromatic scale, despite there being an approximate alternative choice of ratios: 28/27, 15/14, 6/5 (listed from lowest to highest):


28/27
243/224
Chromatic
32/27
'Near-chromatic' (non-Archytean)
6/5
15/14

Archytas' choice is explained by considering how a practising musician could tune their instrument. One could tune *up* by a tone (9/8) and then *down* by a fourth (4/3) to achieve the interval 32/27:


4/3
Tone 9/8
Fourth 4/3
4/3
32/27
hypatē
mesē
paramesē
nētē

---

[1] Barker, *The Science of Harmonics*, pp. 283–5.

[2] Ptolemy, *Harmonics*, § I.13, DÜR 30–1.

[3] Huffman, *Archytas*, pp. 406–7.

A musician would require the note *paramesē* (which Philolaus called *tritē*) in any case, and so tuning the note below *mesē* by descending a fourth from *paramesē* would be economical.[1] The important conclusion here is that though Archytas did seem to respect where possible the pythagorean prescription of ratios of the form $(n + 1)/n$, he gave precedence to his observations of practising musicians. That is, regardless of whether it can be correctly described as empirical, Archytas' musical scale is directed towards the world.[2] In general, Archytas' approach to music seems to have been guided by observation: for example, he gathered and listed a variety of evidence to support his theory of the relation between the pitch of a note and the speed of the motion that produces it.[3]

Archytas' predecessors knew that it is impossible to find a geometric mean in the octave, in the sense that there are no commensurable terms p, q, r with $p/r = 2/1$ that form a geometic mean (that is, there are no whole-number terms p, q, r, with $p/r = 2/1$ and $p/q = q/r$); this is equivalent, in modern terminology, to the irrationality of $\sqrt{2}$. Archytas proved that it is impossible to find a geometric mean in any superparticular ratio $(n + 1)/n$, such as the octave or tone.[4] Thus for Archytas, with his focus on the practice of music, the geometric mean perhaps had less importance in harmonics than the arithmetic or harmonic means.

Unlike for Philolaus, there is no evidence indicating that Archytas' harmonic theory was applied to cosmogony. However, in general terms, Archytas seems to have attempted to take further Philolaus' aim of understanding the world through number. He held that the sciences of astronomy, geometry, arithmetic, and harmonics had an affinity: 'concerning the speed of the stars and their risings and settings as well as concerning geometry and numbers and not least concerning music [...] these sciences seem to be akin'.[5]

Logistic and ethics

Archytas suggested that knowledge of logistic — in the sense of arithmetical calculation — helps inculcate ethical behaviour: 'by means of calculation we will seek reconciliation in our dealings with others';[6] further, it deters those who would commit injustice, for calculation would lead to their being detected. Thus, given Archytas' view that there was no greater evil than bodily pleasure,[7] it is clear that one must endeavour to maintain the ability to calculate without emotional or hedonistic impediment, for such reasoning is vital to ethical behaviour.[8]

Archytas wrote that '[l]ogistic seems to be far superior indeed to the other arts in regard to wisdom and in particular to deal with what it wishes more concretely (clearly) than geometry.'[9] Note that Archytas did not praise absolutely logistic above other sciences, but only with regard to wisdom and concreteness. The superiority of logistic comes

[1] Barker, *The Science of Harmonics*, pp. 288–9.

[2] Huffman, *Archytas*, pp. 63, 412 sqq.

[3] DK, 47B1; orig. Porphyry, *Kommentar zur Harmonielehre des Ptolemaios*, pp. 55–8; trans. Huffman, *Archytas*, pp. 105–7.

[4] DK, 47A19; orig. Boethius, *De Institutione Musica*, ch. III.ii, pp. 285–6; trans. Huffman, *Archytas*, pp. 452–3.

[5] DK, 47B1; trans. Huffman, *Archytas*, pp. 105–6.

[6] DK, 47B3; orig. Iamblichus, *De Communi Mathematica Scientia*, p. 44, ll. 10–17; trans. Huffman, *Archytas*, p. 183.

[7] Cicero, *De Senectute*, XII.39–40; trans. Huffman, *Archytas*, pp. 323–4.

[8] Cf. Huffman, *Archytas*, p. 74.

[9] DK, 47B4; orig. Stobaeus, *Anthologium*, bk I, proem, § 4 ⟦vol. 1, p. 18⟧; trans. Huffman, *Archytas*, p. 225.

about because it is necessary, when dealing with a thing in the real world, to measure or describe it using numbers and proportion: as Polykleitos noted,[*] beauty in sculpture comes about 'through many numbers', not shapes.[1] Archytas' remarks suggest that he held that wisdom had to do with the sensible world, and that logistic was the highest science because it yields such wisdom.[2]

Legacy

Different historiographical trends record entirely opposite relationships between Archytas and Plato:[†] one sees Plato the student learning from Archytas after the death of his first teacher Socrates; the other maintains that Archytas' success as a soldier and statesman were specifically due to his studying under Plato.[3] As the classicist Carl Huffman (1951–) pointed out, these strands can be reconciled: Plato sought out Archytas for his mathematical knowledge, because he was keen to apply mathematics to aid in tackling philosophical problems; Archytas, in turn, may have asked Plato to teach him dialectic.[4]

Plato agreed with Archytas that the mathematical sciences are akin.[5] But Archytas' view that wisdom was concerned with the sensible world stands in direct opposition to Plato's view that wisdom was concerned with knowledge of the Forms. Unlike Archytas, Plato saw mathematics as a route away from the sensible world toward knowledge about the intelligible Forms.[‡] This fundamental disagreement may have influenced Plato's writing of the *Timaeus*. The central figure in this dialogue is Timaeus of Locri, who is otherwise unknown,[6] and who has sometimes been considered a veiled portrayal of Archytas: like Archytas, the character Timaeus is from a southern-Italian city-state, is a statesman and philosopher, and is well-versed in mathematics. Given the fundamental opposition between Plato's and Archytas' positions, it is more probable that Timaeus was a fictional anti-Archytas.[7]

SOCRATES

> ‘Socrates autem primus philosophiam devocavit e caelo et in urbibus collocavit’
>
> — Cicero, *Tusculan Disputations*, § v.iv.

The life and work of Socrates (c. 469–399 BCE) stands as the crisis that brought Greek and thus western philosophy into focus: his precursors, whatever their achievements, remain forever the *pre*-Socratics. It was Socrates who took philosophy from the study of nature into the study of human matters, from which it never withdrew. According to Xenophon (c. 430–355/354 BCE), at least one of his

[*] ▸ pp. 51–2
[1] Huffman, *Archytas*, p. 72.
[2] *Ibid.*, p. 71.
[†] ▸ pp. 58–81
[3] *Ibid.*, pp. 32–40.
[4] *Ibid.*, pp. 40–2.
[5] Plato, *Republic*, STE VII.530d.
[‡] ▸ pp. 74–7
[6] Guthrie, *The Later Plato*, p. 244.
[7] Huffman, *Archytas*, p. 86.

motivations for this new direction was that the cosmologists' speculations were dogmatic and useless and mutually contradictory.[1] 'Useless' here should be construed as irrelevant to what Socrates saw as the proper study of mankind, namely the kind of human concern that he himself discussed exclusively: what is pious, beautiful, just, prudent, or courageous.[2]

Socrates' preëminence over earlier philosophers is of course partly due to his having in Plato a follower with the intellect and the drive to set down, expand on, and disseminate his thought and methods. But Plato's very reverence for Socrates led him to use the character of Socrates as his spokesman, possibly believing that Socrates, had he lived longer, would have come to at least some of the same conclusions. Sifting the thought of Socrates from the dialogues of Plato is a component of the broader problem of the historical Socrates, which has its own complex history.[3]

Perhaps fortunately, it is of little relevance here. With Socrates, examining the notion of beauty became a definitive part of the philosophical enterprise. But the Socrates that existed outside of Plato's dialogues seems unlikely to have influenced the development of the aesthetics of mathematics. As Xenophon related it, Socrates taught that sufficient geometry should be learned for practical purposes such as measuring land, and that it was easy to learn both to practise the art and to understand it.[4] However, despite having knowledge of higher mathematical learning, he disdained it as a distraction:

> 'He was against taking the study of geometry so far as to include the more complicated figures, on the ground that he could not see the use of them. Not that he was himself unfamiliar with them, but he said that they were enough to occupy a lifetime, to the complete exclusion of many other useful studies.'[5]

PLATO

> ‘ SOCRATES: [...] I think I know the meaning of the proverb "beautiful things are difficult". ’
>
> — Plato, *Greater Hippias*, STE 304e
> (trans. H. N. Fowler).

Plato's (c. 427–348/347 BCE) monumental corpus and the enormous wealth of ideas it contains led Alfred North Whitehead (1861–1947) to his famous dictum that philosophy in the western tradition could safely be characterized as 'a series of footnotes to Plato'.[6] It is unsurprising that this epithet applies also to the study of beauty in mathematics: Plato's arguments overshadowed philosophy in this area, certainly up to the eighteenth century, and only definitively ceded preëminence in the twentieth century, to G. H. Hardy's (1877–1947) treatment in *A Mathematician's Apology*.*

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[1] Xenophon, *Memorabilia*, § I.1.13–15.

[2] *Ibid.*, § I.1.16.

[3] Dorion, 'The Rise and Fall of the Socratic Problem'.

[4] Xenophon, *Memorabilia*, § IV.7.2.

[5] *Ibid.*, § IV.7.3.

[6] Whitehead, *Process and Reality*, p. 39.

* ▸ pp. 498–523

Whatever the claims of platonists and neoplatonists, it is a stretch to count Plato as a mathematician.[1] Plato was rather a philosopher who had a deep appreciation for mathematics, who pushed for its advancement by society,[2] and who was fundamentally affected by mathematics as a system and as a way of thinking.[3] Studying mathematics may have resulted in his disillusionment with the Socratic elenchus as a method yielding truth.[4] Possibly it was the experience of 'the feeling of proof'[5] or the beauty of mathematics[6] (which perhaps overlap greatly), that led him to posit eternal immaterial entities and thence to formulate his theory of Forms. He seems to have understood the excitement of mathematical discovery well enough to give a convincing portrayal of Theaetetus describing it.[7] At the very least, it is *not* a stretch to link the prominence he granted mathematics in his work to his appreciation of its beauty.[8]

[According to tradition, there was a sign 'ΑΓΕΩΜΕΤΡΗΤΟΣ ΜΗΔΕΙΣ ΕΙΣΙΤΩ' (roughly, 'let no one ignorant of geometry enter') at the entrance to Plato's Academy; the earliest testimony is a scholium by Sopater of Apamea (*fl.* early 4th century CE), which does not ascribe the sign to Plato himself.[9]]

Plato was the student and possibly the publisher of Socrates. In each dialogue in which he appears, the character of Socrates may reflect the historical Socrates, indicate what Plato thought the historical Socrates would have or should have become, or simply be a spokesman for Plato himself. Perhaps the earlier dialogues are closer to the historical Socrates, or perhaps in every one Socrates is Plato's champion. This is not the place to enter into this scholarly debate. But it does not seem too inaccurate to hold that, if Socrates brought philosophy to earth, Plato, armed with the Socratic method of inquiry and aware of the outlooks of the pythagoreans (Philolaus, Archytas) and atomists (Leucippus, Democritus) and Eleatics (Xenophanes, Parmenides, Empedocles) took it back to the heavens and sought explanations for the world.

Plato's views on beauty can be found throughout his work in connection with rhetoric, poetry, justice, politics, cosmogony, and mathematics. Of course Plato's thought developed during his lifetime, and a degree of caution is needed in building a synthesis from different dialogues. The idea that the good and the beautiful are connected is one that recurs in Plato's work. As a single example, consider what Socrates says in the *Philebus*: 'if we cannot catch the good with the aid of one idea, let us run it down with three — beauty, proportion, and truth [*kallei kai symmetria kai alētheia* κάλλει καὶ ξυμμετρίᾳ καὶ ἀληθείᾳ]'.[10]

A coherent picture emerges regarding Plato's views of beauty in mathematics. The basic idea that mathematics can be a useful tool in the philosophical enterprise is one that Plato inherited from Philolaus and Archytas, one or both of whom he may have met on his first visit to Italy in 388–387 BCE.[11] But some of the functions Plato assigns to

[1] Dantzig, *The Bequest of the Greeks*, p. 25.

[2] Cornford, 'Mathematics and Dialectic II', p. 174.

[3] Hacking, 'What Mathematics Has Done', pp. 91–9.

[4] Vlastos, 'Elenchus and Mathematics'.

[5] Hacking, 'What Mathematics Has Done', pp. 98–9.

[6] Whitehead, *Adventures of Ideas*, p. 275.

[7] Plato, *Theaetetus*, STE 147d–8b; cf. Szabó, *The Beginnings of Greek Mathematics*, pp. 82–3.

[8] Vlastos, 'Elenchus and Mathematics', p. 388.

[9] Saffrey, 'Ἀγεωμέτρητος Μηδεὶς Εἰσίτω', pp. 72–3.

[10] Plato, *Philebus*, STE 65a; cf. Plato, *Gorgias*, STE 474d–5a; Plato, *Symposium*, STE 201c, 204b; Plato, *Cratylus*, STE 440b.

[11] Diogenes Laertius, 'Plato', MEI 6; Guthrie, *Plato: The Man and His Dialogues*, pp. 17–19.

mathematics, notably its role in turning the mind from the sensible world to the intelligible Forms, are his own. Further, he is the first extant author to consider beauty in mathematics generally, rather than just to apply mathematics to explain aesthetic concerns, as in harmonics.

Early explorations of beauty

In his early dialogue the *Greater Hippias* (which, although previously thought spurious, is generally accepted today) Plato had Socrates ask his sophist interlocutor Hippias, on the pretext of wanting to enlighten an argumentative friend, 'what is the beautiful [*to kalon* τὸ καλόν]'?[1] After various false starts of attempts to answer by example, during which Hippias grasps the meaning of the question, Socrates begins to propose various definitions, each of which is tested and found wanting:

Beauty consists in fittingness or appropriateness. This definition is rejected because appropriate clothes can make an ugly person handsome, and so this may only create the appearance of beauty, not actual beauty.[2]

Beauty is that which is useful. This definition is then clarified as 'useful for good', which means 'beneficial' or 'causing good'. But a cause and its effect must be distinct, so this possibility contradicts the identification of the good and beautiful.[3]

Beauty is pleasure through hearing and sight. This definition is also rejected, by a rather complicated argument.[4]

Totally absent here is any hint of beauty in mathematics: pleasure through hearing and sight could have been rejected immediately on this ground. But in this early dialogue, Plato was either staying close to what Socrates said (or what Plato thought he would have said), or else mathematics had yet not been fully assimilated into Plato's philosophy. Absent too are the (at least partly) mathematical criteria whereby beauty and goodness consist in a certain order brought about through measure, proportion, and symmetry.[5]

Beauty and the mathematical

In his later dialogues, Plato's conception of beauty took a turn towards the mathematical: specifically, in the *Philebus* it is agreed that 'measure [*metriotēs* μετριότης] and proportion [*symmetria* συμμετρία] are everywhere identified with beauty [*kallos*] and virtue [*aretē* ἀρετὴ]'.[6] Note also the recurrent adjacency of aesthetic and ethical value, also mentioned in the *Timaeus*, which again posits measure as a condition for beauty: 'All that is good is fair [καλὸν], and the fair is not void of due measure'.[7] Nevertheless, Plato had not abandoned usefulness as an aspect of beauty, for in the *Republic*, his spokesman Socrates maintains that

[1] Plato, *Greater Hippias*, STE 287d.

[2] *Ibid.*, STE 293e sqq.

[3] *Ibid.*, STE 295c sqq.

[4] *Ibid.*, STE 197e sqq.

[5] Guthrie, *Plato: The Man and His Dialogues*, p. 183.

[6] Plato, *Philebus*, STE 64e.

[7] Plato, *Timaeus*, STE 87c.

> ‘the excellence [*aretē*], beauty [*kallos*], and correctness [*orthotēs* ὀρθότης] of each piece of equipment, living creature, and activity has no other purpose than the usage for which each of them has been created or developed by nature’.[1]

The characterizations of or conditions for beauty involving ‘measure’ or ‘proportion’ are not explicitly applied to mathematics in the preceding quotations. But elsewhere, in the *Philebus*, the character of Socrates is much more explicit:

> ‘when I say beauty of form,[2] I am trying to express, not what most people would understand by the words, such as the beauty of animals or of paintings, but [...] the straight line and the circle and the plane and solid figures formed from these by turning-lathes and rulers and patterns of angles; perhaps you understand. For I assert that the beauty of these is not relative, like that of other things; but they are always absolutely beautiful by nature and have peculiar pleasures in no way subject to comparison with the pleasures of scratching’.[3]

[Plato was also careful to distinguish calculation and measurement as practised by builders and traders from arithmetic and geometry as studied by philosophers, the latter being ‘infinitely superior yet in precision and truth’.[4]]

Loci of beauty

The most important source for Platonic loci of beauty in mathematics is the dialogue *Timaeus*, which was almost certainly, though not definitely, composed late in Plato’s career.[5] This dialogue is mainly a lecture on the origin of the universe by Timaeus of Locri, whose character may have been inspired by Archytas and Plato’s disagreement with him.* In the dialogue, Timaeus admitted that one cannot hope to give an exactly correct explanation, but that he would give an account that his listeners should accept as probable.[6] He described how an artificer, the demiurge [*dēmiourgos* δημιουργὸς], shaped the universe. The demiurge did not create the universe, but worked with a primordial world-stuff. Timaeus’ explanation is mathematical, and is openly intended for experts, but he knew his audience would be familiar with the necessary learning.[7]

Circles and spheres. The demiurge had aesthetic goals in forming a universe ‘harmonized by proportion’ and ‘united in identity with itself’,[8] He desired that the universe should be a living creature and that it should be perfect both in its parts and as a whole.[9] There was thus a befitting choice for the shape of the universe: a sphere,

> ‘which comprises within itself all the shapes there are [...] which of all shapes is the most perfect and the most self-similar,

* ▸ p. 57

---

[1] Plato, *Republic*, STE x.601d.

[2] Beauty of shapes [*schēmata* σχημάτων], not of Forms [*eidōn* ειδῶν].

[3] Plato, *Philebus*, STE 51c–d.

[4] *Ibid.*, STE 57d.

[5] Guthrie, *The Later Plato*, pp. 243–4.

[6] Plato, *Timaeus*, STE 29c–d.

[7] *Ibid.*, STE 53d.

[8] *Ibid.*, STE 32d.

[9] *Ibid.*, STE 33a.

> since He deemed that the similar is infinitely fairer [*kallion*] than the dissimilar.'[1]

The phrase 'comprises within itself all the shapes' may refer to the possibility of inscribing the five regular solids in a sphere, which is mentioned later in the dialogue.[2]

Plato's conception of a spherical universe in the *Timaeus* was not *ex nihilo*. Anaximander's cosmology placed the heavenly bodies on circular rings;[*] Philolaus thought the universe spherical.[†] In the earlier dialogue *Phaedo*, Plato had stated that the Earth was a sphere and *implied* that the heavens formed a sphere: a central Earth in a heaven homogeneous in every direction would remain stable, for it would have no tendency to move in any direction.[3] But only in the *Timaeus* did Plato explicitly accede to the aesthetic primacy of the sphere previously asserted by Pythagoras.[‡]

The late neoplatonist philosopher and commentator Simplicius of Cilicia (*fl.* 6th century CE) reported, on the authority of Sosigenes ('the Peripatetic', *fl.* 2nd century CE), that Plato had set the problem: 'on what hypotheses of uniform and ordered motions could the phenomena concerning the motions of the planets be preserved?'[4] The 'uniform and ordered motions' are said to be 'circular motions' a few lines earlier. The phenomena to be explained included the diurnal rising and falling of the planet, the general apparent eastward motion of the planet relative to the 'fixed stars', and the apparent retrograde motion, wherein the usual eastward movement is sometimes interrupted by short periods of apparent westward motion. The problem was to show that these apparently complex motions could be resolved into some system of circular motions.

However, Simplicius wrote nine centuries after Plato flourished, and, though he cited Sosigenes, who possibly derived this information from the history of astronomy written by Eudemus of Rhodes (*fl.* late 4th century BCE), his account would be a shaky foundation on which to erect any argument.[5] Nevertheless, the challenge was certainly implicit in astronomy. Circular motion is not only a reasonably accurate way to model the motion of the planets, but lends itself to straightforward calculation of future motion.[6]

The three means. To create the movement of the heavenly bodies, the demiurge took a strip of world-stuff, divided it, and shaped it into circles to carry them. Before the division, he marked off intervals on the strip. First, he marked off 1, 2, 4, 8 and 3, 9, 27: three intervals with ratio 2/1 and three with ratio 3/1. Within each interval, he marked the harmonic and arithmetic means: this divided each interval 2/1 into intervals 4/3, 9/8, 4/3 and each interval 3/1 into 3/2, 4/3, 3/2. Finally, each interval 4/3 is divided into two intervals 9/8 and an interval 256/243, presumably, though this is not specified, in the same way as Philolaus described.[7] The *Timaeus* does not make clear whether the 1–2–4–8 and 1–3–9–27 divisions were superimposed.

[1] Plato, *Timaeus*, STE 33b–c.

[2] *Ibid.*, 55a.

[*] ▸ pp. 42–3

[†] ▸ p. 49

[3] Plato, *Phaedo*, STE 109a.

[‡] ▸ p. 44

[4] Simplicius, *On Aristotle On the Heavens*, HEI 488.22 ⟦vol. 2.10–14, p. 29⟧ (comment. on Aristotle, *On the Heavens*, BEK II.12.292b10 sqq.).

[5] Vlastos, *Plato's Universe*, Appendix, § L.

[6] Guthrie, *The Later Plato*, p. 296, n. 2.

[7] Plato, *Timaeus*, STE 35b–6c; for a detailed explication, see Cornford, *Plato's Cosmology*, pp. 66–72.

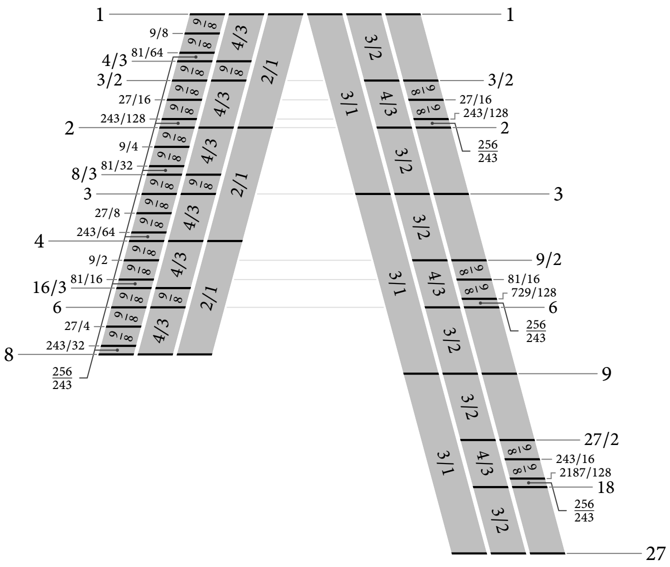

FIGURE 2.5
The intervals that, according to Timaeus, were used by the demiurge in constructing the heavens (Plato, *Timaeus*, STE 35b–6c).

In the early Academy, Crantor (d. c. 275 BCE) and Cleararchus of Soli (b. c. 340 BCE) preferred a scheme where the divisions were marked on a Λ-shaped figure, with 1 at the apex, 2, 4, 8 on one leg and 3, 9, 27 on the other; Theodorus of Soli (*fl.* 4th century BCE) had them marked on a single strip. Both schemes had their historical followers;[1] Figure 2.5 illustrates the divisions using the former layout.

Although this part of the dialogue contains no explicit mention whatsoever of music, the demiurge's division clearly parallels, and is presumably based upon, the Philolaan* and Archytan† harmonic theories. The first division of each 'octave' 2/1 is the division Philolaus described: the octave is made up of a fourth 4/3, a tone 9/8, and another fourth 4/3. Philolaus may not have thought of these divisions in terms of the harmonic and arithmetic means, for he did not think of calculating with intervals, but Archytas defined the three means used in music.‡ The division of the interval 3/1 is the precise mathematical analogue of the division of the octave.

However, the range from 1 to 27 comprises four octaves, a fifth, and a tone, so while building music intervals into the world-soul can be seen as an aesthetic choice, Plato's upper limit did not come from the theory of musical harmonics: the range is far greater than any used in practice.[2] Plato was rather thinking that the world-soul involves a harmony — musical in an abstract mathematical sense — that extends to the solid (that is, to the third powers $2^3 = 8$ and $3^3 = 27$).[3] Taking the powers of both 2 and 3 may be an indication of the pythagorean even–odd duality. Plato elsewhere disparaged an approach to harmonics that investigated empirically which ratios corresponded to consonant intervals.[4] But his criticism does not imply that the intervals the demiurge used were a purely mathematical structure that Plato saw as detached from music, for, later in the *Timaeus*, he

[1] Dillon, *The Heirs of Plato*, pp. 223–4.

* ▸ p. 50

† ▸ pp. 55–6

‡ ▸ pp. 53–5

[2] Barker, *The Science of Harmonics*, p. 321.

[3] Cornford, *Plato's Cosmology*, pp. 67–8.

[4] Plato, *Republic*, STE VII.531b–c.

made use of the analogy between the structure of the heavens and the harmony of audible music.[1] Plato was perhaps aware of the tension between the musical origin of the structure he described and his desire to completely mathematize it; his silence on its Philolaan–Archytan ancestry may be a deliberate result of this tension.[2]

As discussed, the intervals used by the demiurge were constructed using the harmonic and arithmetic means. The third mean, the geometric, played an even more fundamental role, in the making of the primary substances. Plato adopted the Empedoclean doctrine of four primary substances: fire, water, earth, and air. Unlike Empedocles, Plato offered a justification for this doctrine. His justification is a mixture of an appeal to empirical observation and an argument that draws on mathematical aesthetics and symbolism. Empirically, two of the elements are evidenced directly by the world: the visibility of things requires fire, and the solidity of things requires earth. A third substance is required to bond the first two, and for this Plato looked to mathematics, choosing the geometric mean. A fourth substance is required because the cosmos is to be 'solid' (in modern terminology, three-dimensional): for a plane, one geometric mean would suffice to bind together the three components; for a solid, two are required. Plato here was addressing expert readers who could fully unpick his rather abbreviated argument.[3] He used 'plane' and 'solid' to refer to square and cubic numbers.[4] His point was that a single instance of the geometric mean suffices to connect two square numbers p^2 and q^2:

$$p^2 : pq = pq : q^2,$$

but two instances of the geometric mean are necessary to connect two cubic numbers p^3 and q^3:

$$p^3 : p^2q = p^2q : pq^2 = pq^2 : q^3.$$

So to unite the two (solid) primary substances fire and earth, two geometric means are required. In the necessity for two means, there is an echo of the problem of duplicating the cube, which Hippocrates of Chios reduced to finding two geometric means in continued proportion.* There is also a shadow here of the demiurge's use of powers of 2 and 3 in the first stage of marking off intervals: 1, 2, 4, 8; 1, 3, 9, 27. The sequences must reach the third powers to extend to the solid: as a trivial detail — but possibly for Plato a pleasant one — 2 and 4 are the required geometric means linking 1 to 8; and 3 and 9 the means linking 1 to 27. By the mathematical bond the cosmos enjoys unity through *philia* [φιλία],[5] the same quality that Plato saw joining heaven and Earth, gods and men.[6]

Why is the geometric mean used as the bond? The choice is aesthetic: it is the 'fairest [*kallista*] of bonds', because it 'most perfectly unites into one both itself and the things which it binds together'.[7] If p, q, r form a geometric mean, then, as Plato explained it:

[1] Plato, *Timaeus*, STE 47c–e.

[2] Cf. Huffman, *Philolaus*, p. 150.

[3] Cornford, *Plato's Cosmology*, p. 47.

[4] Heath, *Hist. Greek Math.*, vol. 1, p. 89.

* ▸ p. 53

[5] Plato, *Timaeus*, STE 32c.

[6] Plato, *Gorgias*, 508a.

[7] Plato, *Timaeus*, STE 31c.

> ‘the middle term of any three numbers […] is such that as the first term is to it, so is it to the last term,
>
> $$p : q :: q : r$$
>
> — and again, conversely, as the last term is to the middle, so is the middle to the first, —
>
> $$r : q :: q : p$$
>
> then the middle term becomes in turn the first and the last, while the first and last become in turn middle terms,
>
> $$q : r :: p : q$$
>
> and the necessary consequence will be that all the terms are interchangeable, and being interchangeable they all form a unity.’[1]

Interchangeability of the terms gives unity, unity makes the geometric mean merit the aesthetic superlative of ‘fairest’. Thus the geometric mean is seemingly elevated above the arithmetic and harmonic means. The definitions of the arithmetic and harmonic means both require an extra ‘outside’ term: in the arithmetic mean, the same excess; in the harmonic mean, the same ‘part’. Thus they do not bond terms together as cleanly as the geometric mean; this difference may underlie Plato’s preference for it. In celebrating the geometric mean as the fairest, Plato made a perhaps deliberate step away from any parallel with the empirical harmonics he disdained in the *Republic*,[2] for in dividing the octave to devise a musical scale, he made the arithmetic and harmonic means primary.*

Note how Plato’s use of the geometric mean as an aesthetic–mathematical bond between the primary substances of fire and earth immediately recalls Philolaus’ use of numerically-governed harmony to bond limiting things and unlimited things. It seems probable that Plato consciously set up here a more mathematically sophisticated version of Philolaus’ bond.

It is worth observing that, for Plato, those who were aware of the mathematical structure of the world gained a higher kind of enjoyment from listening to music:

> ‘pleasure [*hedonēn* ἡδονὴν] […] to the unintelligent, and to the intelligent that intellectual delight [*euphrosynēn* εὐφροσύνην] which is caused by the imitation of the divine harmony manifested in mortal motions.’[3]

The fairest triangles. The four elements are each made up of bodies, which must now be determined. Aesthetic considerations guide the choice throughout: the bodies of the elements are ‘the four fairest [*kallista*] bodies, dissimilar to one another, but capable in part of being produced out of one another by means of dissolution’,[4] and these turn out to be four of the five regular solids. Bodies must be bounded by surfaces, and rectilinear surfaces are composed of triangles. All

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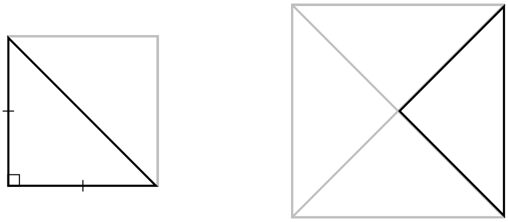
[1] Plato, *Timaeus*, STE 31c–2a.

[2] Plato, *Republic*, STE VII.530d sqq.

* ▸ p. 56

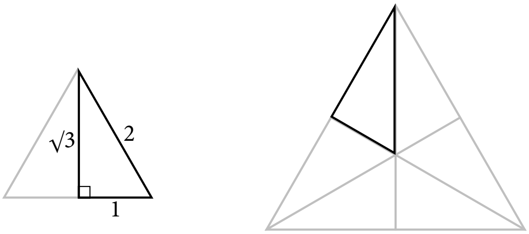
[3] Plato, *Timaeus*, STE 80b.

[4] *Ibid.*, STE 53e.



A right-angled isosceles triangle, forming half of a square.

Assembling four such triangles into a square.



The scalene triangle that forms half of an equilateral triangle.

Assembling six such triangles into an equilateral triangle.

FIGURE 2.6
The isosceles and scalene right-angled triangles used to build the surfaces of the regular solids making up the four elements.

triangles are ultimately derived from two kinds: the right-angled isosceles and the right-angled scalene.[1] The right-angled isosceles triangle has one nature only, and forms half of a square. A right-angled scalene triangle can have any of infinitely many natures: thus it is necessary to choose the most beautiful of these triangles. Plato held that this was the triangle whose sides are in the ratio $1 : \sqrt{3} : 2$, which is half of an equilateral triangle; '[t]he reason why is a longer story'.[2] (See Figure 2.6.) However, he left open the possibility that someone else might discover triangles fairer still, and declared that he would greet such a discoverer with good will and friendship.[3] Plato's reason for thinking this scalene triangle the fairest may have been the small-number ratio $1 : 2 : 3$ of its angles $\pi/6$, $\pi/3$, $\pi/2$.[4]

Plato then proceeded to describe the building of the bodies from these triangles. First of all, the triangles can be combined to form squares and equilateral triangles. Although squares could be built by joining two isosceles triangles along the hypotenuse and equilateral triangles by joining two scalene triangle along the longer leg, Plato took a different route: he posited that four isosceles triangles are joined along their equal legs to make a square and six scalene triangles are joined along the hypotenuses and shorter legs to make an equilaterial triangle (see Figure 2.6).[5] Various reasons have been advanced to explain this choice, for Plato gave none. The assembly of four isosceles triangles to make the square is used in the doubling of the square, which Plato used in the *Meno* to demonstrate that knowledge is recollection,[6]* and the assembly of six scalene triangles is used in the doubling of the equilateral triangle. This possibility may also explain why Plato did not directly use the equilateral triangle

[1] Plato, *Timaeus*, STE 53d.

[2] *Ibid.*, STE 54a.

[3] *Ibid.*, STE 54a–b.

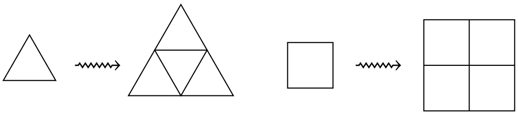
[4] Artmann & Schäfer, 'On Plato's "Fairest Triangles"', pp. 257–8.

[5] Plato, *Timaeus*, STE 54d–e, 55b.

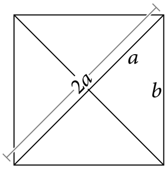
[6] Plato, *Meno*, STE 82b–5b.

* ▸ p. 74

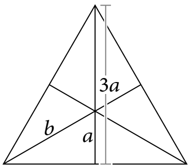
and the square as the primitive elements, for neither of these can be combined in a way that doubles its area, but only in a way that quadruples it:



These constructions may also have influenced Plato's selection of the fairest scalene triangle, especially since there are geometric means — 'fairest of bonds' — within both:[1]



$a : b = b : 2a$



$a : b = b : 3a$

But counting against this possibility is the term 'fairest' only being applied to the chosen scalene triangle, not the isosceles.

But given the importance of aesthetic considerations here and elsewhere in the *Timaeus*, it is not a stretch to say that Plato chose these assemblies because they had greater beauty. Both have symmetry (in both the modern sense and in the ancient sense of *symmetria*), which the two-triangle assemblies lack. Further, 4 is a square number and 6 is a triangular number, and it is not difficult to conceive of the pythagorean in Plato gaining aesthetic pleasure from a symmetrical arrangement of a square number of triangles forming a square, and even more from a symmetrical arrangement of a triangular number of triangles forming an equilateral triangle.

Regular solids. From the equilateral triangles and squares are constructed the solid bodies of the four elements. Four triangles together produce the tetrahedron, the body of fire; eight produce the octahedron, air; twenty produce the icosahedron, water.[2] Six squares together produce the cube, the body of earth.[3] The demiurge used the dodecahedron, the remaining regular solid, for the universe.[4] Plato did not say explicitly that there are exactly five regular solids, nor did he mention the dodecahedron by name. That he knew there were five is implicit in the desire for completeness indicated by his positing a use for the fifth. Further, he mentioned in a different context 'balls which have leather coverings in twelve pieces',[5] which suggests that he knew of the dodecahedron. Unlike for the other four regular solids, he did not describe how to build its pentagonal faces using triangles. It is easy to see that this would have required introducing a third species of triangle, for no combination of the angles of the given two triangles can be put together to form the angle of a regular pentagon: the angles

[1] Artmann & Schäfer, 'On Plato's "Fairest Triangles"', pp. 258–60.

[2] Plato, *Timaeus*, STE 54e–5b, 55e–6b.

[3] *Ibid.*, STE 55c–e.

[4] *Ibid.*, STE 55c.

[5] Plato, *Phaedo*, STE 110b.

of the given triangles are $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, all of which are multiples of $\pi/12$, while the angle of a regular pentagon is $2\pi/5$, which is not a multiple of $\pi/12$.

[Xenocrates (396/5–314/3 BCE), who was the third head of the Academy after Plato and Speusippus[*] (c. 410–c. 340 BCE), found the role assigned to the dodecahedron unsatisfactory and instead made it the body of the aether that Aristotle postulated as a fifth element; giving it this role had the merit of establishing a one-to-one correspondence between the elements and the regular solids.[1]]

The idea that atoms have different shapes goes back to Democritus,[†] who thought the various shapes explained why atoms cling together. But Democritus' atoms were irregular, with angles, hooks, convexity, and concavity, seemingly with no particular system.[2]

The tetrahedron, octahedron, icosahedron, and cube are thus the four fairest bodies satisfying Plato's condition that they are 'capable in part of being produced out of one another by means of dissolution':[3] this condition refers to the possibility of dispersing any of the tetrahedron, octahedron, and icosahedron into their component triangles and reforming them as another.[4] Plato's condition is a restriction on his notion of 'fairest body', not part of its definition.[5] Thus the fairest bodies are, for Plato, the five regular solids, not just those satisfying his criterion.

It has been suggested that the dodecahedron was reserved for the universe on the grounds that it is the closest regular solid to the sphere.[6] [The icosahedron has volume $\sqrt{3(5-\sqrt{5})/10} \approx 0.910\,592$ that of a dodecahedron inscribed in the same sphere.] If book XIII of Euclid's *Elements* is largely due to Theaetetus (c. 417–369 BCE), as scholars seem to agree,[7] then it is likely that it was known in Plato's time which of the icosahedron and dodecahedron inscribed in the same sphere had the greater volume or surface area. [However, it was Apollonius, who lived about two centuries after Plato, who established the result that the same ratio holds between the volumes and surface areas of these polyhedra.[8]] But with five regular solids and four elements, one solid could not have an atomic role. The cube and dodecahedron are both outside the genus of polyhedra whose faces can be constructed from the equilateral triangle alone; thus these two solids were the natural candidates for exclusion from the elementary correspondence. The dodecahedron has much greater resemblance to the sphere and so could be elevated to a universal role. But it seems less likely that Plato *initially* set aside the dodecahedron on the grounds of proximity to the sphere, and then developed the triangle-based system.

Note the contrast between Plato's attitude to the fairest triangles, where he openly admitted that triangles still fairer might yet be discovered, and to the regular solids, where he made no such caveat. The difference is most probably due to the regular solids having been completely classified. The complete classification would have fitted with the preference for the limited over the unlimited that descends from the pythagoreans.[*] Furthermore, the concepts behind the result may have impressed Plato, not least because the notion of a regular solid is not easy to formulate precisely. Scholium 1 to book xiii of Euclid's *Elements* says that the cube, tetrahedron, and dodecahedron were due to the pythagoreans, but the octahedron and icosahedron were due to Theaetetus.[1] This latter geometer was a contemporary of Plato, which points to a late discovery of the concept of a regular solid.

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[*] ▸ pp. 81–4

[1] Dillon, *The Heirs of Plato*, pp. 195–6.

[†] ▸ p. 52

[2] DK, 68A37; orig. Simplicius, *In Aristotelis de Caelo Commentaria*, pp. 294–5; trans. Taylor, *The Atomists*, pp. 70–1 [§ 44a].

[3] Plato, *Timaeus*, STE 53e.

[4] *Ibid.*, STE 54b–d.

[5] This is clearer in trans. Cornford, *Plato's Cosmology*, pp. 213–14.

[6] Gottlieb, *The Dream of Reason*, ch. 11.

[7] Heath, hist. note. to Euclid, *Elements*, bk XIII, p. 438; Knorr, *The Evolution of the Euclidean Elements*, pp. 273–5; Mueller, *Philosophy of Mathematics*, p. 251.

[8] Heath, *Hist. Greek Math.*, vol. 2, p. 192.

There is also linguistic evidence for this: 'cube' [*cubos* κύβος] and 'pyramid' [*pyramis* πῡρᾰμίς] are unsystematic terms, while 'octahedron' [*oktaedron* ὀκτάεδρον], 'dodecahedron' [*dōdekaedron* δωδεκάεδρον], and 'icosahedron' [*eikosaedron* εἰκοσάεδρον] are systematic terms. This difference suggests the latter three shapes were probably not named until after the establishment of theoretical geometry. 'Tetrahedron' later displaced 'pyramid', but the systematic 'hexahedron' [*exaedron* ἐξάεδρον] did not dislodge 'cube'.[2] [The difference may be due to there being an ambiguity about whether *pyramid* refers to the tetrahedron or to the square-based pyramid, but there being no such ambiguity for *cube*.] Iron pyrite can crystallize as dodecahedra, and good examples are found in Italy; this may explain why the dodecahedron came to the attention of the pythagoreans.[3] Icosahedral crystals are much rarer,[4] and so the icosahedron would have been less likely to be noticed outside of a systematic investigation. Although one can form an octahedron by joining two square-based pyramids base-to-base, the resulting shape would not have been deemed special enough to name. The octahedron only gained significance when the *concept* of a regular solid had been identified.[5]

The late formulation of the concept was doubtless a consequence of the difficulty of finding the correct definition of a regular solid. Regularity of the faces is necessary but not sufficient. All of the archimedean solids have regular faces, but of different shapes (see Figures 3.11 and 3.12 on pages 110–11). For example, the cuboctahedron has six square and eight equilateral triangular faces (with two square and two triangular faces meeting at each vertex). But the triangular dipyramid (see Figure 1.7 on page 28), which has six equilateral triangular faces, meets the stronger condition that all faces have the *same* regular shape, but has some vertices where three faces meet and some vertices where four meet. The earliest extant formulation of the definition of a regular solid states that it should have equal faces and be inscribable in a sphere. It is natural to understand Plato as having referred to this definition when he wrote that a regular solid 'divides the whole of the circumscribed sphere into equal and similar parts'.[6] And the definition may be the original one, for it fits with the earlier idea of seeking to understand circles via inscribed polygons.[7]

[The existence of five regular solids and their correspondence to the elements and universe was attributed to Pythagoras by Aëtius

[*] ▸ pp. 46–7

[1] Heath, hist. note. to Euclid, *Elements*, bk xiii, p. 438.

[2] Waterhouse, 'The discovery of the regular solids', p. 217.

[3] *Ibid.*, p. 213.

[4] *Loc. cit.*

[5] *Ibid.*, p. 214.

[6] Plato, *Timaeus*, ste 55a.

[7] Waterhouse, 'The discovery of the regular solids', p. 214.

(*fl.* 1st–2nd centuries CE), based on Theophrastus (c. 371–c. 287 BCE).[1] But the idea is unlikely to be due to Pythagoras himself or his immediate followers, since the list of elements seems to be due to Empedocles, who flourished two or three generations later than Pythagoras. An echo of the idea is found in words attributed to Philolaus,[2] but the fragment has been judged spurious. Even if it is authentic, it refers only to 'bodies', not necessarily mathematical objects.[3]]

Divinity throughout the universe

Like any craftsman,[4] the demiurge introduced order [*taxis* τάξις] into his material and so created a product possessing *kosmos* [κόσμος]. For Plato, this aesthetic order imposed on the world was evidence of the goodness of the demiurge:

> 'Now if so be that this Cosmos is beautiful and its Constructor good, it is plain that he fixed his gaze on the Eternal [...]. But it is clear to everyone that his gaze was on the Eternal; for the Cosmos is the fairest of all that has come into existence, and He the best of all the Causes.'[5]

But there is more: this order indicates the divine mind that is present throughout the world. The idea that mind shapes everything had long been present in Plato's thought. In his account of Socrates' last hours in the *Phaedo*, he wrote that Socrates had learned of this idea from a book by Anaxagoras (c. 500–c. 428 BCE), and thought that

> 'it was somehow a good way of looking at it that the mind is the cause of everything, and I thought if this is right, then the mind in ordering everything orders and arranges each thing in the best possible way'.[6]

But Socrates was disappointed that Anaxagoras did not explain how mind caused things, but rather considered physical causes of things.[7] Anaxagoras did in fact write that mind was the cause of the revolutions of the heavenly bodies, but with little detail of what other aspects of order were imposed by mind.[8]

In contrast, the cosmogony Plato presented in the *Timaeus* is a detailed description of the geometrical and numerical process by which order and thus mind was emplaced in the world by the demiurge. He explicitly referred to the outermost sphere of the universe as 'the intelligence of the Supreme'.[9] The uniform motion of revolution means that the world-soul will think consistently 'identical thoughts about the same objects'.[10] Plato reaffirmed this point later, in what was probably his last dialogue, the *Laws*:

> 'the motion which moves in one place must necessarily move always round some centre, being a copy of the turned wheels [...] has the nearest possible kinship and similarity to the revolution of reason?'[11]

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[1] DK, 44A15; orig. Aëtius, *Placita*, § 2.6.5 ⟦p. 834⟧.

[2] DK, 44B12; orig. Stobaeus, *Anthologium*, bk I, proem, § 5 ⟦vol. 1, pp. 18–19⟧; trans. Huffman, *Philolaus*, p. 392.

[3] Huffman, *Philolaus*, pp. 392–3.

[4] Plato, *Gorgias*, 503e–4a.

[5] Plato, *Timaeus*, STE 29a.

[6] Plato, *Phaedo*, STE 97c.

[7] *Ibid.*, STE 98b–d.

[8] DK, 59B12; orig. Simplicius, *In Physicorum*, p. 164–5; trans. KRS, pp. 363–4 [§ 476].

[9] Plato, *Timaeus*, STE 40a.

[10] *Ibid.*, 40a.

[11] Plato, *Laws*, STE 898a.

Thus, for Plato, the circular structure of the heavens was produced by and reflected the operations of mind.[1] The mechanically perfect actions one witnesses in a machine age lead to the association of unvarying repeated movement with automatism and mindlessness. But in a pre-industrial society of hand-tooling, any exact reproduction was seen as divine or at least divinely inspired.[2] Plato held to this position: to prove the existence of the gods, it was only necessary to point to the heavenly bodies.[3] [This point is made explicitly in the probably spurious dialogue *Epinomis*: ‘men ought to have found proof of the stars and the whole of that travelling system being possessed of mind in the fact that they always do the same things’.[4]]

These aspects of the cosmogonical theory often find precursors in the work of earlier philosophers, from whom Plato borrowed readily what he could fit into his own arguments.[5] Notably, Empedocles also saw the world filled with intellect: ‘mind alone [...] darting through the whole cosmos with swift thoughts’.[6] And the link between circular motion and divinity is present, albeit in a negative sense, in Alcmaeon’s (*fl.* c. early 5th century BCE) attribution of mortality to an *absence* of circular motion.[7] The link between sphericity and divinity of the universe lingered on to Aristotle.[8]

Metaphysics

In a passage already quoted from the *Philebus*, Plato had Socrates describe the beauty of the geometric figures as ‘not relative [...] absolutely beautiful by nature’[9].* This points to the beauty not being in the actual drawn figure, but in the geometrical object that it represents. Plato had long had in mind this distinction. In the early dialogue *Euthydemus*, he pointed out that diagrams play only an instrumental role in mathematics: ‘geometers, astronomers, and calculators [...] are not in each case diagram-makers, but discover the realities of things’.[10] This observation foreshadows important aspects of his later thought, and it has been called ‘the most advanced piece of Platonic thinking’[11] in the entire dialogue.

So, for Plato, geometrical diagrams represented unseen geometrical objects. This is not to belittle the diagrams, for looking upon representations of mathematics in the guise of the motions of the heavenly bodies was the starting-point for the study of the universe, for mathematics, and ultimately for philosophy. Precisely for this reason, sight is the most important of the senses.[12] The mathematical objects themselves do not exist within the sensible world. Their existence is bound up with Plato’s theory of Forms, which are first mentioned in the *Phaedo*.[13] Here Plato first used the term *Form* [*eidos* εἶδος] to denote something that encompassed what he had previously referred to by formulations like ‘absolute beauty’ or ‘the beautiful itself’ [*to kalon auto* το καλόν αυτό].[14] Plato nowhere gave a detailed description of the nature of the Forms, but certain details emerge. The Forms

[1] Guthrie, *The Later Plato*, p. 257.

[2] Sambursky, *The Physical World of the Greeks*, p. 54.

[3] Plato, *Laws*, STE 886d.

[4] ps.-Plato, *Epinomis*, STE 982c.

[5] Guthrie, *The Later Plato*, p. 280.

[6] DK, 31B134; orig. Ammonius, *In Aristotelis de Interpretatione Commentarius*, p. 249; trans. KRS, p. 312 [§ 397].

[7] DK, 24B2; orig. Aristotle, *Problems*, BEK XVII.3.916a33; trans. KRS, p. 347 [§ 455]; see also Guthrie, *Earlier Presocratics*, pp. 351 sqq.

[8] Aristotle, *On the Heavens*, BEK 286a10–b31.

[9] Plato, *Philebus*, STE 51c–d.

* ▸ p. 61

[10] Plato, *Euthydemus*, STE 290c.

[11] Guthrie, *Plato: The Man and His Dialogues*, p. 281.

[12] Plato, *Timaeus*, STE 47a.

[13] Plato, *Phaedo*, STE 102b.

[14] For example, Plato, *Greater Hippias*, STE 292d; Plato, *Euthydemus*, STE 301a.

exist outside of the sensible world and are unchanging, in contrast to the particular things that exist in the sensible world.[1] Each property such as *beautiful* or *good* has a unique corresponding Form. Each Form perfectly exemplifies the corresponding property, and particular things have a given property insofar as they participate in the Form of that property. Thus the Form of the Beautiful is eternally absolutely beautiful, and any other beautiful thing is beautiful because (and only because) it participates in that Form.[2]

Plato's position here stands in opposition to that of Polykleitos, for whom beauty resided in proportion.* Thus for Polykleitos beauty was dependent on certain mathematical criteria. For Plato, it was possible that an object satisfying those criteria could be beautiful, but its beauty would not *consist in* those criteria. Rather, those criteria would serve only as a description of objects that participate in the Form of the Beautiful.

From the existence of the Form of the Beautiful, it follows that beauty is not subjective: the beauty of any thing is determined by the degree of its participation in that unique Form. It remains possible that two different observers might *perceive* different degrees of the beauty in a thing, but the difference is in their abilities to perceive the fixed amount of beauty that it has.

[There is a linguistic ambiguity in the article-and-adjective construction in Greek, just as there is in English: 'the beautiful' may refer to beautiful things or to the quality of being beautiful. Plato was aware of the distinction, but he still used 'the beautiful' and 'beauty' to name the relevant Form.[3]]

In the famous 'divided line' analogy in the *Republic*,[4] Plato attempted to describe how mathematical objects relate to Forms. He wrote of dividing a line into unequal sections, representing the classes of sensible things, which we can perceive through our senses, and intelligible things, which can be perceived only by the intellect. He then subdivided each main division in the same ratio as the line. The lower subdivision of the first main division represents images, including shadows and reflections. The upper subdivision of the first main division represents the actual objects in the world. As regards truth, these two subdivisions are related in proportion: as opinion is to knowledge, so are images to originals.[5]

Note that the subdivision of the first main division — the sensible things — is described in terms of the objects, and only then are the corresponding intellectual faculties discussed. In contrast, the subdivision of the second main division — the class of intelligible things — is phrased in terms of how the objects are studied.[6] In the lower subdivision, the mind must work from hypotheses and use images to make deductions; in the upper subdivision, the mind abandons images and 'constructs its way of operating from the very Forms by themselves'. A subsequent discussion expands on how the first sub-

[1] Plato, *Phaedo*, STE 78d–e.

[2] *Ibid.*, STE 100c.

* ▸ pp. 51–2

[3] Guthrie, *The Later Plato*, p. 138.

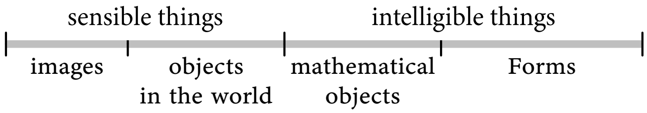
[4] Plato, *Republic*, STE VI.509d–11e.

[5] *Ibid.*, STE VI.510a.

[6] *Ibid.*, STE VI.510b.

division is populated by the objects of thought that mathematicians study through the medium of diagrams.[1]

In summary, the divided line represents various classes of objects, and the relationships between them, as follows:



It is not clear, and has been a matter of scholarly debate,[2] whether Plato viewed mathematical objects as Forms or assigned them to an intermediate position between the Forms and the sensible objects. The subdivision in the same ratio of the two main divisions certainly indicates that *in some sense* he held that:

sensible things are to intelligible things
AS
images are to objects in the world
AS
mathematical objects are to Forms.

Plato may have been thinking solely of the intellectual faculties involved, or he may have been considering the ontological status. One could argue that images comprise a special class of things in the sensible world, and the proportion of the divided line would then indicate that mathematical objects comprise a special class of Forms. Or one could distinguish an image (say, a reflection) from the object that supports it (a mirror), and infer from the proportion that mathematical objects are not Forms. Aristotle reported that Plato held mathematical objects to have an intermediate status between sensible things and Forms. Mathematical objects differed from sensible things in being eternal and unchangeable, and differed from Forms in that there are many of each kind, while each Form is unique.[3] Aristotle here reached a conclusion from an argument that Plato did not explicitly make, but whose premisses are drawn from Plato.[4] Aristotle recorded that there were different views on this point, even in the Academy itself.[5] Regardless of whether Plato held mathematical objects to be Forms, he did hold that mathematical objects exist eternally: later in the *Republic*, the conversants agree that '[g]eometry after all is the knowledge of the eternally real [*aei ontos* ἀεὶ ὄντος]'.[6] Seen in this light, the cosmogony of the *Timaeus* is the stabilizing of the changeable world upon an eternal mathematical structure.

[That Plato viewed mathematical objects as intermediates seems to have become accepted in antiquity, for Proclus* (410/12–485 CE) opened his *Commentary on the First Book of Euclid's Elements* by stating it without demur.[7]]

This is not the place to examine further the debate on Plato's ontology of mathematical objects, for it is of very limited importance for aesthetics. Indeed, later in the *Republic*, when the discussion returns

[1] Plato, *Republic*, STE VI.510c–11a.

[2] See, for example Brentlinger, 'The Divided Line', Annas, 'On the "Intermediates"'.

[3] Aristotle, *Metaphysics*, BEK A.6.987b14–18.

[4] Annas, 'On the "Intermediates"', p. 147.

[5] Aristotle, *Metaphysics*, BEK Z.2.1028b19 sqq.

[6] Plato, *Republic*, STE VII.527b.

* ▸ pp. 149–58

[7] Proclus, *Commentary on Euclid*, FRI 3–5.

to this division, Socrates says that they should not dwell further upon it, and moves the conversation on,[1] suggesting that it was not of the utmost importance for Plato's discussion.[2] But it is very significant that *cognitively* mathematics lies between the study of the sensible things and the study of Forms. Mathematical study leads to a specific type of knowledge called *dianoia* [διάνοια], usually translated 'thought'. Studying the sensible things leads to the lower type of knowledge called *doxa* [δόξᾰ], 'opinion' or *pistis* [πίστις], 'belief'.[3] Studying the Forms leads to the higher type of knowledge called *nous* [νοῦς].[4] The fundamental distinction between the processes of studying mathematics and studying the Forms is that mathematics begins with assumed hypotheses and deduces conclusions. Dialectic aims to remove the need for such hypotheses. But to cease starting from hypotheses is to cease doing mathematics: dialectic is an entirely different mode of thought.[5]

More speculatively, the divided line analogy may apply to beauty. If so, perhaps the beauty of mathematical objects is to the beauty of the Forms as the beauty of an image is to the beauty of the original: the beauty of mathematical objects will reflect imperfectly the beauty of the Forms. But also, the beauty of mathematical objects, which are in the upper main division, will be greater than the beauty of sensible objects.

Ascending to the Forms

In the *Charmides*, Plato argued that living a good life depends on knowledge — but knowledge of what?[6] In the *Republic*, Plato answered: knowledge of the Forms and ultimately of the Form of the Good. Indeed, 'the Form of the Good is the most important thing to learn [*megiston mathēma* μέγιστον μάθημα]'[7] (literally, greatest object of learning), which prompts the next question: *how* can one learn about the Forms?

In a celebrated passage of the *Meno*,[8] Socrates leads a servant boy to the discovery of how to duplicate the square. Since the boy had never learned these things in his present life, Socrates argues that he must be recalling knowledge that he had acquired before he was born into a human body.[9] [Socrates is thus implicitly arguing that mathematical knowledge is not empirical; this passage is the first recognition that mathematical truth is *a priori*.[10]] More generally, learning is simply the process of recovering access to knowledge that one already has; this is the doctrine of *anamnesis* [*anamnēsis* ἀνάμνησις].

The soul's accumulation of knowledge between incarnations is described in the *Phaedrus*: the soul can follow the gods on a journey during which it can gaze on the true reality, although some souls see more than others.[11] '[T]he soul which has never seen the truth can never pass into human form'.[12] Here, 'the truth' can be construed as

[1] Plato, *Republic*, STE VII.533e–4a.

[2] Burnyeat, 'Plato on Why Mathematics is Good for the Soul', pp. 33–5.

[3] Plato, *Republic*, STE VII.510a.

[4] *Ibid.*, STE VII.511c–d.

[5] *Ibid.*, STE VII.511c–d, 533c.

[6] Plato, *Charmides*, STE 173d sqq.

[7] Plato, *Republic*, STE VI.505a.

[8] Plato, *Meno*, STE 82a–5b.

[9] *Ibid.*, 86a.

[10] Cornford, *Before and After Socrates*, p. 72.

[11] Plato, *Phaedrus*, STE 246c–8c.

[12] *Ibid.*, STE 249b.

referring to the Forms.[1] The soul that has seen most truth will indeed be incarnated as 'a philosopher or a lover of beauty, or one of a musical or loving nature'.[2]

So knowledge of the Forms is innate in every person: the challenge is to regain that knowledge. But to do so requires extensive training.[3] In the course of the *Republic*, Plato gave an account of what is necessary. The *Republic* is centred around the meaning of justice, and, in exploring this question, the participants in the dialogue seek insight in the analogy between justice in the individual and justice in the state. Pursuing this analogy leads to the discussion of an ideal state. Rather than imagining a society of perfect individuals, Plato's approach was to try to posit a social order that will make the best of individual human nature as it actually exists.[4] The social order that emerges includes an elite military or 'guardian' class, perfect members of which become the actual rulers of the state.[5] These select guardians must be true paragons, being the only citizens to have knowledge of the Form of the Good.[6] Therefore their education is specially prescribed to lead them to this knowledge.

They are introduced to arithmetic, geometry, and the preliminaries for dialectic when young, learning through play.[7] After military training, they must be trained further in mathematics in order to bring together into a unified view the various pieces of mathematics they previously learned in an unsystematic way.[8] The subjects studied will be arithmetic, plane geometry, solid geometry, astronomy, and harmonics. The ten years allowed for this study implies that they will learn advanced mathematics that takes them to the frontiers of research.[9] This arduous mathematical training precedes the final five years of training in dialectic, which should ultimately, after fifteen years of practical experience in military and government roles, allow them to resume dialectic and finally lead them to knowledge of the Form of the Good and so fitness to rule.[10]

This extensive mathematical training is not aimed at mathematical creativity. Nor is it aimed at any practical end. There is a humorous exchange in the dialogue between Socrates and Glaucon, who speculates about the necessity for a military leader to know arithmetic to organize his troops, geometry to manoeuvre and to plan his camps, and astronomy to predict the changing seasons.[11] But the real purpose of mathematical training is to turn the soul away from the sensible world and direct it towards the truth and the Form of the Good.[12] The purpose is neither a simple redirection of attention nor a mere intellect-building exercise. Plato suggested several times that doing mathematics could have a direct moral effect. In the *Gorgias*, Socrates reproaches Callicles for the dissolute life he leads, blaming it on a neglect of geometry.[13] In the *Republic*, Socrates says that the philosopher who focuses on that which is divine and orderly will develop those qualities.[14] In the *Timaeus*, Plato wrote that, in considering orbits of heavenly bodies,

[1] Guthrie, *Plato: The Man and His Dialogues*, p. 492.

[2] Plato, *Phaedrus*, STE 248d.

[3] Plato, *Parmenides*, STE 135d–e.

[4] Cornford, 'Plato's Commonwealth', pp. 58–9.

[5] Plato, *Republic*, STE III.414a.

[6] *Ibid.*, STE IV.428e–9a.

[7] *Ibid.*, STE VII.536d–e.

[8] *Ibid.*, STE VII.537b–c.

[9] Cornford, 'Mathematics and Dialectic II', p. 175; Burnyeat, 'Plato on Why Mathematics is Good for the Soul', p. 1.

[10] Plato, *Republic*, STE VII.539d–40a.

[11] Plato, *Republic*, STE VII.521d–e, 526c–d, 527d; Burnyeat, 'Plato on Why Mathematics is Good for the Soul', pp. 10–11.

[12] Plato, *Republic*, STE VII.525a–c, 526d–e.

[13] Plato, *Gorgias*, STE 508a–b.

[14] Plato, *Republic*, STE VI.500d.

> ‘learning and sharing in calculations which are correct by their nature, by imitation of the absolutely unvarying revolutions […] we might stabilize the variable revolutions within ourselves.’[1]

This idea directly contrasts Archytas’ view that logistic helps prevent unjust behaviour through fear of detection, not from inner moral development.[*]

[Plato admitted that, whatever its effects on the soul, mathematics has certain ‘incidental topics [*parerga* πάρεργα]’ that are ‘not insignificant’, such as its military applications.[2] But this language makes it clear that skills applicable to practical problems are only *secondary* benefits of studying mathematics.[3]]

The ascent to the Form of the Good through mathematical and then dialectical training has a strong aesthetic element, which is entirely mathematical and non-sensory. Even the study of astronomy will be directed away from the actual motions of the heavenly bodies and towards pure theory, for, like the mathematician’s diagrams, these are but a figure representing true geometry.[4] In particular, although the heavenly bodies are supremely beautiful in the sensible world, they pale in comparison with the mathematical reality:

> ‘these stars that adorn the heavens, since they ornament the visible sky, we think they’re the most beautiful and perfect examples of their kind. And yet they fall far short of the real ones — those courses, represented by real speed and real slowness in real number and in all the real geometrical shapes, which are conveyed in relation to each other and convey what is in them, all of which can be apprehended by reason and intellect, but not by sight.’[5]

The same holds for harmonics: Plato agreed with the pythagoreans, including Archytas,[†] that harmonics is akin to astronomy.[6] But again he criticized the empirical focus: the pythagoreans sought number in musical consonance, but failed to rise to the investigation of which numbers are themselves consonant.[7] In the *Republic*, Glaucon sees the difficulty in this task, but Socrates insists it is ‘a useful one in the search for beauty and goodness [*tou kalou te kai agathou* τοῦ καλοῦ τε καὶ ἀγαθοῦ]’.[8] Seeking consonance in immaterial numbers, independent of any musical application, teaches the student that there are aesthetic judgements of non-sensory things, aesthetics judgements that are not dependent on convention. Aesthetic judgements — value judgements — apply also to the intelligible objects of mathematics, and this is a step on the road towards objective judgement of goodness and knowledge of the Form of the Good.[9]

Here, in the midst of the discussion about shaping the ideally virtuous rulers of the ideal state through raising them to a knowledge of the Forms, Plato linked the good and the beautiful. That he brought it up in this context hints at the alternative method of learning of the

[1] Plato, *Timaeus*, STE 47c.

[*] ▸ p. 56

[2] Plato, *Republic*, STE VII.527c.

[3] Burnyeat, ‘Plato on Why Mathematics is Good for the Soul’, pp. 9–10.

[4] Plato, *Republic*, STE VII.529c–30b.

[5] *Ibid.*, STE VII.529c–d.

[†] ▸ pp. 52–7

[6] *Ibid.*, STE VII.530d.

[7] Plato, *Republic*, STE VII.531b–c; see also Huffman, *Archytas*, p. 84.

[8] Plato, *Republic*, STE VII.531c.

[9] Cf. Barker, *The Science of Harmonics*, pp. 317–18.

Forms he described in the *Symposium*: ascent through the beautiful. Beauty can be a starting-point for the ascent because when one sees something beautiful, the Form of the Beautiful itself shines through; other Forms are only darkly seen in worldly things.[1] The sight of beauty in this world recalls to mind the true beauty that the soul glimpsed before being born into a body.[2]

In a portion of the *Symposium* where he is supposed to make a speech on love, Socrates relates a discussion he had with Diotima, a wise woman from Mantinea. This story appears to be a second-level fiction that enables Socrates to maintain his preferred question-and-answer method, with Diotima taking the Socratic role.[3] In this tale, Diotima tells Socrates how it is possible to ascend to the Form of the Beautiful through appreciation of beauty that is ever more abstract and intellectual:

> 'Beginning from obvious beauties he must for the sake of that highest beauty be ever climbing aloft, as on the rungs of a ladder, from one to two, and from two to all beautiful bodies; from personal beauty he proceeds to beautiful observances, from observance to beautiful learning, and from learning at last to that particular study which is concerned with the beautiful itself and that alone; so that in the end he comes to know the very essence of beauty.'[4]

This ascent to the Form of the Beautiful lies parallel to the course prescribed in the *Republic*,[5] and it is not difficult to imagine them being combined, with 'beautiful learning' being beautiful mathematics. Indeed, the word translated 'learning' is *mathēmata* [μαθήματα], which referred to learning in general and the mathematical sciences in particular. Since the good and the beautiful were closely allied for Plato, attaining knowledge of the Form of the Good would entail knowledge of the Form of the Beautiful and vice versa.

Morality and mathematical beauty

Thus, for Plato, mathematics was bound up with the moral education that is required for rulers. But there was a personal aspect as well: in the contemplation of the Form of the Beautiful a person finds a state where life is worthwhile.[6]

At the end of the *Philebus*, Plato listed five human 'possessions' in decreasing order of value.[7] There are difficulties in interpreting this section,[8] but, in short, these 'possessions' seem to be five good things in and for human life:

1) 'measure, moderation, fitness, and all which is to be considered similar to these';
2) 'proportion, beauty, perfection, sufficiency, and all that belongs to that class';
3) 'mind and wisdom';

[1] Plato, *Phaedrus*, STE 250b–c.

[2] *Ibid.*, STE 249c.

[3] Guthrie, *Plato: The Man and His Dialogues*, p. 385.

[4] Plato, *Symposium*, STE 211c.

[5] Cf. Cornford, *Principium Sapientiæ*, pp. 85–6.

[6] Plato, *Symposium*, STE 211d.

[7] Plato, *Philebus*, STE 66a–c.

[8] Guthrie, *The Later Plato*, pp. 235–6.

4) ‘sciences, arts, and true opinions’;
5) ‘pleasures which we separated and classed as painless, which we called pure pleasures of the soul itself, those which accompany knowledge and, sometimes, perceptions’.

In particular, reason ranks above pleasure, and indeed above ‘empirical’ matters. This is not to say that there is anything wrong with pleasure or that intellectualizing should always be preferred, but pleasure alone is not good for a person. Nor, indeed, is pure intellectualizing. Rather a well-proportioned blend of knowledge and the better pleasures is necessary.[1]

The better pleasures arise from beautiful colours, shapes, odours, and sounds, provided the lack of them is not painful and the satisfaction of them is pleasant.[2] Thus the pleasure of satisfying a physical desire — scratching an itch or sating hunger — does not count. And as returning to the quotation from the *Philebus** makes clear, beautiful shapes are geometric, and their beauty is absolute, not relative. The language in the quotation is similar to that Plato used for Forms, which suggests that the beauty is in the mathematical objects that are embodied in the physical shapes.[3] Guthrie called this idea

> ‘the apotheosis […] of the peculiarly Greek preference […] for “the intelligible, determinate, measurable”,[4] the keen appreciation of mathematical form, symmetry and proportion which pervades classical architecture, sculpture, pottery and literature’.[5]

Speculations

Unlike in the *Republic*, where the ideal state is simply said to be limited to a size that will maintain its unity,[6], in the *Laws* the size of the state is given as 5040 households,[7] for several related mathematical reasons:

1) it is divisible by all numbers from 1 to 10;
2) it has 59 proper divisors (in modern terminology, it is *highly composite*, having more divisors than any smaller number);
3) it can be divided by 12 to give tribes of 20×21, which is again divisible by 12;
4) it is almost divisible by 11, in the sense that $5040 - 2 = 11 \times 458$.[8]

[For comparison, Athens in the fourth century BCE had around 30 000 (by definition, adult and male) citizens, and was one of the largest city-states.[9] However, most city-states were much smaller than Plato’s ideal, even taking into account the uncertainties in their reconstructed sizes.[10]]

The third criterion of being twice divisible by 12 is important because this division into twelve parts reflects the months of the year and so the turnings of the heavens.[11] The *double* instance of this

[1] Plato, *Philebus*, STE 67a sqq.; Guthrie, *The Later Plato*, p. 206.

[2] Plato, *Philebus*, STE 51b.

* ▸ p. 61

[3] Cf. Guthrie, *The Later Plato*, pp. 226–7.

[4] Quoting Fraenkel, *Rome and Greek Culture*, p. 25.

[5] Guthrie, *The Later Plato*, p. 226.

[6] Plato, *Republic*, STE IV.423b.

[7] Plato, *Laws*, STE V.737e sqq.

[8] *Ibid.*, STE V.737e–8b, 738b, 771b, 771c.

[9] Hansen, ‘Attika’, p. 627.

[10] See Hansen, *The Shotgun Method*, esp. Tbl. 1.1.

[11] Plato, *Laws*, STE V.771b.

division is vital, for otherwise Plato could have chosen 2520: it is also divisible by all numbers from 1 to 10 (and indeed is the smallest such number); it has 47 proper divisors (and is also highly composite); it is even more closely divisible by 11, since $2520 - 1 = 11 \times 229$. But it can be divided by 12 *once* to give 10×21, which is not divisible by 12.

It seems more likely that Plato chose 5040 as 7! and then constructed *ex post facto* a set of criteria that distinguished it from any smaller number. There *may* be something aesthetic in the choice of 5040 or in the formulation of the criteria, but there is nothing explicit save a faint echo of the aesthetics of the *Timaeus* in the connection of divisibility by 12 with the motions of the heavens.

Methods of mathematics

According to Theon of Smyrna, when the oracle challenged the Delians to duplicate the cube,* the craftsmen [*architektosin* ἀρχιτέκτοσιν] were baffled about how to do this and so consulted Plato, who said that the god did not wish a new altar, but merely to shame the Greeks for their neglect of mathematics and contempt for geometry.[1] Plutarch's accounts of this tale,[2] like Theon's, seems to be based on Eratosthenes,[3] but Plutarch added that Plato pointed out that doubling a cube required finding two geometric means in continued proportion, and that Eudoxus of Cnidus (c. 400–c. 347 BCE) or Helicon of Cyzicus (*fl.* early 4th century BCE) could do this. Plutarch also recorded Plato's objection to mechanical methods in geometry in solving this problem:

> 'Plato himself reproached Eudoxus and Archytas and Menaechmus for setting out to remove the problem of doubling the cube into the realm of instruments and mechanical devices [...]. In this way, he thought, the advantage of geometry was dissipated and destroyed, since it slipped back into the realm of sense-perception instead of soaring upward and laying hold of the eternal and immaterial images'.[4]

And again, with more emotive language:

> 'Plato was incensed at this, and inveighed against them as corrupters and destroyers of the pure excellence of geometry, which thus turned her back upon the incorporeal things of abstract thought and descended to the things of sense'.[5]

Archytas' solution† was certainly mechanical, requiring defining surfaces by rotating a semicircle and a triangle, and considering the intersection of these surfaces with a hemicylinder.

Plato's hostility to mechanical methods appears to have been linked to an essentially static conception of geometry. This conception is reflected in the language used in Greek geometrical arguments: 'let it have been drawn' [*gegraphthō* γεγράφθω], which de-emphasizes

* ▸ p. 53

† ▸ p. 53

[1] Theon, *Exposition*, p. 5.

[2] Plutarch, *The E at Delphi*, STE 386e–f [§ 6]; Plutarch, *On the Sign of Socrates*, STE 579a–c [§ 7].

[3] Knorr, *The Ancient Tradition of Geometrical Problems*, p. 3.

[4] Plutarch, *Table-Talk*, STE 718e–f [Qu. VIII.2].

[5] Plutarch, *Marcellus*, STE 305 [§ XIV.6].

the *act* of drawing and suggests a static rather than dynamic situation.[1] The very idea that the study of geometry is a road to knowledge of the eternal Forms is dependent on this unchanging nature, and thus introducing mechanical methods, which use motion, pulls one's mind back to the realm of the senses and imperils one's progress toward the Forms. Plato seems not to have been opposed in principle to the use of mechanical devices to understand the world: referring to the motions of the heavens, he said that 'to describe all this without an inspection of models of these movements would be labour in vain'.[2] His objective seems rather to be the preservation of the study of mathematics itself as a technique for elevating the mind away from the sensible world.

In spite of Plato's preference, the problem of the duplication of the cube was generally treated of mechanically and linked to practical problems.[3] [In the nineteenth century, Pierre Wantzel (1814–48) proved that it is impossible to duplicate the cube using straightedge-and-compass constructions.[4]] However, there are indications that even before Plato's time a higher value was placed on straightedge-and-compass constructions than mechanical ones.[5] Proclus credited Oenopides (*fl.* c. 450 BCE) with two results[6] in the first book of Euclid's *Elements*: '*To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line*'[7] and '*On a given straight line and at a point on it to construct a rectilineal angle equal to a given rectilineal angle.*'[8] By mechanical methods, these constructions are easy; the first requires only a set-square. Oenopides' reputation as a geometer was sufficient to earn him not just the attention of Proclus but a passing mention in the pseudo-Platonic but probably fourth-century BCE Academic *The Lovers*.[9] His standing is thus unlikely to rest on such easy results. It is far more plausible that Oenopides discovered straightedge-and-compass methods for these constructions,[10] and he would only be honoured for this if such constructions were already valued. This conclusion does not exclude the possibility that the authority of Plato further augmented the value that straightedge-and-compass methods were assigned.

Legacy

Plato's lauding of the beauty of geometry did not extend to its use in fine art: although he noted the mathematical beauty in diagrams and in objects that embody geometric forms, he did not suggest their use in the creation of artworks — sculptures of the regular solids, for instance.[11]

The notion of mathematics as a route to philosophy and thus to knowledge of the Form of the Good was not well-received by the public in his own time. Aristoxenus (b. c. 370 BCE) wrote that Aristotle often related how the majority of people who attended Plato's lecture 'On the Good' expected to hear about worldly goods such as

[1] Little, 'A Mathematician Reads Plutarch', p. 276; Masià, 'A new reading of Archytas' doubling', p. 197.

[2] Plato, *Timaeus*, STE 40d.

[3] Cuomo, *Pappus of Alexandria*, p. 135.

[4] Wantzel, 'Recherches'.

[5] Heath, *Hist. Greek Math.*, vol. 1, p. 288.

[6] Proclus, *Commentary on Euclid*, FRI 283, 333.

[7] Euclid, *Elements*, Prop. I.12.

[8] *Ibid.*, Prop. I.23.

[9] ps.-Plato, *The Lovers*, STE 132a.

[10] Heath, *Hist. Greek Math.*, vol. 1, pp. 175–6.

[11] Tatarkiewicz, 'Abstract Art and Philosophy', p. 234.

riches or health, but found themselves confronted by mathematical demonstrations, numerical, geometrical, and astronomical.[1]

The reports of this lecture are sources for Plato's 'unwritten doctrines', which are thought to have comprised an account of first principles which was discussed at the Academy but only adumbrated in the dialogues.[2] It is not necessary here to enter into the scholarly discussion of whether there were in fact such esoteric or inner-school teachings.[3] If Plato had such unwritten doctrines, then their reconstruction depends on testimony of Aristotle and later writers like the commentator Alexander of Aphrodisias (*fl.* c. 200 CE), who had Aristotle's now-lost work *On the Good*, seemingly based on Plato's lecture.[4] These accounts indicate that the 'unwritten doctrines' described a number-metaphysics that identified Forms and numbers. The principles of Forms were thus the principles of numbers, namely the One [*to hen* τὸ ἓν] and the Dyad [*hē duas* ἡ δυάς].[5] This idea reaches back to pythagoreanism:* the combination of One–odd–good–limit and Dyad–even–bad–unlimited, and the imposition of the former on the latter, produces the harmony that pervades the world.[6]

If Plato's thought involved such a number-metaphysics, then it follows that the Form of the Beautiful, which is absolutely beautiful, was identified with a number. Consequently, mathematical beauty did not merely appear in the *ascent* to the Forms, but can in some way be found there. Beyond making these bare statements and noting the possible closer integration of pythagorean views of mathematical beauty into Plato's thought, there is little more to be said here. But in various guises the One, the Dyad, and number-metaphysics will reappear in connection with mathematical beauty in the thought of later writers, starting with Plato's successor Speusippus.

SPEUSIPPUS

After Plato's death, his nephew Speusippus (c. 410–c. 340 BCE) succeeded him as head of the Academy, and held the position for eight years. He apparently wrote extensively,[7] but little of his work has survived. In the particular case of Speusippus' metaphysics, the most proximate source for his work is the testimony of Aristotle, but there the scarcity of explicit references has left scholars with the delicate task of building outwards from just two solid anchors, successively attaching passages with views that fit with and perhaps extend the reconstruction so far; the end product seems to be a coherent whole.[8]

According to Aristotle, Speusippus held that substances were divided into a larger number of classes than the threefold Platonic division into sensible things, mathematical objects, and Forms. Speusippus first posited the One [*to hen* τὸ ἓν or *hē monas* ἡ μονάς], numbers, spatial magnitudes, soul, sensible objects.[1] There is no place for Forms: by implication, Speusippus did not agree with Plato's theory.[2] The context of Aristotle's report is important. Aristotle carefully distinguished between different notions of number. There are Forms of numbers — the Form of Two, Form of Three, and so on — each of which, like all of Plato's Forms, is unique and exemplifies itself. Separate from these are *mathematical* numbers, each of which is an assembly of units, but which does not have the uniqueness of a Form. There is no limit on how many instances there can be of the mathematical number 3, which is an assembly of three units and participates in the Form of Three. Two such instances of the mathematical number 3 can be added to give a mathematical number 6, which is an instance that participates in the Form of Six. Speusippus rejected Forms of numbers, but still accepted that mathematical numbers had a separate existence: 2 existed separately from two objects. Mathematical magnitudes and the objects of geometry also had a separate existence, and all these mathematical objects were eternal and unchangeable.[3] [Proclus explained that Speusippus thought that any theoretical mathematical result should be called a 'theorem' [*theōrēma* θεώρημα] and never a 'problem' [*problēma* πρόβλημα], for the former term more accurately reflected the mind's contemplation of an eternal mathematical object.[4]] In any case, Speusippus' metaphysics consisted of an ontological hierarchy whose top levels were i) the One; ii) numbers and magnitudes; iii) soul. One interpretation holds that Speusippus detached Forms in their causative role from the mathematical or geometrical realms, but established the concept at the level of soul, which he thought gave form to physical objects.[5] Regardless of whether Speusippus retained some semblance of the concept of Forms at the soul-level or abandoned it completely, there seems to be an incompatibility with Plato's view that the study of mathematics was vital as a stepping-stone towards knowledge of the Form of the Good,* for even if Speusippus accepted the Form of the Good, it would be at a lower ontological (and thus, presumably, epistemological) level than the objects of mathematics.

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[1] Aristoxenus, *Harmonics*, MEI 30; for discussion, see Gaiser, 'Plato's Enigmatic Lecture "On the Good"'.

[2] Guthrie, *The Later Plato*, pp. 421, 424 sqq.

[3] For a discussion, see Gaiser, 'Plato's Enigmatic Lecture "On the Good"'.

[4] Guthrie, *The Later Plato*, p. 425.

[5] Alexander, *On Aristotle Metaphysics*, HAY 55.20–56.35 ⟦vol. 1, p. 84–5⟧; cf. Aristotle, *Metaphysics*, BEK A.987b34; Guthrie, *The Later Plato*, pp. 435–7.

* ▸ pp. 46–7, 49–50

[6] Guthrie, *The Later Plato*, pp. 429–31; Dillon, *The Heirs of Plato*, pp. 18–20.

[7] Diogenes Laertius, 'Speusippus', MEI 4.4–5.

[8] Dancy, 'Speusippus', § 1.

What, though, did Speusippus think of beauty in mathematics? Aristotle reported that Speusippus thought that the most beautiful [*to kalliston*] and the best are not present at the beginning of things.[6] Rather, 'both the good and the beautiful [*to kalon*] appear only when nature has made some progress'.[7]

At this point, scholarly opinion bifurcates. Leonardo Tarán (1933–2022) argued that this report by Aristotle suffices to eliminate beauty and goodness from the properties that can be found in mathematical numbers and magnitude and geometrical objects, for these are eternal and unchanging and so not subject to progress.[8] It seems very likely that, for Speusippus, goodness and beauty could only be found in souls or sensible objects.[9] [Aristotle also ascribed to the pythagoreans the view that beauty and goodness are not present at the beginning.

[1] Aristotle, *Metaphysics*, BEK Z.2.1028b21–4; Tarán, *Speusippus*, p. 12.

[2] Tarán, *Speusippus*, p. 12; Dancy, 'Speusippus', § 1.

[3] Tarán, *Speusippus*, p. 29.

[4] Proclus, *Commentary on Euclid*, FRI 77–8.

[5] Dillon, *The Heirs of Plato*, pp. 48–52, esp. p. 51.

* ▸ pp. 75–6

[6] Aristotle, *Metaphysics*, BEK Λ.7.1072b31.

[7] *Ibid.*, BEK N.4.1091a34–5.

[8] Tarán, *Speusippus*, pp. 41–2.

[9] *Ibid.*, p. 44.

But this remark is not relevant for early pythagorean views on beauty in mathematics, for, unlike Speusippus, the early pythagoreans had not posited a separate, immutable, existence for number.]

However, an alternative interpretation has been offered for Aristotle's report: that Aristotle was simply unconcerned about the order in which goodness and beauty arose in Speusippus' system, because he was focused on criticizing Speusippus for not ascribing goodness and beauty to the first principle.[1]

This point of contention is just one aspect of a broader question. The philosopher Philip Merlan (1897–1968) presented the thesis that chapter IV of Iamblichus'[*] (c. 240–c. 325 CE) book *On the General Science of Mathematics* was taken, in substance at least, from a work of Speusippus, and that it gave a more detailed account of Speusippus' metaphysics than what was reported by Aristotle.[2] Tarán disagreed with Merlan on the grounds of contradictions between chapter IV of *On the General Science of Mathematics* and what can be gleaned of Speusippus' philosophy from Aristotle and other sources.[3] The classicist and philosopher John Dillon (1939–) thought that all the purported contradictions could be reconciled.[4] This is not the place to embark on an analysis of the debate, but rather to examine what difference each side makes to the conclusions.

As already mentioned, if Tarán is correct in his interpretation of Aristotle's report, Speusippus held that there is no beauty in mathematical numbers or magnitudes or in geometrical objects. If Merlan and Dillon are correct, then Speusippus' views are reflected in chapter IV of *On the General Science of Mathematics*, which says explicitly that there is beauty [*to kalōn* and *kallos*] among mathematical objects.[5] In this account, the One is the cause of beauty in mathematical objects, but is not beautiful in itself: it is above the beautiful and the good and indeed above being. Beauty and goodness appear, rather, at level of mathematical objects. Ugliness and badness appear at the lower levels: souls and perceptible objects.[6] Thus the metaphysics espoused in chapter IV of *On the General Science of Mathematics* has mathematical beauty at its heart.

Thus it is possible to reach two diametrically opposed conclusions about Speusippus' views on whether beauty could be found in mathematical objects. If he did find beauty in number, his remarks about 10 being a 'perfect number' take on an aesthetic flavour and deserve some attention. Speusippus specifically asserted that the perfection of 10 is dependent on its *properties*:

> 'For it has many of the properties which are suitable for a number that is perfect in this way, and it also has many properties which are not peculiar to it, but which a perfect number ought to have.'[7]

The properties that Speusippus enumerated are, to modern eyes, properties of the set $\{1, 2, \ldots, 10\}$, which reflects the Greek conception of number [*arithmos* ἀριθμός] as 'ordered plurality'. A perfect number

[1] Dillon, 'Speusippus in Iamblichus', p. 327.

[*] ▸ pp. 145–8

[2] Merlan, *From Platonism to Neoplatonism*, ch. v.

[3] Tarán, *Speusippus*, ch. v.

[4] Dillon, 'Speusippus in Iamblichus', esp. p. 325.

[5] Iamblichus, *On the General Science of Mathematics*, FES 16.3–4, 18.2, 8–9.

[6] Dancy, 'Speusippus', § 1.

[7] Speusippus, fr. 28, ll. 17–18, in Tarán, *Speusippus*, p. 140. This and later trans. ps.-Iamblichus, *Theology of Arithmetic*, DEF 82–4

must be even, so that the odd and even numbers in it (that is, in the set of numbers less than it) are equinumerous.[1] Further, a perfect number must contain 'an equal amount of prime and incomposite [*prōtous kai asynthetous* πρώτους καὶ ἀσυνθέτους] numbers, and secondary and composite [*deuterous kai synthetous* δευτέρους καὶ συνθέτους] numbers'.[2] For this to make sense, Speusippus must have counted 1 as a prime, so that his primes are 1, 2, 3, 5, 7, and his composite numbers are 4, 6, 8, 9, 10. Speusippus pointed out that while other numbers with this property exist, 10 is the first, which gives it 'a kind of perfection'.[3]

The number 10 also has an equal number of 'multiples and submultiples', apparently in the sense that 1, 2, 3, 5 divide other numbers less than or equal to 10, while 6, 8, 9, 10 are multiples of other numbers less than or equal to 10.[4] Speusippus seems to have excluded 4 from either class, rather than counting it as both multiple of 2 and a submultiple of 8; the equality holds in either case.

All the ratios, 'the equal, and the greater and the less, and the superparticular and all the remaining kinds' are found in 10.[5] Further, plane and solid numbers are found in 10; in particular, '1 is a point, 2 is a line, 3 is a triangle and 4 is a pyramid'. Speusippus also said that the number 10 can be found in the tetrahedron: it has 4 vertices, and each of its triangles has three sides and three angles, which make 6, and 4 + 6 = 10.[6] Speusippus seems to have been reaching here.

EUDOXUS

Eudoxus of Cnidus (c. 400–c. 347 BCE) was probably the greatest Greek mathematician before Archimedes, though only testimonia and fragments of his work remain. He seems to have learned geometry from Archytas,[7] and his own contributions were considerable. He is usually credited with the development of the 'method of exhaustion',[8] in which a sequence of constructions approaches arbitrarily closely to some limiting shape, 'exhausting' the difference. An anonymous scholium to book v of Euclid's *Elements* reports that 'some say' that the general theory of proportion presented therein is due to him.[9] He was the first to prove that the volume of a cone is one-third that of a cylinder with the same base and height and that of a pyramid is one-third that of a prism with the same base and height, although the result may have been discovered but not proved by Democritus.[10][*] Archimedes praised these results as the best of Eudoxus' work on solids.[11] Archimedes also implicitly attributed to him the result that circles are to one another as the squares on their diameters.[12] Eudoxus also tackled the problem of duplicating the cube, though Plato criticized his use of mechanical methods.[†]

[1] Speusippus, fr. 28, ll. 18–21, in Tarán, *Speusippus*, p. 140.

[2] Speusippus, fr. 28, ll. 21–2, in *loc. cit.*

[3] Speusippus, fr. 28, l. 25, in *loc. cit.*

[4] Speusippus, fr. 28, ll. 27–31, in *ibid.*, pp. 140–1.

[5] Speusippus, fr. 28, ll. 31–4, in *ibid.*, p. 141.

[6] Speusippus, fr. 28, ll. 34–5, in *loc. cit.*

[7] Huffman, *Archytas*, p. 7.

[8] Heath, hist. note to Euclid, *Elements*, bk xii, p. 365.

[9] Heath, intro. note. to *ibid.*, bk v, p. 112.

[10] See Archimedes, *The Method*, p. 13; Archimedes, *Sphere and Cylinder*, bk i, Introduction.

[*] ▸ p. 52

[11] Archimedes, *Sphere and Cylinder*, bk i, Introduction.

[12] Heath, hist. note to Euclid, *Elements*, bk xii.

[†] ▸ p. 79

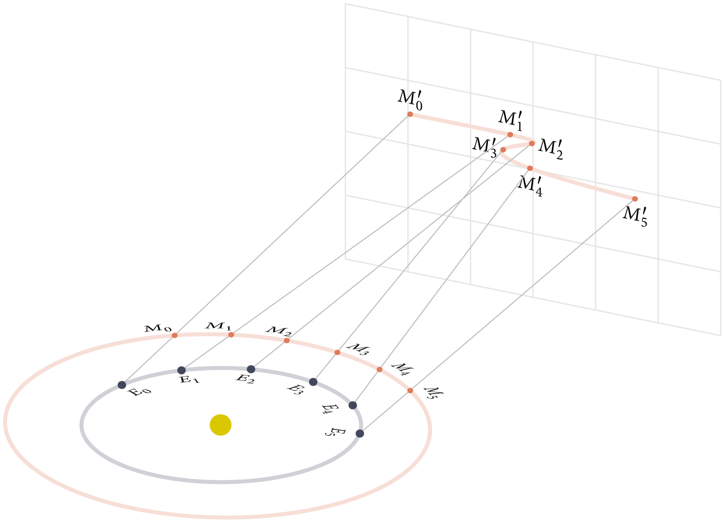

FIGURE 2.7
The relative orbits of Earth and another planet such as Mars produce an apparent retrograde motion, in the opposite direction to the usual progression of the planet as observed against a background of immobile distant stars. When Earth is at position E_i, Mars is at position M_i and is observed against the distant stars at position M'_i.

According to Simplicius, who wrote in the sixth century CE, Eudoxus was the first to devise an explanation of the movements of the planets using circular motions, answering the problem attributed to Plato.[1]* His system was an ingenious arrangement of homocentric spheres rotating on different axes. The motion of each planet was produced by four spheres centred on the (spherical) Earth. The first sphere rotated about the Earth's north–south axis. The second sphere rotated about an axis embedded into the first sphere at a certain angle; the third about an axis embedded in the second; and the fourth about an axis embedded into the third. The planet itself was located on the fourth sphere. The first sphere was responsible for the diurnal movement; the second for the general eastward motion; the third and fourth together created a looped motion that accounted for the apparent retrograde motion.[2] This observed motion of a planet occurs when it is 'overtaken' by the Earth in its orbit, so that an astronomer on Earth sees the planet move in the opposite direction against the background of distant stars (see Figure 2.7). More complicated explanations are required in a geocentric cosmology.

In the broadest sense, Eudoxus' system accounted for many of the features of the observed motions of the planets, though it was very far from being a quantitatively accurate description of them.[3] Callippus (b. c. 370 BCE) refined it by the use of additional spheres,[4] but the main and ultimately fatal flaw of a Eudoxian-type homocentric sphere system is that it cannot account for the varying brightness of the planets.[5]

The Eudoxian system descended, through Plato's prescription of uniform circular motions, from that of Anaximander with its symmetrical rational cosmos and from a pythagorean aesthetic preference for circles and spheres. But the use of circular motion was driven not just by aesthetic concerns, but because circles are easy to handle

[1] Simplicius, *On Aristotle On the Heavens*, HEI 488.20 ⟦vol. 2.10–14, p. 29⟧.

* ▸ p. 62

[2] Heath, *Aristarchus*, pp. 193–211; Kuhn, *The Copernican Revolution*, pp. 55–9.

[3] Neugebauer, *The Exact Sciences in Antiquity*, pp. 154–5.

[4] Heath, *Aristarchus*, pp. 212–6.

[5] Heath, *Aristarchus*, p. 211; Kuhn, *The Copernican Revolution*, p. 59.

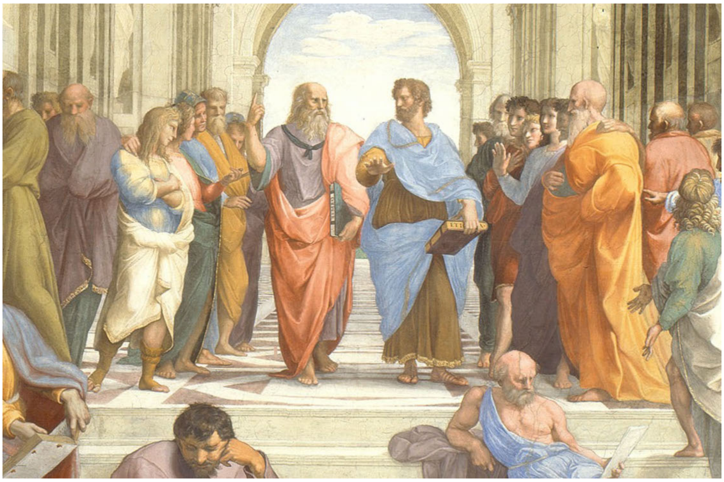

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FIGURE 2.8
Detail from Raphael's fresco *The School of Athens*, showing Plato and Aristotle.

mathematically.[1] It is worth emphasizing that Eudoxus' system was a purely geometrical and theoretical construction for explaining the motions of the planets in terms of circular motions. He did not posit the concentric spheres as physical entities carrying the planets in their movements.[2]

But after Eudoxus, the notion of spheres carrying the planets took hold. Aristotle reified Eudoxus' scheme as a set of contiguous physical spheres and so had to posit additional 'back-rolling' spheres to make the required motions possible. And this idea survived to the Copernican revolution: in Copernicus' *magnum opus* *De revolutionibus orbium cœlestium*, the 'orbs' are not the planets themselves, but spheres upon which the planets are borne.[3]

ARISTOTLE

Aristotle (384–322 BCE) was born in Stagira and came to Athens at the age of seventeen to complete his education. When Aristotle joined the Academy, Plato was on his second visit to Italy and may have left the Academy in the charge of Eudoxus,[4] whom Aristotle came to respect morally, speaking highly of him in the *Nicomachean Ethics*,[5] and intellectually, drawing on the Eudoxian cosmology in constructing his own.[6] A few years later, Plato would have been absent again for his final visit to Italy. Plato himself, though absent during much of Aristotle's formative time, doubtless had a profound influence through his writings.[7]

Though he started from Plato's philosophy, Aristotle's thought was fundamentally different, as was so aptly illustrated in Raphael's masterpiece *The School of Athens*. At the geometric centre of the fresco Plato and Aristotle are depicted in conversation. Plato carries his

[1] Lloyd, *The Ambitions of Curiosity*, p. 39.

[2] Heath, *Aristarchus*, p. 196.

[3] Kuhn, *The Copernican Revolution*, p. 59.

[4] Guthrie, *Aristotle, An Encounter*, p. 20.

[5] Aristotle, *Nicomachean Ethics*, BEK X.2.1172b15.

[6] Aristotle, *Metaphysics*, BEK Λ.8.1073b18 sqq.

[7] Guthrie, *Aristotle, An Encounter*, p. 21.

Timaeus and points upwards towards intelligible things; Aristotle carries a codex on ethics and extends a moderating hand to indicate sensible things (see Figure 2.8).[1] When we try to understand the world, Plato and Aristotle agreed that we must first look to the archetypes of being and goodness and the sciences that study them. But they disagreed on where these archetypes are found: for Plato, they were in the intelligible world accessible to mathematics and dialectic; for Arisototle, they were in the world of sensible things accessible to natural philosophy.[2]

It is therefore natural that Aristotle's work says relatively little about mathematical beauty *per se*, with the only discussion being in book M of the *Metaphysics*. What he says about beauty in general is certainly applicable to mathematical beauty in particular. Aristotle held that the experience of beauty is a kind of pleasure that a person is not temperate with regard to, because it does not involve desire or appetite. That is, the pleasure arises directly from the stimulus itself, not from what the stimulus indicates. This kind of pleasure is peculiar to humans: beasts feel only the appetite to quench thirst or sate hunger. The state of mind is one of rapt attention and enjoyment, as one who is 'charmed by the Sirens'.[3]

Aristotle gave a criterion for beauty in his *Poetics*:

> 'to be beautiful, [...] every whole made up of parts, must not only present a certain order in its arrangement of parts, but also be of a certain definite magnitude. Beauty is a matter of size and order'.[4]

'Order' also appears in the *Metaphysics* as one of three forms of beauty, three forms that are best demonstrated by mathematics.* The mention of 'magnitude' does not exclude mathematical objects, for it does not necessarily refer to magnitude in the physical world. Beautiful objects in mathematics have to satisfy the same criterion as beautiful objects in general: they should neither be of such minute size that they cannot be distinguished, nor of such vast size that they cannot be seen all at once.[5]

[Similarly a story or plot must be long enough to merit attention but short enough to remember;[6] such a condition might also apply to mathematical proofs, but Aristotle nowhere explicitly considered the aesthetics of proofs.]

Aristotle approved of leisure being filled with stimulation for the mind: 'intellectual enjoyment in leisure [...] one of the ways in which it is thought that a freeman should pass his leisure'.[7] His example of such stimulation was music, but mathematics would also fit. He clearly recognized that some people enjoyed mathematics.[8]

Ontological status of mathematical objects

Aristotle's one explicit discussion of beauty in mathematics sits within a discussion of the ontology of mathematical objects; it is thus necessary to recapitulate briefly Aristotle's position and how it differs from Plato's.[1]

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[1] Cf. Clark, *Civilisation*, p. 131; Williams, *Raphael*, p. 88.

[2] Burnyeat, 'Platonism and mathematics', p. 145; cf. Cleary, *Aristotle and Mathematics*, p. 344.

[3] Aristotle, *Eudemian Ethics*, BEK III.2.1230b20 sqq.

[4] Aristotle, *Poetics*, BEK 7.1450b34–7.

* ▸ pp. 89–90

[5] *Ibid.*, BEK 7.1450b37–1a2.

[6] *Ibid.*, BEK 7.1451a5–6.

[7] Aristotle, *Politics*, BEK VIII.3.1338a20–3.

[8] Aristotle, *Nicomachean Ethics*, BEK X.5.1175a33.

Aristotle mentioned the claim that geometry does not deal with perceptible lines: for instance, a perceptible circle and straight line that touch do not meet at a single point as the geometer's circle and straight line do.[2] But it is not clear that Aristotle himself held this view, for it is embedded in an aporetic discussion where he considered arguments that might support the Platonic position.[3] The failure of the geometer's diagram to have lines of absolute straightness or an exact given length does not invalidate the reasoning, for the geometer does not conclude anything from the drawn line, but from what the drawn line represents.[4] Drawing a diagram can be an aid to the student, but is not necessary for the demonstration.[5]

According to Aristotle, Plato held that there are two kinds of eternal substance that are more numerous and more real than sensible things: the objects of mathematics and Forms.[6]

Aristotle set out an exhaustive list of the possibilities for the way in which mathematical objects might exist:[7] they are either

I) in sensible things,
II) separate from sensible things,
III) do not exist, or
IV) exist in some other way.

Aristotle aimed to show that possibility IV was correct by eliminating the alternatives. He did not explicitly treat possibility III, presumably on the basis that a genuine science must have a real object;[8] he immediately went on to say that his discussion was about 'not whether they exist but how they exist'.[9] This left possibilities I and II, which he claimed were held by others, though he did not here specify whom.

Aristotle advanced various arguments against possibilities I and II. Some of the arguments against possibility I he recalled from book B,[10] where they were specifically directed against platonists who accepted Forms and intermediate mathematical objects; Aristotle assumed that the arguments apply to mathematical objects of any type.[11] Some of his arguments against possibility II involve complicated and delicate notions of mathematical and sensible objects being prior or posterior to each other with respect to truth, substance, formula, or generation;[12] fortunately, it is not necessary to analyse them here.

One argument he advanced against possibility II perhaps touched on aesthetic value. The separation of mathematical and sensible objects would lead, argued Aristotle, to a surfeit of mathematical objects. Every sensible solid, plane, line, and point would be separate from a corresponding mathematical solid, plane, line, or point[13] But the planes and lines that bound the mathematical solid would also be mathematical objects on a second level. Then the lines that bound the second-level planes would be third-level mathematical objects. A cumbersome multiplicity of mathematical objects is thus generated. One

[1] Cleary, *Aristotle and Mathematics*, esp. ch. 5; Bostock, 'Aristotle's Philosophy of Mathematics'; Annas, intro. to Aristotle, *Metaphysics M–N*, § 2.

[2] Aristotle, *Metaphysics*, BEK B.2.998a1–4.

[3] Cf. Cleary, *Aristotle and Mathematics*, p. 256.

[4] Aristotle, *Posterior Analytics*, BEK I.10.76b39–77a2.

[5] Aristotle, *Prior Analytics*, BEK I.41.50a1–4.

[6] Aristotle, *Metaphysics*, BEK Z.2.1028b19–21.

[7] *Ibid.*, BEK M.1.1076a32–7.

[8] Cleary, *Aristotle and Mathematics*, p. 280.

[9] Aristotle, *Metaphysics*, BEK M.1.1076a37.

[10] *Ibid.*, BEK B.998a12–19.

[11] Annas, notes to Aristotle, *Metaphysics M–N*, BEK M.2.1076a37–b11.

[12] Aristotle, *Metaphysics*, BEK M.2.1077a15–19, 24–9, a36–b11.

[13] *Ibid.*, BEK M.2.107612–39.

could argue that such an unparsimonious ontology is aesthetically undesirable, although Aristotle did not suggest this.

Having made these arguments, Aristotle thought that possibilities I to III had been refuted. His remaining task was to give an account of how mathematical objects exist in some other way. His position was that mathematics deals with sensible objects, but not *qua* sensible objects, but only *qua* planes or *qua* lines, so that sensible objects are treated in a very general and abstract way.[1] A geometer considers a physical square not *qua* a physical square, but as an object having certain attributes of relevance to geometry, although it can be useful to *think* of the mathematical object as having a separate existence.[2] This position is consistent with the actual practice of mathematicians. But such a *logical* separation of mathematical and sensible objects does not entail an *ontological* separation.[3]

The process of subtracting unneeded attributes lends mathematics greater accuracy and simplicity:

> 'a science which abstracts from the magnitude of things is more precise than one which takes it into account; and a science is most precise if it abstracts from movement, but if it takes account of movement, it is most precise if it deals with the primary movement, for this is the simplest; and of this again uniform movement is the simplest form.'[4]

This parsimony with respect to concepts is at least compatible with an observation Aristotle made elsewhere about parsimony with respect to *assumptions*:

> 'Obviously then it would be better to assume a finite number of principles. They should, in fact, be as few as possible, consistently with proving what has to be proved. This is the common demand of mathematicians, who always assume as principles things finite either in kind or in number.'[5]

Beauty in mathematics

Against this background came Aristotle's one explicit discussion of beauty in mathematics.[6] The passage is worth quoting in full:

> 'Now since the good [*to agathon*] and the beautiful [*to kalon*] are different (for the former always implies conduct as its subject, while the beautiful is found also in motionless things), those who assert that the mathematical sciences say nothing of the beautiful or the good are in error. For these sciences say and prove a very great deal about them; for if they do not expressly mention them, but prove attributes which are their results or their formulae, it is not true to say that they tell us nothing about them. The chief forms of beauty are order [*taxis*

[1] Aristotle, *Metaphysics*, BEK M.3.1077b17--34.

[2] Aristotle, *Metaphysics*, BEK M.3.1078a21–3; cf. Aristotle, *Physics*, BEK II.2.193b30–34.

[3] Cleary, *Aristotle and Mathematics*, p. 329.

[4] Aristotle, *Metaphysics*, BEK M.3.1078a10–13.

[5] Aristotle, *On the Heavens*, BEK III.4.302b26–29.

[6] Heath, *Mathematics in Aristotle*, p. 201 & *passim*.

> τάξις] and symmetry [*symmetria* συμμετρία] and definiteness [*hōrismenon* ὡρισμένον], which the mathematical sciences demonstrate in a special degree. And since these (e.g. order and definiteness) are obviously causes of many things, evidently these sciences must treat this sort of cause also (i.e. the beautiful) as in some sense a cause. But we shall speak more plainly elsewhere about these matters.'[1]

The promise of the last sentence is unfulfilled in Aristotle's surviving writings.[2]

This passage is puzzling: it seems to have no connection with the preceding discussion, unless Aristotle's point was that there is no need to postulate a separate realm of mathematical objects to think of mathematics as being concerned with the good and the beautiful.[3] Further, 'since' does not support the claim to have refuted those who claim mathematics says nothing about goods or evils [*peri agathōn kai kakōn* περὶ ἀγαθῶν καὶ κακῶν]. Aristotle mentioned in book B that some sophists, including Aristippus (c. 435–c. 356 BCE), held this view, in an argument about how final causes are not used in mathematical proofs:

> 'since an end or purpose is the end of some action, and all actions imply change; so that in unchangeable things this principle could not exist nor could there be a good-in-itself. This is why in mathematics nothing is proved by means of this kind of cause, nor is there any demonstration of this kind — "because it is better, or worse"; indeed no one even mentions anything of the kind. And so for this reason some of the Sophists, e.g. Aristippus, ridiculed mathematics; for in the arts, even in handicrafts, e.g. in carpentry and cobbling, the reason always given is "because it is better, or worse", but the mathematical sciences take no account of goods and evils.'[4]

It is *possible* that Aristotle was here thinking of beauty in proofs when he referred back to book B from the discussion of the good and the beautiful in book M, for an aesthetic choice of proof is not something internal to the proof. Counting against this, though, is the absence of 'beautiful' [*kalon*] in the passage from book B, while 'good' [*agathon*] seems here to be restricted to the sense of *morally* good.[5]

The broader content of the passage in book M is a little obscure. It certainly indicates that beauty and goodness are inherent in mathematics, even if not explicitly named. It does not give any hint as to what Aristotle saw as the loci of mathematical beauty. But, in compensation, there is a characterization of beauty and the claim that mathematics fulfils this characterization best: 'order and symmetry and definiteness'. This is the first, albeit partial, attempt to characterize mathematical beauty. It seems not too far away from Plato's conception of mathematical beauty: all the examples of mathematics Plato found beautiful seem to satisfy Aristotle's characterization. It is, after

[1] Aristotle, *Metaphysics*, BEK M.3.1078a31–b6.

[2] Cleary, *Aristotle and Mathematics*, p. 340.

[3] Netz, 'What did Greek Mathematicians Find Beautiful?', p. 146.

[4] Aristotle, *Metaphysics*, BEK B.2.996a26–35.

[5] Cleary, *Aristotle and Mathematics*, pp. 340–1; Ross, comm. on Aristotle, *Metaphysics* (ed. Ross), BEK M.3.1078a31–b6 [vol. II, p. 418].

all, objects that have order, symmetry, and definiteness that would be fitted to dwell unchanging in Plato's heaven.

Autonomy of mathematics

Aristotle prohibited crossing from one genus of knowledge to another in the course of a proof, a step known, following his terminology, as *metabasis eis allo genos* [μετάβασις εἰς ἄλλο γένος], for such a demonstration will neither be secure nor grant understanding.[1] One should not use results from arithmetic to prove results in geometry, for instance.[2] The only exception to this is that one can legitimately apply theorems from one genus to another that is subordinate to it, as optics is subordinate to geometry or harmonics to arithmetic.[3]

This restriction is an early instance of concern for what later became known as *purity* in proof.[4] But it also has a consequence of relevance to aesthetics: one cannot apply the results and techniques of mathematics in making judgements about beauty in mathematics. Aristotle's specific example was that one cannot prove by geometry whether the straight line is the most beautiful line.[5]

Cosmology

Aristotle considered the universe to be spherical on the grounds of the primacy of the sphere.[6] The circle is primary among the plane figures, the sphere among the solids, because they are the simplest shapes in their classes: the circle is bounded by only one line, the sphere by only one surface.[7] Thus the farthest of the heavenly bodies must be carried by a sphere, and, since there must be no voids, each of the other bodies must also be carried by a sphere, which entails a concentric system of spheres.

He adopted the Eudoxian system of spheres,* but added extra spheres between those for one heavenly body and those for the next, to compensate for their different motions.[8] This addition indicates that Aristotle viewed the spheres as physically existing components of the celestial machinery.[9]

Aristotle thus moved away from Plato's abstract mathematical problem of accounting for the phenomena of the motions of the planets using uniform motions and turned Eudoxus' model into an actual mechanism. Thus his preference for circular motion and for spheres is grounded in non-aesthetic concerns. But he seems to have held to the aesthetic primacy of circular motion, albeit in an indirect way. Circular motion is unceasing eternal motion, and is as close as a sensible thing can come to being a prime mover, which moves without itself being moved, and which is the kind of motion of the object of thought and desire.[10] And this object is beautiful: 'the apparent good

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[1] Aristotle, *Posterior Analytics*, BEK I.7.75a28–39.

[2] *Ibid.*, BEK I.7.75b3–4.

[3] *Ibid.*, BEK I.9.75b15–17, 76a22–5.

[4] Detlefsen, 'Purity as an Ideal of Proof'.

[5] Aristotle, *Posterior Analytics*, BEK I.7.75b17–18.

[6] Aristotle, *On the Heavens*, BEK II.4.286b9.

[7] *Ibid.*, BEK II.4.286b11–21.

* ▸ pp. 84–6

[8] Aristotle, *Metaphysics*, BEK Λ.8.1073b16–4a13.

[9] Lloyd, *Early Greek Science*, p. 92.

[10] Aristotle, *Metaphysics*, BEK Λ.7.1072a22–7.

[*to phainomenon kalon* τὸ φαινόμενον καλόν] is the object of appetite, and the real good [*to on kalon* τὸ ὄν καλόν] is the primary object of wish'.[1]

BEFORE EUCLID

Thus, at the end of the classical period, there had been much work that explicitly treated of or implicitly drew on mathematical beauty. Almost all of the extant discussions seem to have located mathematical beauty in the *objects* of mathematics: spheres, polyhedra, circles, ratios, whether seen in the sensible or intelligible worlds. Mathematical beauty was considered a property of mathematical objects and was therefore objective, not subjective. Plato's theoretical grounding for this objectivity of mathematical beauty was the same as for the objectivity of beauty in sensible objects: beauty comes about precisely through participation in the Form of the Beautiful. Aristotle's approach was to give more specific characterizations of beauty, namely order, symmetry, and definiteness, which are certainly found in the most beautiful objects of mathematics, such as the five regular solids, which were one of Plato's loci of mathematical beauty.

The only locus of mathematical beauty that was arguably not an *object* of mathematics was Plato's 'fairest of bonds': the geometric mean. The *concept* of the geometric mean is unlikely to have been thought of as a mathematical object. While ratios could be called mathematical objects, like numbers and the objects of geometry, each of the *means* has a degree of abstraction higher than any specific ratio that instantiates it.

But even admitting that the geometric mean was a non-object locus of mathematical beauty, there is a feature of the discourse that a modern mathematician might find curious. Despite the clear recognition of mathematical beauty in the classical period, *none* of the texts from this time mentions beauty in theorems. Nor has there been mention of beauty in proofs, save for the vaguest hint in Aristotle.

Is it possible that Greek mathematicians of this time did not recognize beauty in results and proofs? Yes: there seems to be no text by a modern mathematician that points to beauty in two of Plato's loci: the scalene $1 : \sqrt{3} : 2$ and isosceles $1 : 1 : \sqrt{2}$ triangles (the 'fairest' triangles) or the geometric mean (the 'fairest' bond). If such loci of mathematical beauty are alien to mathematicians today, it is plausible that Greek mathematicians and philosophers of the classical period did *not* see beauty in results and proofs.

Supposing this to be the case leads to inevitable speculation — and it can only be speculation — about the explanation. Perhaps the notion of proof, which was emerging around this time, was simply still too new or too unsettled. The Greek word for proof is *deiknymi* [δείκνυμι],

[1] Aristotle, *Metaphysics*, BEK Λ.7.1072a28.

but this can also mean 'show', 'point out', 'display',[1] which suggests that proving a result originally meant making its truth visible in some way. This meaning fits with how in the *Meno* Socrates drew diagrams to *show* the servant boy that his initial guesses of how to duplicate the square were incorrect, and to *show* the truth of the correct solution.[2] So perhaps sequential deductive proofs that established results not amenable to being 'shown' had not been fully accepted.

In opposition to such speculation, the historical record shows that actual deductive reasoning was in use when Hippocrates of Chios proved his quadratures of lunes in the middle of the fifth century BCE, when Socrates was a young man. According to a quotation made by Simplicius from the lost *History of Geometry* by Eudemus of Rhodes (*fl.* late 4th century BCE), Hippocrates proceeded by showing that circles have the same ratios as the squares on their diameters.[3] The idea that certain axioms had to be assumed was not yet clearly established in Plato's time, for Plato discussed the 'hypotheses'[4] that geometers assume without giving a justification. On the other hand, Proclus reported that Hippocrates was the first to write an '*Elements of Geometry*',[5] which suggests that he had the conceptual tools to at least begin creating an axiomatic-deductive text in what would later be seen as the Euclidean vein.

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[1] LSJ, 'δείκνυμι'.

[2] Szabó, *The Beginnings of Greek Mathematics*, pp. 189–90; Plato, *Meno*, STE 82a–5b.

[3] Simplicius, *On Aristotle Physics*, HEI 61.4–9 ⟦vol. 1.1–2, p. 105⟧.

[4] Plato, *Republic*, STE VI.510c sqq.

[5] Proclus, *Commentary on Euclid*, FRI 66.

Beauty and Subtlety

3

From Euclid and Archimedes to Pappus

> ‘ those speculations in whose beauty and subtlety
> there is no admixture of the common needs of life. ’
>
> — Plutarch, *Marcellus*, STE 307; § XVII.4
> (trans. T. L. Heath, intro. to Archimedes, *Works*, p. xvi).

> ‘ geometry […] is hard at the beginning and difficult of access,
> delightful in its regularity, full of beauty,
> unsurpassable in its effect. ’
>
> — Aggenus Urbicus, ‘De controversiis agrorum’, p. 25, ll. 17–21
> (trans. S. Cuomo *Pappus of Alexandria*, pp. 24–5).

On the deaths of Alexander (356–323 BCE) and Aristotle (384–322 BCE), once pupil and teacher, Greek culture had been sown throughout the eastern Mediterranean and western Asia by the Hellenizing project of the Macedonian conqueror. From the struggles among Alexander’s successors, Ptolemy I Soter emerged as ruler of Egypt, and he founded in the early third century BCE the fabled Library and Museum in Alexandria. Literature, philology, science, engineering, and mathematics all benefitted from the patronage of the Ptolemies, and Alexandria became preëminent in many fields of research — though not in philosophy, whose crown Athens retained.[1]

So began what the historian of mathematics Gino Loria (1862–1954) called ‘The golden age of Greek geometry’.[2] Alexandria was its centre,[3] as shown by the four giants that dominated that time: Euclid (*fl.* c. 300 BCE), the great synthesizer, was ‘of Alexandria’; Archimedes (c. 287–212 BCE), the greatest mathematician of antiquity, doubtless visited Alexandria and it was to correspondents there that he sent his work;[4] Eratosthenes (c. 276–c. 195 BCE), the polymath, was the second chief librarian at Alexandria;[5] and Apollonius (c. 260–c. 190 BCE), the greatest researcher and synthesizer of conic sections, certainly worked at Alexandria for a time.[6]

But mathematical eminence became the historian’s bane. The mathematical exposition of this period was so successful that it displaced its predecessors. Thus it is from this period that the earliest complete Greek mathematical texts survive; all earlier mathematical history comes through reconstructions and secondary sources. In his *Commentary on Euclid*, written more than seven centuries after Euclid lived, Proclus (410/12–485 CE) wrote a brief historical summary of Greek mathematics, relying on now-lost earlier sources such as the *History of Geometry* by Eudemus of Rhodes (*fl.* late 4th century BCE), who was a student of Aristotle.[1] According to Proclus' account, Hippocrates of Chios (*fl.* late 5th century BCE) was the first to compose an '*Elements of Geometry*'; another was compiled by Leon (*fl.* c. mid 4th century BCE), an otherwise-unknown pupil of Neoclides, an otherwise-unknown figure who was younger than Leodamas of Thasos, an almost-otherwise-unknown contemporary of Plato (c. 427–348/347 BCE).[2] But their works were eclipsed by Euclid's *Elements*: their manuscripts were no longer copied and so were lost to history. And Euclid himself was also a victim: his four books on conic sections are lost, surpassed by Apollonius' eight-book *magnum opus*.

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[1] Lloyd, ‘Hellenistic science’, pp. 322–3; Fraser, *Ptolemaic Alexandria*, vol. I, esp. ch. 7.

[2] Loria, *Le Scienze Esatte nell’ Antica Grecia*, bk 2; trans. C.

[3] Netz, ‘Classical Mathematics in the Classical Mediterranean’.

[4] Clagett, ‘Archimedes’, p. 213.

[5] Dicks, ‘Eratosthenes’, p. 388.

[6] Toomer, ‘Apollonius of Perga’, p. 179.

Important here is that among these surviving texts are indications of the views mathematicians held on the value, including the aesthetic value, of their work. This chapter examines the attitude of mathematicians, focusing first on three of the four titans of the golden age, namely Euclid, Archimedes, and Apollonius, then on the later work of Ptolemy, who, though best known as an astronomer, offered a theory of mathematical beauty. The work of Pappus (*fl.* c. 300–c. 350 CE), in late antiquity, serves as a coda to the chapter. The next chapter will examine the role of mathematical beauty in post-Aristotelian philosophy.

EUCLID

About Euclid of Alexandria (*fl.* c. 300 BCE) little is known save his works, but the *Elements* is monument enough for anyone.[3] In his own time, Euclid was only a minor figure: his fame came not from contemporaries who honoured him for new discoveries, but from later writers who fastened onto his *Elements* as the most systematic account of geometry.[4] He may have no claim to original *results*, but he clearly reshaped existing '*Elements of Geometry*', rearranging the exposition, finding new proofs, adding in new developments. He was so successful in this that no subsequent author displaced him: just as the epoch-maker of western philosophy was Socrates, so that of western mathematics was Euclid.

The status thus conferred on the *Elements* means that it crops up frequently in later discussions of the aesthetics of mathematics, and the work as a whole or individual results it contains have been frequently praised in aesthetic terms from antiquity onwards.[5] And the same holds true of Euclid's other works: Pappus wrote that proofs in the lost work *Porisms* 'have a delicate and natural aspect, cogent and quite universal, and pleasant for people who know how to see, and how to find',[6] and also criticized 'tasteless predecessors'[7] who added extra proofs to it. Further, though the *Elements* says nothing

[1] Heath, *Hist. Greek Math.*, vol. I, p. 118–21; see also Zhmud, *The Origin of the History of Science*, ch. 5.

[2] Proclus, *Commentary on Euclid*, FRI 66.

[3] Heath, *Hist. Greek Math.*, vol. 1, pp. 354 sqq.

[4] Netz, *The Transformation of Mathematics*, p. 10; Knorr, *The Ancient Tradition of Geometrical Problems*, ch. 4.

[5] For example, Proclus, *Commentary on Euclid*, FRI 73–4; Artmann, *Euclid*, p. vi; Kline, *Mathematics in Western Culture*, p. 55.

[6] Pappus, *Collection* 7, § 13.

[7] *Loc. cit.*

directly about aesthetics, the presentation and the deductive structure upon which Euclid settled together hint at aesthetic concerns.

Furthermore, the aesthetic value of Euclid's writing has assumed historiographical significance: the elegance and clarity of his writing have been used as evidence by scholars such as Franz Woepcke (1862–4), Karl von Jan (1836–99), Heinrich Menge (1838–1904), and T. L. Heath (1861–1940) to accept or reject as his work various other texts and fragments.[1]

Compass and straightedge

Euclid's *Elements* is well-known for its straightedge-and-compass approach to geometry. In formal terms, this means that the fundamental permissible geometrical procedures are 1) to draw a straight line between two given points; 2) to extend a straight line; and 3) to draw a circle with a given centre and radius.[2] Other permissible constructions are those that are built up from these. Note that the straightedge is not a *ruler*: it cannot be used to measure. Thus for Euclid, the procedure '*To bisect a given finite straight line*' comprises constructing an equilateral triangle on that line, finding the bisector of the angle at its apex, and showing that this bisector also bisects the original line (see Figure 3.1 (left)).[3] The constructions of the equilateral triangle and angle bisector have been previously built up from the fundamental procedures.[4] (There are other ways to bisect a line using compass and straightedge; see Figure 3.1 (right).)

In fact, book I of Euclid's *Elements* is unique in extant Greek mathematics in setting out an explicit list of the permissible constructions,[5] although the restriction to straightedge and compass may stretch back to Oenopides (*fl.* c. 450 BCE) and is presumably the context of Plato's rejection of mechanical methods in geometry.* It has been suggested that the choice of straightedge and compass was ultimately aesthetic: it was 'motivated by the desire to keep geometry simple, harmonious, and, therefore, aesthetically appealing'.[6] On the other hand, it has also been suggested that the choice is epistemological, driven by the desire to ensure that the concepts introduced actually exist: for instance, that dodecahedra exist as physical objects is clear from nature; that precise regular dodecahedra exist as mathematical objects is *prima facie* questionable. By building up mathematics from the simplest

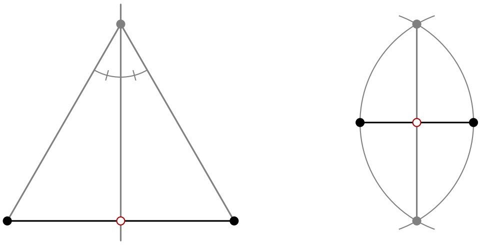
Elements, Prop. I.10.
A direct method.

FIGURE 3.1
Two methods for bisecting a straight line using only a straightedge and compass.

[1] Heath, *Hist. Greek Math.*, vol. 1, pp. 426, 441, 444–5.

[2] Euclid, *Elements*, bk I, Postulates 1–3.

[3] *Ibid.*, Prop. I.10.

[4] *Ibid.*, Props I.1, 9.

[5] Mueller, *Philosophy of Mathematics*, p. 27.

* ▸ pp. 79–80

[6] Kline, *Mathematics in Western Culture*, p. 50.

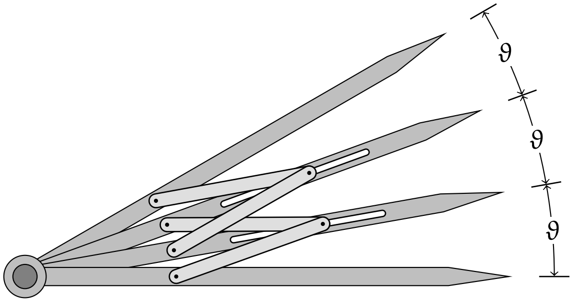

FIGURE 3.2
A ‘trisector compass’ (after Isaacs, ‘Two Mathematical Papers without Words’).

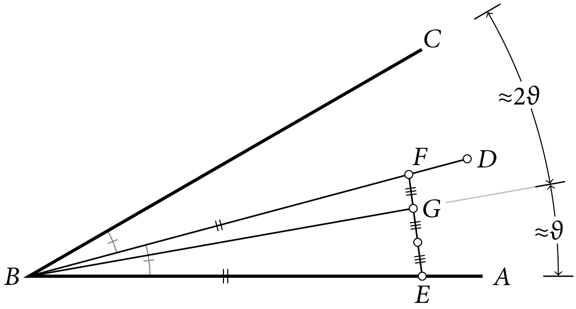

FIGURE 3.3
Rappaport’s method to trisect approximately an angle $\angle ABC$. Let BD be the angle bisector. Let E be on AB. Let BF have the same length as BE. Divide the segment EF into three equal lengths and let G be the division closest to F. Then BG is an approximate trisector.

concepts, there was much less room for dispute about the existence of more complex ones.[1] Euclid himself left no clues as to whether the choice was aesthetic or epistemological or otherwise.

It is now known that there is no straightedge-and-compass procedure for trisecting an arbitrary angle, for duplicating the cube, or for the problem known as ‘quadrature of the circle’ or ‘squaring the circle’, namely constructing a square with the same area as a given circle.

It is important to emphasize, however, that there is no evidence that the straightedge-and-compass restriction was applied beyond elementary mathematics.[2] Some of the highlights of Greek mathematics are free of the strictures of straightedge and compass: consider, for instance, Archytas’ (c. 425–c. 355 BCE) solution of the duplication of the cube.[*] Similarly, several non- straightedge-and-compass methods of angle trisection were known in antiquity.[3]

[For practical purposes, it is simple to trisect an angle by using a ‘trisector compass’ such as that shown in Figure 3.2. The earliest surviving account of such an instrument is in René Descartes’s (1596–1650) manuscript *Cogitationes privatae*,[4] and a discussion of the trisector compass by Henri Brocard[5] (1845–1922) implies that it was independently re-discovered by Charles-Ange Laisant (1841–1920) before 1875. Further, using only a straightedge and compass it is possible to obtain an approximate trisection that is accurate to within $\pi/498$ (or within 0.362°) for any acute angle; see Figure 3.3.[6] Finally, one can use repeated bisection of an angle to obtain an approximate trisection of an angle to an (in principle) arbitrary degree of accuracy, using the fact that $1/3 = \sum_{n=1}^{\infty}(-1)^{n-1}/2^{n}$.[7]]

[1] Kline, *Mathematical Thought*, vol. 1, p. 174.

[2] Dijksterhuis, *Archimedes*, p. 138.

[*] ▸ p. 53

[3] Heath, *Hist. Greek Math.*, vol. 2, pp. 235–44.

[4] Sasaki, *Descartes’s Mathematical Thought*, § 3.2.C.

[5] Brocard, ‘Note sur un compas trisecteur’.

[6] The technique was found by Rappaport, an amateur mathematician whose first name seems to be unrecorded, and was published in Ważewski, ‘Sur une méthode approximative’; see also Domoradzki & Stawiska, ‘Distinguished graduates in mathematics’, p. 106.

[7] Kincanon, ‘Trisection of an Angle’.

Post-Euclid, the same freedom applied: for instance, the curve known as the quadratrix, introduced by a Hellenistic geometer named Hippias (*fl.* 3rd–2nd centuries BCE; not Plato's contemporary the sophist Hippias of Elis) as a method of circle quadrature,[1] is generated by a mechanical process of moving two lines. Similarly, the conchoid, used by Nicomedes (*fl.* c. 3rd century BCE) to trisect the angle and duplicate the cube,[2] is drawn by a special instrument (see Figure 3.4).

In reality, Euclid bent his own rules. In the proofs of Propositions I.4 and I.8 of the *Elements*, concerning equality of triangles, Euclid made use of the technique of superposition of one configuration on another. Superposition is lining up two figures so that angles and lines exactly coincide and drawing the conclusion that the figures are congruent. It is not one of the procedures Euclid declared at the start of book I, but these proofs are the only places he used it.[3] There is a simple way Euclid could have avoided superposition in the proof of Proposition I.8 by moving it to after I.23 and modifying the proofs of three intermediate results that depend on it.[4] Proposition I.4 cannot be deduced from Euclid's first principles and should be adopted as a postulate.[5] On the one hand, the scarcity of places where Euclid used superposition and its absence from his first principles suggest a desire to avoid it. On the other hand, he seems not to have sought hard for an alternative to superposition in the proof of Proposition I.8.[6]

And Euclid often appealed to spatial intuition, in a rather inconsistent way. For example, Proposition III.1 assumes that a straight line and a circle cannot have more than two points in common; Proposition III.2 asserts that a line segment joining two points on a circle lies within that circle, which amounts to the same thing.[7]

'Two blemishes'

Although the use of undeclared techniques could reasonably be held, at least in retrospect, to be deductive flaws, they do not ever seem to have been considered aesthetic flaws. In the seventeenth century, Henry Savile* (1549–1622) wrote, with reference to Euclid's *Elements*, that 'On the most beautiful body of geometry there are two blemishes and no more, so far as I know.'[8] Savile's observation was made in the context of the parallel postulate, and this doubtless

* ▸ pp. 270–2

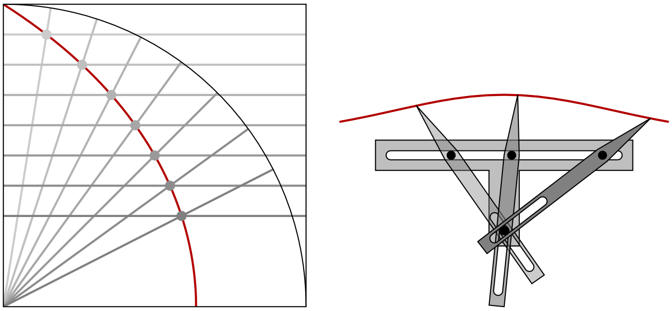

FIGURE 3.4
Left: The quadratrix, generated by the intersection of the horizontal line and the radius, both moving at constant speed.
Right: The conchoid, drawn by the point of a special instrument.

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[1] Knorr, *The Ancient Tradition of Geometrical Problems*, pp. 80–6.

[2] Pappus, *Collection* 4, #26–8.

[3] Cuomo, *Ancient Mathematics*, p. 99.

[4] Mueller, *Philosophy of Mathematics*, pp. 22–3.

[5] *Ibid.*, p. 23.

[6] *Loc. cit.*

[7] *Ibid.*, pp. 178–9.

[8] Savile, *Prælectiones*, p. 140; trans. Goulding, *Defending Hypatia*, p. 177.

one of the blemishes; the other is probably the theory of proportions.[1] Note also his view that there were 'no more' blemishes than these: the rest of the work had no aesthetic flaws.

Consider first the parallel postulate:

> '5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.'[2]

(See Figure 3.5.) Even in antiquity, the idea that this statement was a postulate, not a theorem, came under attack:

> 'This ought to be struck from the postulates altogether. For it is a theorem — one that invites many questions, which Ptolemy proposed to resolve in one of his books — and requires for its demonstration a number of definitions as well as theorems. And the converse is proved by Euclid himself as a theorem.'[3]

Proclus thought it 'ridiculous [*geloion* γελοῖον]'[4] that the converse of the parallel postulate '*In any triangle two angles taken together in any manner are less than two right angles*'[5] was provable but that the postulate itself was not. Proclus perhaps failed to realize both that the postulate was a stronger assertion than the converse, and that it was an indicator of Euclid's skill that he *did* realize this.[6] To those who immediately believed the assertion of the postulate, Proclus wrote:

> 'Geminus has given the proper answer when he said that we have learned from the very founders of this science not to pay attention to plausible imaginings in determining what propositions are to be accepted in geometry. Aristotle likewise says[7] that to accept probable reasoning from a geometer is like demanding proofs from a rhetorician.'[8]

According to al-Nairīzī (*fl.* 900 CE), who preserved in Arabic translation comments of other authors that are lost in the original, the late neoplatonist philosopher Simplicius (*fl.* 6th century CE) also thought that this postulate was not manifest and required proof.[9]

These objections are all concerned *prima facie* with whether the postulate requires proof. But lurking behind them is another concern: if the parallel postulate *can* be proved from the other first principles,

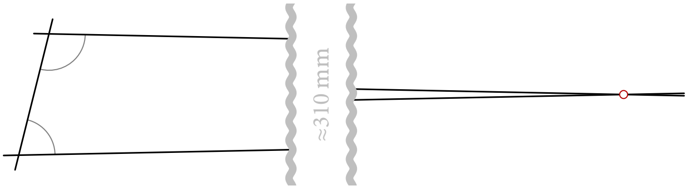

FIGURE 3.5
Illustration of Euclid's parallel postulate. The two marked angles sum to less than two right angles, so the lines intersect on that side.

[1] Goulding, *Defending Hypatia*, p. 177; Hilbert, *Lectures on the Foundations of Geometry*, p. 210; Stäckel & Engel, *Die Theorie der Parallellinien*, p. 22.

[2] Euclid, *Elements*, bk I, Postulate 5.

[3] Proclus, *Commentary on Euclid*, FRI 191–2.

[4] *Ibid.*, FRI 183–4.

[5] Euclid, *Elements*, Prop. I.17.

[6] Mueller, notes to Proclus, *Commentary on Euclid*, FRI 184.2.

[7] Aristotle, *Nicomachean Ethics*, BEK 1094b26 sq.

[8] Proclus, *Commentary on Euclid*, FRI 192.

[9] Heath, notes to Euclid, *Elements*, bk I, Post. 5, p. 203.

why assume it as a postulate? Today, this would be termed a desire for *independence* of one's axioms. Epistemologically and ontologically, there is nothing wrong with using a dependent axiom, provided that the axioms are consistent. But 'one feels that it is *aesthetically* bad to do so'.[1] Although it would be anachronistic to see in the ancient authors the concepts of independence or consistency of axioms, one cannot but sense a certain aesthetic distaste in their complaints about the parallel postulate.

Aristotle seemed to suggest that the theory of parallels as he knew it contained a *petitio principii*;[2] it is reasonable to infer that Euclid was the first to break the circularity by formulating the parallel postulate.[3] There is also evidence in the organization of book I of the *Elements* that Euclid was aware that the parallel postulate was in some way special. The first result whose proof uses the parallel postulate is Proposition I.29. However, if Euclid had been willing, he could have placed Propositions I.23, 27–32 immediately after I.16, and deduced I.17 as a trivial consequence of I.32.[4] This apparent willingness to sacrifice simplicity hints that Euclid wanted to push the content of book I as far as he could without invoking the parallel postulate. Even if Euclid adopted this approach, it would not imply that he thought of the parallel postulate as a blemish. But *if* he considered the parallel postulate a blemish, then he might have proceeded in this way. There is no direct evidence.

The theory of proportions, that other 'blemish', is developed in book V of the *Elements*. The treatment of proportion is complex, and certain results in this book have complicated proofs; these factors suggest that, globally and locally, the mathematics had been reworked little since its original development by Eudoxus (c. 400–c. 347 BCE).[5]

Euclid appears to have recast the proofs that he gathered into the early books of the *Elements* so as to maximize the material he could develop before having to treat of proportion theory and similarity for rectilinear figures.[6] There are peculiarities in the logical structure of book IV that can be plausibly explained as artifacts of the process of reformulation,[7] and many of the theorems would admit structurally simpler (if conceptually more complicated) proofs if proportion theory were developed first. It would be more elegant — or at least efficient — to begin with proportion and similarity and treat congruence as a special case of similarity.[8]

For instance, the proof of Proposition II.4 ('*To construct a square equal to a given rectilineal figure*') involves first constructing a rectangle with the same area as the rectilinear figure (which is possible by Proposition I.45), then the bulk of the proof involves 'manually' constructing a square equal to the rectangle. But this latter process can be accomplished directly using the theory of proportions, as shown in Propositions VI.13, 16, 17.[9] Similarly, Proposition III.36 likewise admits a simpler proof than Euclid's, but one which ultimately relies on the theory developed in book V.[1]

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[1] Wilder, *Evolution of Mathematical Concepts*, p. 9, emphasis in orig.

[2] Aristotle, *Prior Analytics*, BEK II.16.65a4–9.

[3] Heath, *Hist. Greek Math.*, vol. 1, p. 358; Heath, *Mathematics in Aristotle*, pp. 27–30.

[4] Mueller, *Philosophy of Mathematics*, p. 31.

[5] Artmann, *Euclid*, pp. 133–4; Mueller, *Philosophy of Mathematics*, pp. 203–4.

[6] Mueller, *Philosophy of Mathematics*, p. xi; Artmann, *Euclid*, p. 140.

[7] Mueller, *Philosophy of Mathematics*, pp. 197 sqq.

[8] *Ibid.*, p. ix.

[9] Artmann, *Euclid*, p. 140.

The most direct evidence of this reformulation is Proposition I.47, the pythagorean theorem. Proclus wrote that he admired those who discovered this theorem, but he admired Euclid more for establishing it via a 'very lucid proof' and extending it in book VI.[2] This observation implies that the proof in Proposition I.47 was Euclid's own, which in turn suggests that Euclid specially devised the proof so that he could put the theorem in book I rather than let a proportion-dependent proof relegate it to book VI.[3]

But, on the whole, there is a great deal of 'free play' in the arrangement of the *Elements*, with Euclid apparently preferring to group together results by subject-matter rather than deductive sequence. This approach sometimes led to wide separation between statements of lemmata and their applications in proofs.[4]

Perfect numbers

Perfect numbers, which will re-appear later in this history, were defined by Euclid: 'A perfect number is that which is equal to its own parts.'[5] In modern terminology, a perfect number is a positive integer equal to the sum of its proper divisors. Euclid proved that *If the sum* $1 + 2 + 2^2 + \dots + 2^k$ (*which equals* $2^{k+1} - 1$) *is prime, then the product of this prime and* 2^k *is a perfect number.*[6] Nicomachus* (*fl.* c. 100 CE) listed four perfect numbers 6, 28, 496, and 8128, which arise from Euclid's result for $k = 1, 2, 4$, and 6. Since the largest of these requires knowing only that $2^7 - 1 = 127$ is prime, it is reasonable to assume that all four were known in Euclid's time.

The name *perfect number* [*teleios arithmos* τέλειος ἀριθμός] hints at an aesthetic appreciation of this type of number, but only in later authors like Philo of Alexandria† (*fl.* early 1st century CE) and Nicomachus does this appreciation become explicit.

Beauty

Before leaving Euclid, it is worthwhile to examine some exemplars of beautiful or elegant mathematics in the *Elements*. First, Euclid's method of constructing a tangent to a circle through a given point has been seen as elegant because of the surprising indirect approach using a different tangent, which at first seems irrelevant:[7]

PROPOSITION A. *'From a given point to draw a straight line touching a given circle.'*[8]

Proof of A. Let $\mathcal{K}$ be the circle with centre E and A a point. Describe a circle $\mathcal{L}$ with centre E and radius AE. Let D be the intersection of AE and $\mathcal{K}$. Draw a line perpendicular to AE at D. Let F be its intersection with $\mathcal{L}$. Let B be the intersection of EF and $\mathcal{K}$; see Figure 3.6.

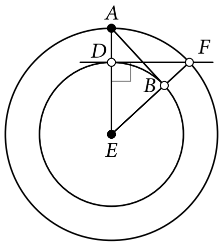

FIGURE 3.6
Construction of a tangent to a circle through a given point (Euclid, *Elements*, Prop. III.17).

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[1] Artmann, *Euclid*, p. 91.

[2] Proclus, *Commentary on Euclid*, FRI 426.

[3] Heath, *Hist. Greek Math.*, vol. 1, pp. 147–8.

[4] Mueller, *Philosophy of Mathematics*, p. 204.

[5] Euclid, *Elements*, Def. VII.22.

[6] *Ibid.*, Prop. IX.36.

* ▸ pp. 138–40

† ▸ pp. 130–3

[7] Artmann, *Euclid*, pp. 83–4.

[8] Euclid, *Elements*, Prop. III.17.

The triangles EDF and EBA are similar (by Proposition I.4; side-angle-side similarity). Therefore $\angle ABE$ is a right angle and so AB is a tangent. $\boxed{\text{A}}$

Another celebrated instance is Euclid's construction of the regular pentagon:[1]

> 'If there is such a thing as beauty in a mathematical proof, I believe that this proof of Euclid's for the construction of the regular pentagon sets the standard for a beautiful proof.'[2]

The proofs below are abbreviated, giving only the constructions in full. The arguments that the constructed figures have the claimed properties are a *tour de force* of application of results from previous books, but the real delight is in the dazzling efficiency of the construction.

First, a preliminary result constructs a isosceles triangle whose base angles (in modern terms) are $2\pi/5$ and whose apex angle is $\pi/5$. The proof appears to be one of Euclid's own formulation, replacing a simpler proof that depended on the theory of proportion.[3]

PROPOSITION B. *'To construct an isosceles triangle having each of the angles at the base double the remaining one.'*[4]

Proof of B. Let AB be a straight line. Divide AB at C so that the rectangle contained by AB and BC is equal to the square on AC (by Proposition II.2; the division is in what Euclid later called 'extreme and mean ratio'). Describe a circle $\mathcal{K}$ with centre A and radius AB. Let BD be the chord of $\mathcal{K}$ with length equal to AC. Describe a circle $\mathcal{L}$ circumscribing the triangle ACD (by Proposition IV.5); see Figure 3.7.

The line BD touches the circle $\mathcal{L}$ by a 'brilliant application'[5] of Proposition III.37. That each of $\angle BDA$ and $\angle DBA$ is double $\angle DAC$ follows by application of Propositions I.5, 6, 32, and III.32.

Thus ABD is the required triangle. $\boxed{\text{B}}$

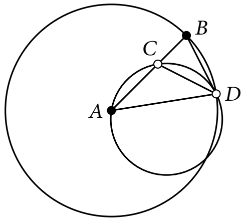

FIGURE 3.7
The construction of an isosceles triangle whose base angles are each double the apex angle (Euclid, *Elements*, Prop. IV.10).

Now comes the actual construction:

PROPOSITION C. *'In a given circle to inscribe an equilateral and equiangular pentagon.'*[6]

Proof of C. Let $\mathcal{K}$ be the given circle. Set out an isosceles triangle FGH having the angles at G, H double the angle at F (by Proposition IV.10). Inscribe in $\mathcal{K}$ a triangle ACD equiangular with the triangle FGH, so that $\angle CAD$ is equal to the angle at F and $\angle ACD$ and $\angle CDA$ are equal to the angles at G and H and so each double $\angle CAD$ (by Proposition IV.2; see page 8). Bisect $\angle ACD$ and $\angle CDA$ (by Proposition I.9) and let E and B be the intersection of these bisectors with $\mathcal{K}$; see Figure 3.8.

The pentagon $ABCDE$ is equilateral by application of Propositions III.26 and 29, and is thus equiangular by application of Proposition III.27. $\boxed{\text{C}}$

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[1] Euclid, *Elements*, Props IV.10–11.

[2] Hartshorne, *Geometry*, p. 50.

[3] Mueller, *Philosophy of Mathematics*, p. 194.

[4] Euclid, *Elements*, Prop. IV.10.

[5] Hartshorne, *Geometry*, p. 47.

[6] Euclid, *Elements*, Prop. IV.11.

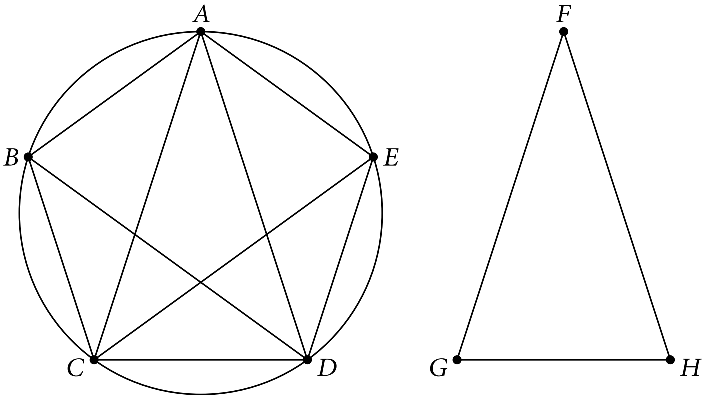

FIGURE 3.8
The construction of a regular polygon in a given circle (Euclid, *Elements*, Prop. IV.11).

Finally, Euclid's theorem asserting the infinitude of the prime numbers, together with its proof, appears so often as an exemplar of beauty that it is almost an obligation to give it here:

PROPOSITION P. *'Prime numbers are more than any assigned multitude of prime numbers.'*[1]

Euclid's original proof shows the existence of a prime outside a given collection of three prime numbers, but it is immediately clear that the technique would work for an arbitrary multitude of prime numbers. The proof that follows is structured like Euclid's, but uses modern notation:

Proof of P. Let $p_1, p_2, \ldots, p_n$ be prime numbers. Let $q = \operatorname{lcm}\{p_1, p_2, \ldots, p_n\}$. Either $q + 1$ is prime or it is not. If is it prime, then $q + 1$ is a prime number outside the given collection $p_1, p_2, \ldots, p_n$. If it is not prime, some prime number r divides it. This prime r cannot be any of the p_i, for if it were, then this number p_i would divide both q and $q + 1$ and hence would divide their difference 1, which is absurd. Thus r is a prime number outside the given collection $p_1, p_2, \ldots, p_n$. $\boxed{\text{P}}$

It is perhaps worth noting that the proof, as originally written, contains a flaw in the form of an unnecessary complication: Euclid considered the least common multiple (or, as he put it, 'the least number measured by') the given primes, instead of their product. Thus the proof could arguably be improved in terms of simplicity.

THE DIVISION OF THE CANON

The Division of the Canon is a short treatise on harmonics, attributed to Euclid by Porphyry (234–c. 305 CE) and ascribed to him by certain manuscripts. Whether he actually wrote it is questionable; whether it even had a single author is debated.[2] It is the

[1] Euclid, *Elements*, Prop. IX.20.

[2] Barbera, *Division*, pp. 3–13; cf. Barker, *The Science of Harmonics*, pp. 366–70.

only text from before Ptolemy's (c. 100–c. 170 CE) time that confronts a problem Plato raised: that of not just determining the ratios of concords, but of finding *what numbers* are consonant and explaining why.[1] Other writers had maintained that there was something special about the ratio of concords, but did not give a mathematical explanation.[2] The explanation offered in the *Division* is that the unity in the blending of the notes of the concord is linked to the unity of the numbers in the ratios that describe those concords:

> 'And we know of notes that some are consonant, some dissonant; and the consonant notes make a single blend from both notes, but dissonant notes do not. This being the case, it is reasonable that consonant notes, since they make a single blend of sound from both notes, are related numerically to one another in one name, since they are either multiple or superparticular.'[3]

This passage is the bridge that links the purely arithmetical propositions with which the *Division* begins with the later musical propositions: it is 'reasonable' that the ratio of any concord be multiple or superparticular (that is, of the form $n/1$ or $(n+1)/n$), which is a view that Ptolemy attributed to the pythagoreans.[4]*

ARCHIMEDES

> ' Il y a une imagination étonnante dans la mathématique pratique; et Archimède avait au moins autant d'imagination qu'Homère. '
>
> — Voltaire, *Dictionnaire Philosophique*, 'Imagination'.

Archimedes (c. 287–212 BCE) is traditionally seen as the first of the three greatest mathematicians in history, Newton and Gauss being the others.[5] In antiquity, Archimedes' reputation was such that, according to Plutarch, when the Roman army was forced back by the war-machines he designed for the defence of Syracuse, their general Marcellus (d. 208 BCE) called him 'this geometrical Briareus',[6] in reference to one of the Hecatoncheires, the hundred-handed, fifty-headed giants who helped Zeus and the Olympians defeat the Titans.[7]

The surviving works of Archimedes are of a very different character from those of Euclid. Archimedes' works are letters describing his own new discoveries; they are analogues of today's research articles. Euclid excelled in compiling and expositing and devising new proofs for previously-known results; the *Elements* is a textbook. Archimedes' letters take us closer to the actual process of mathematical discovery.[8] In particular, in his *Method* he described a process for discovering results and proofs. He imagined attaching figures to a lever in such a way as to produce equilibrium, considered a solid figure as consisting

[1] Plato, *Republic*, STE 531b–c.

[2] Barker, 'Mathematical Beauty Made Audible', p. 406.

[3] ps.-Euclid, *Division of the Canon*, p. 117.

[4] Barker, *The Science of Harmonics*, p. 385.

* ▸ pp. 50–1

[5] See, for example, Klein, *Geometry*, p. 215 [206]; Bell, *Men of Mathematics*, chs 2, 14; Snow, 'Foreword', p. 28.

[6] Plutarch, *Marcellus*, STE 307; § XVII.1.

[7] See Hesiod, *Theogony*, ll. 147 sqq., 713 sqq.

[8] Knorr, 'Archimedes After Dijksterhuis', p. 420.

of all of the sections obtained by cutting it by every plane of some fixed inclination, and similarly considered a plane figure as consisting of all segments obtained by cutting it by every line of some fixed angle. He knew this method did not yield a proof, but did supply useful knowledge that could make finding a proof easier,[1] and he wrote of it with evident pride.[2]

Given this insight into his process of investigation, it is therefore frustrating that — at least in his extant works — Archimedes is virtually silent on philosophy.[3] In particular, none of his surviving works contain any explicit aesthetic discussion. There is, however, evidence of aesthetic concerns in his work, to which particular attention is paid in the ongoing translation of and commentary on his works by Reviel Netz (1968–), who wrote that Archimedes' 'entire science was based on a sense of play and beauty, on hidden meanings'.[4] Archimedes clearly enjoyed teasing the correspondents to whom he sent his works, misleading them about intent and direction, surprising them with unexpected connections.[5] This approach, which often yields beauty in proof (though nowhere did Archimedes use that term), seems to have been widespread among post-Euclidean mathematicians of the early Hellenistic period, who wrote in what Netz has termed the *ludic* genre (as opposed to the paedagogic or survey genres).[6] Later commentators sometimes undid the unexpected and playful forms, producing a text that was simpler, more straightforward, less challenging — but perhaps ultimately less enjoyable.[7]

There has been a historiographical tendency to portray Archimedes as only interested in abstract, unapplied, theoretical mathematics. This tendency bears the patina of the respectability of age: according to Plutarch, writing in the late first or early second century CE, Archimedes saw 'the work of an engineer and every art that ministers to the needs of life as ignoble and vulgar'.[8] And modern writers such as the historian of mathematics Morris Kline[9] (1908–92) and the philosopher Alfred North Whitehead (1861–1947) saw Archimedes as the symbol of the abstract and theoretical:

> 'The death of Archimedes by the hands of a Roman soldier is symbolical of a world change of the first magnitude; the theoretical Greeks, with their love of abstract science, were superseded in leadership of the European world by the practical Romans.'[10]

This is a gross oversimplification. Archimedes' vast reputation as an engineer[11] is evident in the legendary boast that, given a place to stand, he could move the Earth,[12] He also considered empirical matters such as how to measure the angular diameter of the Sun, including the potential errors in observation and how, consequently, it must suffice to give an upper and lower limit.[13] But he was also willing to investigate and write about mathematics that could not have the slightest practical use, such as his study of weights in equilibrium

[1] Dijksterhuis, *Archimedes*, pp. 314, 318–9.

[2] Knorr, 'Archimedes After Dijksterhuis', p. 420.

[3] *Ibid.*, p. 440.

[4] Netz & Noel, *The Archimedes Codex*, p. 58.

[5] Netz, *Ludic Proof*, p. 13–14, ch. 3, *passim*.

[6] *Ibid.*, pp. 110–11.

[7] *Ibid.*, p. 113.

[8] Plutarch, *Marcellus*, STE 307; § xvii.4.

[9] Kline, *Mathematics in Western Culture*, pp. 11–12.

[10] Whitehead, *An Introduction to Mathematics*, ch. III.

[11] Knorr, 'Archimedes After Dijksterhuis', pp. 422–4.

[12] Pappus, *Collection*, bk VIII, § xi.

[13] Dijksterhuis, *Archimedes*, pp. 364–9.

at incommensurable distances from the fulcrum;[1] this work was 'pure mathematical virtuosity'.[2]

Sphere and cylinder

In the twentieth century, Max Dehn (1878–1952) said that Archimedes' discovery that the surface area of a sphere was four times its great circle was the one of the most beautiful results of Greek mathematics.[3] This theorem is one of the highlights of Archimedes' two books *On the Sphere and the Cylinder*. Archimedes himself had a high opinion of the results in this work. He wrote that he

> 'would not hesitate to compare them to the properties investigated by any other geometer, indeed to those which are considered to be by far the best [*hyperechein* ὑπερέχειν] among Eudoxus' investigations concerning solids: that every pyramid is a third part of a prism having the same base as the pyramid and an equal height, and that every cone is a third part of the cylinder having the base the same as the cylinder and an equal height.'[4]

The results Archimedes had in mind were three. The first was the one Dehn so admired: that the surface area of a sphere is four times its great circle.[5] The other two results are that if a cylinder circumscribes a sphere, its volume and surface area are each half as large again as those of the sphere.[6]

According to Plutarch, Archimedes desired that his tomb should be marked by a cylinder enclosing a sphere and an inscription of the ratio of the one to the other;[7] Cicero related how he himself sought out Archimedes' tomb and found a column just so inscribed.[8] That his tomb was so formed suggests that Archimedes, or those close to him, considered his result relating the sphere and cylinder, or his books *On the Sphere and the Cylinder* as a whole, to be his masterpiece. As his translator Netz pointed out, this work is the simplest and highest manifestation of his project of measuring curvilinear figures.[9]

Archimedes in fact described how he formulated the conjecture about the surface area of a sphere, and a conjecture related to the result about the volume of a sphere:

> 'From this theorem, to the effect that a sphere is four times as great as the cone with a great circle of the sphere as base and with height equal to the radius of the sphere, I conceived the notion that the surface of any sphere is four times as great as a great circle in it; for, judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius.'[10]

[1] Dijksterhuis, *Archimedes*, pp. 305–6.

[2] Lloyd, *The Ambitions of Curiosity*, p. 62.

[3] Dehn, 'The Mentality of the Mathematician', pp. 21–2.

[4] Archimedes, *Sphere and Cylinder*, bk 1, Introduction.

[5] *Ibid.*, bk 1, § 33.

[6] *Ibid.*, bk 1, § 34, Corollary.

[7] Plutarch, *Marcellus*, STE 307; § XVII.7.

[8] Cicero, *Tusculan Disputations*, § v.xxv.

[9] Netz, intro. to Archimedes, *Sphere and Cylinder*, p. 19.

[10] Archimedes, *The Method*, after Prop. 2; see also Dijksterhuis, *Archimedes*, pp. 322–3.

Archimedes' conjectures may have been guided by a conscious or unconscious search for beautiful integer ratios between geometric configurations.[1] There is no direct evidence for this motivation, but Archimedes' work seems to exhibit a preference for small integer ratios (as do books XII and XIII of Euclid's *Elements*).[2]

On the other hand, this apparent desire for small-number ratios seems to have been tempered by the aesthetic value of the geometric configuration. Archimedes' results relate the volume of the sphere to the volume of the cone whose base equals the great circle of the sphere and whose height is the radius of the sphere, and the proof indicates that Archimedes thought of the cone as inscribed in the sphere.[3] From the design of his tomb, he thought of a cylinder circumscribing the sphere. That is, he seems to have thought of a configuration like that shown in Figure 3.9 (left), in which the ratio of the volumes of the cylinder, sphere, and cone is 6 : 4 : 1. He could have considered a hemisphere circumscribed by a cylinder and enclosing a cone (Figure 3.9 (middle)): then the ratio of the volumes of cylinder, hemisphere, and cone is 3 : 2 : 1, which is surely a more pleasing ratio. The same ratio could be found by basing the cone at the base of the cylinder and having its height be the diameter of the sphere (Figure 3.9 (right)), but now the cone cuts the sphere. Speculatively, therefore, Archimedes had inherited sufficient aesthetic preference for the sphere so as to resist cutting it in half, and wished to retain the three-level containment of cylinder, sphere, cone.

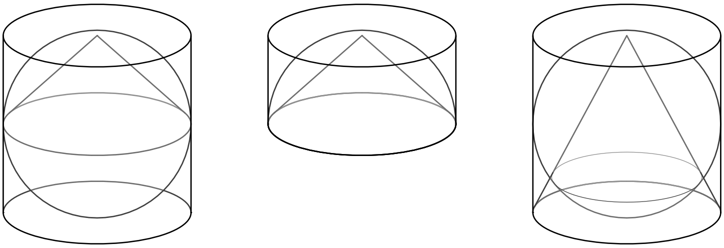

FIGURE 3.9
Configurations of sphere or hemisphere, a circumscribed cylinder, and a cone with base equal to the base of the cylinder and height equal to the radius or diameter of the sphere.

Archimedes' translator Netz pointed out the narrative aesthetic structure of the first book *On the Sphere and the Cylinder*:[4] the announced intention is to prove that the surface of a sphere is four times its great circle, and that the volume of a sphere is two-thirds of a cylinder that exactly encloses it. Yet Archimedes started by proving two-dimensional results: a series of theorems on circles and polygons in proportion (Propositions 2–6). These he followed by results on the surfaces of pyramids and cones (Propositions 7–20), which at least approaches the cylinder if not the sphere. After an interlude of complicated results on the earlier polygons (Propositions 21–2) comes the dramatic shift: he imagined rotating polygons about an axis: all the two-dimensional results suddenly acquire three-dimensional analogues and their applicability to the situation at hand suddenly makes sense to the reader. The reader retrospectively perceives the elegance

[1] Schneider, *Archimedes*, pp. 89–90.

[2] Netz, 'What did Greek Mathematicians Find Beautiful?', pp. 437–40.

[3] Archimedes, *Sphere and Cylinder*, bk 1, § 34; Dijksterhuis, *Archimedes*, ch. x, Prop. 2.

[4] Netz, 'The Aesthetics of Mathematics', pp. 256–7.

and efficiency of the approach: nothing has been superfluous to the proof.[1] There was no need for Archimedes to arrange his exposition so: he could have explained the rotational manoeuvre at the start, but this would have robbed his 'master-stroke'[2] of its brilliance.

There is also a wonderful later shift, though not the same kind of surprise, in how the complexities of the statements of the preceding results give way to the glorious simplicity of Proposition 33:[3] '*The surface of every sphere is four times the greatest circle of the* ⟨*circles*⟩ *in it*.'[4]

* * *

It was not only mathematicians of Archimedes' calibre who appreciated the cone–sphere–cylinder ratio. Three centuries later, the architect Nicon (d. 149/50 CE), father of the philosopher and physician Galen (129–c. 210/217 CE), thought it fitting to point out the ratio of the configuration shown in Figure 3.9 (right) in a public inscription in his city, Pergamon:

> '[...] the cone, the sphere, the cylinder.
> If a cylinder encloses the other two shapes,
> [...]
> Competition the principle and in solids
> the progression 1 : 2 : 3,
> a noble [*gennikē* γεννικὴ], divine equalization,
> but also mutual interdependence
> of the solids, always in the ratio 1 : 2 : 3.
> They should be beautiful [*kala*] and wonderful,
> the three solid shapes,
> because the cube makes an equal ratio
> for all time for solids and surface areas,
> and if a cylinder fits inside a cube,
> and especially if a divine sphere does,
> leading for all, the cube is 42,
> the cylinder 33, and the sphere 22.'[5]

Nicon was clearly impressed by the 1 : 2 : 3 ratio of both the volumes and the surface areas of the cone, sphere, and cylinder, and this admiration extended to the analogous relation of the volumes and surface areas of the sphere, the cylinder, and the cube that fits around them, which is approximated by 22 : 33 : 42 (and which is precisely 1 : $\pi/4$: $\pi/6$). He undoubtedly admired these ratios as an architect: a sphere inside a cylinder brings to mind the Pantheon built by Hadrian (76–138 CE; r. 117–38 CE) at Rome, of which the Temple of Zeus Asclepius Soter in Pergamon was a half-scale copy.[6] These almost archimedean edifices were shaped so that a basically cylindrical rotunda was crowned with a hemispherical dome under which a sphere would fit (see Figure 3.10). Such architectural reification of mathemat-

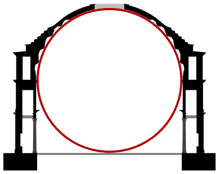

FIGURE 3.10
A sphere would fit under the dome of the Pantheon.

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[1] Netz, intro. to Archimedes, *Sphere and Cylinder*, p. 25.

[2] Netz, 'The Aesthetics of Mathematics', p. 257.

[3] Netz, comm. on Archimedes, *Sphere and Cylinder*, bk 1, § 33.

[4] *Loc. cit.*

[5] Fränkel, *Die Inschriften von Pergamon*, § 333A, ll. 16–28 [vol. 2, p. 246]; trans. Thomas, 'From Text to Building', pp. 141–2.

[6] Thomas, 'From Text to Building', p. 149.

ics fitted with the sentiment in the closing line of Nicon's inscription: 'Geometry shall lead the Muses.'[1]

Archimedean solids

According to Pappus, Archimedes discovered thirteen solids, besides the five regular ones, that were 'contained by equilateral and equiangular, but not similar, polygons'.[2] This condition is not a complete characterization. Pappus then proceeded to enumerate the various solids in terms of the numbers of faces of each type of regular polygon, giving the numbers of edges and vertices of the solid. The solids are shown, with their modern names, in Figures 3.11 and 3.12. Notice that these figures together contain *fourteen* polyhedra, for the rhombicuboctahedron and pseudo-rhombicuboctahedron are non-congruent polyhedra that both have 18 square faces and 8 triangular faces; thus Pappus' account could technically describe either. [Hero of Alexandria (*fl.* 62 CE) recorded that Archimedes said that Plato knew of the cuboctahedron.[3] Hero's report contains inaccuracies: it claims that Archimedes' thirteen solids included the five regular solids and a solid with eight square and three triangular faces. Even if the statement about Plato's knowledge of the cuboctahedron is correct, Archimedes was the first to attempt a systematic study of this class of polyhedra.]

Pappus' account is a discussion of how the sphere is the largest solid with a given surface area, and examines solids that appear 'regular' (in an informal sense, not in the technical sense).[4] Thus it is reasonable to conclude that Archimedes was concerned with solids that can be inscribed in a sphere, in the sense that there is a sphere passing through all of their vertices. This excludes polyhedra such as the triangular dipyramid (see Figure 1.7 on page 28). A scholium to Pappus, which has only partly survived, describes how Archimedes constructed the truncated tetrahedron, cuboctahedron, truncated octahedron, and truncated cube from the tetrahedron, cube, octahedron, and cube respectively, by truncating them at each vertex by a specified amount.[5] The scholium breaks off before the construction of the rhombicuboctahedron or pseudo-rhombicuboctahedron, the fifth solid in Pappus' order.[6] It is plausible, though, that Archimedes constructed the rhombicuboctahedron by truncating the cube along each edge, then truncating those vertices of the resulting shape that lay on the diagonals of the original cube.[7]

It is straightforward to see that if truncation of a regular solid is uniform, in the sense of being invariant under the symmetries of the original solid, then it will yield another solid that can be inscribed in a sphere. So such truncations would have supplied Archimedes with examples of 'regular' solids inscribable in a sphere. And some of these would have regular faces. But this does not account for two of the thirteen solids he discovered: the snub cube and snub dodecahedron

[1] Fränkel, *Die Inschriften von Pergamon*, § 333A, l. 41 [vol. 2, p. 246]; trans. Thomas, 'From Text to Building', p. 142.

[2] Pappus, *Collection*, § v.xix; trans. Dijksterhuis, *Archimedes*, p. 405.

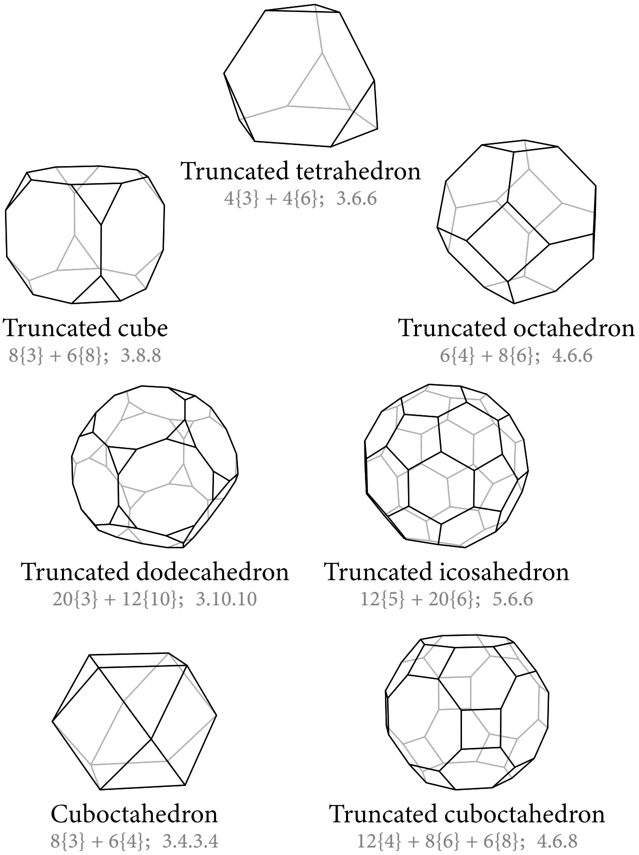
[3] Hero, *Definitiones*, § 104, p. 66.

[4] Pappus, *Collection*, § v.xix.

[5] Dijksterhuis, *Archimedes*, p. 407.

[6] *Loc. cit.*

[7] *Ibid.*, p. 408.



Truncated tetrahedron
4{3} + 4{6}; 3.6.6

Truncated cube
8{3} + 6{8}; 3.8.8

Truncated octahedron
6{4} + 8{6}; 4.6.6

Truncated dodecahedron
20{3} + 12{10}; 3.10.10

Truncated icosahedron
12{5} + 20{6}; 5.6.6

Cuboctahedron
8{3} + 6{4}; 3.4.3.4

Truncated cuboctahedron
12{4} + 8{6} + 6{8}; 4.6.8

FIGURE 3.11
Seven of the fourteen solids whose faces are regular polygons and where the arrangement of faces around each vertex is the same, other than the regular solids, prisms, and anti-prisms. These seven solids include the five obtainable by truncating the regular solids. The expressions below each name describe the faces and vertex configurations of that solid.

For instance, for the truncated tetrahedron, '4{3} + 4{6}' signifies 4 triangular (3-gonal) and 4 hexagonal (6-gonal) faces, while '3.6.6' signifies that clockwise around each vertex there is a triangle followed by a hexagon followed by a hexagon.

cannot be obtained by such a process of truncation.[1] Thus Archimedes cannot have found them in this way. But the solids that he could have found by uniform truncations would have had the same configuration of faces at each vertex.

This conclusion naturally suggests that Archimedes employed the same method as Johannes Kepler (1571–1630) used over eighteen centuries later: enumerating the possible solids by considering the possible configurations of regular polygons at each vertex: there can be only a limited number of possibilities, for there must be at least three regular polygons at each vertex and the internal angles must sum to less than 2π. Thus Archimedes would have enumerated the possibilities, found thirteen candidate configurations, and then constructed a corresponding sphere-inscribable solid using truncation and other methods. Such an approach implies that the class of solids Archimedes studied was defined by the following criterion:

> All faces are regular polygons, and the arrangement
> of faces around each vertex is the same. } (L)

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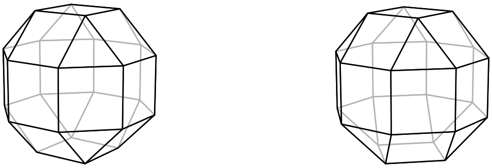
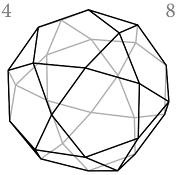
[1] Cromwell, *Polyhedra*, p. 84.

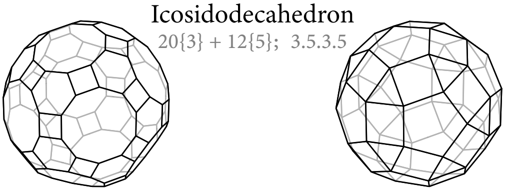


Rhombicuboctahedron
8{3} + 18{4}; 3.4.4.4

Pseudo-rhombicuboctahedron
8{3} + 18{4}; 3.4.4.4

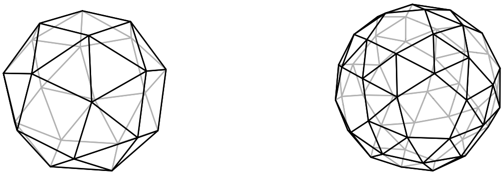


Icosidodecahedron
20{3} + 12{5}; 3.5.3.5



Truncated icosidodecahedron
30{4} + 20{6} + 12{10}; 4.6.10

Rhombicosidodecahedron
20{3} + 30{4} + 12{5}; 3.4.5.4



Snub cube
32{3} + 6{4}; 3.3.3.3.4

Snub dodecahedron
80{3} + 12{5}; 3.3.3.3.5

FIGURE 3.12
Seven of the fourteen solids whose faces are regular polygons and where the arrangement of faces around each vertex is the same, other than the regular solids, prisms, and anti-prisms.

Both the rhombicuboctahedron and the pseudo-rhombicuboctahedron have eight triangular faces and eighteen square faces, and both have three square faces and one triangular face meeting at each vertex. But these two polyhedra are distinct: the pseudo-rhombicuboctahedron has a unique band of eight squares, which form an 'equator'; the rhombicuboctahedron has three such (orthogonal) bands.

The class of solids that satisfy criterion (L) includes the two infinite classes of prisms (two congruent *n*-gonal faces joined by a band of *n* square faces) and anti-prisms (two congruent *n*-gonal faces joined by a band of 2*n* equilateral triangular faces); see Figure 3.13. It is reasonable to presume that Archimedes found the classes of prisms and anti-prisms during his exploration of possible configurations. It is also reasonable to suppose that he excluded them as 'disc-like' rather than 'sphere-like', as Kepler would do.* Outside of these classes, there are thirteen possible vertex configurations to consider, but there are fourteen solids that satisfy criterion (L); see Figures 3.11 and 3.12. The vertex configuration 3.4.4.4 (around the vertex, one triangle and three squares) corresponds to two distinct solids: the rhombicuboctahedron and the pseudo-rhombicuboctahedron (see the first row of Figure 3.12).

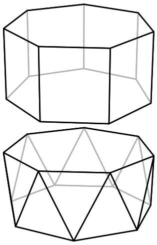


FIGURE 3.13
A heptagonal prism and anti-prism.

Thus, if Archimedes did follow the vertex-enumeration procedure, he seems to have missed one of the fourteen solids satisfying (L). Pappus' list of solids mentions only one with eight triangular and eighteen square faces,[1] which leads to the conclusion that Archimedes missed either the rhombicuboctahedron or pseudo-rhombicuboctahedron.

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* ▸ pp. 290–2

[1] Pappus, *Collection*, § v.xix.

This omission suggests in turn that he did not imagine that the same vertex configuration could give rise to distinct solids. In this, he may have been misled by the analogue of the regular solids, where there is a one-to-one correspondence between vertex configurations and solids (see Figure 1.5). Archimedes seems to have thought that criterion (L) was equivalent to the following:

> All faces are regular polygons, and for each pair of vertices, some symmetry carries one vertex to the other. (G)

(The wording is anachronistic, since at the time *symmetria* did not signify 'symmetry' in the modern sense of 'transformation', but Archimedes would at least have seen that the rhombicuboctahedron would 'look the same' when rotated in a way that made the pseudo-rhombicuboctahedron 'look different'.) With the insertion of 'congruent' before 'regular', criteria (L) and (G) are equivalent characterizations of the regular solids. But the criteria (L) and (G) themselves are not equivalent: clearly (G) implies (L), but the converse does not hold. To see that the pseudo-rhombicuboctahedron does not satisfy (G), observe that there is a unique band of eight squares in the pseudo-rhombicuboctahedron (forming an 'equator' in the diagram in Figure 3.12), which must be carried to itself by a symmetry.

Archimedes may have been misdirected by aesthetic concerns. He may have assumed that the class of solids satisfying (L) would have the symmetry properties of the regular solids that so pleased Plato. One cannot condemn him for this, for the confusion over the criteria (L) and (G) has persisted ever since, and authors make different and sometimes inconsistent use of them to define 'archimedean' or 'semi-regular' solids.[1] And aesthetic value is sometimes the ultimate arbiter that favours (G) over (L), so as to exclude the pseudo-rhombicuboctahedron:

> 'The true Archimedean solids, like the Platonic solids, have an aesthetic quality which Miller's solid[2] does not possess. This attractiveness comes from their high degree of symmetry — a property that is easily appreciated and understood on an intuitive level.'[3]

But all of this argument depends on the assumption that Archimedes sought solids satisfying (L). It is just as reasonable to think that Archimedes had in mind (G) as his criterion. With this premiss, he was not misled by aesthetic value at all: he was guided by it. Even if he used enumeration of vertex configurations in his search, he would have used the symmetries of the solid to contruct it as a whole. Thus he would have counted only the rhombicuboctahedron and not the pseudo-rhombicuboctahedron, and his list of thirteen solids would have been complete. In short, if Archimedes was guided by aesthetic judgement or a sense of symmetry, his classification is complete; if he was guided by a technical definition, it is incomplete.

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[1] Grünbaum, 'An enduring error'.

[2] The pseudo-rhombicuboctahedron; here named for J. C. P. Miller (1906–81), who accidentally rediscovered it.

[3] Cromwell, *Polyhedra*, p. 91.

APOLLONIUS

Conic sections, or simply *conics*, are the shapes that are formed by the intersection of a cone and a plane. If the angle of the plane is less than the angle of the cone, the resulting shape is an ellipse; if equal, a parabola; if greater, a hyperbola (see Figure 3.14).

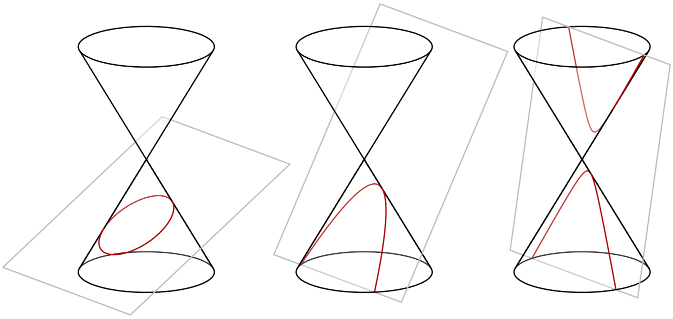

FIGURE 3.14
The three kinds of conic sections: the ellipse, where the angle of the section is less than the slope of the cone; the parabola, where they are equal; the hyperbola, where the angle of the section is greater.

Conic sections were first studied by Menaechmus[1] (*fl.* mid 4th century BCE), and Euclid wrote four books on the subject.[2] But all of this earlier work has been lost, displaced by the *Conics* of Apollonius of Perga (c. 260–c. 190 BCE), which became the standard authority.[3]

In the preface to the first book of the *Conics*, which serves as a preface to the whole work, Apollonius wrote, addressing Eudemus of Pergamon (*fl.* third century BCE):

> ‘The third book contains many incredible [*paradoxa* παράδοξα] theorems of use for the construction of solid loci and for limits of possibility of which the greatest part and the most beautiful [*kallista* κάλλιστα] are new. And when we have grasped these, we knew that the three-line and four-line locus had not been constructed by Euclid, but only a chance part of it and that not very happily. For it was not possible for this construction to be completed without the additional things found by us.’[4]

This quotation is triply important in the historiography of mathematical beauty: it is the earliest extant description of a mathematical *theorem* as ‘beautiful’; it is the *earliest* extant application of the term ‘beautiful’ to mathematics by a mathematician; and it is the *unique* extant use of the term ‘beautiful’ to describe theorems by an ancient Greek mathematician.[5] There is another use of ‘beautiful’ in the preface to the seventh book, but this is not clearly applied to theorems:

> ‘In this book are many wonderful and beautiful [*ḥasana* حَسَنَة] things on the topic of diameters and the figures constructed on them, set out in detail.’[6]

A modicum of caution is required since books 5–7 survive only in an Arabic translation by Thābit ibn Qurra (836–901 CE). Nevertheless, with the assumption that 'beautiful' here tracks back to *kalos* [καλός], this quotation at least serves to reinforce that Apollonius applied the term 'beautiful' in mathematics to things other than mathematical objects. [Thābit's Arabic translation of the passage from the preface to the first book quoted above does not contain any term indicating beauty, so no comparison of aesthetic vocabulary is possible.[1]]

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[1] Heath, *Hist. Greek Math.*, vol. I, p. 251.

[2] *Ibid.*, vol. I, p. 438–9.

[3] *Ibid.*, vol. II, p. 126.

[4] Apollonius, *Coniques*, vol. 1.2, p. 4, ll. 5–11; bk 1, preface; trans. Apollonius, *Conics* (Ed. Taliaferro), p. 603.

[5] Netz, ‘What did Greek Mathematicians Find Beautiful?’, p. 428.

[6] Apollonius, *Coniques*, vol. 4, p. 351, ll. 6–7; bk 1, preface; trans. Apollonius, *Conics* (Ed. Toomer), p. 382.

Note also that in the quotation from book 1, there is at least a hint of a connection between unexpectedness and beauty, since the word *paradoxa* [παράδοξα], translated 'incredible' above, has the sense of 'contrary to expectation'.[2] This connection is an idea that will recur.

According to Apollonius, the new and beautiful theorems are those that are necessary for the three-line and four-line locus. A *four-line locus* is the set of points whose distances d_1, d_2, d_3, d_4 to four given straight lines satisfy an equation $d_1 d_2 = k d_3 d_4$ for some given k. A *three-line locus* is the set of points where the distances satisfy an equation $d_1 d_2 = k d_3^2$. Three- and four-line loci are always conic sections. For the three-line locus, this fact can indeed be proven using the last three results in book 3 of the *Conics*,[3] which thus seem to be the theorems that Apollonius called 'most beautiful'. In Apollonius' original formulation, they seem unwieldy to modern eyes; witness, for instance, Proposition III.54:

THEOREM A. *'If two tangents to a section of a cone or to a circumference of a circle met, and through the points of contact parallels to the tangents are drawn, and from the points, of contact to the same point of the line of the section, straight lines are drawn across cutting the parallels, then the rectangle contained by the straight lines cut off to the square on the straight line joining the points of contact has a ratio compounded of the ratio which the inside segment line joining the point of meeting of the tangents and the midpoint of the straight line joining the points of contact has in square to the remainder, and of the ratio which the rectangle contained by the tangents has to the fourth part of the square on the straight line joining the points of contact.'*[4] $\boxed{\text{A}}$

Theorem A is more easily understood as follows: Given a conic section and points A, Γ and Θ on it, let the tangents at A and Γ meet at Δ. Let E be the midpoint of $A\Gamma$ and let ΔE intersect the conic section at B. Let Z be the intersection of $\Gamma\Theta$ and the line through A parallel to the tangent at Γ, and let H be the intersection of $A\Theta$ and the line through Γ parallel to the tangent at A (see Figure 3.15). Then

$$\left.\begin{array}{l} \mathrm{rect}(AZ, \Gamma H) : \mathrm{sq}(A\Gamma) \text{ is composed of} \\ \qquad\qquad \mathrm{sq}(E\Delta) : \mathrm{sq}(B\Delta) \\ \qquad\qquad\quad \text{and} \\ \qquad\qquad \mathrm{rect}(A\Delta, \Delta\Gamma) : \text{fourth part of } \mathrm{sq}(A\Gamma). \end{array}\right\} \quad \text{(A)}$$

Interpreting each line segment as its length, which would have been alien to Apollonius' thought, (A) becomes:

$$(AZ \cdot \Gamma H)/A\Gamma^2 = (E\Delta^2/B\Delta^2) \cdot \big((A\Delta \cdot \Delta\Gamma)/(A\Gamma/4)\big).$$

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[1] Apollonius, *Coniques*, vol. 1.1, p. 253, ll. 7–10; bk 1, preface.

[2] LSJ, 'παράδοξος'.

[3] Taliaferro, 'Appendix to Apollonius' *Conics*'.

[4] Apollonius, *Coniques*, vol. 2.3, p. 312; Prop. III.54; trans. Apollonius, *Conics* (Ed. Taliaferro), pp. 793–4.

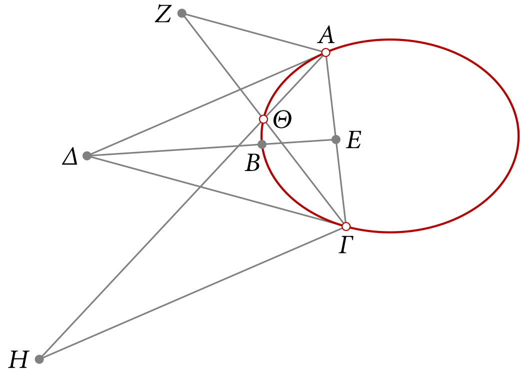

FIGURE 3.15
Illustration of Apollonius, *Coniques*, Prop. III.54, for an ellipse.

[Reviel Netz pointed out that book 3 is rather anticlimactic, in that Apollonius could have ended by proving the three-line locus property. This would have illustrated his claim in the preface that, in order to do so, his results were necessary, and would have given a more archimedean flavour to a text that, at least in retrospect, has a particular aim. Instead, Apollonius left a seemingly aimless and labyrinthine text.[1]]

[In his much later commentary on the *Conics*, Eutocius (b. c. 480 CE) praised Apollonius' account of intersections of conic sections in book 4 as 'elegant [*charien* χαρίεν] and clear'.[2]]

PTOLEMY

Claudius Ptolemy's (c. 100–c. 170 CE) legacy and influence as an astronomer is indicated in the label 'Ptolemaic' for the entire class of cosmologies that attempt to use circular motions to model the movement of the heavenly bodies around the Earth. He was the epoch-maker of geocentric astronomy. His great astronomical treatise was originally titled *Mathēmatikē Syntaxis* [Μαθηματικὴ Σύνταξις], but both of the extant Arabic translations, by al-Ḥajjāj (*fl.* 786–830 CE) and by Thābit ibn Qurra (836–901 CE), simply title it *al-mjsṭy* [المجسطي], derived from the Greek *megistē* [μεγίστη], signifying 'The Greatest [Treatise]' whence derives the modern title '*Almagest*'.[3] The technical promise of his theory and the philosophical and aesthetic preference for circular motion were such that there was no fundamental conceptual innovation in astronomy until the sixteenth and seventeenth centuries CE, when Nicolaus Copernicus (1473–1543) re-introduced heliocentrism* and Johannes Kepler (1571–1630) gave elliptical motion to the planets.†

Although his fame derives from his work in astronomy, he is one of the preëminent figures in the history of mathematical beauty. It is clear

[1] Netz, *Ludic Proof*, pp. 90–1.

[2] Eutocius, *Comm. Conica*, 168.5–6; trans. C.

[3] Toomer, intro. to Ptolemy, *Almagest*, p. 2.

* ▸ pp. 274–5

† ▸ pp. 285–7

that he recognized mathematical beauty: he wrote in the introduction to the *Almagest* that

> 'we thought it fitting [...] never to forget, even in ordinary affairs, to strive for a noble [*kalēn*] and disciplined disposition, but to devote most of our time to intellectual matters, in order to teach theories, which are so many and beautiful [*kalōn*], and especially those to which the epithet "mathematical" is particularly applied.'[1]

But Ptolemy's importance here derives not from his recognition of mathematical beauty, but that he was the only mathematician in antiquity to leave a theory to account for it. He gave an explanation of *why* there is beauty in mathematics, or, more precisely, why the same kind of experience can arise in doing mathematics as in listening to beautiful music or seeing a beautiful thing.

Circles, spheres, and astronomy

Before plunging into Ptolemy's theory of beauty, it is necessary to recapitulate some of the features of his geocentric cosmology and its antecessors, for they will have a role to play later in this history.

An enduring theme thus far has been the aesthetic primacy afforded to circles and spheres, which dates back to Pythagoras.* Plato held that the universe was spherical because the sphere is more beautiful than other solid figures.† He was also reported to have insisted that any treatment of the apparent movement of the planets should be based on circular motion.‡ The system of concentric spheres that Eudoxus had proposed and Aristotle had developed§ is compatible with the aesthetically-motivated desire for circular motions and a spherical cosmos.

The preference for circles and spheres was not confined to geocentric systems. Aristarchus (c. 310–c. 230 BCE) was the first to propose a heliocentric cosmos,[2] and he put the Earth on a circular path around the Sun. The earliest testimony is from Archimedes' *Sand-reckoner*:

> 'Aristarchus of Samos has, however, enunciated certain hypotheses [...]. As a matter of fact, he supposes that the fixed stars and the sun do not move, but that the earth revolves in the circumference of a circle about the sun, which lies in the middle of the orbit, and that the sphere of the fixed stars, situated about the same centre as the sun, is so great that the circle in which the earth is supposed to revolve has the same ratio to the distance of the fixed stars as the centre of a sphere to its surface.'[3]

The heliocentric hypothesis never took hold in antiquity. Although it explained gross phenomena such as the apparent retrograde motion

[1] Ptolemy, *Almagest*, § I.1, HEI 4–5.

* ▸ p. 44

† ▸ pp. 61–2

‡ ▸ p. 62

§ ▸ pp. 84–6, 91–2

[2] Stahl, 'Aristarchus of Samos', pp. 246–7.

[3] Archimedes, *Arenarius*, p. 219; trans. Dijksterhuis, *Archimedes*, pp. 362–3.

of the planets,* it was resisted because, despite ever more precise observations being made at Alexandria, no parallax could be observed in the fixed stars.[1] (Parallax is the change in the relative positions of observed objects caused by the motion of the observer. The distance to even the nearest stars other than the Sun is so great that the motion of the Earth produces only a minuscule parallax.)

In the Hellenistic period and in later antiquity, the aesthetic foundation of the preference for circles and spheres reappeared in writings on cosmology and astronomy. For instance, the astronomer, geographer, and mathematician Posidonius (c. 135–c. 51 BCE) thought the universe spherical for aesthetic reasons:

> 'The heaven is beautiful [*kalos*]. This is evident from its shape, its colour, its size, and the variety of the heavenly bodies that adorn it. For heaven is spherical, (a shape) which takes the first place among all shapes, for it alone corresponds to its own parts, since it is round and so are its parts.'[2]

For the astronomer Cleomedes (*fl.* not earlier than 1st century BCE and not later than early 2nd century CE[3]), the reason for the cosmos being spherical was that

> 'the sphere is the most complete of all shapes. For the sphere can enclose every shape that has the same diameter as it, but no other shape can enclose a sphere that has a diameter equal to it.'[4]

The term 'most complete' seems almost aesthetic here, although the reason the explanation Cleomedes then gave is purely mathematical: the sphere being the greatest shape with a given diameter is a formal mathematical statement, not an aesthetic one.

[A passage in the writings of the celebrated rhetorician Quintilian (b. c. 35 CE) suggests that the idea of the aesthetic primacy of the circle was not held only by mathematicians. Quintilian at least seems to have hinted at it when he wrote that

> 'the more perfect [perfectissima] the shape, the greater the area it contains. [...] a circle [...] is the most perfect [maxime perfecta] plane figure.'[5]

Whatever knowledge Quintilian had of mathematics, he appreciated its utility. For instance, it could be used to detect falsehoods, such as the incorrect premiss that two figures with the same perimeter have the same area,[6] which was implicitly used when historians described areas of islands in terms of the time taken to circumnavigate them.[7]]

Even when Aristarchus worked in the third century BCE, the theory of conic sections had been established by Menaechmus and had been developed to a point where Euclid could write four books, no longer extant, on the topic. Thus for Aristarchus and later authors, the ellipse would have been available as a concept, but no-one made the leap to using it to model planetary motion until Kepler in the seventeenth century. Instead, devices such as epicycles, eccentrics, and equants were conceived:

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* ▸ p. 85

[1] Stahl, 'Aristarchus of Samos', p. 248.

[2] Aëtius, *Placita*, § 1.6.1; = Arnim, *Stoicorum Veterum Fragmenta*, vol. II, fr. 1009.

[3] Dicks, 'Cleomedes', p. 318.

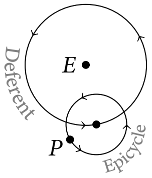
[4] Cleomedes, *Lectures on Astronomy*, TOD I.5.140–1.

[5] Quintilian, *The Orator's Education*, § 1.10.41.

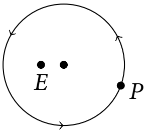
[6] *Ibid.*, § 1.10.40–1.

[7] For example, Thucydides, *The Peloponnesian War*, § VI.1; Polybius, *The Histories*, § IX.26a.

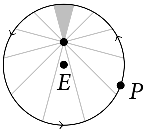
Epicycle and deferent. An epicycle is a circular path in which a point moves with constant speed, and this path is centred on a point that itself moves with constant speed in a circular path. This path itself may be another epicycle, or may be centred on the Earth, in which case it is called the deferent.[1]



Eccentric. An eccentric is a circular path, not centred on the Earth, in which the planet moves with contant speed. The centre of the eccentric may be fixed or may itself move in a circular path.[2]



Equant. An equant is a circular path, centred on the Earth, in which the planet moves with constant angular speed relative to (or constant apparent speed seen from) some point displaced away than the Earth.[3]



It is not known when epicycles and eccentrics were first conceived.[4] Simplicius quoted Sosigenes (‘the Peripatetic’, *fl.* 2nd century CE) as recording that Polemarchus of Cyzicus (*fl.* 4th century BCE), a follower of Eudoxus (c. 400–c. 347 BCE), was aware of the eccentric but rejected it because he preferred the spheres to have the same centre as the universe.[5]

Hipparchus (first quarter of 2nd century–after 127 BCE) seems to have been the first to seek to derive from observation numerical parameters for his model.[6] His work does not survive, but it is known that he used both eccentrics and epicycles and discussed their equivalence, in the sense that any motion produced by an eccentric can be produced by a suitable epicycle, and vice versa.[7] Apollonius proved this equivalence,[8] and Ptolemy also gave a proof, likely his own, in the *Almagest*.[9]

Ptolemy inherited the epicycles and eccentrics and introduced the equant and used these devices to build his system, which eclipsed all earlier attempts. His methods and results were such that the problem of modelling the motions of a planet became reduced to finding the best system of epicycles, deferents, eccentrics, and equants.[10]

Harmonic facts

Ptolemy subscribed to the view that beauty [*to kalon* τὸ καλόν] comprised forms of *symmetria* [συμμετρία], which, according to Plotinus (c. 204–270 CE), was widely held.* Ptolemy made no

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[1] Kuhn, *The Copernican Revolution*, p. 59.

[2] *Ibid.*, p. 70.

[3] *Ibid.*, p. 71.

[4] Neugebauer, *A History of Ancient Mathematical Astronomy*, [vol. 1] 263.

[5] Simplicius, *On Aristotle On the Heavens*, HEI 505.19 sqq. ⟦vol. 2.10–14, p. 44⟧.

[6] Toomer, ‘Hipparchus’, p. 211.

[7] *Loc. cit.*

[8] Neugebauer, *A History of Ancient Mathematical Astronomy*, [vol. 1] 263–4.

[9] Toomer, notes to Ptolemy, *Almagest*, HEI 452, n. 3.

[10] Kuhn, *The Copernican Revolution*, pp. 72–3.

* ▸ p. 142

special issue of switching his discussion from *symmetria* to 'beautiful things', suggesting a close link between the two.[1]

The explanation Ptolemy gave for the experience of beauty in mathematics was interwoven with his harmonic theory, which sat in the tradition descending from early pythagoreanism in that it recognized that consonant intervals correspond to numerical ratios. For Ptolemy, the three most excellent classes of intervals were the homophonic, the concordant, and the melodic.[2] The homophones were those which sound like a single note, like octaves and those composed of octaves. The concords formed a more general class, including the homophones and also those intervals closest to the homophones, like the fifth and fourth and intervals composed of these. The melodic intervals were still pleasing, but less so than the concords. There was, for Ptolemy, a parallel between the musical beauty of intervals and the quality of the ratios:

> 'just as there are two primary classes of unequal-toned notes, that of the concords and that of the discords, and that of the concords is finer [*kallion*], so there are also two primary distinct classes of ratio between unequal numbers, one being that of what are called "epimeric" or "number to number" ratios, the other being that of the epimorics and multiples; and of these the latter is better than the former on account of the simplicity of the comparison, since in this class the difference, in the case of epimorics, is a simple part, while in the multiples the smaller is a simple part of the greater.'[3]

That is, in an epimoric (that is, superparticular) ratio, the difference between the two numbers is 1, while in a multiple ratio, the smaller number divides the greater. And the superiority of these ratios was the due to the 'simplicity of comparison' of ratios with these properties.

The octave was the 'finest [*kallistē*]' of the concords, because it is 'nearest to the equal-toned', and 2/1 was the 'best [*aristos* ἄριστος]' of the ratios, because the difference between the terms is equal to the smaller term.[4] Note that the reason Ptolemy supplied for the beauty of the octave was empirical — two tones separated by an octave are the closest to *sounding* like two equal tones — while his reason for excellence of the ratio 2/1 was mathematical. Among the other concords are the fifth and fourth, corresponding to the epimoric ratios 3/2 and 4/3; among the melodics are the intervals corresponding to the epimoric ratios after 4/3, namely 5/4, 6/5, and so on.[5] As one progresses in this sequence, there is decreasing 'simplicity of comparison' of the ratios because the common measure of the terms becomes an ever-smaller fraction of the two terms.[6]

[1] Barker, 'Mathematical Beauty Made Audible', pp. 409, 419.

[2] Ptolemy, *Harmonics*, § I.7, DÜR 15.

[3] *Ibid.*, § I.5, DÜR 11.

[4] *Ibid.*, § I.5, DÜR 11.

[5] *Ibid.*, § I.7, DÜR 16.

[6] Barker, 'Mathematical Beauty Made Audible', p. 417.

Harmonic explanation

For Ptolemy, the very aim of harmonics, like the other theoretical sciences, lay in *demonstrating* the agreement between the dictates of reason and the evidence of the senses:[1]

> ‘The aim of the student of Harmonics must be to preserve in all respects the rational postulates of the *kanōn*, as never in any way conflicting with the perceptions that correspond to most people’s estimation, just as the astronomer’s aim is to preserve the postulates concerning the movements of the heavenly bodies in concord with their carefully observed courses, these postulates themselves having been taken from the obvious and rough and ready phenomena, but finding the points of detail as accurately as is possible through reason. For in everything it is the proper task of the theoretical scientist to show that the works of nature are crafted with reason and with an orderly cause, and that nothing is produced by nature at random or just anyhow, especially in its most beautiful constructions, the kinds that belong to the more rational of the senses, sight and hearing.’[2]

Therefore it was not sufficient merely to describe the numerical ratios corresponding to pleasing musical intervals. Some explanation had to be given as to *why* there is a mathematical structure to the phenomena that please us. Ptolemy was aware of the attempt in *The Division of the Canon* to connect purely arithmetical propositions with the empirical observation that concords are described by numerical ratios.[3]* But the *Division* only postulated the connection, without giving it a foundation. Ptolemy offered a connection that was grounded in philosophy.

Ptolemy held that *harmonia* [ἁρμονία] ‘makes correct the ordering [*taxis* τάξις] that exists among things’, namely ‘melodiousness’, via *symmetria*.[4] He gave a detailed analysis of how our conception of *symmetria* corresponds to different aspects of *harmonia*. The first principles [*archai* ἀρχαί] of things are matter, which corresponds to ‘what underlies a thing and what it comes from’; movement, which corresponds to ‘cause and agency’; and form, which corresponds to ‘the end and purpose’. Ptolemy thought that *harmonia* was a kind of agency. Agencies are divided into three kinds: one corresponds to nature [*physis* φύσις], ‘concerned only with being’, one to reason [*logos* λόγος], ‘concerned only with being good’, and one to God, ‘concerned with good and eternal being’. Ptolemy considered *harmonia* to be a type of reason. There is then another threefold division, of reason: one aspect is intellect [*nous* νοῦς]; one is practical art or skill [*technē* τέχνη]; one is habituation [*ethos* ἔθος].[5] Reason is responsible for order and *symmetria*, and harmonic reason creates correct audible order or melodiousness [*emmeleia* ἐμμέλεια]: intellect is used in the theoretical discovery of *symmetria*, practical skill is used in its prac-

[1] Barker, ‘Mathematical Beauty Made Audible’, pp. 410–11.

[2] Ptolemy, *Harmonics*, § I.2, DÜR 5.

[3] Barker, ‘Mathematical Beauty Made Audible’, p. 407.

* ▸ pp. 103–4

[4] Ptolemy, *Harmonics*, § III.3, DÜR 92–3.

[5] *Ibid.*, § III.3, DÜR 92.

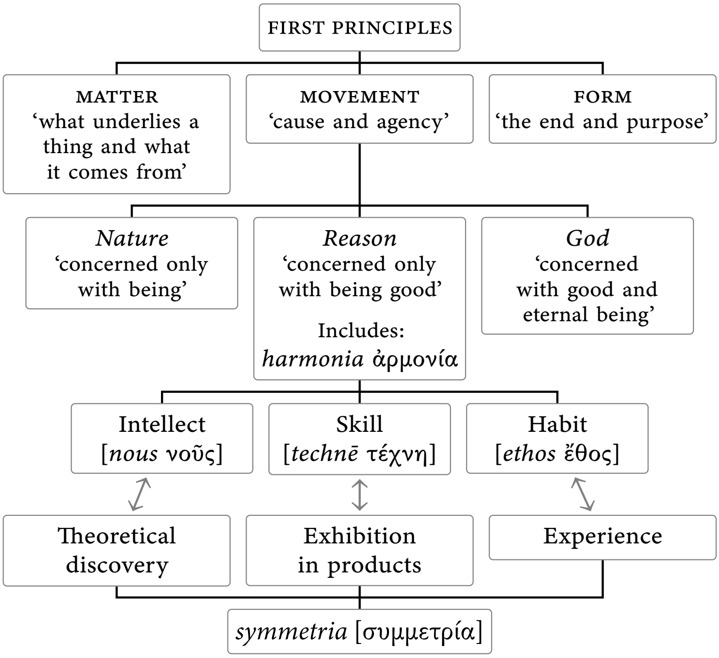
FIRST PRINCIPLES
MATTER
'what underlies a thing and what it comes from'
MOVEMENT
'cause and agency'
FORM
'the end and purpose'
Nature
'concerned only with being'
Reason
'concerned only with being good'
Includes:
harmonia ἁρμονία
God
'concerned with good and eternal being'
Intellect
[nous νοῦς]
Skill
[technē τέχνη]
Habit
[ethos ἔθος]
Theoretical discovery
Exhibition in products
Experience
symmetria [συμμετρία]

FIGURE 3.16
Ptolemy's location of *harmonia* in the division of first principles and the correspondence of its aspects with those of *symmetria* (Ptolemy, *Harmonics*, § III.3, DÜR 92).

tical exhibition, and habituation in the experience of following it (see Figure 3.16).[1]

But none of this was special to musical beauty:

> 'When we consider this — that reason in general also discovers what is good, establishes in practice what it has understood, and brings the underlying material into conformity with this by habituation — it is to be expected that the science that embraces all the species of science that rely on reason, which has the special name "mathematics", is not limited solely by a theoretical grasp of beautiful things [*tōn kalōn* τῶν καλῶν], as some people would suppose, but includes at the same time their exhibition and practice, which arise out of the very act of understanding.'[2]

Note that Ptolemy wrote that the subject-matter of mathematics was 'beautiful things', not the abstract notions of quantity or space. Beautiful things have a structure that can be described using numbers or geometry, but mathematics was only concerned with such descriptions that are beautiful, not with arbitrary numerical or geometrical configurations. *Mathematics is the science of beauty.*[3]

Ptolemy's general point here was that beauty *anywhere* was governed by *harmonia*, which was *not* specific to music. *Harmonia* was a kind of reason, and the science of reason was mathematics, so mathematics had a privileged role in beauty. But the beauty found there was based on *harmonia* in precisely the same way as beauty in music, the heavens, the mind, or morals.[4]

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[1] Ptolemy, *Harmonics*, § III.3, DÜR 92–3; Barker, 'Mathematical Beauty Made Audible', pp. 418–19.

[2] Ptolemy, *Harmonics*, § III.3, DÜR 92–3.

[3] Barker, *Scientific Method in Ptolemy's* Harmonics, p. 264.

[4] *Ibid.*, pp. 28–9.

This also explains why sight and hearing are the only senses that appraise objects not only in terms of pleasure, but also in terms of beauty. A smell or touch or taste may be found agreeable or disagreeable, but not beautiful or ugly.[1] Reason makes use of sight and hearing, which are closest to the 'ruling principle [*hegemonikon* ἡγεμονικόν]' and engage with 'the rational part of the soul';[2] the mathematical structure of a heard interval thus plays a role and so the act of appraisal is aesthetic. But holding a 3 kg mass in one hand and a 2 kg mass in the other does not give an aesthetic experience analogous to hearing two notes in a 3/2 ratio. The difference is not in the absence of the mathematical structure, but because the sense of touch is not so closely linked to the 'ruling principle', is not used by reason, and so is not sensitive to the mathematical structure of the thing experienced.[3]

Ptolemy in fact thought that mathematics was the highest part of theoretical philosophy, which would fit with the importance of its role in beauty. He considered mathematical objects as intermediate between physical and theological. He made this point in his *Harmonics*, writing that ('the mathematical genus is involved to a high degree both in the natural and in the theological'[4]), but repeated and justified it in the *Almagest* (which was probably written later[5]). After defining theology as the study of the first cause, which can be considered to be an unmoving deity, and physics as the study of matter and moving nature, he wrote

> 'That division [of theoretical philosophy] which determines the nature involved in forms and motion from place to place, and which serves to investigate shape, number, size, and place, time and suchlike, one may define as "mathematics". Its subject-matter falls as it were in the middle between the other two, since, firstly, it can be conceived of both with and without the aid of the senses, and, secondly, it is an attribute of all existing things without exception, both mortal and immortal: for those things which are perpetually changing in their inseparable form, it changes with them, while for eternal things which have an aethereal nature, it keeps their unchanging form unchanged.'[6]

So mathematical objects can be perceived by the senses, like the objects of physics, or by the intellect alone, like the subject of theology. Further, mathematical objects inhere in both the common objects of the sublunar world and in the heavens (to which the word 'aethereal [αἰθερώδης]' refers in aristotelian physics). Ptolemy ascribed this division of theoretical philosophy into three to Aristotle,[7] but the relationship between the three parts does not fit with Aristotle's view. In particular, Ptolemy gave the highest status to mathematics, not to theology,[8] because only mathematics yields certainty:

> 'the first two divisions of theoretical philosophy should rather be called guesswork than knowledge [...]; hence there is no

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[1] Ptolemy, *Harmonics*, § III.3, DÜR 93.

[2] *Ibid.*, § III.3, DÜR 93–4.

[3] Barker, *Scientific Method in Ptolemy's* Harmonics, p. 265.

[4] Ptolemy, *Harmonics*, § III.6, DÜR 98.

[5] Feke, *Ptolemy's Philosophy*, pp. 7–8.

[6] Ptolemy, *Almagest*, § I.1, HEI 5–6; ins. in ed.

[7] Cf. Aristotle, *Metaphysics*, BEK E.1.1026a18 sqq.

[8] Taub, *Ptolemy's Universe*, pp. 24–6.

> hope that philosophers will ever be agreed about them; and that only mathematics can provide sure and unshakeable knowledge to its devotees, provided one approaches it rigorously. For its kind of proof proceeds by indisputable methods, namely arithmetic and geometry.'[1]

Ptolemy seems to have been the first to so elevate mathematics. His position that true knowledge could only come from mathematics, and not, for instance, physics or theology, was without precedent in the history of Greek thought; it would doubtless have been very controversial.[2]

Ethical considerations

Like Plato* before him, Ptolemy thought that the study of mathematics could be a part of shaping one's character. For Ptolemy, it was *harmonia* and *symmetria* that creates 'good order [*eutaxian* εὐταξίαν]' in the soul:

> 'Let this be enough to show that the power of *harmonia* is a form of the cause corresponding to reason, the form that concerns itself with the proportions [*symmetrias* συμμετρίας] of movements, and that the theoretical science of *harmonia* is a form of mathematics, the form concerned with the ratios of differences between things heard, this form itself contributing to the good order that comes from theoretical study and understanding to people habituated in it.'[3]

He made the same point in the *Almagest*, pointing to the mathematical structure of the motion of the heavens as inculcating in the scholar a similarly ordered state of mind:

> 'physics, mathematics can make a significant contribution. [...] With regard to virtuous conduct in practical actions and character, this science, above all things, could make men see clearly; from the constancy, order [*eutaxias* εὐταξίας], symmetry [*symmetrias* συμμετρίας] and calm which are associated with the divine, it makes its followers lovers of this divine beauty, accustoming them and reforming their natures, as it were, to a similar spiritual state.'[4]

PAPPUS

The life and work of Pappus of Alexandria (*fl.* c. 300–c. 350 CE) can reasonably be said to mark the end of Greek theoretical mathematics, at least for the historian. Pappus knew that he lived in

[1] Ptolemy, *Almagest*, § I.1, HEI 6.

[2] Feke, *Ptolemy's Philosophy*, p. 4.

* ▸ pp. 77–8

[3] Ptolemy, *Harmonics*, § III.4, DÜR 94–5.

[4] Ptolemy, *Almagest*, § I.1, HEI 7.

a time in which mathematical research could not compare with the golden age that had ended with Apollonius.[1] Outside of astronomy, geometrical research had languished; such new works as emerged were textbooks unworthy of the tradition from which they descended. As a commentator and expositor — and in some ways an original thinker — Pappus seems to have tried to revive interest in geometrical research.[2] His *Collection* [*Synagogē* Συναγωγή] was a handbook, meant to be read in conjunction with the original theoretical texts. Although only partly extant, it is an important source for the history of earlier mathematics.

Only at one point did Pappus explicitly discuss beauty, when he seemed to agree with, or at least not dissent from, the aesthetic primacy of the sphere:

> 'The philosophers say that the first god [...] has opportunely given to the cosmos the shape of a sphere, choosing the most beautiful [*kalliston*] among things that are, and declare the natural properties inherent to the sphere and add that the sphere is the greatest among solid figures with the same area.'[3]

[This passage introduces the section where Pappus described the archimedean solids.*]

But Pappus' legacy in the history of mathematical beauty is greater than a single quotation, for the results in book iv of his *Collection* concerning the space between two semicircles, a shape known as the *arbēlos* [ἄρβηλος], or 'shoemaker's knife',[4] had a lasting influence. The following theorem, which is Proposition 16 of book iv, is just one example:

THEOREM P. *Divide a diameter AC of a semicircle ABC at E. Erect semicircles ADE and CEF inside ABC. Let* O_1 *be the centre of the circle tangent to all three semicircles; and iteratively let* O_{k+1} *be the circle tangent to the semicircle ABC, ADE and the circle with centre* O_k*. Then for each i, the distance from* O_k *to AC is k times the diameter of the circle with centre* O_k*.* (See Figure 3.17.) $\boxed{\text{P}}$

Results such as this were in later times seen as elegant,[5] and, as will be discussed later, became a noteworthy locus of mathematical beauty in mediaeval Islamicate mathematics.†

RETROSPECTIVE

Greek mathematicians are far closer to their modern descendants than the scribes of ancient Egypt or Mesopotamia. They are 'Fellows of another college',[6] as J. E. Littlewood (1885–1977) once put it. And the modern reader can still experience beauty in reading their theorems and proofs. But fundamentally Greek mathematics

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[1] Jones, 'Later Greek and Byzantine mathematics', p. 65.

[2] Heath, *Hist. Greek Math.*, vol. ii, p. 355.

[3] Pappus, *Collection*, § v.xix; trans. Cuomo, *Pappus of Alexandria*, p. 81.

* ▸ pp. 109–12

[4] Pappus, *Collection* 4, Props 13–18.

[5] See Bankoff, 'How Did Pappus Do It?', p. 118; Bulmer-Thomas, 'Pappus of Alexandria', p. 295.

† ▸ pp. 212–13

[6] Littlewood, quot. in Hardy, *Apology*, § 8.

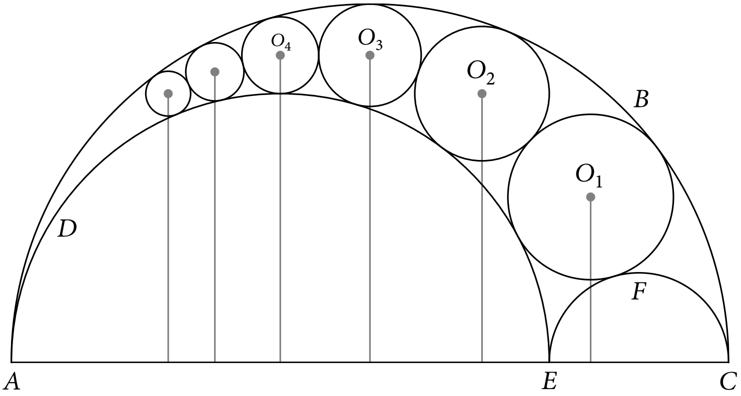

FIGURE 3.17
Illustration of Pappus' theorem on circles inscribed in an *arbēlos*.

is admired as a museum-piece, coloured by the reverence for the achievements of a culture that has traditionally been awarded the epithet of *classical*. The Greek standards of exposition and notation and rigour are vastly different from today's. Just as no-one sculpts or composes music in the Greek mode, save as a conscious archaism, no-one does mathematics as the Greeks did it, regardless of what beauty is still seen there.

◆

Intelligible Beauty

4

Aesthetics of mathematics in later ancient philosophy

> ‘beauty as something intelligible,
> a kind of mathematical quality’
>
> — Umberto Eco, *Art and Beauty in the Middle Ages*, p. 43 (trans. H. Bredin).

> ‘as apparent beauty shining most manifestly,
> is seen through the clearest of the senses [...]
> so likewise intelligible beauty
> appears to the intellect of the soul shining intelligibly.’
>
> — Proclus, *Theology of Plato*, bk III, ch. 22 ⟦vol. 1, p. 205⟧ (trans. Thomas Taylor).

In the Hellenistic period, philosophy continued to flourish, with the emergence of competing schools such as the Cyrenaics, Cynics, Stoics, and Epicureans alongside Plato's Academy and the Peripatetic school of Aristotle. Athens was the philosophical centre of the Hellenistic and Roman worlds, just as Alexandria was their scientific centre. The imperial capitals at Rome and later Constantinople broadly seem to have preferred to import philosophy rather than to produce it, but the state supported philosophy until the decline of pagan thought with the rise of Christianity and Justinian I's (482–565 CE; r. 527–65 CE) edict against teaching philosophy.[1]

While there seems to have been no general system of aesthetics developed in the half-millennium between Aristotle (384–322 BCE) and Plotinus (c. 204–270 CE),[2] thinkers associated to the various new schools made important contributions to philosophical aesthetics.

Consider, for example, Epicurean philosophy, which was founded upon two principles: an atomistic and resolutely materialist account of reality, and the tenet that pleasure, construed as an absence of bodily or spiritual pain or distress, is the only goal.[3] Therefore, perhaps unsurprisingly, Epicurean aesthetics associated beauty and pleasure. One fragment of Epicurus (341–270 BCE) has him *identifying* beauty with pleasure, the distinction being no more than linguistic. In another fragment he did not equate them, but admitted that there was beauty that did not give pleasure; such beauty had no value and those who admired it should be scorned.[4] While there was space in the Epicurean aesthetic theory for mathematical beauty as a source of

[1] Watts, *City and School*, pp. 131–8.

[2] Tatarkiewicz, *Hist. Aesthetics*, vol. I, p. 332; cf. Gilbert & Kuhn, *A History of Esthetics*, pp. 87–9.

[3] Morel, ‘Epicurean atomism’; Woolf, ‘Pleasure and desire’.

[4] Tatarkiewicz, *Hist. Aesthetics*, vol. I, pp. 175–6.

pleasure, it was not explicitly addressed. A contributing factor may have been the potential conflict between theoretical geometry and Epicurean atomism, with its minimal magnitudes that are mathematically indivisible.[1]

Epicurus' statements form an instance of what could be called a 'negative theme' in later antiquity: philosophical aesthetic theories that were compatible with mathematical beauty but did not directly engage with it. Most of the philosophers who wrote on mathematical beauty and whose work survives in some form were platonists. The aesthetics of the other schools seems to have said nothing explicit on the subject. Thus the general stature of the philosophers of this period does not necessarily accord with their importance in this history of mathematical beauty.

Plotinus, for example, who in retrospect is seen as the originator of the neoplatonic tradition, was in a sense the most influential philosopher of post-Aristotelian antiquity, for he transformed platonism, and all later platonists — the *neo*platonists — drew on his fundamental postulates, though they also looked to elements of Aristotle's thought that Plotinus would have rejected. Directly, or through his neoplatonic successors, Plotinus left his mark on later pagan and mediaeval Christian and Islamic thought.[2] But while Plotinus does feature in the history of philosophers' attitudes to mathematical beauty, he is less significant there for what he wrote than for a lingering and nebulous effect in the Middle Ages. In the context of the present study, Plotinus' work is less important than that of Plutarch (before 50–after 120 CE) and Proclus (410/12–485 CE), who wrote centuries earlier and later.

A recurring theme in this chapter is *arithmology*. This word has varying use in the literature, but is here used in a narrow sense to denote a genre of non-mathematical writing about the first ten natural numbers, including their symbolical, mystical, or aesthetic properties.[3] It is thus a particular type of writing about the broader cultural phenomenon of number symbolism or numerology. Some authors have argued that arithmology originated in the early Academy and that it was part of Plato's legacy.[4] But philological reconstruction of the extant arithmological texts suggests that they all descend from a lost text written in the late second or early first centuries BCE[5] and now referred to as the *Anonymus arithmologicus*.[6] Through the works of the thinkers discussed here, the arithmological tradition had a lasting legacy.

CICERO

It is perhaps surprising that the earliest author of the post-Euclidean period who left an extant philosophical discussion of mathematical beauty wrote in Latin. The statesman, orator, and

[1] Vlastos, 'Minimal Parts in Epicurean Atomism'.

[2] Emilsson, *Plotinus*, pp. 373–5.

[3] Zhmud, 'Greek Arithmology', pp. 321–2.

[4] *Ibid.*, p. 322.

[5] Robbins, 'The Tradition of Greek Arithmology', esp. p. 123, Fig. 1; see also Robbins, 'Posidonius and the Sources of Pythagorean Arithmology'.

[6] Zhmud, 'Greek Arithmology', p. 322.

jurist Marcus Tullius Cicero (106–43 BCE) was one of the most prolific authors of antiquity. He was well-read in Greek philosophy and, in addition to his famous speeches and works on rhetoric, left behind a substantial body of writing on political philosophy, ethics, epistemology, and theology.[1] Among his philosophical works, and the focus here, is a partial Latin translation (STE 27d–47b) of Plato's *Timaeus*.

Cicero's version of the *Timaeus* is not an exact translation. The participants in the dialogue became Publius Nigidius Figulus (early 1st century–45 BCE), a scholar, mystic, and self-proclaimed pythagorean;[2] Cratippus (*fl.* 1st century BCE), a Peripatetic of whom Cicero thought highly and under whom he sent his son to study;[3] and the narrator, who may be Cicero himself.[4] It therefore seems likely that the translation offers at least some insight into Cicero's own philosophical views. But it can yield at best limited insight into what Cicero thought about mathematical beauty, for Cicero's partial translation covers only some of Plato's loci of beauty.[*] In particular, the Ciceronian *Timaeus* does not reach Plato's discussion of the fairest triangles[†] or the regular solids.[‡]

But the translation does cover Plato's discussion of the three means, arithmetic, geometric, and harmonic. Of these means, Plato considered the geometric to be the 'fairest [*kallistos* κάλλιστος]';[5] Cicero rendered this evaluation as 'most apt and most beautiful [aptissimum atque pulcherrimum]'.[6] Cicero was doubtless trying to capture some of the senses of the Greek *kallistos*, which would include 'most useful'.[§] But Cicero perhaps thought that here Plato's meaning encompassed what were — for Cicero — both non-aesthetic and aesthetic senses. If so, Cicero's choice of Latin terminology suggests that he saw aptness as a separate and possibly non-aesthetic value which was not an aspect of beauty. The Latin word *aptus*, of which *aptissimum* is the superlative, indicates, like the English *apt*, that a thing is suitable or proper. Cicero thus seemed to acknowledge the possibility that the geometric mean was both suitable for Plato's purpose and had the pure and autonomous aesthetic value denoted by *pulcherrimum*.

In Cicero's *Timaeus*, the shape of the cosmos was the sphere, because this shape was 'most similar and fitting [maxime cognatam et decoram]'.[7] Cicero's translation then diverges from Plato's text in his description of the sphere as 'having no roughness, no bumps, no cut angles, no crookedness, no protrusions, no lacunae'.[8] This description might seem like a step away from the notion that the sphere has beauty, in particular because Cicero made no direct translation of Plato's phrase 'the similar is infinitely fairer [*kallion*] than the dissimilar'.[9] But Cicero's own dialogue *De Natura Deorum* contains a passage that resonates with his translation, a passage where he seems to agree with Plato's view of the aesthetic primacy of the sphere:

> 'You say that you think a cone, a cylinder and a pyramid more beautiful than a sphere. [...] However, assuming that the figures which you mention are more beautiful to the eye

[1] Schofield, 'Writing Philosophy'.

[2] Rawson, *Intellectual life in the late Roman Republic*, p. 94.

[3] *Ibid.*, p. 82.

[4] Hoenig, *Plato's* Timaeus *and the Latin Tradition*, pp. 48–50.

[*] ▸ pp. 61–70

[†] ▸ pp. 65–7

[‡] ▸ pp. 67–70

[5] Plato, *Timaeus*, STE 31c.

[6] Cicero, *Timaeus*, p. 186; trans. C.

[§] ▸ pp. 40–1

[7] *Ibid.*, p. 190; trans. C.

[8] *Ibid.*, p. 192; trans. C.

[9] Plato, *Timaeus*, STE 33b.

> — though for my part I don't think them so, for what can be more beautiful than the figure that encircles and encloses in itself all other figures, and that can possess no roughness or point of collision on its surface, no indentation or concavity, no protuberance or depression? There are two forms that excel all others, among solid bodies the globe (for so we may translate the Greek *sphaera*), and among plane figures the round or circle, the Greek *kyklos*; well then, these two forms alone possess the property of absolute uniformity in all their parts and of having every point on the circumference equidistant from the centre'.[1]

Thus for Cicero the beauty of the sphere depends both on mathematical ('encircles and encloses in itself all other figures') and non-mathematical properties ('no roughness').

VITRUVIUS

The name of the Roman architect and engineer Vitruvius (early 1st century–c. 25 BCE) will appear again in this history, for he was cited often by later authors. From a chronological perspective, this is the natural place to make a brief study of him, and, despite his profession, he fits better among the philosophers than the mathematicians, for his system of aesthetics is pythagorean. But the conclusion must be that, in short, his well-deserved legacy in architecture is mirrored by his unwarranted reputation in mathematical beauty.

Vitruvius is remembered for his *Ten Books on Architecture*, which had continuing influence in the architecture of the European Renaissance and beyond. In this work, Vitruvius emphasized that an architect must have knowledge of mathematics and in particular of mathematical harmonics and acoustics.[2] But those who have sufficient ability to gain a deep knowledge of geometry, astronomy, and musical theory become mathematicians, not architects.[3]

Architecture, for Vitruvius, depended on various qualities that partly recall Aristotle's criteria for beauty:* among them are ordering (which Vitruvius noted was *taxis* τάξις in Greek), design (*diathesis* διάθεσις), shapeliness, symmetry, correctness, and allocation (*oikonomia* οἰκονομία).[4] But Vitruvius explained these notions only with reference to architecture: shapeliness is

> 'an attractive appearance and a coherent aspect in the composition of the elements. It is achieved when [...] every element corresponds in its dimensions to the total measure of the whole';[5]

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[1] Cicero, *De Natura Deorum*, § II.xviii ⟦pp. 167–9⟧.

[2] Vitruvius, *Ten Books on Architecture*, §§ I.i.8–9, V.iii.8 ⟦pp. 23, 66⟧.

[3] *Ibid.*, § I.i.17 ⟦p. 24⟧.

* ▸ pp. 89–91

[4] *Ibid.*, § I.ii.1 ⟦p. 24⟧.

[5] *Ibid.*, § I.ii.3 ⟦p. 25⟧.

symmetry is 'the proportioned correspondence of the elements of the work itself'.[1] But symmetry also depends on proportion (*analogia* ἀναλογία): just as the magnitudes of the parts of the human body stand in certain proportion to each other, so should the parts of a temple.[2]

One root for such a system of proportions is 10, the choice of which as the 'perfect number' Vitruvius attributed to '[t]he ancients';[3] here he evoked the pythagorean view of 10 as perfect.[*] Vitruvius also presented an alternative view: 'mathematicians [...] have said that the number which is called six is perfect, because that number has six components, all of which agree in their ratios with the number six'.[4] Thus it can be presumed that Vitruvius had heard of the Euclidean notion of a perfect number (a number that is equal to the sum of its proper divisors[†]), but he seems not to have understood it. In Euclid's sense, 6 = 1 + 2 + 3 is indeed perfect, but Vitruvius' discussion of the perfection of 6 comprises little more than trivialities: 'One-sixth of six equals one. One-third of six equals two. Half of six equals three. Two-thirds of six equal four. Five-sixths of six equal five, and the complete number, the perfect number, is six'.[5] His lack of mathematical understanding is cemented by his writing that 16 is the most 'perfect' number because it is 10 + 6.[6]

Although Vitruvius incorporated mathematical properties into a defence of a system of visual proportion, his scant comprehension of the underlying principles makes his role in the history of mathematical beauty important only in hindsight: later authors, in spite of their own more secure grounding in mathematics, continued to cite him as an authority.

PHILO

Philo of Alexandria (*fl.* early 1st century CE), who is also called Philo Judaeus, was a Hellenized Jewish thinker who sought to reconcile Mosaic revelation and Greek philosophy. His attempt to assimilate philosophical and Biblical scholarship seems to have had little contemporaneous impact in either Judaic or Greek thought, but came to be appreciated by later Christian thinkers.

Little is known of Philo's life, although his family was wealthy and one of the most prominent in Alexandria.[7] He was educated and widely read in Greek philosophy and culture. In particular, he had knowledge of mathematics and a lively appreciation of mathematical beauty. He related how, when first attracted to philosophy, he spent time in the company of one of Philosophy's handmaids, 'namely Geometry, and was charmed with her beauty [*kallous* κάλλους], for she shewed symmetry and proportion in every part'.[8]

[1] Vitruvius, *Ten Books on Architecture*, § I.ii.4 ⟦p. 25⟧.

[2] *Ibid.*, § III.i.1–3 ⟦p. 47⟧.

[3] *Ibid.*, § III.i.5 ⟦p. 47⟧.

[*] ▸ pp. 47–8

[4] *Ibid.*, § III.i.6 ⟦p. 47⟧.

[†] ▸ p. 101

[5] *Ibid.*, § III.i.6 ⟦p. 47⟧; parenthetical insertions in trans. omitted.

[6] *Ibid.*, § III.i.8 ⟦p. 48⟧.

[7] Schwartz, 'Philo', pp. 9–10.

[8] Philo, *On Mating with the Preliminary Studies*, ch. XIV, § 75 ⟦p. 495⟧.

Like Plato, Philo saw training in mathematics as an essential part of education. He did not characterize it in terms of raising the soul's attention to the Forms, but certainly thought that mathematical education would have a moral effect: 'Geometry will sow in the soul that loves to learn the seeds of equality and proportion, and by the charm of its logical continuity will raise from those seeds a zeal for justice.'[1] In any case, Philo differed from Plato in placing 'the absolutely pure and unadulterated intellect of the universe' above the Form of the Good and the Beautiful.[2]

The only loci of beauty in mathematics that Philo examined are numbers, or at least are closely connected to numbers. Much of his discussion of mathematical beauty is part of the arithmological tradition. The main source is his treatise *On the Creation of the Cosmos according to Moses*; the creation story itself was, for Philo, of such surpassing beauty that words cannot convey it.[3] Part of its beauty comes from the number of days involved. The creation took six days, not because God required this time,[4] but because number is inherent in the order that is imposed upon the created things, and 6 is the 'most generative of numbers'.[5] The number 6 is the first perfect number: $6 = 1 + 2 + 3$. Philo also pointed out that 6 also equals the product of its proper divisors: $6 = 1 \cdot 2 \cdot 3$. It also forms a harmonic union of the odd and even, the male and female; Philo here seems to have been thinking of the pythagorean cognates.[*] It was thus right that the cosmos should be built in accordance with the number 6.[6] The order thus inculcated was necessary for beauty, 'for there is no beauty in disorder'.[7]

This seems to be the earliest connection of perfect numbers in the Euclidean sense with aesthetic value. But although Philo clearly knew the Euclidean concept, he also applied Euclid's term *perfect* [*teleios* τέλειος] to the number 4. One reason for the perfection of 4 is that when the numbers from 1 to 4 are added, they yield 10, which he called 'all-perfect [*panteleias* παντελείας]',[8] which recalls the pythagorean view of 10 as perfect.[†] Another reason 4 is perfect is that it contains the most important musical intervals (the fourth $4 : 3$, the fifth $3 : 2$, the octave $2 : 1$, and the double octave $4 : 1$);[9] the number 10 is all-perfect because it contains these intervals and also $9 : 8$.[10] These factors clearly recall the tetractys of pythagoreanism.[‡] But more: 4 is 'the first number that has revealed the nature of the solid', for one is a point, while two points define a line, and three define a surface.[11] Also, 4 is a product of two equal numbers and the sum of the same two equal numbers. Such a 'splendid form of concord […] not possessed by any other number' shows that the number 4 'is a measure of fairness and equality'.[12] The four elements that make up the cosmos — fire, earth, air, water — arise from the number 4.[13]

Philo also praised 7, though only partly in aesthetic terms. When a sequence is formed by starting with the unit and multiplying each term by some constant to give the next, the seventh term is both a square and a cube.[1] To modern eyes, this fact is mathematically rather trivial: if the constant term is k, then the seventh term in the sequence is k^6, which is the square of k^3 and the cube of k^2. But without modern notational conventions, it is more mysterious, and Philo said such seventh terms had 'both incorporeal and corporeal being, the former through the surface produced by squares, the latter through the solidity produced by cubes';[2] more generally, 7 'is the starting-point of geometry and stereometry and — to summarize — of both incorporeal and bodily reality'.[3] But 7 consists of 1, 2, and 4, and so contains the octave and the double octave. The number 7 can also be expressed as 1 + 6, 2 + 5, or 3 + 4, which give the ratios 6 : 1, the greatest interval according to Philo's harmonics, the ratio 5 : 2, which almost rivals the octave, and 4 : 3, the fourth.[4] Further, another of the beauties of 7 is that it consists of 3 and 4, the length of the legs of the 3–4–5 triangle.[5] And 7 is the only number in the range from 1 to 10 that is neither a multiple nor a non-trivial divisor of another number in that range: it 'has the nature neither to generate or be generated'.[6]

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[1] Philo, *On Mating with the Preliminary Studies*, ch. iv, § 16 ⟦p. 465⟧.

[2] Philo, *On the Creation*, ch. 2, § 8 ⟦p. 48⟧.

[3] *Ibid.*, ch. 1, §§ 4, 6 ⟦pp. 47, 48⟧.

[4] *Ibid.*, ch. 3, § 13, ch. 5, § 26 ⟦pp. 49, 52⟧.

[5] *Ibid.*, ch. 3, § 13 ⟦p. 49⟧.

[*] ▸ pp. 46–7

[6] *Ibid.*, ch. 3, §§ 13–14, ch. 15, § 89 ⟦pp. 49, 70⟧.

[7] *Ibid.*, ch. 5, § 28 ⟦p. 52⟧.

[8] See, for example, Philo, *On the Creation*, ch. 9, § 47 ⟦p. 57⟧; Philo, *On the Decalogue*, ch. vi, § 20 ⟦p. 15⟧.

[†] ▸ pp. 47–8

[9] Philo, *On the Creation*, ch. 9, § 48 ⟦p. 58⟧.

[10] Philo, *On the Special Laws*, bk ii, ch. xxxii, § 200 ⟦pp. 431–3⟧.

[‡] ▸ pp. 47–8

[11] Philo, *On the Creation*, ch. 9, § 49 ⟦p. 58⟧.

[12] *Ibid.*, ch. 9, § 51 ⟦p. 58⟧.

[13] *Ibid.*, ch. 9, § 52 ⟦p. 58⟧.

The number 7 is 'harmonious in the highest degree' and the source of 'the most beautiful table' which Philo said contained the fourth, the fifth, the octave, and the arithmetic, geometric, and harmonic means.* The scheme comprises the numbers 6, 8, 9, 12, and what connection to 7 Philo saw is unclear. In these numbers are found the fourth (8 : 6), the fifth (9 : 6), and the octave (12 : 6), and examples of the arithmetic mean (6, 9, 12) and the harmonic mean (6, 8, 12). Philo also claimed that the geometric mean can be found here. For the standard notion of geometric mean, this claim is incorrect, save in the trivial sense that three equal numbers form a geometric mean. No combination of different numbers from {6, 8, 9, 12} forms a geometric mean. But what Philo *called* the geometric mean in this context was actually two ratios in proportion, namely 12 : 9 :: 8 : 6.[7] Philo's assertion seems not to have been based upon any real misunderstanding, for he gave the correct definition of the geometric mean in another work.[8] It must thus be seen as a deliberate distortion to force this number-scheme into a prior aesthetic paradigm.

Philo saw the number 7 throughout the cosmos: 'what section of the cosmos is not philhebdomadic [*philhebdomon* φιλέβδομον], overpowered by love and desire for the seven?'.[9] The heavens contain seven circles (arctic, antarctic, summer and winter solstice, equator, zodiac, and the milky way); there are seven planets (Sun, Moon, Mercury, Venus, Mars, Jupiter, Saturn); the constellation Ursa Major contains seven bright stars; the Pleiades cluster, whose annual rising and setting marked the harvest and sowing seasons, was likewise considered to contain seven stars (this number being perhaps unique to the Greek tradition);[10] seven days take a new moon to a half-moon, a half-moon to a full moon, and so on.[11] The number 7 is also found in the divisions of the soul and the structures and processes of the human body.[12]

* ▸ pp. 53–5

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[1] Philo, *On the Creation*, ch. 15, §§ 91–2 ⟦p. 71⟧; cf. ch. 15, § 106 ⟦p. 75⟧.

[2] *Ibid.*, ch. 15, § 92 ⟦p. 71⟧.

[3] *Ibid.*, ch. 15, § 98 ⟦p. 72⟧.

[4] *Ibid.*, ch. 15, §§ 95–6 ⟦pp. 71–2⟧.

[5] *Ibid.*, ch. 15, § 97 ⟦p. 72⟧.

[6] *Ibid.*, ch. 15, § 99–100 ⟦pp. 72–3⟧.

[7] *Ibid.*, ch. 15, §§ 107–9 ⟦pp. 75–6⟧.

[8] Philo, *On the Decalogue*, ch. VII, § 21 ⟦p. 19⟧.

[9] Philo, *On the Creation*, ch. 15, § 111 ⟦p. 76⟧.

[10] Puhvel, 'Names and Numbers of the Pleiad', esp. p. 1244.

[11] Philo, *On the Creation*, § 112–15, 101.

[12] *Ibid.*, ch. 15, §§ 117–25 ⟦pp. 77–9⟧.

The same kind of analysis appears elsewhere in Philo's work. To take a single example, consider his explanation for Abraham's being eighty-six years old when his son Ishmael was born. Philo said that 86 is the sum of 6 and 80, both of which are special. The number 6 is the first perfect number. The number 80 is 'the most harmonious of numbers, consisting of two most excellent scales'. The first of the 'scales' is 6, 8, 9, 12, which sum to 35; the second is 6, 9, 12, 18, which sum to 45. And $35 + 45 = 80$. These 'scales' seem to be significant because $8 : 6 :: 12 : 8$ is the fourth and $9 : 6 :: 18 : 12$ the fifth. Philo repeated in this context his incorrect statement about all the means being present.[1]

Philo's views on mathematical beauty were in some sense paradoxical. He was clearly educated in mathematics: he knew of the three means and the Euclidean notion of perfect number, and by his own account appreciated the beauty of geometry. Yet his actual engagement with mathematical beauty primarily reached back to the pythagorean view of certain numbers as embodying musical harmonies, and he was clearly willing to bend mathematical correctness to accommodate his aesthetic requirements. While it is entangled with mathematically less sophisticated ideas, the one occasion when he considered perfection in the Euclidean sense *is* important as the earliest extant connection of this notion with aesthetic value. (The discussion by Vitruvius* does not count, since he did not really understand the concept).

PLUTARCH

The philosopher and biographer Plutarch (before 50–after 120 CE) is most commonly remembered for his collection *Parallel Lives of Noble Greeks and Romans*, which used forty-eight life-stories grouped into twenty-four pairs (one now lost), each comprising one Greek and one Roman, to explore individual greatness and virtue in the political and military spheres. His other extant works, wide-ranging and often more explicitly philosophical, have come down, together with some spurious treatises, under the collective title *Moralia*.

Plutarch studied at the Academy in Athens and regarded himself as a platonist, though his thought was shaped also by pythagorean influences.[2] As discussed here, his opinions on beauty in mathematics are primarily in a platonic vein, reaching back to the *Timaeus*, but with a greater role for number symbolism.

Pleasure in mathematics

Plutarch knew that mathematics and science can be sources of pleasure for those who practise them. Other intellectual endeavours like history also grant pleasure, but there is something *different* about mathematics and science. The attraction of history is ‘uniform and equable’, but

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[1] Philo, *Questions on Genesis*, bk III, § 38 ⟦pp. 225–6⟧.

* ▸ pp. 129–30

[2] Dillon, 'Plutarch and Platonism', p. 61.

> ‘the pleasures of geometry and astronomy and harmonics have a pungent and multifarious enticement that gives them all the potency of a love-charm as they draw us with the strong compulsion of their theorems’.[1]

Similarly, the pleasure that Euclid, Archimedes, Apollonius, Aristarchus found in their mathematical and scientific discoveries is incomparably greater than the pleasures of food.[2] Plutarch recounted the famous story of Archimedes, in the ecstasy of discovery, leaping out of his bath and shouting ‘Eureka’, and pointed out that ‘no glutton have we ever heard that he shouted with similar rapture “I ate it” and of no gallant that he shouted “I kissed her,”’ despite how many such people there are in the world. It was actually distasteful, thought Plutarch, to hear recounted the pleasures of food or flesh; in contrast, it was enjoyable to hear of the pleasures of mathematical or scientific discovery.[3]

Beauty of mathematics

For Plutarch, at least part of the pleasure to be found in mathematics was aesthetic, for humans are ‘born with a love of art and beauty’ and are therefore ‘disposed to welcome and cherish every product or action that bears the stamp of mind and reason’.[4] Mathematics sits alongside ‘self-mastery, the pursuit of wisdom, the knowing what we should about the gods’ as being superior to what wealth can buy, having ‘luminousness [...] and a surpassing radiance’, and allowing the soul to understand the Good:[5]

> ‘Such is the nature of virtue, truth, the beauty of mathematics — geometry and astronomy — ; [*mathēmaton kallos geōmetrikōn astrologikōn* μαθημάτων κάλλος γεωμετρικῶν ἀστρολογικῶν] and with what of these do your trappings of wealth, your necklaces, your girlish baubles, compare?’[6]

This passage seems to be the earliest mention of the beauty of *mathematics* as a whole, rather than of specific mathematical objects or theorems.

A fragment of Plutarch’s lost work *On the Art of Prophecy* suggests a partial explanation for assigning beauty to the whole of mathematics. There are subjects that

> ‘men study to acquire and treat with honour because they love the logic, the precision, and the purity of thought that belongs to them: such are arithmetic, geometry, all the theory of music, and astronomy, arts which Plato says “flourish perforce by their own charm [*charitos* χάριτος],”[7] even if neglected.’[8]

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[1] Plutarch, *That Epicurus*, STE 1093d [§ 11].

[2] *Ibid.*, STE 1093d–e [§ 11].

[3] *Ibid.*, STE 1094c–d [§ 11].

[4] Plutarch, *Table-Talk*, STE 673e [Qu. v.1, § 1].

[5] Plutarch, *On Love of Wealth*, STE 527f–8a [§ 10].

[6] *Ibid.*, STE 528a [§ 10].

[7] See Plato, *Republic*, STE VII.528c.

[8] Plutarch, ‘Fragments from Other Named Works’, fr. 147, p. 273.

The term *charitos* is an inflection of *charis* [χάρις], which indicates outward grace or delight, and which certainly has an aesthetically positive connotation. Plutarch here pointed to the subjects that make up mathematics — arithmetic, musical theory, geometry, astronomy — rather than any particular mathematical object or result. These subjects he saw as possessing logic, precision, and purity of thought, and he connected this, using the quotation from Plato's *Republic*, to their charm or grace.

Regular solids

Plutarch shared in the aesthetic appreciation of the regular solids, calling them 'the five most beautiful [*kallista*] and complete [*teleōtata* τελεώτατα] forms among those found in Nature'[1] and 'the fairest [*kallista*] and foremost forms'.[2] He also gave the earliest extant explanation for their beauty:

> 'said I, "Theodorus of Soli seems to follow up the subject not ineptly [...]. He follows it up in this way: a pyramid, an octahedron, an icosahedron, and a dodecahedron [...] are all beautiful [*kala*] because of the symmetries and equalities in their relations, and nothing superior or even like to these has been left for Nature to compose and fit together. [...]"'.[3]

Though Plutarch wrote that Theodorus of Soli (an author who is only known through Plutarch's citations) made this explanation, he thought that Theodorus did so 'not ineptly', suggesting that Plutarch broadly agreed. The explanation in terms of 'symmetries and equalities' indicates that the beauty of the regular solids was due to the equal side lengths, angles, and vertex configurations and not to their being inscribable in a sphere or to the roles they were assigned by Plato in the *Timaeus*.

The role of the solids in the *Timaeus* arose in the famous passage in book VIII of *Table-Talk* where the speakers discuss what Plato meant by 'God is always doing geometry'.[4] (This phrase is often quoted in the form 'God ever geometrizes'.) Plutarch acknowledged that Plato never made this precise assertion, but that it fitted with his thought.[5] The character Tyndares thought that Plato held that the aim of geometry was to raise our minds above the world of sense-perception. Another disputant, Autobulus, thought that the aim of geometry was to impose order and harmony on matter: the four regular solids that form the elements (tetrahedron, cube, octahedron, icosahedron) *imprison* the 'shapeless and disorganized' matter.[6] Plutarch emphasized that in the *Timaeus*, the demiurge — or, in Plutarch's language, God — was the creator not of matter, but only of symmetry, beauty, similarity. Plutarch wrote that Plato taught 'that god was father and artificer not of body in the absolute sense, that is to say not of mass and matter, but of symmetry in body and of beauty [*kallous*] and similarity'.[7] That is, in imposing geometry on the primordial world-stuff, the demiurge was creating beauty in the universe.

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[1] Plutarch, *The E at Delphi*, STE 390a [§ 11].

[2] Plutarch, *The Obsolescence of Oracles*, STE 422f [§ 23].

[3] *Ibid.*, STE 427a–e [§ 32].

[4] Plutarch, *Table-Talk*, STE 718c [Qu. VIII.2, § 1].

[5] *Ibid.*, STE 718c [Qu. VIII.2, § 1].

[6] *Ibid.*, STE 719c–e [Qu. VIII.2, § 3].

[7] Plutarch, *On the Generation of the Soul in the Timaeus*, STE 1017a [§ 9].

Elegant theorem

Plutarch never described a result as beautiful, but later in the famous passage in *Table-Talk* he described a result as elegant and in so doing gave the earliest explicit aesthetic comparison between two theorems:

> 'I said, "[...] Now among the most characteristic theorems, or rather problems, of geometry is this: given two figures, to construct a third equal to one and similar to the other. They say, in fact, that Pythagoras offered sacrifice when he solved this problem; for it is surely much more elegant [*glaphyrōteron* γλαφυρώτερον] and inspired than that famous theorem which gave the proof that the square on the hypotenuse is equal to the sum of the squares on the sides enclosing the right angle."'[1]

The latter, less elegant, result is of course the pythagorean theorem; the former, more elegant, result is Proposition vi.25 of Euclid's *Elements*:

THEOREM C. *'To construct one and the same figure similar to a given rectilinear figure and equal to another given rectilinear figure.'* $\boxed{\text{C}}$

[The proof in the *Elements* only discusses the case where the given 'similar' figure is a triangle, but the construction depends on Proposition vi.18, which deals with general rectilinear figures. There may have been textual corruption in the transmission of the proof.[2]]

For what reason was Theorem C more elegant than the pythagorean theorem? It is possible that it gained some value via an analogy Plutarch saw with the demiurge's forming of the universe. The demiurge could work matter, 'the least ordered of substances', and form, 'the most beautiful [*kalliston*] of patterns', and intended 'to reduce nature to a cosmos by the use of proportion and measure and number'.[3] And the cosmos is that which 'is *equal* to matter and *similar* to form'.[4] Thus the demiurge's task — the creation of beauty in the universe — is analogous to that of the geometer who uses Theorem C.

Triangles

Plutarch disagreed with Plato's view that the $1 : 1 : \sqrt{2}$ isosceles triangle and the $1 : \sqrt{3} : 2$ scalene triangle are the most beautiful,[*] though he did acknowledge these triangles as 'primary'.[5] Plutarch described the $3 : 4 : 5$ triangle as 'the most beautiful [*kalliston*] of the triangles'[6] and again as 'the most beautiful [*kalliston*] of the right-angled triangles'.[7] The reason for this assessment seems to have been primarily symbolic. Conjecturing that the Egyptians honoured this triangle, he wrote:

[1] Plutarch, *Table-Talk*, ste 720a [Qu. viii.2, § 4].

[2] Heath, notes to Euclid, *Elements*, Prop. vi.28.

[3] Plutarch, *Table-Talk*, ste 720b [Qu. viii.2, § 4].

[4] *Ibid.*, ste 720b [Qu. viii.2, § 4], emphases added.

[*] ▸ pp. 65–7

[5] Plutarch, *The Obsolescence of Oracles*, ste 427d [§ 32]; Plutarch, *Platonic Questions*, ste 1003f [Qu. v, § 1].

[6] Plutarch, *Isis and Osiris*, ste 373f [§ 56].

[7] Plutarch, *The Obsolescence of Oracles*, ste 429e [§ 36].

> ‘This triangle has its upright of three units, its base of four, and its hypotenuse of five, whose power is equal to that of the other two sides. The upright, therefore, may be likened to the male, the base to the female, and the hypotenuse to the child of both, and so Osiris may be regarded as the origin, Isis as the recipient, and Horus as perfected result. Three is the first perfect [*teleios*] odd number: four is a square whose side is the even number two; but five is in some ways like to its father, and in some ways like to its mother, being made up of three and two.’[1]

The term *perfect* is here being used in a pythagorean rather than a Euclidean sense. It seems that, for Plutarch, 3 was perfect in the same way that 10 was perfect in early pythagoreanism: it is a sum of consecutive numbers starting at 1.

For Plutarch, there seems to have been a link between this most beautiful triangle and the most beautiful solids — the regular solids — via the number 5. This number was the length of the hypotenuse of this most beautiful triangle, and is the number of regular solids. And there is the correspondence of the regular solids with the elements (which must have included the aether*), which ‘the Master’ (probably Pythagoras, possibly Plato) had hinted at, and which Plutarch thought was difficult to understand.[2]

The number 5 ‘represents neither a triangle nor a square, nor is it a perfect number nor a cube, nor does it seem to present any other subtlety for those who love and admire such speculations’.[3] But it seems to have a kind of unity through connection to other numbers. It has special significance as both the sum of the first two square numbers; as the sum of the first even and first odd numbers (the unit 1 being considered neither even nor odd here);[4] and as the first number whose square is equal to the sum of the squares of the two numbers that immediately precede it.[5] [It is straightforward to prove that, within the scope of natural numbers, *only* 5 has this last property.]

[Plutarch’s dialogue *The* E *at Delphi* is a discussion of the sign ‘E’ (majuscule epsilon) at the temple of Apollo at Delphi, in which the speakers offer interpretations of the sign and reasons for its presence. The character Eustrophus is the first to broach the idea that the symbol E actually denotes 5 in Greek numeration, ‘great and sovereign number’.[6] The character of Plutarch, who was enthusiastic about mathematics at the time, expanded this into a explanation using pythagorean number symbolism.[7]]

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[1] Plutarch, *Isis and Osiris*, STE 373f–4a [§ 56].

* ▸ p. 68

[2] Plutarch, *The Obsolescence of Oracles*, STE 426f–7a [§ 31].

[3] *Ibid.*, STE 426f [§ 31].

[4] Plutarch, *The* E *at Delphi*, STE 388a [§ 8].

[5] Plutarch, *The Obsolescence of Oracles*, STE 429d–e [§ 36].

[6] Plutarch, *The* E *at Delphi*, STE 387e [§ 7].

[7] *Ibid.*, STE 387f sqq.[§ 7 sqq.]

ALCINOUS

Alcinous was the author of an introductory *Handbook of Platonism*, which, because of what has been lost to the ages, is an important source for platonic thought of the period.[1] Except for the *Handbook*, he is an empty name who probably wrote later than Plutarch.

Like Plato, Alcinous saw mathematics as best used to elevate the mind towards contemplating the Forms.[2] His discussion occasionally touched on mathematical beauty. For instance, he held to Plato's original view that the isosceles right-angled triangle and the $1 : \sqrt{3} : 2$ triangle were the most beautiful,[3][*] so Plutarch's disagreement with Plato on this point seems not to reflect a universal change of opinion by platonists.[†]

The aesthetic primacy of the sphere also appears, though with different terminology. In discussing the creation of the world, Alcinous wrote:

> 'By way of shape, he bestowed on it sphericity, seeing as that is the fairest of shapes and the most capacious and mobile.'[4]

The word 'fairest' here translates *eumorphotaton* [εὐμορφότατον],[5] meaning 'most well-shaped'.[6] The term 'most capacious' points to the sphere having the greatest volume for a shape with a given surface area. Alcinous thus took Plato's phrase 'comprises within itself all the shapes'[7][‡] and gave it a precise mathematical interpretation.[8] But Alcinous did *not* make this mathematical property a contributor to the sphere being 'fairest of shapes'; rather, it is a second reason, besides its beauty, for the demiurge having chosen it.

NICOMACHUS

Though he seems to have been famous in his day, little is known about the life of Nicomachus of Gerasa (*fl.* c. 100 CE).[9] Two of his works are extant: an *Introduction to Arithmetic* and a *Handbook of Harmonics*. The *Introduction* was effectively a text-book for students of arithmetic, in the sense of theoretical mathematical investigation rather than of practical calculation.

Nicomachus was in no sense a great mathematician and the mathematical content of the *Introduction* is quite limited.[10] Only a few trivial results, in which Nicomachus displayed a wretched pride, are original.[11] Nevertheless, the *Introduction* remained influential until the sixteenth century.[12] Like his *Handbook*, it has a particular historiographical value precisely because it was written not by a giant but by a lesser intellect who was interested in arithmology.[13]

Nicomachus agreed with Plato that mathematics should be studied as a way of elevating the mind to awareness of higher things.[14] He classified the four mathematical sciences — namely arithmetic, music, geometry, and astronomy (or spherics) — in a scheme which seems to descend from Plato's curriculum for the education of the guardians in the *Republic*:* arithmetic studies absolute quantity; music, relative quantity; geometry, magnitude at rest; astronomy, magnitude in motion and revolution.[1] Arithmetic should be studied first, because it is the origin of the other mathematical sciences, and because it existed in the mind of God as the plan by which the world was shaped.[2] And the order and regularity of numbers is mirrored in the universe[3] and in particular in the musical harmonies.[4]

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[1] Dillon, intro. to Alcinous, *Handbook of Platonism*, §§ 1, 4.

[2] *Ibid.*, HER 161–2 [§ 7.2–4].

[3] *Ibid.*, HER 168 [§ 13.2].

[*] ▸ pp. 65–7

[†] ▸ pp. 136–7

[4] *Ibid.*, HER 167.46–168.2 [§ 12.3].

[5] Alcinoou, *Tōn Platonis Dramatōn*, p. 167, l. 37.

[6] LSJ, 'εὔμορφ-ος'.

[7] Plato, *Timaeus*, STE 33b.

[‡] ▸ pp. 61–2

[8] Mansfeld, *Prolegomena Mathematica*, p. 111.

[9] Robbins, 'The Life of Nicomachus', p. 71; Tarán, 'Nicomachus of Gerasa', p. 112.

[10] Heath, *Hist. Greek Math.*, vol. 1, p. 99.

[11] Robbins, 'The Development of the Greek Arithmetic Before Nicomachus', p. 16.

[12] Tarán, 'Nicomachus of Gerasa', p. 112.

[13] Barker, 'Pythagorean harmonics', p. 201.

[14] Nicomachus, *Introduction to Arithmetic*, §§ 1.3.4–7.

Nicomachus seems to have thought perfect numbers beautiful. He started his discussion of this topic[5] by contrasting a perfect number with an abundant number [*hypertelēs arithmos* ὑπερτελὴς ἀριθμός] — a positive integer whose proper divisors sum to more than that number — and a deficient number [*ellipous arithmos* ἐλλιποῦς ἀριχμος] — a positive integer whose proper divisors sum to less than that number. Between the greater and the less lie 'virtues, wealth, moderation, propriety, beauty [*kallē*], and the like, to which the aforesaid form of number, the perfect, is most akin'.[6]

There are few perfect numbers and Nicomachus thought that they were all produced by Euclid's result† and were thus easy to find.[7] And Nicomachus said this after noting that 'fair [*kala*] and excellent things are few and easily enumerated, while ugly [*aischra* αἰσχρὰ] and evil ones are widespread'.[8] Since he also found abundant and deficient numbers ugly, likening them to monstrous creatures with too many or too few limbs, mouths, eyes,[9] his contrast between beautiful/rare and ugly/numerous is clear evidence that he thought that perfect numbers were beautiful in the sense of *kalos*.

Further, he thought that perfect numbers were distributed regularly: one in the units (namely 6), another in the tens (28), another in the hundreds (496), another in the thousands (8128). And they were always even and end alternately in 6 or 8.[10] The fifth perfect number 33 550 336 breaks the first pattern and thus is unlikely to have been known in antiquity, though it could have been discovered using Euclid's result, since checking that $2^{13} - 1 = 8191$ is prime is within reach of manual calculation. [Knowledge of the fifth perfect number is first attested in a fifteenth century CE manuscript.[11]] The pattern of alternating last digits 6 and 8 breaks down with the sixth perfect number 8 589 869 056.[12]

Nicomachus described Euclid's result on perfect numbers as elegant [*glaphyros*].[13] This word was a term of approbation which he favoured for results and procedures.[14]

As already mentioned, Nicomachus claimed that Euclid's result produced *all* the perfect numbers, but even today no proof is known of this assertion. It is tempting to speculate that it was Nicomachus' desire for regularity and order among the perfect numbers that led him to this hasty claim. [All *even* perfect numbers are generated by Euclid's result, but the earliest known proof of this fact is in a posthumously-published paper by Leonhard Euler (1707–83),[1] although René Descartes (1596–1650) had earlier thought that it was possible to prove the result.[2] Whether there exist any odd perfect numbers remains an open question.]

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* ▸ pp. 75–6

[1] Nicomachus, *Introduction to Arithmetic*, §§ I.3.2–3.

[2] *Ibid.*, §§ I.4.1–2, I.6.1.

[3] *Ibid.*, §§ I.6.1, II.19.1.

[4] *Ibid.*, § I.5.1.

[5] *Ibid.*, chs I.14–16.

[6] *Ibid.*, § I.14.2.

† ▸ p. 101

[7] *Ibid.*, § I.16.4.

[8] *Ibid.*, § I.16.3.

[9] *Ibid.*, §§ I.14.3, I.15.1.

[10] *Ibid.*, § I.16.3.

[11] Dickson, *Hist. Theory of Numbers*, vol. I, p. 6.

[12] OEIS, A000396.

[13] Nicomachus, *Introduction to Arithmetic*, § I.16.4.

[14] Dillon, *The Heirs of Plato*, p. 60, n. 71.

Nicomachus was certainly aware of the early pythagorean view of 10 as a 'perfect' number,[*] explicitly mentioning it in the second book of the *Introduction*.[3]

Philo had earlier connected perfect numbers in the Euclidean sense with aesthetic value.[†] Given the dates when they flourished, it is possible that Nicomachus followed Philo in making this connection. But their texts both descended from the *Anonymus arithmologicus*,[4] and so it is also possible either that the earlier author saw beauty in perfect numbers, or that Philo and Nicomachus perceived it independently. If the former possibility is the correct one, it is likely to have been a bare assertion of beauty in perfect numbers, since Philo's and Nicomachus' discussions do not otherwise overlap.

PLOTINUS

History sees Plotinus (c. 204–270 CE) as the founder of neoplatonism, but he saw himself as little more than an interpreter of Plato, the philosopher whom he thought had come closest to truth, to the point where he seems to have conflated solving philosophical problems with finding the correct interpretation of Plato's writings.[5] But he drew openly on other philosophical traditions in seeking such interpretations: his work incorporates selected elements from Aristotle and the Stoics.[6] Plotinus' writings and associations, and those of his friend and disciple Porphyry (234–c. 305 CE), served as the fountainhead of a school of philosophy that showed enough originality that nineteenth-century writers started to distinguish it as a new form of platonism.

The works of Plotinus survive as Porphyry edited them into *The Enneads*. Porphyry divided some treatises to bring their number to fifty-four and thus to be able to group them into six thematic 'enneads', or sets of nine. The division of the treatises was artificial and disadvantageous, scattering four parts of one important treatise across three different enneads.[7] If Porphyry recognized the problems, he thought them worth what he sought:

> 'I hit on the *pleasing* idea of dividing the 54 books of Plotinus into six "enneads" — groups of nine multiplied by the perfect number 6.'[8]

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[1] Dickson, *Hist. Theory of Numbers*, vol. I, p. 19.

[2] Descartes, letter to Mersenne, dated 15 Nov. 1638, repr. in Descartes, *Œuvres*, vol. II, p. 429.

[*] ▸ pp. 47–8

[3] Nicomachus, *Introduction to Arithmetic*, § II.22.1.

[†] ▸ p. 131

[4] Robbins, 'The Tradition of Greek Arithmology', p. 123, Fig. 1.

[5] O'Meara, *Plotinus*, p. 6.

[6] Emilsson, *Plotinus*, pp. 31–2.

[7] O'Meara, *Plotinus*, p. 9.

[8] Porphyry, 'On the Life of Plotinus', § 24, emphasis added.

Whatever problems Porphyry's editing created, it is likely that the mathematical neatness of the arrangement is one reason for its survival intact, without treatises being omitted or new ones inserted.[1]

Return to Porphyry's reference to 6 being a perfect number. He doubtless considered it perfect because $6 = 1 + 2 + 3$. But what significance did this have for Porphyry? He may have thought of 1, 2, 3 as the proper divisors of 6, in which case he thought that 6 was perfect in the Euclidean sense, or as the first three natural numbers, in which case 6 was perfect in a pythagorean sense.

The latter is more likely, because the number 9 — the ennead — was also a special number in pythagoreanism: in the *Arithmetical Theology** it 'is greatest of the numbers within the decad and is an unsurpassable limit'.[2] Porphyry thus seems to have been drawing on pythagorean notions to which Plotinus himself did not pay much attention.[3] It is thus unlikely that Plotinus would have found pleasing the structure that Porphyry imposed on his works.

Ontological foundations

Before examining Plotinus' theory of beauty, it is necessary to outline the ontology presented in *The Enneads*, which is also relevant to the thought of later neoplatonists.

Plotinus followed Plato in positing a fundamental divide between the sensible and intelligible worlds. But there are differences in the detail. For Plotinus, the intelligible world comprised three fundamental principles: the One [*to hen* τὸ ἕν], Intellect [*nous* νοῦς], and Soul [*psychē* ψῡχή]. Plotinus also referred to the One as the Good, using the latter term almost as often as the former.[4] The One is primary, and generates Intellect, which in turn generates Soul, which indicates individual human souls.[5] The One, having produced the indefinite Dyad, which seems to be identified with Intellect, imposes definiteness upon it, and so brings about Number.[6] And 'Soul [...] is Number; for the first things are neither masses nor magnitudes.'[7]

In reasoning — and in particular in reasoning about whether something is just or beautiful — the soul makes use of Intellect, which originated in the intelligible realm.[8] And one should turn away from the sensible world and seek to ascend back to the intelligible world.[9] Even if Plotinus did not actually despise the sensible world, he thought it had relatively little value compared to the intelligible world, of whose principles it was at best a pale reflection.[10]

Mathematics has its part to play in achieving this end, for Plotinus agreed with Plato that mathematical studies were necessary to gain 'a grasp of and confidence in the incorporeal' and to prepare the way for dialectic.[11] According to Porphyry, Plotinus had a broad knowledge of geometry and arithmetic,[12] so there was no hypocrisy in what Plotinus wrote here. But *The Enneads* do not include any deep discussion of mathematical themes.[13]

[1] Kalligas, *Comm. Enneads*, vol. 1, p. 89.

* ▸ pp. 144–5

[2] ps.-Iamblichus, *Theology of Arithmetic*, p. 76.

[3] Slaveva-Griffin, *Plotinus on Number*, pp. 138–9.

[4] Emilsson, *Plotinus*, p. 73.

[5] *Ibid.*, p. 38.

[6] Plotinus, *Enneads*, §§ 5.1.3, 5.

[7] *Ibid.*, §§ 5.1.5.

[8] *Ibid.*, § 5.1.11.

[9] *Loc. cit.*

[10] Emilsson, *Plotinus*, p. 39.

[11] Plotinus, *Enneads*, § 1.3.3.

[12] Porphyry, 'On the Life of Plotinus', § 14.

[13] Kalligas, *Comm. Enneads*, vol. 1, p. 58.

Theory of beauty

The starting-point for Plotinus' theory of beauty is the sixth treatise of the first ennead, called 'On beauty', which became the best-known part of his work.[1] Plotinus set out the problem of beauty in several questions: What is it that makes a visual or audible thing beautiful? How are things that depend directly on souls beautiful? Is there one sort of beauty in these things, or different sorts?[2]

For Plotinus, beauty was found mainly in the objects of sight, but also in music, and, by those who do not focus their attention there, in habits, actions, knowledge [*epistēmai* ἐπιστῆμαί], and virtues.[3] Thus Plotinus certainly recognized intellectual beauty. This recognition played into his argument against the view, held with particular tenacity by the Stoics,[4] that 'in regard to the objects of sight and all other things, their beauty consists in their symmetry [*to symmetrois* τὸ συμμέτροις] or measure'.[5] A thing has symmetry if there is a concordance or commensurability or harmony amongst its parts.* Consequently, beauty cannot exist in partless things but only in composite things.[6] But this argument leads to a *reductio ad absurdum*, precisely because of the existence of intellectual beauty, for intellectual things are partless:[7]

> 'what does it mean to say that there is proportion in beautiful practices or customs or studies or types of scientific understanding? For how could theorems be proportional to each other? If it is because they are in concord, it is also the case that there is agreement or concord among bad theorems.'[8]

(The word 'theorems' here does not refer specifically to *mathematical* propositions.) Therefore symmetry can only be a *characteristic* of beautiful things: 'beauty is that which shines from symmetry, rather than the symmetry itself'.[9] Those who think the symmetry *produces* the beauty have simply confused the cause of beauty with an effect.[10]

But more: if a thing in the sensible world is beautiful, it is so because of its relation to a thing in the intelligible world. Plotinus' conception of this relation was a departure from Plato's view that things are beautiful through their participation in the Form of the Beautiful. Plotinus asked the question and explained the relation: 'How, then, are things here and there both beautiful? We say that these are beautiful by participation in Form.'[11] The participation is not in the Form of the Beautiful, but in *Form*. Things are beautiful insofar as they participate in the intelligible world.[12] And since it is the One that produces Intellect, it is the ultimate source of beauty.

A more detailed discussion of beauty as Intellect awaits in the eighth treatise of the fifth ennead, titled 'On the Intelligible Beauty', which is in fact the second part of a larger work that Porphyry split into four.[13] In the treatise that formed the third part of the original work, Plotinus distinguished between beauty and the Good, both in terms of our reactions to them[14] and in terms of their metaphysical

[1] Emilsson, *Plotinus*, p. 20; O'Meara, *Plotinus*, p. 89.

[2] Plotinus, *Enneads*, § 1.6.1, ll. 7–13.

[3] *Ibid.*, § 1.6.1, ll. 1–6.

[4] Kalligas, *Comm.* Enneads, vol. 1, pp. 196–7.

[5] Plotinus, *Enneads*, § 1.6.1, ll. 24–5.

* ▸ p. 41

[6] *Ibid.*, § 1.6.1, ll. 25–7.

[7] Konstan, *Beauty*, pp. 124–5.

[8] Plotinus, *Enneads*, § 1.6.1, ll. 43–6.

[9] *Ibid.*, § 6.7.22.

[10] Kalligas, *Comm.* Enneads, vol. 1, p. 197.

[11] Plotinus, *Enneads*, § 1.6.2, ll. 13–14.

[12] O'Meara, *Plotinus*, p. 91.

[13] Emilsson, *Plotinus*, p. 21.

[14] Plotinus, *Enneads*, § 5.5.12, ll. 9–30.

status: ‘the Good is not in need of that which is beautiful, whereas that which is beautiful needs it’.[1] This difference fits with the beauty being participation in Intellect, which is secondary to the One, which, as already mentioned, Plotinus also called the Good.

Beauty in mathematics?

Despite intelligible beauty being fundamental to his whole theory of beauty, and his evident familiarity with mathematics, Plotinus tiptoed around beauty in mathematics without ever delving into it, and the one passage in his work that might refer to beauty in mathematics is ambiguous. When discussing beauty in nature, as opposed to beauty in the products of craft, Plotinus noted that

> ‘the beauty is not in magnitude, but in intellectual pursuits [*tois mathēmasi* τοῖς μαθήμασι] and in practices and, generally, in souls.’[2]

The word *mathēmasi* is the dative plural of *mathēma* [μάθημᾰ].* With the narrow interpretation of the meaning, Plotinus could have referred here to beauty in the mathematical sciences, rather than more broadly in ‘intellectual pursuits’.[3] The reader is left uncertain. But this phrase had a legacy: the author of the partial Arabic translation-cum-paraphrase of *The Enneads* that was known in the mediaeval Islamicate world interpreted it as referring specifically to the objects of mathematics.†

AESTHETICS AND ARITHMOLOGY

In the arithmological tradition, in which non-mathematical texts discussed various properties of the numbers 1 to 10, there are occasional glimpses of an aesthetic appreciation of numbers. Although arithmology was distinct from mathematics, there was an influence in at least one direction: arithmologists would have borrowed such mathematics as they found useful to show the importance of particular numbers. Arithmologist writers did not hesitate to incorporate whole passages from previous authors, and this makes it difficult to determine the precise origin of particular ideas.[4]

Some of the texts from the arithmological tradition describe numbers using aesthetic terms. Sometimes these descriptors are grounded solely in the allegorical or symbolic meaning of each number, but sometimes also in their mathematical properties. As an example of the former, consider Theon of Smyrna’s (*fl.* 127–32 CE[5]) *Explanation of Mathematics Useful for Understanding Plato’s Writings*, which mentions beauty of number in only one passage, a paean to unity. Unity is unchanging, in particular since 1 × 1 = 1, and in it can be found

[1] Plotinus, *Enneads*, § 5.5.12, ll. 9–30.

[2] *Ibid.*, § 5.8.2, ll. 37–9.

* ▸ p. 21

[3] See, for example, Adamson, ‘The Arabic Plotinus’, p. 117.

† ▸ pp. 193–4

[4] Robbins, ‘The Tradition of Greek Arithmology’.

[5] Heath, *Hist. Greek Math.*, vol. 2, p. 239.

intelligible things such as the beautiful [*to kalon* τὸ καλὸν] and the good [*to agathon* τὸ ἀγαθὸν].[1] It is difficult to raise this to an assertion of mathematical beauty.

Arithmology also exercised some influence on Christian writers. Methodius of Olympus (d. c. 310 CE), who was central to third-century Christian theology and who had evidently been educated in Greek philosophy,[2] considered both 10 and 6 to be perfect. He seems to have echoed the significance Philo of Alexandria[*] saw in the number of days of creation, both in terms of the harmony and generative power of the perfect number 6:

> ‘the entire creation of the world was achieved out of the harmony of the number 6: *for in six days the Lord made heaven and earth and all the things that are in them*,[3] since the creative power of the Word contains the number 6 insofar as it produces bodies.’[4]

But Methodius saw a deeper symbolic meaning in the Euclidean notion of perfection: the extraction of the proper divisors of a perfect number and their addition back into the *same* whole pointed to how the Incarnation did not detract from God’s perfection or unity.[5]

The *Theologoumena tēs arithmetikēs* [Θεολογούμενα τῆς ἀριθμητικῆς] is a richer source. The word *theologoumena* denotes ‘theological doctrines’; the title is perhaps best translated as *Arithmetical Theology*.[6] It was preserved in Iamblichus’ corpus, and was long attributed to him, but was probably the work of some other unknown author, who composed it around the middle of the fourth century CE. It contains explicit quotations from previous writers, notably Anatolius of Alexandria (*fl.* c. 269 CE), Nicomachus,[†] and Speusippus[‡] (c. 410–c. 340 BCE), as well as passages that were clearly influenced by earlier texts. It reads like a set of written-up notes.[7]

The text touches on mathematical beauty only in arithmological terms. The number 3 (the triad) was particularly beautiful:

> ‘The triad has a special beauty [*kallos* κάλλος] and fairness [*euprepeian* εὐπρέπειαν] beyond all numbers, primarily because it is the very first to make actual the potentialities of the monad — oddness, perfection [*teleiotēta* τελειότητα], proportionality, unification, limit.’[8]

The triad also seems to have gained something from being the only number that is sum of all the numbers that precede it.[9]

The phrase ‘make actual the potentialities’ suggests that the number 1 (the monad) has the properties listed in some trivial sense. For example, an odd number — in the sense of an ‘ordered plurality’ — has a middle element. But 1 only has a middle element and nothing else. Perfection here seems to have been in the sense of being a sum of consecutive natural numbers starting at 1, or, equivalently, being a triangular number: again, 1 has this property in a trivial sense.

[1] Theon, *Exposition*, p. 164, ll. 2–9.

[2] Musurillo, intro. to Methodius, *Symposium*, p. 3.

[*] ▸ p. 131

[3] Exodus 20:11.

[4] *Ibid.*, § 8.11.

[5] *Loc. cit.*

[6] Mansfeld, *Prolegomena Mathematica*, p. 19, n. 60.

[†] ▸ pp. 138–40

[‡] ▸ pp. 81–4

[7] Waterfield, intro. to ps.-Iamblichus, *Theology of Arithmetic*, p. 23.

[8] *Ibid.*, DEF 14.

[9] *Ibid.*, DEF 15.

The author of the *Arithmetical Theology* applied a new aesthetic term to number, namely the word *euprepeian* [εὐπρέπειαν], which carries the meaning of *goodly appearance*, *comeliness*, *dignity*; it is derived from *eu* [εὖ], meaning *well*, and *prepō* [πρέπω], signifying *to be clearly seen* or *conspicuous*.[1]

The word *teleiotēta* [τελειότητα] is derived from *teleios*, which was the term Euclid used for perfect numbers. But later, the number 6 was said to be the first perfect [*teleios*] number,[2] and 28 to have a 'perfect nature [*teleian physin* τελείαν φύσιν]'.[3] So the author of *Arithmetical Theology* certainly accepted the Euclidean notion of perfection, having encountered it at least in Nicomachus' work.

The author also accepted a pythagorean notion of perfection that applied to triangular numbers, such as the decad 10 being 'perfect in all manners of perfection' and having 'a natural equilibration and commensurability and wholeness'.[4] And the perfection of triangular numbers extended to numbers related to them. The ennead 9 is also 'perfect' because it arises out of the perfect number 3.[5]

And, for the author, perfection seems to have been connected to beauty, since the discussion of the decad, and indeed *Arithmetical Theology* as a whole, ends with a quotation from the work of Anatolius of Alexandria, presumably from his lost work *Introduction to Arithmetic*,[6] that the number 10 'generates' 55, in the sense that $1 + 2 + \dots + 10 = 55$, and this number 'encompasses wonderful beauties [*kallē*]'. It is the sum of the first four powers of two and of three: $(1 + 2 + 4 + 8) + (1 + 3 + 9 + 27) = 55$, and these powers played a key role in Plato's *Timaeus*.* Also, the sum of the squares of the numbers in the decad is $1^2 + 2^2 + \dots + 10^2 = 385 = 7 \times 55$. Finally, the Greek word for 'one', *hen* [ἕν], when read as a number in Greek numeration, is $\varepsilon + \nu = 5 + 50 = 55$.[7]

IAMBLICHUS

Plotinus may have been the founder of neoplatonism, but Iamblichus (c. 250–c. 330 CE) had a lasting effect on the school, in particular because he venerated Pythagoras just as much as Plato, and joined neoplatonic and pythagorean ideas in his works. Indeed, he looked beyond both men to what he believed was a far more ancient wisdom, that of Hermes Trismegistus ('thrice-great'), a name that originated in the identification of the Greek god Hermes with the Egyptian scribe of the gods Thoth. To this Hermes was attributed a Greek-language philosophical, scientific, and magical literature that was supposed to embody Egyptian philosophy, and from Iamblichus descended a Hermetic thread in platonism that re-emerged in the Renaissance.[8]†

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[1] LSJ, 'εὖ', 'εὐπρέπεια', 'πρέπ-ω'.

[2] ps.-Iamblichus, *Theology of Arithmetic*, DEF 43.

[3] *Ibid.*, DEF 62.

[4] *Ibid.*, DEF 79.

[5] *Ibid.*, DEF 78.

[6] Kieffer, 'Anatolius of Alexandria', p. 148.

* ▸ pp. 62–3

[7] ps.-Iamblichus, *Theology of Arithmetic*, DEF 86–7.

[8] Ebeling, *The Secret History of Hermes Trismegistus*, pp. 19–21; Copenhaver, *Hermetica*, p. xvi.

† ▸ pp. 242–3

Iamblichus seems to have written widely, but very little of his work is extant and what has survived is of little mathematical importance.[1] However, he has a noteworthy place in the history of mathematical beauty, for his book *On the General Science of Mathematics* specifically discussed the role of beauty in mathematics and connected it to the ontological status of mathematical objects.

There is evidence that Iamblichus studied under Porphyry,[2] and his ontology shows a clear Plotinian influence. For Iamblichus, the One [*to hen* τὸ ἕν] was simple and was the principle of all things. From the One and the principle of plurality [*plēthos* πλῆθος] come about all numbers.[3] Iamblichus seems to have foreseen a tendency to think of the One as beautiful and good and of the principle of plurality as bad and ugly. He thought neither view was correct. First, the combination of the two brings about 'beautiful elements in numbers', and an attempt to attribute responsibility for this only to the One leads to an absurdity, for the principle of plurality is praiseworthy for its receptivity to the praiseworthy One.[4] Second, it is simply incorrect to call the One beautiful or good, for it is superior to them.[5]

For Iamblichus, the universe consisted of five levels of being descending from the One: i) numbers, ii) geometrical objects, iii) soul, then either iv) animate and v) inanimate beings, or perhaps iv) rational and v) irrational living beings.[6] Beauty first appeared at the level of numbers and was also present at the level of geometrical objects. At these levels, there was no ugliness or badness, which appeared only at the fourth and fifth levels.[7] Goodness first appeared at a lower level than beauty;[8] that is, below the level of numbers. Iamblichus did not indicate whether goodness could appear at the level of geometrical objects, but it seems more likely that he thought that goodness first appears at the level of soul.[9]

As discussed in Chapter 2,[*] there is a contested thesis that the fourth chapter of *On the General Science of Mathematics*, which described this ontology, may have been derived in substance from Speusippus (c. 410–c. 340 BCE), who was Plato's nephew and his successor as head of the Academy. Regardless of whether Iamblichus developed an original position or mirrored that of Speusippus, in this chapter he set out the view he accepted: that there was beauty in numbers and in geometrical objects, and indeed that mathematical objects constituted the highest levels of being in which beauty could appear.

If Iamblichus did not adopt Speusippus' view, then it is likely that he scoured Aristotle's *Metaphysics* to develop a theory that encompassed the ontology of mathematical objects and their relation to beauty and goodness.[10] In particular, Aristotle saw beauty in mathematical objects, but did not see goodness there, since in his one explicit discussion of the subject[†] he maintained that 'the good [...] always implies conduct'.[11] The Aristotelian absence of goodness in mathematical objects was compatible with the Iamblichan thesis that, in the descent from the One, beauty appears first in numbers and

[1] Lloyd, 'Iamblichus'.

[2] Chiaradonna & Lecerf, 'Iamblichus', § 1.1.

[3] Iamblichus, *On the General Science of Mathematics*, FES 15.6–23.

[4] *Ibid.*, FES 16.2–9.

[5] *Ibid.*, FES 16.10–11.

[6] Dillon, notes to *ibid.*, FES 18.1–13, n. 49.

[7] *Ibid.*, FES 18.5–10.

[8] *Ibid.*, FES 16.12–14.

[9] Cf. Tarán, *Speusippus*, p. 105.

[*] ▸ p. 83

[10] *Ibid.*, p. 107.

[†] ▸ pp. 89–91

[11] Aristotle, *Metaphysics*, BEK M.3.1078a31–2.

geometrical objects, while goodness appears lower in the hierarchy of things.

Iamblichus thought that among the sciences, mathematics is 'easily foremost in beauty and in exactness'.[1] The subject-matter of mathematics comprises those theorems that can be applied in arithmetic, geometry, harmonics, astronomy, and other areas. These theorems included results about ratios and equality and inequality,[2] but also includes those concerning

> 'order and beauty in the species of mathematics from the scientific aspect, without further stipulating a particular type of beauty; for that sort of thing already relates to some subordinate science.'[3]

Here Iamblichus seems to have been saying that mathematical results should concern a *general* mathematical beauty, and not one specific to a particular other science.

Mathematics as a practice subsumed certain 'capacities [*dynameis* δυνάμεις]', which studied different aspects of the mathematical enterprise. Some treated of what is common throughout all branches of mathematics, such as theories of symmetry and asymmetry.[4] Others studied what is special to particular branches, such as studying numbers alone or magnitudes alone. Other capacities 'survey the beauty and the measure of mathematical entities, as also their harmony and symmetry'.[5] Thus, studying the beauty of mathematical objects was a part of the practice of mathematics. And the branches of mathematics were ordered not only by the objects that they studied, but also 'with regard to the importance they give to beauty, if some study the superior and supreme beauty, others that which is inferior and incomplete'.[6] That is, some branches of study involved the application of the capacities of mathematics to worldly objects that are subject to change; these branches were therefore annexed to mathematics. But their objects of study were not mathematical entities and so do not have the superior beauty of the highest levels of being that descend from the One.

For Iamblichus, the objects of mathematics were both ontologically and aesthetically between intelligible beings and sensible ones. They have an 'incorporeal and self-subsistent' mode of existence, inferior to that of intelligible beings but superior to that of visible ones,[7] while

> 'they are in beauty [*kallei*], in order [*taxei* τάξει], and in exactitude [*akribeia* ἀκριβεία] superior to things visible but inferior to the intelligible, and similarly they have an intermediate symmetry and compatibility'.[8]

For specific instances of mathematical objects having beauty, one must turn to other works by Iamblichus. In his commentary on Nicomachus' *Introduction to Arithmetic*, there is an implicit indication that Iamblichus, like Nicomachus, saw beauty in the number 6, where 'beauty [*kallos*]' is mentioned in the context of a discussion of the pythagorean virtues and mathematical properties of the number 6.[1]

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[1] Iamblichus, *On the General Science of Mathematics*, FES 83.24–84.1.

[2] *Ibid.*, FES 18.26 sqq.

[3] *Ibid.*, FES 19.10–14.

[4] *Ibid.*, FES 46.14–21.

[5] *Ibid.*, FES 47.2–3.

[6] *Ibid.*, FES 47.26–8.

[7] *Ibid.*, FES 10.8–16.

[8] *Ibid.*, FES 10.16–19.

One of the fragments of Iamblichus' lost commentary on Plato's *Timaeus* lists various reasons Iamblichus saw for the demiurge to give the cosmos a spherical shape. The fifth reason suggests that Iamblichus agreed that the sphere was the most beautiful form:

> 'the cosmos assimilates itself to the Essential Intelligible Beauty [*kallos*] through this shape. For how will that which is uniform and symmetrical and alike (in all its parts) not be outstandingly beautiful? If therefore it was necessary that it should be the most beautiful [*kalliston*] of all sensible objects, it was necessary that it should have such a shape as this, equal on all sides and bounded and exact.'[2]

Iamblichus also referred to 'the beauty and order of the theorems of mathematics'.[3] Through this beauty and order, mathematics reflected intelligible things, and so the study of mathematics helped train the mind to turn away from bodies, granting confidence to reason about intelligible things.[4] Thus the very motivation for pursuing theoretical mathematics was beauty and goodness:

> 'The mathematics of the Pythagoreans is not such as others practice it; for that is for the most part applied [*technikē* τεχνικὴ], without a sole target, and not aiming at the fine [*kalou*] and the good. But that of the Pythagoreans is especially pure [*theōrētikē* θεωρήτική] and relates its insights to a single goal; it relates all its own arguments to the Fine [*kalō*] and the Good, and uses them as means of access to the realm of true being.'[5]

MARTIANUS CAPELLA

The seven liberal arts traditionally comprise the *trivium* of grammar, rhetoric, logic (or 'dialectic') and the *quadrivium* of arithmetic, geometry, music, astronomy. The word *quadrivium* seems to have been coined by Boethius* (c. 480–c. 524 CE) and the analogous term *trivium* emerged during the Carolingian period.[6] But it was Martianus Capella (*fl.* 5th century CE) who set down the 'definitive scheme of the seven liberal arts', in the encyclopaedic work known — perhaps not quite accurately — by the title *The Marriage of Philology and Mercury* [*De Nvptiis Philologiae et Mercvrii*].[7]

This work was perhaps the most popular schoolbook in the mediaeval Latin world.[8] As a compilation of knowledge, it is generally unimpressive. Its mathematical content is especially weak. For instance, the book on geometry, after an opening allegory concerning

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[1] Iamblichus, *In Nicomachi*, p. 34.

[2] Iamblichus, *In Timaeum*, fr. 49, in Iamblichus, *In Platonis Dialogos Commentariorum Fragmenta*, p. 155.

[3] Iamblichus, *On the General Science of Mathematics*, FES 55.12–13.

[4] *Ibid.*, FES 55.13–19.

[5] *Ibid.*, FES 91.3–11.

* ▸ pp. 166–8

[6] Guillaumin, 'Boethius's *De Institutione Arithmetica*', pp. 136–7.

[7] Kristeller, 'The Modern System of the Arts (I)', p. 505; Stahl, *The Quadrivium of Martianus Capella*, pp. 21–2, n. 2.

[8] Taylor, *The Classical Heritage of the Middle Ages*, p. 49.

the personification of Geometry,[1] is largely devoted to geography,[2] with a brief summary of geometrical definitions and postulates,[3] and an allegorical peroration.[4]

The book on arithmetic is marginally stronger, at least quoting some results that can be found in Euclid's *Elements*, but, as might be expected, it also devotes much space to arithmology. The triad had to be 'regarded as perfect'; 6 was 'perfect and proportional' because it is perfect in the Euclidean sense; 8 was also a 'perfect number'; 9 was 'also perfect and called more than perfect, since it is perfected from the multiplication of the perfect triad'.[5] In each case, the descriptor is a form of *perfectus*. For Martianus, mathematical (Euclidean) perfection seems to have been of no greater significance than the other arithmological perfections.

Yet closely following all these perfections is a curious passage that demands closer inspection. On the subject of prime numbers, Martianus wrote that:

> 'Arising in themselves, they beget other numbers from themselves, since even numbers are begotten from odd numbers, but an odd number cannot be begotten from even numbers. Therefore prime numbers must of necessity be regarded as beautiful [primi numeri necessario habendi pulchrique].'[6]

This passage seems to be the unique instance of an ancient author calling prime numbers beautiful.[7] In light of Martianus' overloading of the word 'perfect', it is important to note that he had an accurate conception of 'prime' and indeed of 'relatively prime', and discussed both in a generally correct way. But it is curious that he alone in the arithmological tradition should have ascribed beauty to *prime* numbers. Authors in this tradition tended to attribute beauty to small numbers, or else to perfect numbers; in either case, to *very few* numbers. Martianus undoubtedly knew the result that '*Prime numbers are more than any assigned multitude of prime numbers*'.[8] He would therefore have known that he was praising as beautiful an unlimited multitude of numbers. Of course, the attribution of beauty to prime numbers may not have been original to Martianus; perhaps he drew it from an earlier text that is now lost.

PROCLUS

The work of Proclus (410/12–485 CE) marks the end-point of the discussion of mathematical beauty in antiquity. From his vantage-point, he could look back on nine or ten centuries of development of mathematics and philosophy, and his views on mathematical beauty descend from those of both Plato and Aristotle.

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[1] Martianvs, *De Nvptiis Philologiae et Mercvrii*, §§ 567–89.

[2] *Ibid.*, §§ 590–704.

[3] *Ibid.*, §§ 705–723.

[4] *Ibid.*, § 724.

[5] Martianvs, *De Nvptiis Philologiae et Mercvrii*, §§ 731–41; trans. Martianus, *The Marriage of Philology and Mercury*, §§ 731–41.

[6] Martianvs, *De Nvptiis Philologiae et Mercvrii*, § 744; trans. Martianus, *The Marriage of Philology and Mercury*, § 744.

[7] Stahl, notes to Martianus, *The Marriage of Philology and Mercury*, p. 286, n. 79.

[8] Euclid, *Elements*, Prop. IX.20.

Proclus lived in a period of great upheaval. About the time of his birth, Alaric led the Visigoths to the siege and sack of Rome; in the middle of his life, Genseric and the Vandals pillaged the city again; when he died, the western Empire was extinct. The eastern Empire was in its transition from paganism to Christian ascendancy. The emperors at Constantinople restricted the practice of the polytheistic religions and the civic role of pagans.[1] Philosophy, less readily assimilated to Christianity than rhetoric or grammar, was in decline.[2]

But Proclus flourished as a philosopher. He studied first in Alexandria, then, from age nineteen, in Athens. His teachers were Plutarch of Athens (d. 432 CE; not to be confused with the 1st century CE Plutarch[*]) and Syrianus of Alexandria (d. 437 CE), respectively the retired and acting heads of the Academy. Proclus succeeded to the headship himself at the age of twenty-five, and held it until his death. At least his Alexandrian studies included mathematics,[3] and he esteemed the rigour and methods of mathematics, for his *Elements of Theology* was nothing less than an attempt at an axiomatic-deductive treatment of neoplatonic metaphysics,[4] while his *Elements of Physics* recast books VI and VII of Aristotle's *Physics* in a theorem–proof form.[5]

Proclus wrote prolifically, and about a third of his work survives, comprising over a million words.[6] Of this immense and complex corpus, only two works are considered here: his commentary on book I of Euclid's *Elements* and his commentary on Plato's *Timaeus*. Only a part of the latter work is extant.[7] The surviving portion includes discussion of the beauty of spheres and the geometric mean, but not of Plato's fairest triangles or of the regular solids.

His *Commentary on Euclid* includes a two-part prologue before it embarks on considering book I of the *Elements*. He may have composed this prologue as an introduction to a more general mathematical work, perhaps including a commentary on Iamblichus,[†] for it nowhere mentions Euclid and shows a clear Iamblichan influence, following the arrangement of topics in *On the General Science of Mathematics*.[8] In this prologue, he classified the parts of mathematics using a scheme which he attributed to the pythagoreans, but which perhaps echoed Nicomachus' division:[‡]

> 'Arithmetic, then, studies quantity as such, music the relations between quantities, geometry magnitude at rest, spherics[9] magnitude inherently moving.'[10]

Defending beauty

For Proclus, beauty was essential to mathematics. At the start of his *Commentary on Euclid*, he listed those aspects common to all of mathematics, which served to delineate the subject; among them were beauty [*kallos*] and order [*taxis* τάχις].[11]

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[1] Brown, 'Christianization and Religious Conflict', pp. 638 sqq.

[2] Mueller, foreword to Proclus, *Commentary on Euclid*, p. xiii.

[*] ▸ pp. 133–7

[3] Marinus, *Proclus*, § 9.

[4] Chlup, *Proclus*, p. 37.

[5] Morrow, 'Proclus', p. 161.

[6] Siorvanes, 'Proclus' life', p. 44.

[7] Baltzly & Tarrant, Gen. intro. to Proclus, *Commentary on Plato's* Timaeus, vol. I, p. 8.

[†] ▸ pp. 145–8

[8] Mueller, foreword to Proclus, *Commentary on Euclid*, p. xv; Morrow, notes to Proclus, *Commentary on Euclid*, pp. 344–5; Gertz, intro. to Iamblichus, *On the General Science of Mathematics*, pp. 8–9.

[‡] ▸ pp. 138–9

[9] Astronomy.

[10] Proclus, *Commentary on Euclid*, FRI 35–6.

[11] *Ibid.*, FRI 8.

To those who denied the 'beauty [*to kalon*] and excellence [*to agathon*]' of mathematics since nothing on these matters is found in its discourses,[1] Proclus wrote that:

> 'we can reply by exhibiting the beauty of mathematics on the principles by which Aristotle attempts to persuade us. Three things, he says, are especially conducive to beauty [*kallos*] of body or soul: order, symmetry, and definiteness. […] These characters we find preeminently in mathematical science.'[2]

The entire discussion echoes Aristotle in book M of the *Metaphysics*,[*] although it is possible that Proclus referred to another work, no longer extant.[3] But there is a difference: Aristotle used the term *kalos* and its cognates, while Proclus used *kallos*, which signifies beauty in a narrower sense.[†] This difference suggests that the mathematical beauty had become more a strongly aesthetic quality, more distinct from simple pleasure in mathematics. But, unlike Aristotle, Proclus was also clear that aesthetic judgements are not 'the business of the mathematician'.[4] Of course, a mathematician could make a judgement of beauty *about* mathematics, but such a judgement was not *part of* mathematics. Mathematics did not study beauty, but exemplified it.

Proclus departed both from the basic platonist position that beauty consists in participation in the Form of the Beautiful and from Plotinus' view that beauty derived from a participation in the intelligible world.[‡] Participation in the Form or the intelligible world does not supply criteria that can be used to examine the loci and nature of mathematical beauty.[5] Looking to Aristotle gave him a surer starting-point. Indeed, Proclus expanded on the Aristotelian criteria: bodily ugliness came from disorder, lack of shapeliness, symmetry, and outline; mental ugliness from unreason, irregular and disorderly movement, not in harmony with reason. Bodily beauty must be the opposite of these; hence it was characterized by order, symmetry, and definiteness.[6] By analogy, therefore, intellectual beauty should be similarly characterized. In mathematics, order was seen in the deduction of more complicated theorems from simpler ones and from the hypotheses; symmetry in the mutual consistency of the proofs and 'in their common reference back to Nous'; definiteness in the unvarying and certain ideas, for the objects of mathematics did not waver like the objects of sense or opinion, but were fixed by the Forms.[7] Proclus thus defended the notion of mathematical beauty, using Aristotle's principles, by drawing a parallel with physical beauty.

The proofs' 'reference back to Nous' involved in part an imitation of Nous in their structure. Theorems or problems announced the proof's destination, and the proof ended with either QED [*quod erat demonstrandum*; *hoper edei deixai* ὅπερ ἔδει δεῖξαι] or QEF [*quod erat faciendum*; *hoper edei poiēsai* ὅπερ ἔδει ποιῆσαι] and often, in Euclid's case, with a recapitulation of the statement of the theorem or problem. This repetition joined the end back to the beginning 'in imitation of the Nous that unfolds itself and then returns to its starting-point'.[8] [The

[1] Proclus, *Commentary on Euclid*, FRI 25.

[2] *Ibid.*, FRI 26.

[*] ▸ pp. 89–91

[3] *Ibid.*, FRI 26, n. 49.

[†] ▸ pp. 40–1

[4] *Ibid.*, FRI 35.

[‡] ▸ pp. 140–3

[5] Goulding, *Defending Hypatia*, p. 147.

[6] Proclus, *Commentary on Euclid*, FRI 26.

[7] *Ibid.*, FRI 26–7.

[8] *Ibid.*, FRI 210.

notion of basing beauty on this perceived symmetry in the structure of the proof can be found centuries later in the application of the literary forms of chiasmus and ring-composition to proofs.[*]]

So symmetry was a property of individual proofs and their structures. As already mentioned, it was also seen in the mutual consistency of the proofs. Order was seen in how the results are deduced from others, and thus involved both results and the application of results in proofs. Symmetry was static, concerning the correctness and compatibility of established results; while order was dynamic, concerning how results follow from one another. Definiteness was a property of mathematical ideas and objects.

Beauty in proofs and theories

Proclus never quite gave an aesthetic judgement of an individual proof or solution, limiting himself to expressions such as 'admirable [θαυμαστὴ]',[1] which, even then, specifically related to the precision with which conditions were stated or the fullness with which cases were covered.[2] In his commentary on the *Timaeus*, there is a hint of aesthetic preference in the discussion of the problem of finding two mean proportionals, where he wrote that he would give Archytas' solution[†] at the end of the work. He chose this method on the grounds that Menaechmus' method used conic sections and that of Eratosthenes was a mechanical solution involving moving straightedges in a frame.[3] Archytas' method used no geometric concept more complicated than a hemicylinder. Was Proclus' preference an aesthetic one? Perhaps, though it seems safer just to call it a preference for simplicity.

He stepped much closer to an aesthetic judgement of Euclid's *Elements* as a whole:

> 'We mark also the coherence of its results, the economy and orderliness in its arrangement of primary and corollary propositions, and the cogency with which all the several parts are presented.'[4]

He listed the various achievements of different compilers of '*Elements of Geometry*': shorter proofs; longer treatments; avoidance of *reductio ad absurdum*; avoidance of using the theory of proportions; preëmptive defence against criticism of the axioms.[5] An '*Elements*', thought Proclus, should avoid superfluous topics and comprise material that was 'coherent and conducive to the end proposed'; it should be clear, concise, and general. Some of these properties are arguably aesthetic, but Proclus motivated them purely in terms of how they help the student gain understanding.[6] He thought that Euclid, when judged by these criteria, was superior to all his rivals.[7]

But Proclus had an exaggerated view of the 'coherence' of Euclid's *Elements*: he held that it was directed towards the construction of the five regular solids.[8] This interpretation was incorrect, but under-

[*] ▸ pp. 648–50

[1] Proclus, *Commentary on Euclid*, FRI 232.

[2] Netz, 'Deuteronomic Texts', p. 276.

[†] ▸ p. 53

[3] Proclus, *Commentary on Plato's* Timaeus, DHL II.33.29–34.4 ⟦vol. III, p. 80⟧; Heath, *Hist. Greek Math.*, vol. 1, pp. 251–5, 258–60.

[4] Proclus, *Commentary on Euclid*, FRI 69.

[5] *Ibid.*, FRI 73.

[6] *Ibid.*, FRI 73–4.

[7] *Ibid.*, FRI 74.

[8] *Ibid.*, FRI 68.

standable: book xiii of the *Elements*, where the regular solids are constructed, depends on a large part of the theory Euclid had built up, specifically books i–vi, x, and xi. Proclus was also drawn to this conclusion by his enthusiasm for Plato and particularly for the *Timaeus*[1] and his concomitant desire to associate Euclid with Plato.[2]

Circles and spheres

For Proclus, the primary locus of beauty in mathematics was the circle. He emphasized that the circle was superior to all other figures: it was simpler than solid figures and stood first among plane figures because of its 'homogeneity and self-identity'.[3] This primacy extended to beauty, since, although Proclus was never quite explicit on this point, it is implicit in his example of a non-mathematical assertion: 'the circle is more beautiful than the straight line'.[4]

In his commentary on Plato's *Timaeus*, he raised the sphere to a comparable status among solids. He quoted Plato's statement that the demiurge thought that 'the similar is infinitely fairer than the dissimilar'.[5] He seems to have agreed with this view, suggesting amazement at any platonist who thought that difference was superior to sameness.[6] Thus for Proclus the sphere was of superior beauty, because 'it is both most perfect or complete [...] and also most similar'.[7]

This beauty informed his discussion of the reasons Plato gave for the demiurge shaping the universe like a sphere. The three demonstrations Proclus discerned — in a very brief passage of Plato[8] — were derived from the One, from intellectual beauty, and from intellectual creation or kinship.[9] He expanded upon the second demonstration, saying that it came

> 'from the beautiful and what is fitting. For the spherical shape is fitting to the one who receives, to the one who gives, and to the paradigm. (i) It is fitting to the one who receives. Because it is perfect or most complete, it is amicable [...] to the most perfect of the shapes.'[10]

Proclus clearly endorsed Plato's position here, and went on to cite in support Parmenides,[*] Empedocles,[†] and Plato's later dialogue the *Laws*.[11]

Proclus also suggested that Plato thought that the sphere was a fitting choice because all the regular solids can be inscribed in it, and the spherical universe would encompass matter, which was composed of the five regular solids.[12]

The superior beauties of the circle and sphere were also bound up with Proclus' view of the epistemology and ontology of mathematics. The circle sat in a higher realm of being and knowledge than the straight line. The soul touched both realms: it moved both in a circle when it partook of intelligible nature and in a straight line when it partook of sensible nature.[13] The study of the circle is 'a higher form of being and knowledge', but our mortal intellects could cope better with the study of the straight line.[1]

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[1] Morrow, intro. to Proclus, *Commentary on Euclid*, pp. l–li.

[2] Mueller, *Philosophy of Mathematics*, pp. 302–3; cf. Proclus, *Commentary on Euclid*, FRI 68.

[3] Proclus, *Commentary on Euclid*, FRI 146–7.

[4] *Ibid.*, FRI 35.

[5] Plato, *Timaeus*, STE 33b–c.

[6] Proclus, *Commentary on Plato's* Timaeus, DHL II.78.21–4 ⟦vol. III, p. 138⟧.

[7] *Ibid.*, DHL II.71.32–3 ⟦vol. III, p. 129⟧.

[8] Plato, *Timaeus*, STE 33b.

[9] Proclus, *Commentary on Plato's* Timaeus, DHL II.68.24–6, 69.28 ⟦vol. III, pp. 125, 126⟧.

[10] *Ibid.*, DHL II.69.9–11 ⟦vol. III, p. 126⟧.

[*] ▸ p. 39

[†] ▸ p. 49

[11] *Ibid.*, DHL II69.15–28 ⟦vol. III, p. 126⟧.

[12] *Ibid.*, DHL II.71.7–23 ⟦vol. III, p. 128⟧.

[13] Proclus, *Commentary on Euclid*, FRI 109.

The sphere's beauty depended on its inherent similarity, which in turn derived from its incorporeality. Dissimilarity was almost a kind of contaminant that 'insinuates itself into beings from their matter'; similarity, in contrast, 'comes from [F]orm alone and from intelligible causes'.[2] Therefore the beauty of the sphere ultimately derived from its immaculate intellectual nature.

From the heights of the circle descended beauty: 'it dispenses beauty, homogeneity, shapeliness, and perfection'.[3] This dispensing explained why the equilateral triangle was, for Proclus, the most beautiful among triangles: it was most like the circle.[4] In this context, Proclus pointed to the constant radius of the circle, which suggests that likeness of the equilateral triangle to the circle included all the vertices of the triangle being the same distance from its centre. These remarks were part of his commentary on Proposition I.1 of the *Elements*: *On a given finite straight line to construct an equilateral triangle.* The diagram of the corresponding construction (see Figure 4.1) shows how the triangle is inscribed in the common segment of two circles; for Proclus, this showed 'how the things that proceed from first principles receive perfection, identity, and equality from these principles'.[5] Thus the beauty dispensed by the circle was communicated through the mathematical construction that yielded the equilateral triangle. In declaring the equilateral triangle the most beautiful, Proclus was not actually opposed to Plato's view of the $1 : \sqrt{3} : 2$ triangle as fairest,[*] for Plato's triangle was chosen only from among the *right-angled scalene* triangles.

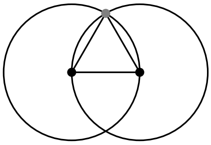

FIGURE 4.1
Construction of an equilateral triangle on a given segment.

Immediately following his statement that the circle dispenses beauty, Proclus gave an example of a numerical result that the circle 'controls': *Among the numbers from* 1 *to* 10*, only the middle numbers* 5 *and* 6 *have squares that end with themselves.*[6] This point also arose in his discussion of the demiurge's division of the strip of world-stuff.[†] He mentioned a treatise that passed under the name of Timaeus of Locri called *De Natura Mundi et Animae*, which is now considered to be a first-century CE paraphrase of the *Timaeus*. In this work, the demiurge's division proceeded in such a way as to obtain thirty-six divisions.[7] Proclus cited this possible division[8] and thought that it would be fitting for the soul, for 36 is the square of 6 ('generated by 6 proceeding into itself'), and 6 was dedicated to the soul partly because 6, like 5, was 'circular'.[9] Thus even the seemingly purely numerical division scheme is, for Proclus, ultimately bound to the circle.

[The notion of 5 and 6 as being 'circular' appeared earlier, in Nicomachus' *Introduction to Arithmetic*, where they are called 'spherical' and 'recurrent [ἀποκαταστατικοί]'.[10] The circularity of 6 was mentioned, without the idea being given a name, in the *Arithmetical Theology*.[11][‡] Proclus' approximate contemporary Martianus Capella[§]

[1] Proclus, *Commentary on Euclid*, FRI 82.

[2] Proclus, *Commentary on Plato's* Timaeus, DHL II.78.24–8 ⟦vol. III, p. 138⟧.

[3] Proclus, *Commentary on Euclid*, FRI 150.

[4] *Ibid.*, FRI 213.

[5] *Ibid.*, FRI 213.

[*] ▸ pp. 65–7

[6] *Ibid.*, FRI 150.

[†] ▸ pp. 62–3

[7] Timaeus Locrus, *De Natura Mundi et Animae*, STE 96b.

[8] Proclus, *Commentary on Plato's* Timaeus, DHL II.188.8–10 ⟦vol. IV, p. 162⟧.

[9] *Ibid.*, DHL II.233.10–18 ⟦vol. IV, p. 217⟧.

[10] Nicomachus, *Introduction to Arithmetic*, § II.17.7.

[11] ps.-Iamblichus, *Theology of Arithmetic*, DEF 42–3.

[‡] ▸ pp. 144–5

[§] ▸ pp. 148–9

called 5 ‘recurrent’, though for slightly broader reasons.[1] None of these discussions suggests any perception of aesthetic value in the notion.]

Combining all these comments, one concludes that, for Proclus, there was a kind of self-reinforcing perception of beauty in the circle, sphere, and regular polygons and solids. The circle and sphere had the supreme beauty because of their pure similarity. This beauty descended to regular polygons and solids, which were most like the circle and sphere, respectively. But the five regular solids — those among all solids that have the highest degree of similarity — can all be inscribed in the sphere. This property confirms, but does not *cause*, the beauty of the sphere.

Astronomy

Proclus admired the ingenuity of the astronomical system that used eccentrics and epicycles as a way of predicting the motions of the planets, going so far as to call it beautiful, but he rejected any physical reality for the system:

> ‘If some people have made use of some sort of epicycles or eccentrics, positing regular motions in order that they might discover the numbers of the motions arising from their combination [...] this is a beautiful [*kalē*] thought and one that is fitting to rational souls, but as far as the nature of wholes (which only Plato apprehended) is concerned, it misses the mark badly.’[2]

Why did Proclus reject their reality? Partly, through appeal to authority: Plato never considered epicycles or eccentrics.[3] And, even leaving this point aside, they were combined in a ‘haphazard’ way to fit the observational evidence.[4] Proclus’ implicit premiss was that the truth could not be so haphazard. Underlying his criticism was the faith of a platonist who knew that the mathematical structure of the cosmos must be beautiful.

Geometric mean

Proclus devoted a long passage of his commentary on the *Timaeus* to analysing the geometric mean and how the demiurge used it as a bond in shaping the cosmos. Only a few parts of his analysis deal with aesthetic value. They show that Proclus agreed with Plato on the beauty of the geometric mean, and that, as a bond, it brought about unity through *philia* [φιλία], but also that he seemed to see it as a cause or at least as a conduit of beauty: ‘it is correctly said that the bond brings with it beauty, harmonious association and unification’.[5] And Proclus emphasized the role of the bond’s beauty in creating the order and unity:

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[1] Martianvs, *De Nvptiis Philologiae et Mercvrii*, § 735.

[2] Proclus, *Commentary on Plato’s* Timaeus, DHL III.96.27–32 ⟦vol. V, p. 176⟧; trans. mod.

[3] *Ibid.*, DHL III.146.15–17 ⟦vol. V, p. 249⟧.

[4] Proclus, *Hypotyposis*, § VII.50; trans. Sambursky, *The Physical World of Late Antiquity*, p. 149.

[5] Proclus, *Commentary on Plato’s* Timaeus, DHL II.18.16–17 ⟦vol. III, p. 61⟧.

> 'the Demiurge has brought forth Aphrodite in order that her beauty might shine forth and bring order, harmony and communion to all the things within the cosmos.'[1]

The goddess Aphrodite was associated with and here symbolized love and beauty.

Elegance and elementary propositions

Proclus pointed out that some theorems were called 'elements' [*stoicheia* στοιχεῖα], some 'elementary propositions' [*stoicheiōdē* στοιχειώδη], and some neither. Elements were theorems that led to broader understanding; they were the primary results from which all others followed.[2] Thus elements were *epistemologically* distinguished among theorems by their foundational role. All other theorems, including elementary propositions, were not so fundamental, and applied only to a part of the subject. Elementary propositions were marked off from the mass of theorems by applicability and by aesthetic considerations:

> '"Elementary" propositions are those that are simple [*haploun* ἁπλοῦν] and elegant [*charien* χαρίεν] and have a variety of applications but do not rank as elements because knowledge of them is not pertinent to the whole of the science [...]. Propositions whose understanding is not relevant to a multitude of others or which do not exhibit any grace [*glaphyron*] or elegance [*charien*] — these do not have the force of elementary propositions.'[3]

Proclus' example of a theorem that was not an elementary proposition was *The altitudes of a triangle meet at a single point* (see Figure 4.2). This result does not appear in Euclid's *Elements*. It was used (at least in the case of acute triangles) in Proposition 5 of the *Lemmas*, which was associated with the works of Archimedes. Although this text cannot have been written by Archimedes in its extant form, the result in question is attributed to him.[4] In any case, Proclus must have expected the result to be familiar to his readers, and must have thought it elegant enough to qualify as an elementary proposition.

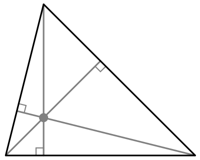

FIGURE 4.2
The altitudes of a triangle meet at the orthocentre.

Beauty and the goal of mathematics

For Proclus, the utility of mathematics was not to be measured by looking only at how it satisfied human needs. Theoretical knowledge had value outside of this; in particular, mathematical knowledge was worth pursuing for its own sake.[5] If it was necessary to evaluate mathematical knowledge with reference to something outside itself, its value came from the intellectual benefits that accrued.[6]

These benefits involved raising the mind to the appreciation of higher things. For Proclus, the domain of mathematics stood intermediate between being and becoming, and this intermediate status was so fundamental that he stated it at the very start of his *Commentary on Euclid*.[1] The objects of being were the Forms, which were apprehended through the faculty (or mode of thought) known as *nous*. Those of becoming are sensible things. The objects of mathematics, which sat above becoming and below being, are *logoi* and were apprehended by the faculty known as *dianoia* (διάνοια).[2] They were less simple than the Forms, but more precise than sensible things.[3] Proclus actually deployed the very beauty of mathematical objects as evidence for their existence. The beauty perceived in them — like certain other properties — was unvarying, and this constancy indicated that it was actually the beauty *of a thing*, not of a mere ephemeral thought: 'how could the precision, beauty, and orderliness of these figures be preserved if they were merely abstractions?'[4]

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[1] Proclus, *Commentary on Plato's* Timaeus, DHL II.54.21–24 ⟦vol. III, p. 106⟧.

[2] Proclus, *Commentary on Euclid*, FRI 72.

[3] *Ibid.*, FRI 72.

[4] Dijksterhuis, *Archimedes*, pp. 400–1, 403.

[5] Proclus, *Commentary on Euclid*, FRI 27–8.

[6] *Ibid.*, FRI 28.

For neoplatonists like Proclus, the goal of a human was to ascend to apprehension of the One, which sat above being. Education should lead the student in this ascent via a programme of study of physics, mathematics, and theology; mathematical education is an important part of this ascent.[5] Knowledge of mathematics prepares the way, 'purifying the eye of the soul and removing the hindrances that the senses present to our knowing the whole of things'.[6] Then 'each theorem lays the basis for a step upward and draws the soul to the higher world'.[7] Proclus credited Pythagoras with transforming mathematics into a plan for liberal education.[8] In explaining how mathematical education plays its role, Proclus gave first place to beauty:

> 'For the beauty and order of mathematical discourse, and the abiding and steadfast character of this science, bring us into contact with the intelligible world itself and establish us firmly in the company of things that are always fixed, always resplendent with divine beauty, and ever in the same relationships to one another. [...] whence does the philosophic nature get its impulse toward intellectual understanding and its awakening to genuine being and truth? [...] he must be given mathematics, says Plotinus,[9] if he is to become familiar with immaterial nature'.[10]

Contemplation of the objects of mathematics bestowed both the motive and the ability to reach the Forms and in particular the Forms of mathematical objects, such as 'the circle more partless than any center' and 'the triangle without extension'.[11]

The role of beauty is manifest in Proclus's emphasis that the demiurge's division of the strip of world-stuff was 'by means of most beautiful terms', and he explained that this beautiful division was chosen 'in order that it may be made similar to the causes that are elder and more originary'.[12] That is, Proclus saw the beauty of demiurge's division as making the cosmos partake more of the intelligible Forms. There is a nexus of thought here: beauty as a route to the Forms; mathematics as a route to the Forms; the beauty of the mathematical structure of the

[1] Proclus, *Commentary on Euclid*, FRI 3.

[2] Mueller, foreword to *ibid.*, p. xviii.

[3] *Ibid.*, FRI 4.

[4] *Ibid.*, FRI 139–40.

[5] *Ibid.*, FRI 46–7.

[6] *Ibid.*, FRI 28.

[7] *Ibid.*, FRI 84.

[8] *Ibid.*, FRI 65.

[9] See Plotinus, *Enneads*, § 1.3.3.

[10] Proclus, *Commentary on Euclid*, FRI 20–1.

[11] *Ibid.*, FRI 141–2.

[12] Proclus, *Commentary on Plato's* Timaeus, DHL II.194.8–11 ⟦vol. IV, p. 169⟧.

cosmos. But in the cultivation of one's soul, mathematics was only a first step. In order to liberate oneself from the *appearance* of harmony in the world and reach the true harmony of the intelligible realm, one had to go further and consider what Plato's text says about the nature of the soul.[1] Here again is an echo of Plato's programme of education: mathematics as training for the ascent to the Forms.

Indeed, Proclus observed, citing Aristotle (though he may have been thinking of Plato[2]), that some people pursued mathematics at the expense of other activities, even though they gain little materially; thus 'those who despise mathematical knowledge are they that have no taste for the pleasures it affords'.[3] This seems to be the earliest explicit normative statement that marked off those who are capable of enjoying mathematics from those who cannot, and labelled the latter as philistines.

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[1] Proclus, *Commentary on Plato's* Timaeus, DHL II.195.12–24 ⟦vol. IV, pp. 170–1⟧.

[2] Proclus, *Commentary on Euclid*, FRI 28, n. 52.

[3] *Ibid.*, FRI 28.

5

Harmony in Proportion

Mathematical beauty in mediaeval Christendom

> ‘Harmony in proportion is what beauty is’
>
> — Robert Grosseteste, *On the Six Days of Creation*, § II.x.4, p. 99
> (trans. Christopher F. J. Martin).

> ‘One should realize that the harmonies discussed here
> were not invented by the human mind’
>
> — Regino of Prüm, ‘De Harmonica Institutione’, p. 239
> (trans. Adam Czerniawski & Ann Czerniawski,
> quot. in Tatarkiewicz, *Hist. Aesthetics*, vol. II, p. 132).

Where once was seen a decline and epochal fall of the Roman Empire, modern historiography sees a long transition as the western Empire gradually weakened and lost political cohesion, and where the culture — *sensu latissimo* — of the Empire survived in the successor states. The extinction of the Empire in the west in 476 CE was not an earthquake, but a symbol of a process that had started long before and still had far to run.

Nevertheless, comparing the state of the European territories of the Empire during the reigns of the ‘five good emperors’ in 96–180 CE with their condition a half-millennium later — and even recognizing that the Empire already faced problems that the ‘good emperors’ had to confront — shows that during this time there had been a manifest change for the worse. However much continuity can be perceived, there had been, during these centuries, an enormous retreat in economic and cultural activity. In particular, the originally high level of literacy, evidenced by the quite casual uses of writing in notes and signs and graffiti, had declined catastrophically.[1]

The Roman interest in mathematics had generally been confined to its practical applications. Individual Romans did of course read the works of Euclid or Archimedes, but they did so in Greek. The Greek tradition of theoretical mathematics never gained traction in the Latin-speaking western Empire. Through the translations of Boethius (c. 480–c. 524 CE), the Latin world inherited a sketch of Greek arithmetic and fragments of geometry. The barbarian polities that took over the territory of the former western Empire had no mathematical

[1] Ward-Perkins, *The Fall of Rome and the End of Civilization*, esp. ch. VII.

traditions of their own, but little mathematics was lost, for there had been little there before.[1]

In the Greek-speaking eastern Empire — for convenience distinguished as the Byzantine Empire — the scholarly texts of antiquity were available and accessible, and were copied and studied and annotated and commented upon.[2] But the Byzantine emperors did not establish or maintain scholarly institutions like the Library of the Ptolemies at Alexandria (unless one counts hospitals that also functioned as medical schools). The Byzantine scholar was not expected to specialize in and advance a field of knowledge, but to display wide erudition and ability to dispute and offer commentary.[3]

There is little evidence of mathematics being pursued for its own sake in the early Middle Ages, in either the Latin or the Greek worlds.[4] The monumental architecture of the Greek east rested on applied mathematics; at least some of its art, arguably, depended on the mathematical theory of optics.[5] In the Latin west, such arithmetic and geometry as survived were used only for practical ends.[6] (Indeed, Hugh of Saint Victor (late 11th century–1141) seems to have been the first to draw a distinction between theoretical and practical geometry.[7]) Calculation, necessary for the functioning even of the simpler post-Imperial economy, reached a high level of sophistication.[8] In the Greek east, various periods produced new editions, commentaries, and encyclopaedias,[9] but little that was truly original.

In the later Middle Ages, the revival of mathematics in the Latin world began when the texts of antiquity started to become available again. But the *ex nihilo* nature of the re-introduction of classical Greek mathematics into a Latin intellectual world that lacked any comparable tradition led to fundamental problems of assimilation. On a purely technical level, mediaeval scholars learned to read and understand the mathematics of antiquity. But they applied it to their own religious, philosophical, governmental, and educational concerns, and did not develop it for its own sake. The real end of the Middle Ages of mathematics came as Renaissance mathematicians began to think as the Greeks did.[10]

Throughout the Middle Ages, there was an ongoing association of numbers and proportion with beauty, an association that generally did not seem to locate beauty *in* numbers. The Bible says that, during and after the creation, God had looked upon his work and found it beautiful [καλά in the Greek text].[11] In the book of Wisdom, a Greek-language text probably composed in the late first century BCE or early first century CE, which in the Catholic and Eastern Orthodox traditions is counted canonical but in Protestantism is generally thought apocryphal, is found the famous and almost platonic phrase: 'you have arranged all things by measure and number and weight [*metrō kai arithmō kai stathmō* μέτρῳ καὶ ἀριθμῷ καὶ σταθμῷ]'.[12] These properties were interpreted as aesthetic[13] and led to a lasting pythagorean–platonic association of number with beauty. To take a few examples:

[1] Mahoney, 'Mathematics', pp. 145–6.

[2] Tihon, 'Science in the Byzantine Empire', p. 206.

[3] *Ibid.*, p. 190.

[4] Molland, 'Mathematics', p. 512.

[5] Mathew, *Byzantine Aesthetics*, ch. 3.

[6] Mahoney, 'Mathematics', pp. 146–7.

[7] Baron, 'Sur l'introduction en Occident des termes "geometria theorica et practica"'.

[8] Molland, 'Mathematics', p. 516.

[9] Tihon, 'Science in the Byzantine Empire', p. 191.

[10] Mahoney, 'Mathematics', pp. 145–6.

[11] Genesis 1:31

[12] Wisdom 11:20.

[13] Eco, *Art and Beauty in the Middle Ages*, p. 17.

William of Auxerre (1150–1230) wrote that '[t]he beauty of a thing consists in these three attributes (species, number and order)'.[1] Albertus Magnus (c. 1200–1280) formulated a conception of beauty in things that encompassed notions like measure–number–weight.[2] Bonaventure's (Giovanni di Fidanza; 1221–74) conception of beauty involved proportion, which in turn rested upon numerical equality.[3]

AUGUSTINE

Augustine of Hippo (354–430 CE) was an intellectual bridge between classical Greek and western Christian thought. He belonged to antiquity by background and education and to the Christian Middle Ages by conversion and choice. He was born into a family of no special distinction and received a good education in rhetoric. Reading Cicero's (106 BCE–43 CE) now-lost dialogue *Hortensius* inspired in him an interest in philosophy.[4] To the chagrin of his mother, who was Christian, he joined the Manichaeans, who believed in a dualistic world in which good and evil vied for supremacy.

While still influenced by Manichaeism, he wrote his first work on an aesthetic subject, *On the Beautiful and Fitting* [*De Pulchro et Apto*] (c. 380 CE), the loss of which he did not regret in later life.[5] This book dealt with the Monad [monas], which was essence of virtue, truth, and goodness, and the Dyad [duas], which explained evil as a separation from unity. These concepts were construed in accordance with Manichaean dualism, but Augustine's position here also reflected a thread of neoplatonism in which the infinite Dyad was the principle of evil.[6]

In the background to Augustine's conversion to Christianity in 386 CE was his exposure to neoplatonic thought, which he absorbed through the work of Plotinus. Especially in the years after his conversion, he was heavily influenced by Plato's thought, particularly the *Timaeus*, which he read in Cicero's translation.[7] Throughout his life, he saw Plato and the platonists as representing an intellectual current that had come very close to Christianity, but which, lacking knowledge of or faith in the Incarnation, remained incomplete.[8]

From earlier writers on beauty, Augustine absorbed certain fundamental ideas, notably that beauty is an objective property, and is not dependent on the beholder.[9] He did, however, question a basic point that thitherto seems to have been accepted: 'whether things are beautiful because they give pleasure, or give pleasure because they are beautiful'.[10] His answer was that they please because they are beautiful.[11]

Soon after his conversion, in the period 386–7 CE, Augustine wrote the four texts that became known as the Cassiciacum dialogues after their place of composition. These dialogues indicate that Augustine

[1] William of Auxerre, quot. in Eco, *Art and Beauty in the Middle Ages*, p. 19.

[2] *Ibid.*, p. 25.

[3] Tatarkiewicz, *Hist. Aesthetics*, vol. II, p. 233; De Bruyne, *Études d'esthétique médiévale*, vol. 2, pp. 199–207.

[4] Augustine, *The Confessions*, bk III, ch. iv, § 7.

[5] *Ibid.*, bk IV, chs xx–xxvii.

[6] Janby, 'Christ and Pythagoras', pp. 119–20.

[7] Hoenig, *Plato's* Timaeus *and the Latin Tradition*, ch. 5.

[8] Augustine, *The Confessions*, bk VII, ch. ix, § 13; Augustine, *The City of God*, bk VIII, §§ 5–9, 11.

[9] Tatarkiewicz, *Hist. Aesthetics*, vol. II, p. 49.

[10] Augustine, *Of True Religion*, § xxxii.59.

[11] Tatarkiewicz, *Hist. Aesthetics*, vol. II, p. 49.

was both familiar with Plotinus' analysis of beauty* and impressed by the mathematical structure of the cosmos.[1]

In the third of the Cassiciacum dialogues, *On Order* [*De Ordine*], Augustine — as a character in his own composition — said that all intellectual disciplines are governed by reason.[2] After words and grammar were completed, reason led to logic and rhetoric,[3] whence

> 'reason wished to take itself to the happiest contemplation of the divine things themselves. [...] For it longed for a beauty that, alone and unmixed, it could gaze upon without these eyes of ours; [yet] it was being impeded by the senses.'[4]

Reason therefore confronted the senses in order to better understand them. Examining the functioning of hearing led Augustine to the understanding that *number* lies behind the pitch, timing, rhythm, and modulation of music and of the human voice. Thus reason

> 'was enduring most reluctantly the fact that the splendor and serenity of these things were being discolored by the bodily matter of voices. [...] what the mind sees is always present and is acknowledged to be immortal (with numbers appearing to be of this class)'.[5]

Thus, however pleasing the music is to the ear, it is just a pale shadow of the true beauty of the numerical relations that lie behind it. Similarly, and more explicitly, when it comes to sight, reason looked at the world and

> 'it perceived that nothing other than beauty pleased it, and in beauty were shapes, in shapes dimensions, and in dimensions numbers. And then it asked itself whether such a line and such a curvature (or whatever other shape and figure) were such that the intellect could contain them. It discovered that they were far inferior and that nothing which the eyes see can in any way be compared to what the mind discerns. It also reduced these distinct and arranged realities to a discipline and called it "geometry."'[6]

Thus the objects of geometry are not the shapes of the physical objects that present themselves to sight, but the abstract perfect forms that can be perceived by the mind. [Augustine also argued that, because the objects of geometry are intellectual constructs, and because mathematical objects are immortal, so is the soul.[7]]

Note, however, that Augustine held that the mind creates geometry through abstraction. When it came to the role of mathematics in the cosmos, he had a more pythagorean view that placed *number* in first place. In *On the Free Choice of the Will* [*De libero arbitrio voluntatis*] (387/8–91/5 CE), he wrote that 'He gave numbers to all things, even to the lowliest placed at the very end.'[8] It is perhaps unsurprising that Augustine frequently cited the 'measure and number and weight' phrase from the book of Wisdom[1] in support of his position that number has a role in the structure of the world.[2]

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* ▸ pp. 140–3

[1] Augustine, *On Order*, § II.5.14.

[2] *Ibid.*, § II.12.35.

[3] *Ibid.*, § II.12.35–13.38.

[4] *Ibid.*, § II.14.39; ins. '[yet]' in orig.

[5] *Ibid.*, § II.14.41; note marks omitted.

[6] *Ibid.*, § II.15.42; note marks omitted.

[7] Augustine, *Soliloquies*, § II.19.33; cf. Plato, *Meno*, STE 86a.

[8] Augustine, *On the Free Choice of the Will*, § 2.11.31.

Another verse that Augustine cited often[3] connected number and wisdom:

> 'I am glad to see that the intelligible structure and truth of number struck you as the best example when you wanted to answer my question. It is no accident that number is linked to wisdom in Scripture: "My heart and I have gone around so that I might search out and think about and know wisdom and number".[4]'[5]

Indeed, Augustine thought that contemplation of number produced a similar response to contemplation of wisdom:

> 'For when I reflect on the unchangeable truth of numbers and their lair (so to speak) and their inner sanctuary or realm — or any other suitable name we can find to refer to the dwelling-place and residence of numbers — I am far removed from the body. Perhaps I even find something to think about, but not something I could put into words. Eventually I return in exhaustion to familiar things, so that I am able to say something or other, and I talk in the usual way about the things right in front of me. This also happens to me when I think as carefully and intently as I can about wisdom. Thus I am quite surprised, since wisdom and number are linked together in the most hidden and certain truth (with the approval of the scriptural passage I mentioned in which they are conjoined), I am quite surprised, as I said, why wisdom is precious to most people and number of little value.'[6]

But Augustine's philosophy of number had several central themes: assurance of certainty; achievement of order; aesthetics; ethics; education; and number symbolism.[7] The core point in his philosophy of number is a distinction between sensible counting numbers and intelligible numbers; this separation fits with the general trend in his early works of pressing his readers to turn from sensible to intelligible realities.[8] The fundamental difference between intelligible number and sensible number is that the former can increase infinitely but cannot be diminished to less than the unit; the latter cannot be increased beyond the capacity of the senses, but can be diminished indefinitely:

> 'number, as cognisable by the understanding, is susceptible of infinite augmentation, but not of infinite diminution, because we cannot reduce it lower than to the units, number, as cognisable by the senses [...], is on the contrary susceptible of infinite diminution, but has a limit to its augmentation? [...] For whoever understands these numbers loves nothing so much as the unit; and no wonder, seeing that it is through it that all the other numbers can be loved by him.'[9]

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[1] Wisdom 11:20.

[2] Janby, 'Christ and Pythagoras', p. 118.

[3] *Loc. cit.*

[4] Ecclesiastes 7:25.

[5] Augustine, *On the Free Choice of the Will*, § 2.8.24.

[6] *Ibid.*, § 2.11.30.

[7] Horn, 'Augustins Philosophie der Zahlen', pp. 390–1.

[8] Horn, 'Augustins Philosophie der Zahlen', p. 391; Janby, 'Christ and Pythagoras', p. 119.

[9] Augustine, letter to Nebridius, dated 387 CE, repr. in Augustine, *Works*, vol. VI, p. 6 [ltr III].

In 386 CE, Augustine began a project of composing a series of textbooks on the liberal arts, which would show how their study would guide the soul to contemplate number and so to philosophy and unity. There is a clear platonic note here: ascension from sensible to intellectual things, with mathematics as a stepping-stone.[1] But by the time he wrote *On the Free Choice of the Will*, he pushed for the contemplation of number in a much more direct way.[2] Of particular relevance here is the passage where he wrote that the creators of works of visual art used numbers; the best art was that which 'conforms as much as possible to the inward light of numbers and, using sense as the go-between, it pleases the internal judge who looks upon the numbers above'.[3] Thus any putatively sensory aesthetic judgement of a work of art seems to have depended on the intellectual aesthetic judgement of its numbers; the senses serve only as a bridge. In a sculpture, the numbers are static; in a moving body, they are connected to time.[4] 'So rise above even the mind of the craftsman to see everlasting number!',[5] Augustine exhorted. The implication here is that contemplation of the intellectual numbers was superior to the aesthetic judgement of the numbers involved in a physical body.

In the dialogue *On Music* [*De Musica*] (388–91 CE), Augustine discussed in more detail the role of number in auditory beauty. He recognized that number was a part of beauty, alongside form, unity, and order, but did not reduce beauty to numerical relations.[6] He thought that numbers [*numeri*] appear in sound in four ways: in the heard sound, in the sense of hearing, in the act of the production of the sound, and in the memory of the sound.[7] More importantly, the participants in the dialogue in *On Music* realize that there is a fifth kind of number: 'a kind in the natural judgment of perceiving when we are delighted by the equality of numbers or offended at a flaw in them'.[8] This kind of number was the best and was called 'judicial' [*numeri judiciales*].[9] They were 'pre-eminent by virtue of the beauty of ratio [*ratio*]'.[10] (There are two plays on words in the Latin text here: the word *numeri* can signify *numbers* or *rhythm*; the word *ratio* can signify *ratio* or *reason*.)

Augustine was clear that the perception of the intellectual numbers, which 'have a beauty of their own'[11] has a moral effect: 'the soul is made better through lack of those numbers it receives through the body, when it turns away from the carnal senses and is reformed by the divine numbers of wisdom'.[12] The soul is thus purified: 'with a restored delight in reason's numbers, our whole life is turned to God'.[13] In his *Retractions*, written towards the end of his life, he maintained this view than numbers can guide the intellect towards eternal things.[14]

Numbers could vary in how beautiful they are but they never completely lacked beauty. Number was 'beautiful in equality and likeness'.[15] The role of equality in beauty also extended to geometry:

> '[*Augustine*]. Which figure has the more beautiful and striking proportions — the one whose sides are equal or unequal?

[1] Janby, 'Christ and Pythagoras', p. 123; cf. Augustine, *On Order*, § 11.18.47–19.51.

[2] Janby, 'Christ and Pythagoras', p. 124.

[3] Augustine, *On the Free Choice of the Will*, § 2.16.42.

[4] *Loc. cit.*

[5] *Ibid.*, p. 2.16.42.

[6] Chapman, 'Some Aspects of St. Augustine's Philosophy of Beauty', p. 49.

[7] Augustine, *On Music*, bk 6, ch. 2, § 2.

[8] *Ibid.*, bk 6, ch. 4, § 5.

[9] *Ibid.*, bk 6, ch. 6, § 16.

[10] *Ibid.*, bk 6, ch. 11, § 31.

[11] *Ibid.*, bk 6, ch. 11, § 33.

[12] *Ibid.*, bk 6, ch. 4, § 7.

[13] *Ibid.*, bk 6, ch. 11, § 33.

[14] Augustine, *The Retractions*, § 1.10.2.

[15] Augustine, *On Music*, bk 6, ch. 17, § 56.

> [*Evodius*]. Undoubtedly the one that excels in equality is more perfect.
>
> *A.* So you prefer equality to the lack of it.
>
> *E.* Certainly, like everyone else.'[1]

Thus an equilateral triangle was more beautiful than a scalene triangle.[2] A square had greater equality than a triangle, and was thus more beautiful, since each side is opposite another (equal) side and each angle is opposite another (equal) angle.[3] The circle had the greatest equality and thus the greatest beauty of figures, for its boundary is everywhere alike, being unbroken by angles, and everywhere the same distance from its centre.[4]

The perception of perfect equality was not via the senses, but by the mind.[5] Thus the perfect square that the mind could conceive, which has absolute equality among its sides and its angles, will have greater beauty than any square seen by the eyes. This distinction between intellectual and sensory apprehension also points to Augustine's discussion of aesthetic experience, about which he is more concerned than any preceding author: he related *how* perceiving the equality leads him to *delight* in it.[6]

It is possible that this liking for equality was connected to the preferences Augustine exhibited for certain numbers in his later works. In *The City of God* (413–26/7 CE), he wrote that the world was created in six days because 6 is a perfect number, being equal to the sum of its proper divisors,[*] and this property signified the perfection of Creation,[7] and was commended by Scripture.[8] Philo of Alexandria[†] (*fl.* early 1st century CE) had posited earlier the association of the six days of creation and the perfect number 6,[‡] and Augustine may have known Philo's association, perhaps via an intermediary such as Ambrose of Milan (c. 339–c. 397 CE), who greatly influenced him, and in whose works references to Philo abound.[9] It seems less likely that he knew of the remarks of Methodius of Olympus[§] (d. c. 310 CE), since he did not follow Methodius in drawing any symbolic parallel between the concept of perfect number and the Incarnation. And, of course, Augustine may have made the connection independently of either Philo or Methodius. Augustine also praised the number 7, 'which is also a perfect one, though for another reason'. The reason for the non-Euclidean perfection of 7 is rather more trivial and arithmological: $7 = 3+4$, and 3 and 4 are respectively the first odd and even numbers.[10] (Augustine presumably excluded 1 as not a 'number' and 2 as not really even, since it is part of the definition of evenness.)

Although Augustine was clearly influenced by a tradition of beauty in mathematics that stretched back to Plato, he did not assimilate it unchanged. While he cited Plato and the *Timaeus* when he argued that 'all contraries are reduced to unity by some middle factor',[11] and while there is a platonic resonance in these words, Augustine's metaphysical foundations are entirely different. In the *Timaeus*, the geometric mean

[1] Augustine, *The Magnitude of the Soul*, ch. 8–9.

[2] *Ibid.*, ch. 9.

[3] *Loc. cit.*

[4] *Ibid.*, ch. 11.

[5] Augustine, *Of True Religion*, § xxxi.57.

[6] *Loc. cit.*

[*] ▸ p. 101

[7] Augustine, *The City of God*, bk xi, ch. 30.

[8] Augustine, *On the Trinity*, bk IV, ch. iv, § 7.

[†] ▸ pp. 130–3

[‡] ▸ p. 131

[9] Dudden, *Life of St. Ambrose*, vol. I, p. 113, n. 2.

[§] ▸ p. 144

[10] Augustine, *The City of God*, bk IV, ch. 31.

[11] Augustine, *The Harmony of the Gospels*, bk I, ch. xxxv, § 53.

— the 'fairest [κάλλιστα] of bonds',[1] which Cicero called 'most apt and most beautiful [aptissimum atque pulcherrimum]'[2] — served as the mediator that bound things together. For Augustine, the abstract aesthetic concept had no importance: his argument was that Christ served as mediator between the human and the divine.[3]

BOETHIUS AND THE LEGACY OF GREEK MATHEMATICS

The Renaissance humanist Lorenzo Valla (c. 1407–1457) called Boethius (c. 480–c. 524 CE) 'the last of the erudite'.[4] The epithet is distressingly revealing: Boethius can hardly have been said to have contributed anything truly new, but he was the last writer who tried to preserve anything of Greek learning for the Latin west.[5]

Boethius was born shortly after the last titular emperor in the west had been deposed. For most of his lifetime, Rome and Italy were ruled by the Ostrogothic king Theodoric (454–526 CE, r. 471–526 CE), who allowed Roman aristocratic families like Boethius' to maintain their traditions. Boethius thus had the education and leisure to begin at an early age a programme of translating into Latin the classics of Greek philosophy and mathematics.[6] He gained a reputation as a man of wide learning and around 522 CE was appointed by Theodoric to the senior administrative post of *magister officiorum*. Perhaps inexperienced in the machinations of the royal court, he became the target of a number of accusations and was removed from office, imprisoned, and subsequently executed.[7] Whilst incarcerated and facing death, he composed his one truly great work: *The Consolation of Philosophy*, an imagined dialogue between his prisoner self and the personification of philosophy.

In philosophy, Boethius' most important contributions were his Latin translations of Aristotle's logical works (*Categories*, *On Interpretation*, *Prior Analytics*, *Topics*, *Sophistical Refutations*);[8] these editions survived throughout the Middle Ages. He also translated Porphyry's *Isagoge*, an introduction to Aristotle's *Categories*, and wrote multiple commentaries, some now lost, on Aristotle's works, as well as on Porphyry's *Isagoge* and Cicero's *Topics*. In mathematics, two of his books are extant: his *De Institutione Arithmetica* and his *De Institutione Musica*. He may also have written now-lost books on geometry and astronomy. The originality of his mathematical works was negligible: *De Institutione Arithmetica* followed Nicomachus' *Introduction to Arithmetic*, and *De Institutione Musica* was based on Ptolemy's *Harmonics* and a lost work by Nicomachus.[9] In *De Institutione Arithmetica*, Boethius developed some parts where he thought Nicomachus had been too concise, and introduced into Latin some terms thitherto known only in Greek. Yet he also omitted some of the more advanced of

[1] Plato, *Timaeus*, STE 31C.

[2] Cicero, *Timaeus*, p. 186; trans. C.

[3] Augustine, *The Harmony of the Gospels*, bk I, ch. XXXV, § 53; Hoenig, *Plato's* Timaeus *and the Latin Tradition*, pp. 257–8.

[4] Valla, *Opera*, p. 646.

[5] Lindberg, 'The Transmission of Greek and Arabic Learning', p. 54.

[6] Moorhead, 'Boethius' life', p. 15.

[7] *Ibid.*, pp. 18–19.

[8] Magee & Marenbon, 'Boethius' Works', pp. 303–5.

[9] *Ibid.*, pp. 303–4, 308–9.

Boethius and the legacy of Greek mathematics

Nicomachus' propositions.[1] The importance of these books lay not in anything new they tried to contribute, but in the part of the older knowledge they preserved.

De Institutione Musica saved the theoretical knowledge of music for the Latin world. Like the predecessors on whose works he drew, for Boethius *music* meant the science of the numerical laws that underlay the auditory phenomena. A true musician was not a performer, but a *theorist*: 'How much nobler, then, is the study of music as a rational discipline than as composition and performance! […] Reason exercises authority and leads to what is right'.[2] For all Boethius' apparent focus on Aristotle, such a desire to seek beauty beyond the phenomena indicates an influence of platonism. Perhaps, in an age of crisis, there was a certain appeal in seeking refuge in abstraction and eternal values.[3]

Through Boethius' mathematical works, the Latin Middle Ages inherited, not just arithmetic, but the ghost of a platonic-pythagorean attitude to mathematics. In particular, Boethius wrote about the idea, stated clearly by Nicomachus,[*] that arithmetic was the first of the mathematical sciences, and was in the mind of the creator when the world was shaped:

> 'This is arithmetic. It is prior to all not only because God the creator of the massive structure of the world considered this first discipline as the exemplar of his own thought and established all things in accord with it; or that through numbers of an assigned order all things exhibiting the logic of their maker found concord […].
>
> […]
>
> From the beginning, all things whatever which have been created may be seen by the nature of things to be formed by reason of numbers. Number was the principal exemplar in the mind of the creator.'[4]

This idea was a foundation for the mediaeval platonists' recognition of harmony in the world, based on a mathematical order, and deriving ultimately from arithmetical ideas that were prior to the creation.[†] But mediaeval thinkers in the Latin world did not have access to Nicomachus' or Iamblichus' works, and were, after all, Christians. Thus they did not give primacy to the numbers, but rather to the divine mind that conceived them.

In at least one respect, Boethius seems to have been the last man in antiquity to see beauty in an abstract, highly theoretical, class of numbers. Boethius followed Nicomachus[‡] in seeing an aesthetic contrast between perfect numbers and deficient and abundant numbers. Unlike Nicomachus, he compared non-perfect numbers to specific monstrous beings. Deficient numbers, whose proper divisors sum to less than the whole, were like the one-eyed Cyclopes; abundant numbers, whose proper divisors sum to more than the whole, were like the triple-headed or -bodied Geryon.[5] And Nicomachus had drawn parallels both between the common abundant and deficient numbers and ‘ugly [*aischra* αἰσχρὰ] and evil’ things on the one hand and between the rare perfect numbers and ‘fair [*kala* καλὰ] and excellent things’[1] on the other. Boethius, in contrast, saw a mainly *moral* analogy in perfect numbers. The perfect numbers were a virtuous middle between two extremes.[2] There is a ‘great similarity to the virtues and vices’ in the rarity and regular order of the perfect numbers and the plenitude and lack of order in the abundant and deficient numbers.[3]

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[1] Minio-Paluello, 'Boethius, Anicius Manlius Severinus', p. 233.

[2] Boethius, *Fundamentals of Music*, § 1.34 ⟦p. 50⟧.

[3] Eco, *Art and Beauty in the Middle Ages*, p. 30.

[*] ▸ p. 139

[4] Boethius, *De Institutione Arithmetica*, § 1.1–2; trans. Boethius, *Number Theory*, pp. 74, 75–6; typos corrected; note markers omitted.

[†] ▸ pp. 172–5

[‡] ▸ pp. 139–40

[5] Boethius, *De Institutione Arithmetica*, § 1.19.

CASSIODORUS AND CIRCULAR NUMBERS

The abstract Euclidean concept of perfect number was partly supplanted by simpler but sometimes more visually appealing numerical concepts. For instance, Boethius also discussed *circular* or *cyclical* or *spherical numbers*, which had earlier appeared in the works of Nicomachus and Proclus,* and which had a lasting legacy in the later Middle Ages and Renaissance. He noted that 5 × 5 = 25, that multiplying again by 5 gives 125, and that continuing to multiply by 5 always yields a number ending in 5. (Boethius, like all those who wrote in Latin, used Roman numerals; thus he observed that the multiples of v always ended in v: xxv; cxxv.) He noted that 6 had the same property. He connected this property to circularity or sphericity because a circle and a sphere are formed by rotating respectively a point or a semicircle through a circle and back to its starting point.[4] This comparison seems to have been a connection between arithmology and the geometrical aesthetic admiration of circles and spheres.

The connection was made much more explicit in the *Institutions of Divine and Secular Learning*, a textbook for Christian education by Cassiodorus (c. 485–c. 585 CE), a scholar who was appointed *magister officiorum* after Boethius’ ouster. In this book, Cassiodorus seems to have followed his predecessor’s *De Institutione Arithmetica* in his discussion of arithmetic as one of the seven liberal arts. On circular numbers, he wrote that

> ‘A circular number is one that when it is multiplied by itself, beginning from itself turns back to itself, for example 5 times 5 is 25 as *the diagram indicates*’.[5]

The diagram that Cassiodorus supplied is similar to that shown in Figure 5.1. The inclusion of the diagram suggests that, for Cassiodorus, there was a very tight connection between the arithmetical property of being a circular number and the geometrical circle. But perhaps one should not read too much into this passage, for Cassiodorus seems to

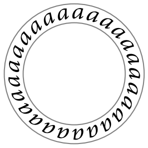

FIGURE 5.1
Cassiodorus’ diagram illustrating how 25 is a circular number.

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[1] Nicomachus, *Introduction to Arithmetic*, § i.16.3.

[2] Boethius, *De Institutione Arithmetica*, § i.19.

[3] *Ibid.*, § i.20.

* ▸ pp. 154–5

[4] *Ibid.*, § ii.30.

[5] Cassiodorus, *Institutions of Divine and Secular Learning*, § ii.iv.6, p. 214, emphasis added.

have been unclear on whether 5 or 25 was the circular number, since he wrote in the same section that

> ‘a spherical number is one that multiplied by a circular number starting from itself returns to itself, for example 5 times 5 is 25; this circular number when it is multiplied by itself, makes a sphere, ⟨i.e.⟩ 5 times 25 is 125.’[1]

Modern mathematicians tend to dismiss concepts such as circular numbers as having only recreational value, for circularity is not an intrinsic property of a number: unlike perfection or primality, it depends on the decimal representation.[2] For example, in hexadecimal notation, $5 \times 5 = 19_{16}$ and $5 \times 5 \times 5 = 7D_{16}$. But circularity was — or at least became — important in the Middle Ages, especially for its role in number symbolism.*

ISIDORE OF SEVILLE

Isidore of Seville (c. 560–636 CE) was a prolific author who wrote in fields including theology, history, and science, and is remembered as a compiler and systematizer of ancient knowledge. His encyclopaedic *Etymologies* was an important and influential book that was cited by countless scholars throughout the Middle Ages.[3] It contained the most representative account of the quadrivium, which at the time was the closest thing to a core of scientific learning. The account of mathematics given by Isidore comprises a brief description of the divisions and contents of the subject, essentially little more than definitions, often imprecise.[4] His definition of perfect numbers was correct, though he listed only the first three perfect numbers as examples,[5] omitting the fourth, 8128, which Boethius knew.[6]

Like others before him, Isidore also considered 10 to be a perfect number.[7] And this idea had a fundamental effect on his aesthetic theory. In his *Etymologies*, he asserted that Latin aesthetic terms like *decorus* (becoming), *decens* (fitting), *decibilis* (decent), were derived from *decem*, the perfect number 10.[8] (In fact, *decorus*, *decens*, *decibilis* are related, but *decem* descends independently from a different Proto-Indo-European root.[9]) For Isidore, a word’s etymology was not supplementary to the definition, but was its *foundation*.[10] Thus the aesthetic terms, and the judgements they expressed, were grounded in the number 10. Since the term *decor*, for Isidore, comprised being beautiful and apt,[11] it was the pythagorean perfection of 10 that undergirded beauty itself.

But, for Isidore, perfection was not limited to Euclidean perfection or to the pythagorean perfection of 10. In his *Liber Numerorum*, besides recapitulating the perfection of 6 and 10, he also declared 3, 4, 7, 8, and 9 to be perfect numbers. The first perfect number is 3,

[1] Cassiodorus, *Institutions of Divine and Secular Learning*, § ii.iv.6, p. 215.

[2] Cf. Hardy, *Apology*, § 15.

* ▸ pp. 170–2

[3] Elfassi, ‘Isidore of Seville and the *Etymologies*’, pp. 269–71.

[4] Grant, *Foundations of Modern Science in the Middle Ages*, pp. 15–16.

[5] Isidorus, *Etymologiarvm sive Originvm*, § iii.v.11. The translation by Barney et al. (*The Etymologies*, ⟦p. 91⟧) partly obscures Isidore’s illustration of the definition using 6.

[6] Boethius, *De Institutione Arithmetica*, § i.20.

[7] Isidorus, *Etymologiarvm sive Originvm*, § xi.i.70.

[8] *Ibid.*, § x.d.68.

[9] Vaan, *Etym. Dict. Latin*, ‘decem’, ‘decet’ [pp. 164–5, 165].

[10] Elfassi, ‘Isidore of Seville and the *Etymologies*’, p. 258.

[11] Isidorus, *Sententiae*, § i.8.18 ⟦pp. 24–5⟧.

because it has a beginning, middle, and end.[1] The 'number of solidity' 4 has, at least, 'a certain perfection'.[2] The number 7 is perfect because it equals 3 + 4 and so is composed of the first odd and first even numbers (excluding 1 as not truly a number and 2 as part of the definition of evenness).[3] The number 8 is perfect since it is the cube of 2, which is the 'first step'.[4] The number 9 is 'a perfect number in itself', and is called 'more perfect' since it is 3 × 3, the product of the first perfect number with itself.[5]

Isidore seems to have followed Augustine in his discussion of the perfection of 7,[*] a number which, as he elsewhere pointed out, appeared often in Scripture to indicate perfection.[6] His arguments about 8 and (especially) 9 seem rather to have echoed Martianus Capella[†] (*fl.* 5th century CE). Like Martianus — but unlike Boethius — Isidore seems not to have specially distinguished Euclidean from pythagorean or arithmological perfection. But there is an important difference. Whatever Martianus' religious affiliation, his writing is not markedly Christian. Isidore was a Christian writer, and the *Liber Numerorum* is filled with symbolic and Scriptural meanings of numbers, and thus imported into the Christian tradition the undifferentiated perfections of numbers.

NUMBER SYMBOLISM AND AESTHETICS

As the work of Isidore shows, in mediaeval Christian thought number symbolism sometimes connected to aesthetic value. Mediaeval number symbolism owed something to the arithmology of the ancient world, but was developed to a vastly higher, broader, and more intricate degree.

The words *symbol* and *symbolism* must be used with care here, for today's reader might have a tendency to dismiss the phenomena they describe, perhaps unconsciously inserting the adjective 'mere'. They point to what were, in mediaeval thought, inevitable associations of ideas that were often connected with Scripture and so were seen as having God's imprimatur.[7] The Byzantine monk and theologian Maximus the Confessor (c. 580–662 CE) said that the whole spiritual world was imprinted on the sensible world in symbol form; symbolic contemplation is the understanding of invisible things through the visible.[8] The theologian and Biblical exegete Hugh of Saint Victor (late 11th century–1141) wrote that

> 'all visible things have their symbolical sense, that is, they are given figuratively to denote and explain invisible things [...]. They are signs of invisible things and images of those things which dwell in the perfect and incomprehensible nature of the Deity'.[9]

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[1] Isidorus, *Liber Numerorum*, § IV.13 ⟦p. 223⟧.

[2] *Ibid.*, § V.19 ⟦p. 224⟧; trans. C.

[3] *Ibid.*, § VIII.35 ⟦p. 228–9⟧; trans. C.

[4] *Ibid.*, § IX.48 ⟦p. 232⟧; trans. C.

[5] *Ibid.*, § X.52 ⟦p. 233⟧; trans. C.

[*] ▸ p. 165

[6] Isidorus, *De Ecclesiasticis Officiis*, § I.xxxii.

[†] ▸ p. 149

[7] Hopper, *Medieval Number Symbolism*, p. vii.

[8] Maximus, *Mystagogia*, ll. 241–57.

[9] Hugo de S. Victore, *Hierarch. Cœlest.*, col. 954; trans. mod. Tatarkiewicz, *Hist. Aesthetics*, vol. II, p. 200.

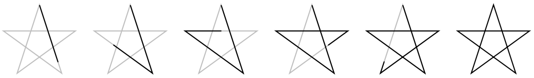
Number symbolism and aesthetics

The symbolic meaning of numbers had a special importance. Hugh made the broader point that the surface meaning of things was elucidated by the trivium, but to understand the deeper allegorical or tropological meaning required the quadrivium.[1] Isidore said that a knowledge of arithmetic made clearer, or even made possible, understanding of Scripture.[2] Augustine had earlier pointed out that '[a]n unfamiliarity with numbers makes unintelligible many things that are said figuratively and mystically in scripture'.[3] The theologian Rabanus Maurus (c. 780–856 CE) quoted Augustine with approval and also thought that a knowledge of geometry was conducive to spiritual understanding.[4]

Number symbolism could be simple but extensive. The number 3 was connected with the Trinity, the gifts of the Magi, Peter's three-fold denial, the three days between the death and resurrection of Christ.[5] The number 4 was associated to the four cardinal directions, four elements, four phases of the Moon, four seasons, four letters in the name 'Adam'.[6] Aldhelm of Malmesbury (c. 639–709 CE) enumerated at length the significance of the number 7 in Scripture.[7]

Conversely, the symbolism could be intricate but focused, such as Honorius of Autun's (c. 1050–c. 1140) detailed analysis of the numbers of various steps of the Mass. A small part of his discussion will suffice as an illustration. In total, the sign of the cross was made 23 times, 10 for the law of the Old Testament; $13 = 10 + 3$ for the additional faith in the Trinity in the New. The sign was made with 3 fingers, and $3 \times 23 = 69$, which, with the 3 more signs made over the chalice during the *Pax Domini*, makes $72 = 69 + 3$, which corresponded to the languages of the world.[8] This explanation was far more subtle than anything foreseen by the original creators of the Mass.[9] To a modern reader, this analysis might seem little more than a numerical game, but at the time it was valuable, and there is no denying the intellectual effort required to take a pre-existing ceremony and devise a necessarily sophisticated symbolical interpretation for all the numbers it involved.

Number symbolism had a major influence on the composition of literature in the Middle Ages.[10] To take a single example, the number 5, which Boethius had called a circular number,[*] appears throughout the Bible: the Pentateuch, the five wounds of Christ.[11] It also appears in the late mediaeval poem *Sir Gawain and the Green Knight* (c. late 13th century[12]). Here it was used as symbol of perfection and eternity: the poet emphasized that Gawain's virtues were five and many times five,[13] and they were linked to the pentagram or pentangle, the five-pointed star, which was Gawain's emblem. At line $625 = 5 \times 5 \times 5 \times 5$, the poet says that the pentagram was a symbol set up by Solomon;[14] it was known as 'þe endeles knot'.[15] This name presumably refers to how the pentagram can be drawn in a single unbroken stroke:



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[1] Hugh of St Victor, *On the Sacraments of the Christian Faith*, bk I, prologue, § vi.

[2] Isidorus, *Etymologiarvm sive Originvm*, §§ III.iv.1–3.

[3] Augustine, *De Doctrina Christiana*, § II.62 ⟦p. 85⟧.

[4] Hrabanus Maurus, *On the Formation of Clergy*, §§ 3.22–3.

[5] Hopper, *Medieval Number Symbolism*, p. 70.

[6] *Ibid.*, p. 84.

[7] Aldhelmus, 'De Metris et Enigmatibus ac Pedim Regulis', pp. 62–73.

[8] Honorius, *Gemma Animæ*, bk I, ch. lvii.

[9] Hopper, *Medieval Number Symbolism*, p. 155.

[10] Curtius, *European Literature and the Latin Middle Ages*, Exc. xv.

[*] ▸ pp. 168–9

[11] Eco, *Art and Beauty in the Middle Ages*, pp. 35–6.

[12] Tolkien, Gordon & Davis, *Sir Gawain and the Green Knight*, pp. xxv–xxvi.

[13] *Ibid.*, l. 632.

[14] *Ibid.*, l. 625.

[15] *Ibid.*, l. 629–30.

Even more subtly, the first line of the poem (‘Siþen þe sege and þe assaut watz sesed at Troye’[1]) is echoed at line 2525 — or, rather, line 25-25, whose repetition is at least discernible in the Roman numerals MMDXXV — (‘After þe segge and þe asaute watz sesed at Troye’[2]).

MEDIAEVAL LATIN PLATONISM

No less than their beginning, the intellectual end of the Middle Ages in Europe was a process rather than an event. Scholarly life gradually recovered. The Carolingian kings’ ascent to power, their conquests, and their reforms, created a realm of stability. These rulers were men of culture and they supported scholars, leading to what is often called the Carolingian renaissance. Charlemagne (748–814 CE; r. 768–814 CE) promoted scholarship as a means to educate better the clergy and so prepare it for its task. The patronage of Charlemagne and his successors attracted men of letters from all over western Europe and led to new translations, commentaries, glosses, and original literary works. Ecclesiastical scholars throughout the Carolingian domains turned cathedrals and monasteries into centres of learning. And in scriptoria everywhere, copyists laboured to produce the needed manuscripts.[3]

Among the scholars who were drawn to the court of Charles the Bald (823–77 CE; r. 840–77 CE), the grandson of Charlemagne, was John Scottus Eriugena (*fl.* early 9th century–after 870 CE), whose philosophical work, in context in ninth-century Christendom, ‘stands out like a lofty rock in the midst of a plain’[4] as the historian of philosophy Frederick Copleston (1907–94) memorably put it. Unlike his contemporaries, who might comment on and gloss philosophical texts, Eriugena engaged in original speculation and created the first great system of the Middle Ages. He undertook to translate from Greek into Latin various neoplatonic texts.

In Eriugena’s time and later in the Middle Ages, platonism in the Latin world was distorted by the lack of access to Plato’s works. The only text by Plato that was available in Latin was Calcidius’ (*fl.* c. late 4th/early 5th century CE) partial translation of the *Timaeus*. Aside from this fragment, the mediaeval Latin world’s knowledge of Plato’s thought was filtered through Calcidius’ own commentary on part of the *Timaeus*, Macrobius’ (*fl.* c. 400 CE) *Commentary on the Dream of Scipio*, and Martianus Capella’s* (*fl.* 5th century CE) *The Marriage of Philology and Mercury*.[5]

Calcidius’ translation of and commentary on the *Timaeus* both stop (at STE 53c) just before Plato’s discussion of fairest triangles† and the regular solids.[6]‡ Thus it is unsurprising that the mathematical aspects of platonism in the Latin world placed less emphasis on the beauty of geometrical objects. Nor did the other sources give reasons

[1] Tolkien, Gordon & Davis, *Sir Gawain and the Green Knight*, l. 1.

[2] *Ibid.*, l. 2525.

[3] Contreni, ‘The Carolingian renaissance’, pp. 709–11.

[4] Copleston, *A History of Philosophy*, vol. II, p. 112.

* ▸ pp. 148–9

[5] Reydams-Schils, *Calcidius on Plato’s Timaeus*, p. 1.

† ▸ pp. 65–7

[6] *Ibid.*, p. 1, n. 1.

‡ ▸ pp. 67–70

to look there for beauty. Macrobius' *Commentary* does not discuss the regular solids or the triangles that make them up. He did discuss the harmony induced by various proportions, but the connecting role of the geometric mean — the fairest of bonds* — has been reduced, in a translation of an excerpt from the *Timaeus*, to an unspecified 'mean which was best [optimum]' for connecting things.[1] The chosen term lacks any aesthetic significance. Finally, the geometrical content of Martianus' *The Marriage of Philology and Mercury* is shallow.†

It was Eriugena who first brought new ideas from Greek texts into Latin platonism. Through his translation efforts, he became familiar with post-Plotinian neoplatonic thought that had been thitherto unknown in the Latin world.[2] In his greatest work, the *Periphyseon*, also known as the *De Divisione Naturae*, he attempted a synthesis of eastern and western neoplatonism, including Augustinian thought, in a Christian context.[3]

For Eriugena, all sensory and intellectual beauty originated from God: 'He is the Cause of every form and all beauty, whether that form and that beauty are perceived by the mind or the sense'.[4] The created universe was based on the incorporation of numerical proportions so as to ensure harmony:

> 'as between the numerical proportions there are the proportionalities, that is to say, similar principles of proportion, in the same way the Wisdom that is the Creator of all things has constituted between the participations of the natural orders marvellous and ineffable harmonies'.[5]

The beauty of the world is dependent on these proportions and the harmony they ensure:

> 'the perfection and the beauty of the whole of nature, and provide the perfect harmony of the whole visible and invisible world, in which no discordant note is heard'.[6]

Eriugena emphasized that the numbers which underpin the beauty of the world are abstract and subsist separately from the world or anything in it: they are 'intellectual, invisible, incorporeal' and 'not of those which we count, but of those by which we count'.[7]

This idea of the world as a mathematically-structured composition was certainly compatible with the book of Wisdom and with Augustinian thought. Boethius' poem 'O qui perpetua', from the third book of *The Consolation of Philosophy*, was a paean to creation as described in the *Timaeus*, and was a frequently glossed text in the Middle Ages.[8] It refers to God shaping the world to be beautiful:

> 'From within yourself, ungrudging, you brought out the
>   pattern of all
> that is good, inasmuch as it partakes of your own goodness.
> Its beauty is your beauty; your mind is the source of its
>   grandeur'[9]

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* ▸ pp. 64–5

[1] Macrobius, *Commentariorvm in Somnivm Scipionis*, § I.vi.28–31; trans. Macrobius, *Commentary*.

† ▸ pp. 148–9

[2] Carabine, *John Scottus Eriugena*, p. 17.

[3] *Ibid.*, pp. 22–6.

[4] Eriugena, *Periphyseon*, FLO 827D ⟦p. 484⟧.

[5] *Ibid.*, FLO 630D ⟦p. 247⟧.

[6] *Ibid.*, FLO 965C ⟦p. 647⟧.

[7] *Ibid.*, FLO 651A, B ⟦p. 272⟧.

[8] Wetherbee, *Platonism and Poetry*, p. 30.

[9] Boethius, *The Consolation of Philosophy*, bk III, metrum IX.

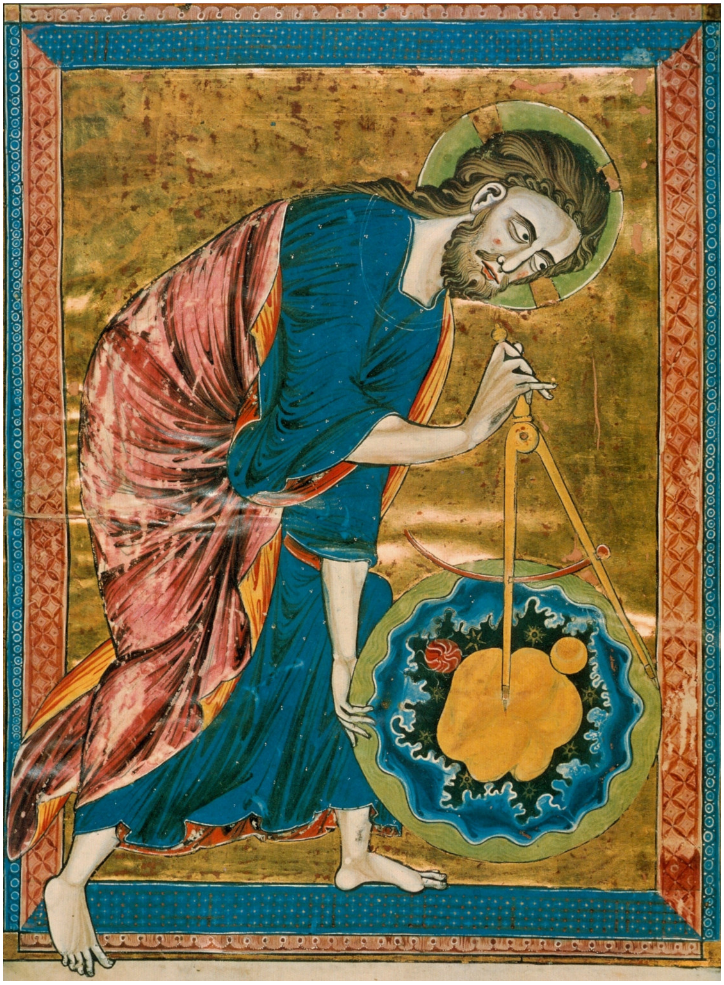

FIGURE 5.2
God as a geometer, from Codex Vindobonensis 2554, a Bible moralisée (1220–30).

[Boethius had contrasted instruments like the compass, which yields an exact shape, with tools like the adze, which does not.[1] When late mediaeval art depicted God creating the universe according to mathematical laws, he was often shown wielding a compass; see Figure 5.2.]

And later, the poet and theologian Walter of Châtillon (*fl.* late 12th century) recognized the mathematical structure of the universe:

> 'All subjects serve the Creator,
> under perfect measure, number, and weight.
> To the invisible, understood through these things,
> direct reason pulls man upwards.'[2]

Walter seems here to have agreed with the view, descending from Plato, that the study of mathematics could elevate the mind.

The beginnings of the School of Chartres were in the late tenth century, but it made its main philosophical contributions in the twelfth.[3] [While the very existence of a 'School of Chartres' has been challenged,

[1] Boethius, *Fundamentals of Music*, § v.2.

[2] Wright, *The Latin poems commonly attributed to Walter Mapes*, 'Prædicatio Goliæ', ll. 31–5 [p. 32]; trans. C.; for the attrib. to Walter of Châtillon, see Wilmart, 'Poémes de Gautier de Châtillon', p. 128.

[3] Copleston, *A History of Philosophy*, vol. II, ch. xvi.

there is affinity enough between the thinkers traditionally associated thereto to treat them as a group.[1]] It did not examine the allegorical sense of number.[2] Rather, it took from platonism the view that the Creator had deliberately shaped the world using mathematics. They saw the world as deriving its beauty from the mathematical principles by which it was deliberately constructed so as to mirror an eternal harmonic idea. The Creator was an artist or craftsman who had worked using mathematical rules.[3] Thierry of Chartres (*fl.* c. 1130–c. 1150) saw the arithmetical and geometrical structure of the world, and also tried to explain the mystery of the Trinity in geometrical terms.[4] William of Conches (c. 1080–c. 1154) thought that the mathematical division of the strip of world-stuff in the *Timaeus*[*] illustrated its perfection.[5]

But the works of the members of the School of Chartres only hint at an appreciation of beauty *in* mathematics. Indeed, it is quite possible that none of these thinkers had any in-depth knowledge of mathematics. Hugh of Saint Victor's (late 11th century–1141) *Practical Geometry* [*Practica Geometriae*] is concerned only with altimetry, planimetry, and cosmimetry, which use instruments, and not with theoretical geometry, which uses mental reflection.[6] John of Salisbury (d. 1180), who was bishop of Chartres from 1176 until his death, wrote that geometry was not much in use in the Latin-speaking world.[7] But John seems to have had some grounding in the quadrivium, at least to the extent of understanding the connection between mathematics and philosophy and its application to specific philosophical questions.[8] So although members of the School of Chartres saw mathematical beauty in the world, they may have used only elementary arithmetical and geometrical concepts, perhaps not even reaching the level of aesthetic application of the Euclidean notion of perfect number.

THE TWELFTH-CENTURY RENAISSANCE AND THE RISE OF THE UNIVERSITIES

The Carolingian revival of letters did not mark the beginning of a steady recovery that led seamlessly into the Renaissance. A corner had been turned, but subsequent progress was fitful. The realm that Charlemagne had built, and the stability that it brought, did not last. The empire disintegrated after the death of his son Louis I 'the Pious' (778–840 CE; r. 813–40 CE). Vikings had already begun to raid; Magyars and others would follow. There was, if not a loss, then at least a period of stagnation in learning. The number of schools did not greatly decrease, but the copying of manuscripts slowed. There were isolated bursts of activity, such as in the creation of art and literature at the imperial court in the late tenth century, in what is sometimes called the Ottonian renaissance. But, for what came later, the more

[1] Otten, *From Paradise to Paradigm*, § 1.VI, esp. p. 37.

[2] De Bruyne, *Études d'esthétique médiévale*, vol. 2, p. 649.

[3] Tatarkiewicz, *Hist. Aesthetics*, vol. II, p. 204–5; see also De Bruyne, *Études d'esthétique médiévale*, vol. 1, pp. 627, 649.

[4] Simson, *The Gothic Cathedral*, p. 27.

[*] ▸ pp. 62–3

[5] Wetherbee, *Platonism and Poetry*, p. 46.

[6] Hugh of St. Victor, *Practical Geometry*, pp. 33–4; see also Rorem, *Hugh of Saint Victor*, pp. 42–5.

[7] John of Salisbury, *The Metalogicon*, ch. IV.6.

[8] Evans, 'John of Salisbury and Boethius on Arithmetic', *passim*; see also John of Salisbury, *The Metalogicon*, ch. I.24.

The twelfth-century renaissance

significant development was the foundation of reformed Benedictine monasteries, especially those that grew from the well-spring at Cluny. The ecclesiastical network comprising these monasteries constituted the centre of cultural life in the Latin world and paved the way for a period of renewed vigour in intellectual development:[1] the time that has been known, since the mediaevalist Charles Haskins (1870–1937) coined the phrase, as the renaissance of the twelfth century.[2]

While the progress of learning in the Carolingian period was given its impetus by kings and emperors, the developments of the twelfth century had a broader base. Besides the monasteries, the rise of towns and of bureaucratic states drove a revival of jurisprudence, which required a new engagement with Roman legal texts. In cathedral schools were the beginnings of what would become scholasticism. There was an influx of newly-available texts, translated from Greek, Hebrew, or Arabic, so that the learning available to Latin-speaking Europe expanded immeasurably between the middle of the twelfth and the end of the thirteenth centuries. The difference was especially dramatic in science. At the start of this period, little more scientific writing was available in Latin than what had survived through Boethius' *De Institutione Arithmetica* and *De Institutione Musica*[*] and Isidore's *Etymologies*.[†] By its end, the range of translations encompassed the mathematical and scientific works of Aristotle, Euclid, Archimedes, Ptolemy, Galen, Hippocrates, and Ibn Sīnā.

The progress of translation from the Arabic was partly aided by the ongoing Christian *reconquista* of the Muslim territories in the Iberian peninsula.[3] Euclid's *Elements* reappeared in the west when Adelard of Bath (1116–42) presented his Latin version (or possibly versions), translated from the Arabic.[4] Robert of Chester (*fl.* 12th century) translated al-Khwārizmī's algebra text *The Compendious Book on the Calculation of Restoring and Balancing*.[5][‡]

Translations from the Greek were initially concentrated in Italy. The south of Italy had remained part of the Byzantine Empire until the eleventh century, and retained a sizeable Greek-speaking population, while the northern Italian states maintained diplomatic and commercial relations with Constantinople.[6] Preëminent among translators from the Greek, both for the quality and the quantity of his work, was William of Moerbeke (1215/35–c. 1286), who in particular translated the known corpus of Archimedes from Greek into Latin in 1269.[7]

Some of this vast trove of translated knowledge began to be used for teaching in the new universities. The first European universities emerged out of communities of masters and students that coalesced in certain towns and cities and which gradually developed greater institutional structure and accrued greater independence from local lay or religious authorities.[8] At some point, perhaps difficult to pinpoint exactly, each achieved a degree of coherence and self-governance that can be taken to define its founding. Universities were formed at Bologna at the end of the twelfth century, at Paris, Oxford, Montpellier

[1] Treadgold, 'Introduction: Renaissances and Dark Ages', pp. 16–17.

[2] Ladner, 'Terms and Ideas of Renewal', p. 1.

[*] ▸ pp. 166–8

[†] ▸ pp. 169

[3] Lindberg, 'The Transmission of Greek and Arabic Learning', pp. 62–7.

[4] Clagett, 'Adelard of Bath', p. 63.

[5] Lindberg, 'The Transmission of Greek and Arabic Learning', pp. 63–4.

[‡] ▸ p. 191

[6] *Ibid.*, pp. 70–3.

[7] Lindberg, 'The Transmission of Greek and Arabic Learning', pp. 73–4; Clagett, *The Arabo-Latin Tradition*, p. 4; Clagett, *The Translations from the Greek by William of Moerbeke*, § 1.i.

[8] Verger, 'Patterns', pp. 37–9.

at the start of the thirteenth century, and in many other places during the next centuries.[1] Most universities comprised several faculties for different disciplines. The classical division was fourfold: arts, theology, law, and medicine.[2] It was in the faculty of arts that were studied the trivium of grammar, rhetoric, logic and the quadrivium of arithmetic, geometry, music, astronomy.

The teaching of arithmetic seems to have been based on Boethius' *De Institutione Arithmetica*, supplemented by texts on calculation and Hindu–Arabic numbers by Jordanus de Nemore (*fl.* c. 1220). The standard geometry text was Euclid's *Elements*, perhaps augmented with Hugh of Saint Victor's *Practical Geometry*. The limits of mathematics education are perhaps indicated by a requirement at Oxford that a *teaching master* should have studied the first six books of the *Elements* (that is, to the end of the treatment of plane geometry).[3] But even this circumscribed mathematics could be enjoyed: earlier, Adelard had said that there was intellectual pleasure [iocunditatem] to be found in parts of Boethius' arithmetic.[4]

ROBERT GROSSETESTE

As a teacher at and later chancellor of the University of Oxford, Robert Grosseteste (c. 1168–1253) helped introduce the recovered knowledge into the curriculum. As a philosopher, he has been claimed as one of the founders of modern science because of his placing of mathematics and experiment at the core of the philosophy of nature,[5] though his 'experiment' was empirical experience, in the sense of gathered evidence, including the accounts of others, rather than the later idea of *controlled* experiment.[6]

Grosseteste lived in an intellectual world that had long viewed the cosmos as the product of intelligence and purpose, and thus exhibiting 'measure and number and weight'.[7] Where Grosseteste seems to have been original in Christian thought is that he conceived of God as a mathematician, and the acts of creation as being mathematical. This produces an expectation of a sophisticated mathematical structure in the world.[8] Thus it is natural that he maintained that it is only through knowledge of 'lines, angles, and figures' that natural philosophy is possible.[9]

That beauty consisted of harmony of parts was accepted among scholastics in a qualitative sense. Grosseteste seems to have returned to mathematical notions of harmony.[10] He saw beauty in the created universe,[11] and his student Thomas of York (*fl.* 1255) at least drew a parallel with the beauty of the circle:

> 'The beauty of the world is explained in many ways: first, with regard to its Creator [...]; secondly, the shape of the circle bears

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[1] Verger, 'Patterns', pp. 62–3.

[2] *Ibid.*, p. 40.

[3] Kibre & Siraisi, 'The Institutional Setting', p. 129.

[4] Adelard of Bath, *De eodem et diverso*, pp. 48, 49.

[5] Crombie, *Robert Grosseteste*, esp. ch. v.

[6] McEvoy, *The Philosophy of Robert Grosseteste*, pp. 206–11.

[7] Wisdom 11:20

[8] *Ibid.*, pp. 214–15.

[9] Grosseteste, 'De lineis angulis', pp. 59–60; trans. C.

[10] Tatarkiewicz, *Hist. Aesthetics*, vol. II, p. 226.

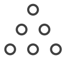
[11] McEvoy, *The Philosophy of Robert Grosseteste*, pp. 412–13; Tatarkiewicz, *Hist. Aesthetics*, vol. II, p. 227.

> witness to the beauty of the world […]; thirdly, the beauty of the world is proved by its component parts.’[1]

Grosseteste himself connected the perfection of the world with the number of days of creation as described in Genesis:

> ‘This brings us in conclusion to all that has gone before the perfection of heaven and earth and of all their adornment. For their perfection follows from the number six, in which number [of days] they were made. […] The perfection of the number six consists in that it is equal to its multiplicative parts added together. […] The universality of creation, the world, demands this perfection, i.e. its equality to itself and the absence of superfluity or diminution. […] the world had to receive this perfection from the number six. […]
>
> 2. The perfection of the world follows from the perfection of the number six as an effect from its cause.’[2]

Further, in this connection, considering 6 as a triangular number also fits with Genesis, for in the form



the first row corresponds to the creation of light; the second row to the creation of the firmament and the Earth; the third to the adornment of the world with stars, with birds and fish, and with animals.[3]

Grosseteste also adhered to the pythagorean tradition that 10 was ‘the perfect number’ as the sum of the first four numbers. But he gave it a new basis, in which the first four numbers correspond to the constituents of bodies. The number 1 corresponds to the form of a thing; 2 corresponds to matter, which has both density and the ability to take on a form; 3 corresponds to composition, which contains matter and form and what is distinctive about the composition; 4 corresponds to the composite proper, which is above these three constituents.[4]

> ‘From these considerations it is clear that ten is the perfect number in the universe, because every perfect whole has something in it corresponding to form and unity, and something corresponding to matter and duality, something corresponding to composition and trinity, and something corresponding to the composite and quaternity. Nor is it possible to add a fifth to these four. For this reason every perfect whole is ten.’[5]

Thus only the proportions found among the numbers 1, 2, 3, 4 can produce harmony in music and in movement.[6]

[1] Thomas of York, *Sapientale*, unedited MSS quot. in Pouillon, ‘La Beauté, Propriété Transcendantale’, p. 326; trans. Tatarkiewicz, *Hist. Aesthetics*, vol. II, p. 230.

[2] Grosseteste, *On the Six Days of Creation*, §§ IX.i.1–2 ⟦p. 269⟧; ins. in ed.

[3] *Ibid.*, § IX.i.2 ⟦p. 270⟧.

[4] Grosseteste, *On Light*, p. 17.

[5] *Loc. cit.*

[6] *Loc. cit.*

LEONARDO PISANO

Leonardo Pisano (c. 1170–after 1240), dubbed 'Fibonacci', was perhaps the greatest European mathematician between antiquity and the Renaissance, and the heritor of both the Greek and Islamicate traditions.[1] In his youth, he accompanied his father, a government official, to Béjaïa in the Almohad caliphate, where he was instructed in the use of Hindu–Arabic numerals. Mathematics appealed to him and he studied it further on later business travels around the Mediterranean.[2] He promoted the use of Hindu–Arabic numerals in his *Liber Abaci*, which brought him to the attention of the Holy Roman Emperor Frederick II, the *stupor mundi*, who was famed for seeking and promoting knowledge.

Leonardo's surviving work, which has been praised for its elegance,[3] shows both that he had an aesthetic appreciation of mathematics and in at least one respect connected aesthetic value with efficiency. This connection can be seen in Leonardo's repeated use of the adjective *pulcer* and related terms to describe choosing forms of fractions that aid calculation. In his translation of Leonardo's *Liber Abaci*, L. E. Sigler (1928–97) consistently preferred to render *pulcer* as 'elegant';[4] perhaps this captures better Leonardo's connection with computational ease than the more conventional translation 'beautiful'. Leonardo, for example, said that it is more elegant/beautiful to express a quantity of pounds and ounces in terms of ounces than to express it as pounds and fractions of pounds, for the latter alternative gives fractions with larger denominators.[5]

In his *De Practica Geometrie*, Leonardo called beautiful [*pulcra*] Archimedes' demonstration that the 'ratio of the circumference of a circle to its diameter is three and a seventh'.[6] Leonardo here referred to Proposition 3 in Archimedes' *Measurement of the Circle*, which states that[7]

$$3\tfrac{10}{71} < \pi < 3\tfrac{1}{7}.$$

Leonardo would have had access to this work, which had been translated by Gerard of Cremona (c. 1114–1187) from Arabic into Latin.[8] Archimedes' celebrated proof proceeds by calculating approximate ratios of the perimeters of 96-gons circumscribed about and inscribed in a circle to the diameter of that circle (see Figure 5.3), implicitly starting with dodecagons and repeatedly bisecting edges to obtain 24-, 48-, and then 96-gons (though Archimedes did not state his argument in these terms). The results of these calculations supply upper and lower bounds for the ratio of the circumference of a circle to its diameter.[9]

Leonardo's aesthetic evaluation of Archimedes' proof is important, for it seems to be the earliest extant description of a *proof* as beautiful in the European tradition. [Al-Nasawī (*fl.* 1029–44) had earlier described a proof as beautiful.*] Antiquity saw beauty first in

[1] Sigler, 'Leonardo Pisano', p. xx; cf. McClenon, 'Leonardo of Pisa', p. 8.

[2] Grimm, 'The Autobiography of Leonardo Pisano'.

[3] For example, McClenon, 'Leonardo of Pisa', p. 2.

[4] For example Fibonacci, *Liber Abaci*, pp. 194–5; cf. Leonardo, *Liber Abaci* (Orig. Latin), p. 126.

[5] Fibonacci, *Liber Abaci*, p. 194.

[6] Fibonacci, *De Practica Geometrie*, ch. 3, § 194, p. 154.

[7] Dijksterhuis, *Archimedes*, pp. 223–4.

[8] Clagett, *The Arabo-Latin Tradition*, p. 4.

[9] Dijksterhuis, *Archimedes*, pp. 224–9.

* ▸ p. 212

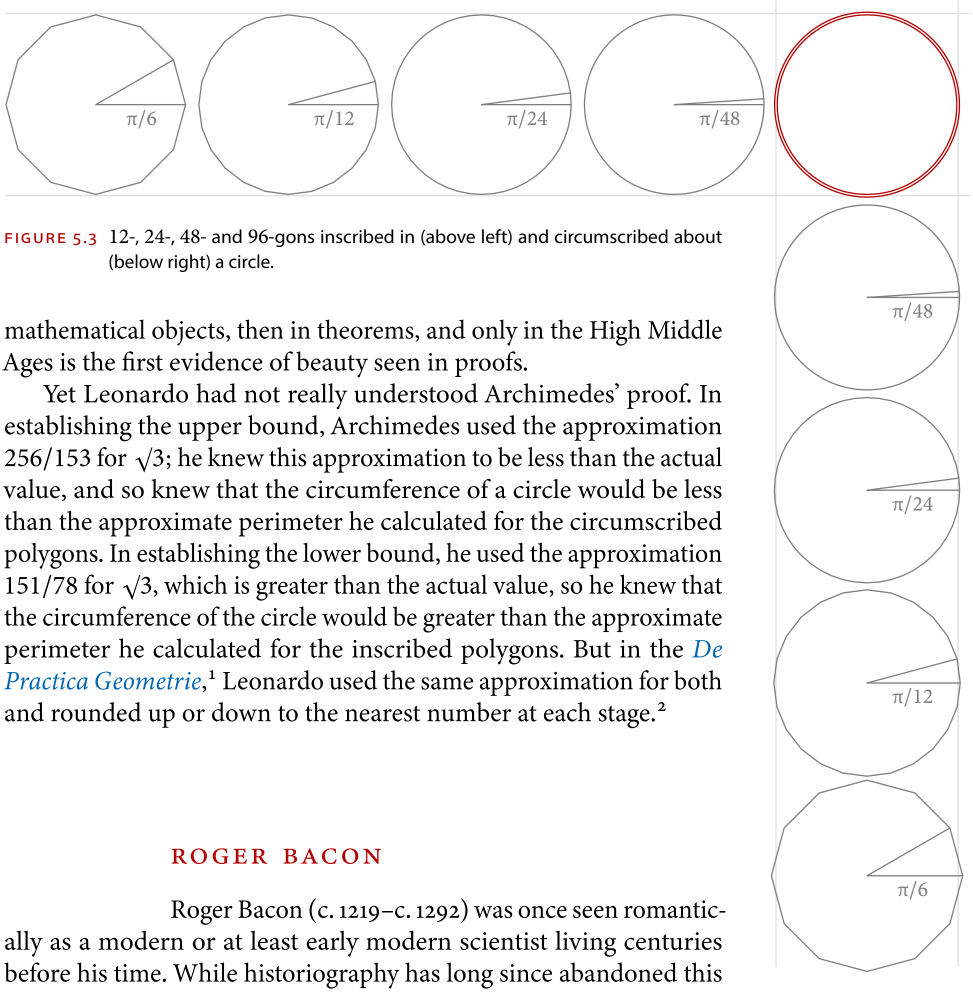


FIGURE 5.3 12-, 24-, 48- and 96-gons inscribed in (above left) and circumscribed about (below right) a circle.

mathematical objects, then in theorems, and only in the High Middle Ages is the first evidence of beauty seen in proofs.

Yet Leonardo had not really understood Archimedes' proof. In establishing the upper bound, Archimedes used the approximation 256/153 for $\sqrt{3}$; he knew this approximation to be less than the actual value, and so knew that the circumference of a circle would be less than the approximate perimeter he calculated for the circumscribed polygons. In establishing the lower bound, he used the approximation 151/78 for $\sqrt{3}$, which is greater than the actual value, so he knew that the circumference of the circle would be greater than the approximate perimeter he calculated for the inscribed polygons. But in the *De Practica Geometrie*,[1] Leonardo used the same approximation for both and rounded up or down to the nearest number at each stage.[2]

ROGER BACON

Roger Bacon (c. 1219–c. 1292) was once seen romantically as a modern or at least early modern scientist living centuries before his time. While historiography has long since abandoned this myth, Bacon was without doubt an important figure in the long gestation of the scientific method. He seems to have become interested in natural philosophy and mathematics when he first studied at Oxford, and he lectured at Paris on Aristotle's *Physics* and *Metaphysics*. But his real devotion to mathematics and science seems to date from his return to Oxford about 1247–8, when he met Grosseteste.[3]

Bacon thought that an ignorance of mathematics would preclude any understanding of the other sciences, while a knowledge of it prepared the mind for higher true understanding.[4] It was the 'the gateway and the key' to the higher sciences.[5] But more: he expressed great enthusiasm for mathematics.[6] His interest seems to have been derived in part from his faith in the vast utility of the subject; in his writing, he often applied mathematics far away from what would usually be considered mathematical topics.[7] And part of the motivation for his *Opus Tertium* was to demonstrate that mathematics was useful, since people who see no utility in mathematics tended to recoil from its study.[8]

[1] See Fibonacci, *De Practica Geometrie*, ch. 3, §§ 194–200, pp. 154–8.

[2] Clagett, *The Fate of the Medieval Archimedes* (Part III), p. 432.

[3] Crombie & North, 'Bacon, Roger', p. 377; Hackett, 'Roger Bacon', §§ 2, 4.2, 6.

[4] Bacon, *Opus Majus*, pt IV, first distinction, ch. 1 ⟦vol. I, pp. 97–8⟧.

[5] *Ibid.*, pt IV, first distinction, ch. 1 ⟦vol. I, pp. 97⟧; trans. C.

[6] Molland, 'Roger Bacon's Knowledge of Mathematics', p. 151.

[7] *Ibid.*, pp. 151–2.

[8] Bacon, *Opus Tertium*, ch. VI, p. 21.

But besides utility, Bacon seems to have seen beauty in mathematics. In his 'Geometria Speculativa', he wrote that

> 'demonstrations of works are stronger and more difficult than of bare truths, as appears in the *Book of Burning Mirrors*, which is one of the practical parts of geometry, which has the subtlest and most beautiful demonstrations [subtilissimas operum demonstrationes et pulcherrimas].'[1]

[The book to which Bacon referred was Ibn al-Haytham's* (965–c. 1040) *On Parabolic Burning Mirrors* [*Maqāla fī al-Marāyā al-muḥriqa bi-al-quṭū*ʿ مقالة في المرايا المحرقة بالقطوع].[2]] Thus Bacon saw beauty in mathematical demonstrations.

Also, at the start of part five of his *Opus Majus*, concerning *perspectiva*, which follows part four on mathematics and the applications of mathematics, Bacon wrote that 'if our deliberations to this point have been beautiful and delightful [pulchra et delectabilis], the matters now to be considered are far more beautiful and delightful.'[3] This may point to him seeing beauty in mathematics more broadly.

But perhaps there is nothing special about mathematics, for Bacon also attributed beauty to knowledge and wisdom generally: 'The unspeakable beauty of wisdom naturally attracts and holds our minds with full admiration.'[4] And other sciences, such as physiognomy, were beautiful.[5]

Bacon saw beauty and usefulness in knowledge; the beauty came from the fragmentary glimpse that knowledge affords of the whole work of God.[6] This same reason applied in mathematics: in the use of fractions and whole numbers, 'the beautiful lights of wisdom shine forth.'[7] And geometry is also connected with divine wisdom:

> 'Oh how the indescribable beauty of divine wisdom would shine and infinite profit would accrue, if this geometry, which is contained in the Scriptures, were put before our eyes as physical figures.'[8]

THOMAS BRADWARDINE

In the fourteenth century, a group of scholars at Oxford studied problems relating to motion and produced works whose influence on the development of science can be seen at least two hundred years later. These thinkers, known to history as the 'Oxford calculators', variously made contributions to science from natural philosophy, from logic, or, in the case of Thomas Bradwardine (1290/1300–1349), from mathematics.[9] For Bradwardine, mathematics was 'the revelatrix of untainted truth', without which it was futile to seek to understand natural philosophy.[10]

[1] Bacon, 'Geometria Speculativa', pp. 271–3.

* ▸ pp. 207–9

[2] See Ibn al-Haitam, 'Schrift über parabolische Hohlspiegel'.

[3] Bacon, *Opus Majus*, pt v, first distinction, ch. 1 ⟦vol. ii, p. 2⟧; trans. Power, *Roger Bacon*, p. 113.

[4] Bacon, *Compendium Studii Philosophiæ*, p. 393; trans. Easton, *Roger Bacon*, p. 76.

[5] Williams, 'Roger Bacon and the *Secret of Secrets*', p. 384.

[6] Easton, *Roger Bacon*, p. 179.

[7] Bacon, *Opus Majus*, pt iv, 'Mathematicae in divinis utilitas' ⟦vol. i, p. 220⟧, trans. C.

[8] Bacon, *Opus Majus*, pt iv, 'Mathematicae in divinis utilitas' ⟦vol. i, p. 211⟧; trans. Power, *Roger Bacon*, p. 158.

[9] Sylla, 'The Oxford Calculators', esp. pp. 548–55.

[10] Bradwardine, quot. in Murdoch, '*Mathesis in Philosophiam Scholasticam Introducta*', pp. 215–16.

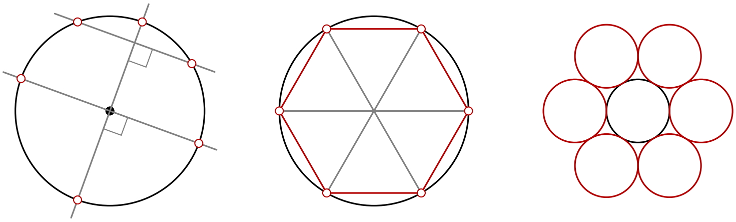


FIGURE 5.4
Results Bradwardine used as evidence for the perfection of the circle.

Bradwardine wrote a number of mathematical texts, the most important of which was his *Treatise on the relative speeds in motions* [*Tractatus de proportionibus velocitatum in motibus*], which considers the problem of relating the variation of the speed of a moving object to the variation of its causes, namely the forces and resistances acting upon it.[1] His *Speculativa geometry* [*Geometria speculativa*] and *Speculative arithmetic* [*Arithmetica speculativa*] seem to be elementary texts for students. The latter work appears to comprise the most basic parts of Boethian arithmetic.[2] The *Speculative geometry* includes some elementary subjects that are not covered in Euclid's *Elements*, and Bradwardine seems to have selected its contents on the basis of its possible relevance to philosophy.[3]

Most important here is his attitude to the circle in the *Speculative geometry*:

> '[The circle] is the first and most perfect of figures, the simplest and most regular, the most capacious and the most beautiful of figures'.[4]

This assertion is recognizable as part of a tradition that descended from Pythagoras. But Bradwardine then presented evidence, in the form of three conclusions that attested to the perfection of the circle: the construction to find the centre of a circle by bisecting a diameter found as the perpendicular bisector of a chord; that the intersections of six equally-spaced radii with the circumference define a regular hexagon; that exactly six circles of equal size can touch a given circle.[5] The perfection of the circle was thus linked to the perfection of the number 6: the construction involves six intersections with the circle; the hexagon is made up of six lines; the third result involves six outer circles (see Figure 5.4).[6]

LIBER DE CURVIS SUPERFICIEBUS

The *Liber de curvis superficiebus* [*Book of curved surfaces*] was an exposition of Archimedes' work on cones. It was attributed to Johannes de Tinumue, but it is not clear who he was or when he wrote.[7] The text itself probably existed in some form in late antiquity or the early Middle Ages,[8] and was certainly known to thirteenth-century scholars.[9]

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[1] Murdoch, 'Bradwardine, Thomas', p. 391.

[2] *Ibid.*, p. 395.

[3] *Loc. cit.*

[4] Bradwardine, quot. in Molland, 'An Examination of Bradwardine's Geometry', p. 143.

[5] *Loc. cit.*

[6] *Loc. cit.*

[7] Clagett, *The Arabo-Latin Tradition*, pp. 439–43.

[8] *Ibid.*, pp. 440–1.

[9] *Ibid.*, p. 444.

Its author — whether Johannes or not — recognized beauty and elegance in mathematics, and indicated that an elegant corollary could increase the beauty of a theorem:

> 'this theorem, sufficiently elegant and having been proposed with sufficient elegance, made beautiful [venustatur] by a sufficiently elegant corollary.'[1]

The verb *venustō* means 'to beautify' or 'to make beautiful'.

The theorem to which the author referred is:

THEOREM A. *'The lateral surface of any* [*right circular*] *cone is equal to a right triangle, one of whose two sides containing the right angle is equal to the slant height of the cone, while the other* [*is equal*] *to the circumference of the base.'*[2] $\boxed{\text{A}}$

The corollary to Theorem A is:

COROLLARY B. *'*[*T*]*he ratio of the lateral surface of a cone to its base is the same as that of its slant height to the radius of its base.'*[3] $\boxed{\text{B}}$

In modern notation, Theorem A says that the lateral surface area A_L of the cone (that is, the surface area not including the base) is given by $A_L = \pi r\ell$, where r is the radius of the base and ℓ is the slant length (the distance from the apex to any point on the circumference of the base). Corollary B says that $A_L/\pi r^2 = \ell/r$. When set down thus, the elegance that the author of the *Liber de curvis superficiebus* saw in the corollary fades into near-triviality.

The remark about elegance and beauty in the *Liber de curvis superficiebus* is important because it seems to be the earliest to include a *reason* for the beauty of a result: that it has an elegant corollary. In antiquity, there had been explanations of why the *objects* of mathematics are beautiful, but Apollonius was the only author to have claimed results as beautiful, and he said nothing explicit about why he judged them so.[*]

DANTE

The works of the supreme poet Dante Alighieri (1265–1321) show him to have been a very learned man. After the death of his beloved Beatrice in 1290, he sought solace in Boethius' *Consolation of Philosophy* and then in Cicero's *De Amicitia* [*On Friendship*]. This encounter with philosophy revealed to him fresh intellectual possibilities, and he began studying it in earnest and attended theological and philosophical schools, where he would have become acquainted with scholasticism.[4] He certainly became familiar, then or at another time, with Euclidean geometry, for in the *Commedia* he effectively stated that an angle in a semicircle must be right (which is part of

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[1] Johannes de Tinumue, *De curvis superficiebus*, p. 459.

[2] *Ibid.*, p. 451; insertions in ed.

[3] *Ibid.*, pp. 459–61.

[*] ▸ pp. 113–14

[4] Dante, *Convivio*, bk II, ch. 12; Mazzotta, 'Life of Dante', p. 7.

Proposition III.31 of the *Elements*; see Figure 5.5)[1] and implicitly referred to the result that the angles of a triangle are twice a right angle (which is part of Proposition I.32 of the *Elements*).[2]

In his creative choices as a poet, Dante looked both to the geometry of antiquity and to a selective and moderated version of the number symbolism of the mediaeval period. His works are ruled by symmetry and numerical exactitude.[3]

Number symbolism is evident from the very beginning of his 1294 autobiographical work *La Vita Nuova*, which describes his love for Beatrice and which frames an anthology of his early poetry. In this work, Dante associated Beatrice, whose earthly perfection he so admired, with the number 9, which had been one of the most meaningful of numbers in the mediaeval period:

> ‘Nine times the heaven of the light had revolved in its own movement since my birth […] the beginning of her ninth year […] When exactly nine years had passed […] It was exactly the ninth hour of day when she gave me her sweet greeting. […] the first of the last nine hours of the night.’[4]

And Dante made the same point more directly: ‘she herself was this number nine; I mean this as an analogy, as I will explain.’[5] The square root of 9 is 3, which is the number of the Trinity. Thus 9 was associated with Beatrice, because she was a miracle, and the source of miracles was the Trinity.[6] But for Dante the ‘perfect number’ was 10, as in the pythagorean tradition.[7]

It is hardly necessary to recall that Dante’s immortal *Commedia* has a threefold structure: the descent into the depths of hell in the *Inferno*, the climb up the mountain of *Purgatorio*, through both of which Dante is guided by Virgil at Beatrice’s bidding, and the ascent through and beyond the celestial spheres in the *Paradiso*, accompanied by Beatrice herself. But the numerical structure based upon 3 is repeated at different levels. The *Commedia* as a whole comprises 100 cantos, placed into the three canticles of *Inferno*, *Purgatorio*, *Paradiso*. If the first canto of the *Inferno*, before Dante’s journey begins, is taken to be an introduction to the work as a whole, then the division is 1 + 33 + 33 + 33: three equal parts and a prolegomenon. Each stanza has three lines, the metre is hendecasyllabic; thus each stanza has 33 syllables. The numbers 3 and 9 recur often throughout the work, such as in the nine spheres of heaven, and even in the nine circles of hell, which pointed to the divine handiwork behind it (‘JUSTICE MOVED MY MAKER ON HIGH. / DIVINE POWER MADE ME’[8]).

In the Middle Ages, there was a widely-recognized association of the seven liberal arts to the seven planets of the geocentric cosmos.[9] In his *Convivio*, Dante described the association:[10]

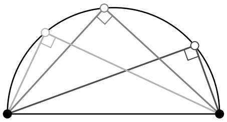

FIGURE 5.5
An angle in a semicircle must be right.

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[1] Dante, *Commedia*, *Para.* XIII, ll. 101–2.

[2] *Ibid.*, *Para.* XVII, ll. 14–15.

[3] See, for example, Curtius, *European Literature and the Latin Middle Ages*, § 17.5.

[4] Dante, *La Vita Nuova*, pp. II–III.

[5] *Ibid.*, XXIX, ll. 18–19.

[6] *Ibid.*, XXIX, ll. 20–9.

[7] *Ibid.*, XXIX, l. 8.

[8] Dante, *Commedia*, *Inf.* III, ll. 4–6.

[9] Klibansky, Panofsky & Saxl, *Saturn and Melancholy*, p. 333.

[10] Dante, *Convivio*, bk II, ch. 14.

Trivium	grammar	dialectic	rhetoric	
	Moon	Mercury	Venus	
Quadrivium	arithmetic	music	geometry	astronomy
	Sun	Mars	Jupiter	Saturn

Dante then gave an explanation for each pairing in the form of a description of two properties through which the planet resembles the corresponding art. In particular, the association of the Sun with arithmetic was

> ‘because of two properties: one is that all the other stars are informed by its light; the other is that the eye cannot look at it. And these two properties are found in Arithmetic: for by its light all sciences are illuminated, because all their subjects are considered under some numerical aspect, and in considering them we always proceed by number.’[1]

Take this passage in conjunction with the *Commedia*, whose entirety can be seen as a journey towards light, where light symbolizes goodness and mind, where the Empyrean beyond the heavenly spheres is ‘pure light, / light intellectual [pura luce: / luce intellettüal]’.[2] The combination is a hint, and no more than a hint, that Dante may have considered the whole of arithmetic to be beautiful.

Turning to the pairing of Jupiter with geometry, Dante wrote that Jupiter moves between Mars and Saturn, which are ‘antithetical to its fine temperance’, just as geometry moves between the point and the circle. In this context, the circle, meaning ‘anything round, whether a solid body or a surface’,[3] was ‘the most perfect figure’.[4] [Dante incorrectly attributed the idea of the circle as perfect to Euclid,[5] who appears in the *Commedia*, dwelling with the other virtuous pagans in Limbo.[6]]

The cosmos of the *Commedia* is a bounded, finite universe built out of nested circles and spheres. It can be interpreted as being a 3-sphere (that is, a three-dimensional surface comprising points in four-dimensional space equidistant from a centre).[7] While it is unlikely that Dante had such a precise mathematical concept in mind, he seems to have genuinely wanted to portray something *more* than a sphere. In the *Commedia*, the Earth sits at the centre of the revolving spheres of the planets, the fixed stars, and the *primum mobile* that impels them. But the *primum mobile* is itself surrounded by the Empyrean, which, in an apparent geometrical impossibility, also comprises a sphere around nine concentric circles of angels turning about God.[8]

And the circle has another key imaginative role in the *Commedia*: at the end of his journey, Dante gazes upon God. He sees three circles with equal size, differing only in colour, reflecting each other; these circles are — or are at least how his mortal mind perceives — the Trinity.[9]

The difficulty of comprehending the final vision in the *Paradiso* is compared to the difficulty of squaring the circle:[10] understanding

[1] Dante, *Convivio*, bk ii, ch. 14.

[2] Dante, *Commedia*, *Para.* xxx, ll. 39–40.

[3] Dante, *Convivio*, bk ii, ch. 13.

[4] *Loc. cit.*

[5] *Ibid.*, ch. 13.

[6] Dante, *Commedia*, *Inf.* iv, l. 142.

[7] Peterson, *Galileo’s Muse*, ch. 3.

[8] Dante, *Commedia*, *Para.* xxviii, ll. 13–78.

[9] *Ibid.*, *Para.* xxxiii, ll. 115–120.

[10] *Ibid.*, *Para.* xxxiii, ll. 133–38.

the perfect circle using the straight line is analogous to processing the perception of God using human thought; both the straight line and human thought are inadequate to the assigned tasks.[1]

Dante's view of beauty was scholastic. Besides sensory beauty, there was spiritual beauty, and the only perfect beauty was God's:[2] 'Divine Goodness [...] unfolds Eternal Beauty'.[3] The association of circles and spheres with God and what God created, in addition to the direct assertion of perfection, can thus be explained simply: Dante saw them as beautiful.

◆

[1] Cf. Herzman & Towsley, 'Squaring the Circle', pp. 112–15.

[2] Tatarkiewicz, *Hist. Aesthetics*, vol. II, p. 279.

[3] Dante, *Commedia*, *Para.* VII, ll. 64–6.

6

Splendour Most Elevated

Beauty and mediaeval Islamicate mathematics

> ‘There cannot exist beauty or splendour more elevated
> than where the essence is exclusively intellectual and pure good,
> free from any kind of imperfection and unique in all its aspects.’
>
> — Ibn Sīnā, *Kitāb al-Najāt*, p. 281
> (trans. Valérie Gonzalez, *Beauty and Islam*, p. 14).

> ‘The splendor, perfection, and beauty
> he saw in the essence of that sphere
> were too magnificent to be described’
>
> — Ibn Ṭufayl, *Hayy Ibn Yaqzān*, p. 152
> (trans. L. E. Goodman).

When Heraclius (c. 575–641 CE) was crowned at Constantinople in 610 CE, the Byzantine Empire was imperilled by the Avars and the Slavs in the north and the Persians in the east. Control over the Balkans was gradually lost and war with the Persians to-ed and fro-ed. At the Byzantine nadir, an Avar-led army besieged the imperial city while Persian soldiers stood on the opposite shore of the Bosphorus. In time, having made a truce with the Avars and an alliance with the Türkic khaganate that dominated the western part of central Asia, Heraclius achieved a victory over the Persian Empire in 628 CE that on paper restored the *status quo ante bellum*; in reality, both states were exhausted and vulnerable.[1] But for internal strife aggravated by climactic change, the Türks might have profited from the weakness of their Byzantine comrades and Persian foes.[2] Instead, a new power arose.

By the time of his death, Muḥammad (c. 570–632 CE) had united the tribes of Arabia under the banner of Islam, and his first successors, the Rashidun caliphs (632–61 CE), conquered the whole of the Persian Empire and much of the Byzantine: thus were united, for the first time since Alexander, the ancient cities of Egypt, Judea, Mesopotamia, and Persia. After its expansion under the Umayyad caliphs (661–750 CE) the Muslim world stretched from the Punjab to the Pyrenees; ‘two hundred days’ journey from east to west’.[3]

This continent-spanning domain did not remain under a single political or religious leadership. Disunity lurked even during the

[1] Louth, ‘The Byzantine Empire in the seventh century’, pp. 294–6.

[2] Sinor, ‘The establishment and dissolution of the Türk empire’, pp. 309–10; Frankopan, *The Earth Transformed*, ch. 11.

[3] Gibbon, *The Decline and Fall of the Roman Empire*, ch. LI.

golden age of the ʿAbbāsid caliphate, when there was an extraordinary cultural and scientific efflorescence centred on the new capital at Baghdad. When al-Saffāḥ (*fl.* 721/2–754 CE; r. 750–4 CE) became the first of the ʿAbbāsid caliphs, an Umayyad emir in al-Andalus maintained his independence and claimed to be the legitimate caliph. Later, other rival caliphates emerged, the greatest being the Fāṭimid caliphate, which dominated north Africa.

Nevertheless, the whole area remained culturally under the influence of Islam, and the usual language of learned writing was Arabic, with its prestige as the language in which the Qurʾan had been set down. But the 'Islamic world' was not monolithically Islamic, or Arabic. Christians and Jews lived there, and scholars might write in Syriac or Persian or Hebrew. Thus the historian Marshall Hodgson (1922–68) coined for it the term *Islamicate*, on the model of *Italianate*: in each word, the doubly-adjectival ending indicates that it refers not to the culture itself, but to the whole sphere that felt its influence.[1]

Scholarship in general and mathematics in particular were valued highly throughout the mediaeval Islamic period. Geometry was esteemed in part because of its relationship to calligraphy.[2][*] But geometry was not just important: it was *prestigious*. Rulers were known to boast of their geometrical prowess.[3] Al-Mu'taman, king of the taifa of Zaragoza (d. 1085, r. 1081–5), was credited with writing, probably before his accession, a wide-ranging mathematical text, apparently a compendium addressed to those cultivated in mathematics rather than to researchers.[4]

The Islamicate world inherited from Greek thought the idea that beauty was grounded in proportion, and this idea sat in the background of all contemporaneous discussions of philosophical aesthetics.[5] Perhaps the only constant in the aesthetics of this period was the association of beauty and pleasure.[6] This connection does not exclude notions such as moral beauty, for pleasure can be caused and experienced in non-sensory ways: as al-Fārābī put it, '"pleasurable" means nothing other than that one is apprehending most excellently a most excellent object of apprehension'.[7] Intellectual beauty could also fit within this approach, and intellectual and intelligible beauty form a recurring theme. But direct philosophical engagement with *mathematical* beauty was rare: it was compatible, and was certainly not excluded, but only occasionally was it explicitly discussed.

Thus, while this chapter examines the limited role of mathematics in philosophers' considerations of beauty, the main focus is on the attitude of mathematicians to mathematical beauty. The quite distinct problem of the relationship between Islamic geometrical art and mathematical beauty will be addressed in the next chapter.

[1] Hodgson, *The Venture of Islam*, vol. I, p. 59.

[2] Bier, 'Geometry in Islamic Art', p. 2060.

[*] ▸ pp. 219–20

[3] Necipoğlu, *The Topkapı Scroll*, p. 132.

[4] Rashed, *Hist. Arabic Sci. Math.*, vol. 1, p. 721–2, 726.

[5] Behrens-Abouseif, *Beauty in Arabic Culture*, p. 37.

[6] *Ibid.*, p. 42.

[7] al-Farabi, *Philosophy of Plato and Aristotle*, p. 73.

THE GREEK HERITAGE AND THE TRANSLATION MOVEMENT

Around the middle of the eighth century CE there emerged in the Islamicate world a movement that worked to translate Greek texts into Arabic. This tendency had wide support from most of the societal elite, regardless of ethnic or religious divisions,[1] and benefitted from state sponsorship. The caliph al-Manṣūr (714–75 CE; r. 754–75 CE), the second of the ʿAbbāsid dynasty, moved the capital east from Damascus to a newly-founded city at Baghdad, and desired to portray his dynasty as the true heirs of the House of Sasan, which had ruled recently-conquered Persia. He thus adopted Sasanian imperial ideology, which had included a translation movement based upon the notion that all knowledge derived from the Zoroastrian canon, and thus that translation was the recovery of lost ancient knowledge.[2]

The *bayt al-ḥikma* [بيت الحكمة], the celebrated House of Wisdom, whose founding is often attributed to the caliphate,[3] was likely a part of the Sasanian administration which was adopted by the ʿAbbāsid caliphs. Although it has often been characterized as a research centre, its main function was to translate Persian texts into Arabic and to house the results.[4] Nevertheless, the *bayt al-ḥikma* was an exemplar and an institutionalization of translation culture.[5]

The seventh ʿAbbāsid caliph, al-Ma'mun (786–833 CE; r. 813–33 CE), had come to power as a result of a civil war and consequently needed to secure his position. One pillar of his strategy was to establish himself as the champion of Islam; thus he made war against the Christian Byzantine Empire.[6] In a *prima facie* paradoxical way, the ideological justification for this war involved philhellenism: the Byzantines were portrayed as culturally inferior both to their Greek ancestors and their Muslim opponents. They were claimed to have rejected the philosophy and science of ancient Greece because of Christianity, whereas the Muslims had embraced them. The translation movement was rooted in and served to reinforce this admiration and appreciation for ancient Greek thought.[7] The scholar al-Jāḥiẓ (776–868/9 CE), who was recruited as a propagandist by al-Ma'mun, listed some of the greatest Greek authors — Aristotle, Ptolemy, Euclid, Galen, Democritus, Hippocrates, Plato — pointing out in turn that each was 'neither Byzantine nor Christian'.[8]

This political motivation for the translation movement would have had effects upon its scholarly outputs. There would have been a desire to present a unified version of Greek philosophy and science, and little reason to draw attention to the debates that characterized Greek thought. In particular, there was an insistence upon an essential harmony between platonism and aristotelianism.[9]

By the time the movement's activity declined towards the end of the tenth century, virtually all secular, non-literary, non-historical, Greek texts that were available in the Near East had been translated into Arabic.[1] Essentially all works of ancient philosophy and mathematics that had survived to the late Hellenistic period were still extant in this time and were thus translated.[2]

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[1] Gutas, *Greek Thought, Arabic Culture*, pp. 2, 5.

[2] *Ibid.*, pp. 42–3.

[3] *Ibid.*, p. 55.

[4] *Ibid.*, pp. 56–9.

[5] *Ibid.*, p. 59.

[6] *Ibid.*, pp. 75–82.

[7] *Ibid.*, pp. 83–6.

[8] Quot. in *ibid.*, p. 87.

[9] Adamson, 'The Arabic Plotinus', p. 4.

During the flourishing of the translation movement, the ʿAbbāsid caliphs seem to have adopted Muʿtazilism as the basis of their ideology. The Muʿtazilah was not an organized movement, but a disparate group who introduced into Islamic thought the application of the methods of Greek philosophy to theological problems. The official approval of Muʿtazilism doubtless further aided in the acceptance of Greek philosophy into Islamic culture.[3] And such acceptance was not guaranteed, for philosophy was seen as a cultural import, as indicated by the Arabic word for philosophy being *falsafa* [فَلْسَفَة], a loan-word from the Greek *philosophia* [φιλοσοφία].[4]

MATHEMATICAL HIGHLIGHTS

Before delving into what philosophers and mathematicians of the period said about mathematical beauty, it seems worthwhile to set out some context by touching upon some of the key achievements of mathematics in the Islamicate world.

Geometry

Geometers stood upon the shoulders of Euclid and Archimedes and Apollonius: there were over fifty Arabic-language commentaries on Euclid's *Elements* alone.[5] The question of the status of Euclid's parallel postulate[*] received particular attention. There were attempts to prove it in the ninth century, including two by so eminent a mathematician as Thābit ibn Qurra[†] (836–901 CE). Later, Ibn al-Haytham[‡] (965–c. 1040) tackled the problem; his second try was the first that was logically sound, in the sense that it gave a valid proof based on Euclid's other postulates *and* a different axiom in place of the parallel postulate. Al-Khayyāmī (1048–1131) — also known as Omar Khayyám, of *Rubáiyát* fame — later gave another such proof.[6]

Much work was done on plane and solid geometry, including new analyses of regular polygons. The construction of the regular heptagon was a focus of research, for there was a feeling that it had not been successfully constructed by the Greeks. The kudos that would accrue to the one who claimed this prize led to a dispute between al-Sijzī (c. 945–c. 1020) and al-Jūd (*fl.* 10th century CE) over their respective claims.[§]

[1] Gutas, *Greek Thought, Arabic Culture*, p. 1.

[2] Rosenthal, *The Classical Heritage in Islam*, p. 10.

[3] *Ibid.*, pp. 4–5.

[4] Adamson, *Al-Kindī*, p. 21.

[5] Hogendijk, 'Mathematics in Islam', p. 2787.

[*] ▸ p. 99

[†] ▸ p. 197

[‡] ▸ pp. 207–9

[6] Rosenfeld, 'Geometry in Islamic Mathematics', pp. 2067–8.

[§] ▸ pp. 209–10

Astronomy

In the Islamicate world, astronomy was counted as one of the mathematical sciences, just as it had been in Greece. Astronomy was one of the main fields to which theoretical mathematics was applied; 'pure' mathematics and astronomy were — at least historiographically — inseparable.[1]

Astronomers of the time were followers of Ptolemy, but not uncritical ones. Ibn al-Haytham and al-Ṭūsi (1201–74) both castigated Ptolemy for betraying the principle that a circular motion should be seen as uniform from its own centre. This objection, which was arguably partly aesthetic, was aimed at the equant.[2]* Eccentrics and epicycles could be tolerated, but the equant too great a step.[3] One way out was what was later called the *al-Ṭūsi couple* after its discoverer: a point on a circle of radius r rolling around the inside of a circle of radius $2r$ moves back and forth along a diameter of the larger circle (see Figure 6.1); such oscillatory motion could help eliminate equants, while still being based upon uniform circular motion.

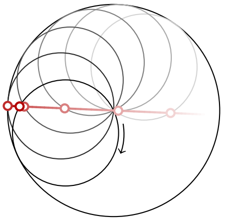

FIGURE 6.1
An al-Ṭūsi couple.

Algebra

The very word *algebra* is derived from the Arabic term *al-Jabr* [الجبر], signifying 'restoring', from the name of al-Khwārizmī's (before 800–after 847 CE) *Compendious Book on the Calculation of Restoring and Balancing* [*al-Kitāb al-Mukhtaṣar fī Ḥisāb al-Jabr wal-Muqābalah* الكتاب المختصر في حساب الجبر والمقابلة], a manual on solving quadratic equations.[4] Mathematicians were quick to adopt the algebraic language of al-Khwārizmī,[5] and in the eleventh and twelfth centuries the theoretical extension of algebra led to its reconceptualization as 'generalized arithmetic', where techniques from arithmetic were applied to unknown quantities.[6] But algebra was still hampered by the necessity of writing out expressions in words; modern algebraic notation was still in the distant future.[7]

Another thread in the development of algebra was the solution of cubic equations using conic sections. Drawing on the work of Archimedes, several eleventh-century mathematicians had seen the connection between the algebraic and geometric concepts. Al-Khayyāmī was disappointed that he could not find algebraic methods for solving cubic equations such as al-Khwārizmī had given for quadratic equations, but he presented a classification of the cubic equations and described the conic sections that could be used to solve equations of each kind.[8]

Magic squares

A magic square of order n is an $n \times n$ array containing the numbers $1, 2, \ldots, n^2$ in which the entries in each row, column, and main diagonal have the same sum, which a simple argument shows

---

[1] Hogendijk, 'Mathematics in Islam', p. 2788.

[2] Pedersen, *Early Physics and Astronomy*, pp. 239–40; Sabra, 'An Eleventh-Century Refutation of Ptolemy's Planetary Theory'.

* ▸ p. 118

[3] Ragep, intro. to al-Ṭūsī, *Memoir on Astronomy*, p. 48.

[4] Rashed, *Classical Mathematics From Al-Khwārizmī to Descartes*, pp. 107–9.

[5] Katz & Parshall, *Taming the Unknown*, p. 147.

[6] *Ibid.*, pp. 158–62.

[7] *Ibid.*, p. 164.

[8] *Ibid.*, pp. 166–7.

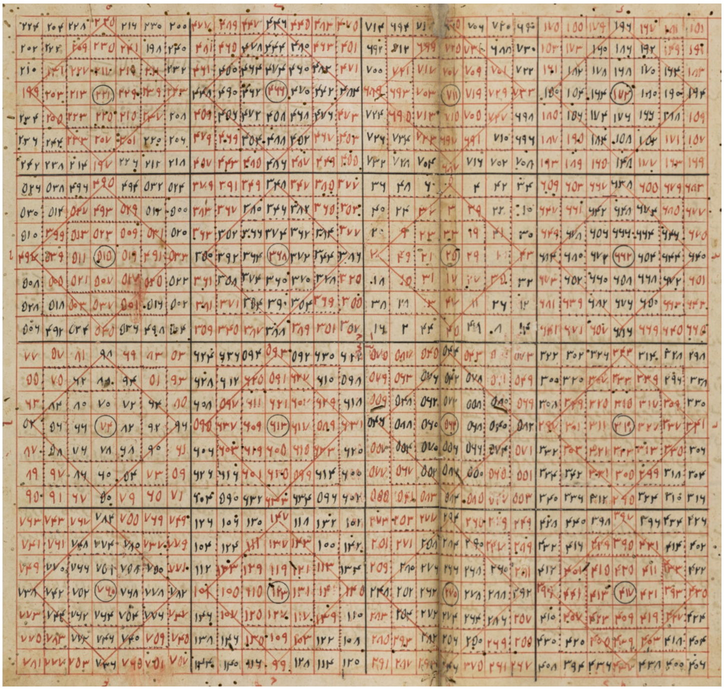

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FIGURE 6.2
A composite magic square of order 28, from an anonymous treatise written in the fifteenth or sixteenth century CE.

must be $n(n^2+1)/2$. Examples of magic squares of orders 3, 4, and 5 are:

	4	9	2	→ 15
	3	5	7	→ 15
	8	1	6	→ 15
↙ 15	↓ 15	↓ 15	↓ 15	↘ 15

	2	11	13	8	→ 34
	5	16	10	3	→ 34
	12	1	7	14	→ 34
	15	6	4	9	→ 34
↙ 34	↓ 34	↓ 34	↓ 34	↓ 34	↘ 34

	11	24	7	20	3	→ 65
	4	12	25	8	16	→ 65
	17	5	13	21	9	→ 65
	10	18	1	14	22	→ 65
	23	6	19	2	15	→ 65
↙ 65	↓ 65	↓ 65	↓ 65	↓ 65	↓ 65	↘ 65

Historical evidence places magic squares of order 3 in first-century CE China[1] and mid-sixth century CE India.[2] The first mention by an Islamicate author is in the *Book of Balances*, attributed to Jābir ibn Ḥayyān (*fl.* late 8th century–early 9th century? CE), which ascribed the given magic square to the neopythagorean Apollonius of Tyana (*fl.* 1st century CE),[3] but this attribution seems to be spurious.[4] Thābit ibn Qurra referred to a magic square in Nicomachus' (*fl.* c. 100 CE) *Introduction to Arithmetic*,[5] but neither that book nor its Arabic translation contains any magic squares;[6] there is, however, a possible allusion to a magic square by Iamblichus (c. 250–c. 330 CE).[7]

Islamicate mathematicians developed the theory of magic squares to a high degree of sophistication (see Figure 6.2). The Ikhwān al-Ṣafā'* considered magic squares of order greater than 3; while they were very unlikely to have been the first to do so, they seem to be the earliest whose writings are extant.[8]

[1] Ho, 'Magic Squares in China', p. 2599.

[2] Hayashi, 'Magic Squares in India', p. 2600.

[3] Djâber, *Le Livre des Balances*, pp. 150–1.

[4] Cammann, 'Islamic and Indian Magic Squares I', pp. 189–90.

[5] Thābit, quot. in Sesiano, *Magic Squares*, pp. 8–9.

[6] *Ibid.*, p. 9, n. 5.

[7] See Vinel, 'Un Carré Magique Pythagoricien?'

* ▸ pp. 200–3

[8] Hallum, 'New Light on Early Arabic *Awfāq* Literature', p. 86.

In mediaeval Arabic, the equal sum of each row, column, and diagonal was often simply called the *harmony* [*wafq* لوفق] or ‘harmonious number’ [*ʿadad al-wafq* العدد الوفق].[1] The adjective *wafq* has the sense of *harmonious* or *fitting*, which hints at aesthetic appreciation. But magic squares were not studied solely for their mathematical interest or for aesthetic reasons, but also because they were *magic*. This terminology, common in European languages (*mágico*, *magique*, *magisches*, *magisk*, *magiczny*, …), comes from the road by which they were introduced to European thought: a treatise written by the astronomer Ibn al-Zarqālluh (al-Zarqālī/Azarquiel; d. 1100) on their use in talismanic magic.[2] The dual nature of magic squares was long-lasting: the sixteenth-century Ottoman chronicler Taşköprüzāde (1494–1561) classified the study of magic squares as both mathematical and magical.[3]

THE ‘ARABIC PLOTINUS’

The story of mediaeval Islamicate philosophers’ examination of beauty begins with a treatise known as the *Theology of Aristotle* [*Uthūlūjiyā Arisṭū* أثولوجيا أرسطو], which is actually an Arabic version of parts of Plotinus’ (c. 204–270 CE) fourth, fifth, and sixth *Enneads*.[*] The text is somewhere between a translation and a paraphrase, with numerous interpolations of original material.[4] The *Theology*, together with a shorter text and some fragments, were based upon a lost earlier Arabic paraphrase of Plotinus, which may or may not have been directly adapted from the original Greek text.[5]

In the system of the *Theology of Aristotle*, the human soul was mid-way between the physical world and a higher intellectual one. This intellectual world was beautiful; indeed, the fourth book of the *Theology* is entitled ‘On the nobility and beauty [*ḥasana* حسنه] of the world of the intellect’. Using the intellect, the human soul could overcome the limitations imposed by the body, which was part of the physical world, and perceive the beauty of the intellectual world. The sensible physical world imitates the intellectual world; thus the beauty of every thing in the physical world descends from the intellectual world.[6] This division between sensible and intellectual beauty is the well-spring of a neoplatonist current that runs throughout mediaeval Islamicate aesthetics.

One passage from the *Theology* paraphrases the part of the eighth treatise of the fifth ennead in which Plotinus *may* have pointed to beauty in mathematics:[†]

> ‘We say that we find the beautiful form not in bodies, like the mathematical forms, for they are not bodily, but are simply figures having linearity’.[7]

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[1] Hallum, ‘New Light on Early Arabic *Awfāq* Literature’, p. 88, n. 80.

[2] Samsó, ‘al-Zarḳālī’, p. 462; Hallum, ‘New Light on Early Arabic *Awfāq* Literature’, p. 61.

[3] Melvin-Koushki, ‘Powers of One’, p. 175, inc. n. 171.

[*] ▸ pp. 140–3

[4] Adamson, ‘The Arabic Plotinus’, pp. 9–10.

[5] *Ibid.*, p. 12.

[6] Puerta Vílchez, *Aesthetics in Arabic Thought*, pp. 575–7.

[†] ▸ p. 143

[7] Quot. in Adamson, ‘The Arabic Plotinus’, p. 117.

The unknown paraphraser, therefore, seems to have interpreted Plotinus' *tois mathēmasi* as referring to mathematics, and not to 'intellectual pursuits' more broadly. But the paraphraser also shifted the meaning, from beauty in mathematics generally to beauty in mathematical objects, and seemingly specifically in geometrical objects.

AL-KINDĪ

Al-Kindī (c. 801–c. 866 CE), although remembered as 'the Philosopher of the Arabs',[1] is better thought of as a polymath. He had a broad view of the sciences, and aimed to build on and complete the inheritance of Greek science.[2] Nevertheless, he seems to have been the first thinker in the Islamic tradition to self-consciously identify himself as a practitioner of philosophy — a *faylasūf* [فَيْلَسُوف] — and so as an intellectual heir of the Greek philosophical tradition.[3] Al-Kindī's aim was to bring philosophy into Islamic culture, not as an alternative to indigenous intellectual disciplines, such as Qur'anic exegesis [*tafsīr* تَفْسِير], Islamic jurisprudence [*fiqh* فِقْه], or theology [*kalām* كَلَام] but as a new means to the same ends.[4] He had a thoroughgoing and lasting influence within the Islamic philosophical tradition, in the sense that he shaped it. Yet there is limited, and often critical, direct engagement with his thought by later Islamic philosophers.[5]

Mathematics, for al-Kindī, was vital to the philosophical and scientific enterprise. He aimed to come as close as possible to mathematical rigour in his argumentation.[6] He wrote a treatise, no longer extant, 'on the subject that philosophy cannot be acquired except with a knowledge of mathematics',[7] and in his surviving work 'On the Quantity of Aristotle's Books' wrote that a 'perfect philosopher' should read the works of Aristotle after having learned the mathematical sciences, which were a necessary prelude.[8] It may have been al-Kindī's interest in mathematics that led him to engage with philosophy.[9]

For al-Kindī, the study of quantity involved two sciences: arithmetic, which concerned quantity in itself, including calculation; and harmonics, which concerned the relation between numbers. The study of 'quality' likewise involved two sciences: geometry, which concerned 'stable' (that is, unmoving) quality; and astronomy, which concerned moving quality and the form of the universe.[10] These four subjects made up the quadrivium,* though al-Kindī's division seems to have descended from that given by Nicomachus† (*fl.* c. 100 CE) and Proclus‡ (410/12–485 CE) and attributed by the latter to the pythagoreans.

Each of the four sciences of quantity and quality sat in a hierarchy 'with respect to order, to the methods it uses [to reach] its end, and to the precedence it deserves'.[11] The first — most fundamental — level is arithmetic, for the other four sciences depend on the concept of num-

[1] al-Nadīm, *The Fihrist*, p. 615.

[2] Jolivet & Rashed, 'al-Kindī', pp. 263–4.

[3] Klein-Franke, 'Al-Kindī', p. 165.

[4] Adamson, *Al-Kindī*, pp. 44–5.

[5] *Ibid.*, p. 12.

[6] Rashed, 'Al-Kindī's Commentary', pp. 8–9; Gutas, *Greek Thought, Arabic Culture*, p. 120.

[7] al-Nadīm, *The Fihrist*, p. 616.

[8] al-Kindī, 'On the Quantity of Aristotle's Books', § IV.1.

[9] See Gutas, 'Geometry and the Rebirth of Philosophy'; Endress, 'Mathematics and Philosophy', esp. pp. 127–31.

[10] al-Kindī, 'On the Quantity of Aristotle's Books', § VII.2.

* ▸ p. 148
† ▸ pp. 138–9
‡ ▸ p. 150

[11] *Ibid.*, § VII.3; insertion in ed.

ber; the second is geometry, for its method of demonstration; the third is astronomy, 'which is composed from arithmetic and geometry'.[1]

> 'The fourth is harmonics, which is composed from arithmetic, geometry, and astronomy. For harmony is in everything, though it is most obvious in sounds, the composition of the universe, and of human souls'.[2]

The important points are these: harmonics includes the other three mathematical sciences, and harmony is seen everywhere.

In seeing this mathematical harmony throughout the universe, al-Kindī was certainly influenced by the pythagorean tradition, with which he was certainly familiar, at minimum via Nicomachus' *Introduction to Arithmetic*, a translation of which he corrected — in the sense of polishing the Arabic text — and glossed.[3] It is now accepted that al-Kindī could not read Greek, but did direct those in his circle to undertake translations.[4] He certainly had access to Arabic summaries of some of Plato's dialogues[5] and in particular may have read a translation of the *Timaeus*.[6]

Unlike Plato in the *Timaeus*, al-Kindī saw the five regular solids as made up of the shapes of their faces: squares for the cube; equilateral triangles for the tetrahedron, octahedron, and icosahedron; pentagons for the dodecahedron. He thus ignored Plato's fairest triangles, isosceles and scalene, from which the equilateral triangles and square faces were made up.[7][*]

Al-Kindī thought that the element earth corresponded to the cube because the cube is the unique regular solid whose faces have an even number of edges; even numbers are associated to femininity, 'the feminine is below (*taḥta*) the masculine', and earth is at the bottom — that is, the centre — of the universe.[8] The association of 'even' with 'feminine' goes back to the 'table of ten opposites' that Aristotle ascribed to certain pythagoreans.[†]

The cube is unique among the regular solids in having a perfect number of faces; the others have numbers of faces that are either deficient (4, 8) or abundant (12, 20). Thus the cube was also related to earth because it has 6 faces, and 6 is a 'perfect and stable number', reflecting the stability of earth at the centre of the cosmos.[9] Presumably al-Kindī did not feel it necessary to note explicitly that 6 was the number of days of the creation.[10][‡]

The ratio 12 : 6 is an octave, and so the 12-sided solid, the dodecahedron, corresponded to the heavens, which are most distant from the Earth.[11] But there was another, more general, numerical measure of distance, based upon 'how far' a number is from being a perfect number. Formally, this measure is the difference between a number n and the sum of divisors of n, divided by n (in modern notation, $|n - \sigma(n)|/n$). List the five regular solids in descending order of this measure applied to the number of faces, and list the elements by their natural levels (in descending order: the heavens, fire, air, water, and earth). The resulting pairing again associates the dodecahedron to the

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[1] al-Kindī, 'On the Quantity of Aristotle's Books', § vii.3.

[2] *Loc. cit.*

[3] Adamson, *Al-Kindī*, p. 173, inc. n. 27.

[4] *Ibid.*, pp. 25–6.

[5] *Ibid.*, p. 26.

[6] *Ibid.*, p. 176.

[7] al-Kindī, 'On Why the Ancients', §§ 6, 9.

[*] ▸ pp. 65–7

[8] *Ibid.*, § 6; parentheses in trans.

[†] ▸ pp. 46–7

[9] *Ibid.*, § 6.

[10] Qur'an 11:7, 32:4.

[‡] ▸ pp. 131, 165

[11] *Ibid.*, § 7.

	Regular solid	Number of faces n	'Imperfection' $\|n - \sigma(n)\|/n$	Element
	Dodecahedron	12	1/3	Heavens
	Tetrahedron	4	1/4	Fire
	Octahedron	8	1/8	Air
	Icosahedron	20	1/10	Water
	Cube	6	0	Earth

TABLE 6.1 Al-Kindī's explanation of the association of elements to regular solids, by ordering the regular solids by their 'imperfection' — that is, 'how far' the number of faces is from being a perfect number — and by ordering the elements by their natural levels.

heavens, the cube to earth, the tetrahedron to fire, the octahedron to air, and the icosahedron to water (see Table 6.1).[1]

Al-Kindī's view on the role of mathematics and harmony endured. His follower Abu'l-Ḥasan al-ʿĀmirī's (d. 992 CE) best-known work, entitled *An Exposition on the Merits of Islam* [*Iʿlām bi-manāqib al-Islām* الإعلام بمناقب الإسلام], argued for the utility of both the secular sciences and the religious sciences. The latter were superior, but the former were compatible with them and could be put in their service.[2] There are echoes of al-Kindī's views: the mathematical sciences pointed the mind toward intellectual pleasure; the science of harmonics makes evident the harmonies in the physical world, in the heavens, and in the spiritual world.[3]

And the poet Sanāʾī (1080–1131/41) suggested that without the mathematician's ability to work with abstractions, one would be unable to perceive divinity,[4] writing that:

> 'You only see with your imagination and your senses,
> When you have not learned about lines, planes and points.'[5]

Explicating this point, Sanāʾī's commentator ʿAbd al-Laṭīf (d. 1638/9) put it even more bluntly:

> 'You know not the very elements of geometry and of common knowledge; how then can you attain to a knowledge of God, whom thought and sense cannot find out?'[6]

For Sanāʾī, therefore, geometry was not just a means to train the mind towards contemplating the intelligible world or divinity, but a necessary precondition.

Centuries later, the philosopher and historian Ibn Khaldūn[*] (1332–1406) made a perhaps related observation. He considered geometry to be admirable for its ability to 'cleanse' the mind of error: the precision and order of geometrical reasoning is such that a person who has studied geometry will become accustomed to thinking in a way free of error.[7]

[1] al-Kindī, 'On Why the Ancients', §§ 14–15.

[2] Rowson, 'Al-ʿĀmirī', pp. 217–18.

[3] Endress, 'Mathematics and Philosophy', p. 134.

[4] de Bruijn, *Of Piety and Poetry*, pp. 237–8.

[5] Sanāʾī, *Ḥadīqat al-Ḥaqīqa*, fol. 4a, l. 12, trans. *ibid.*, p. 238.

[6] ʿAbd al-Laṭīf, quot. in Sanāʾī of Ghazna, *The First Book*, p. 12, n. 2; trans. mod. to remove archaisms.

[*] ▸ pp. 226–7

[7] Ibn Khaldûn, *Muqaddimah*, vol. 3, p. 130–1.

Finally, there is the contribution to the history of mathematical beauty that comes from al-Kindī's role in the translation movement. A group of translators he led produced Arabic versions of Greek philosophical works by Aristotle, Alexander of Aphrodisias, Plotinus, and Proclus.[1] In particular, this circle produced a translation of books α to M of Aristotle's *Metaphysics*[2] and the paraphrased Arabic version of *The Enneads* known as the *Theology of Aristotle*.[3][*] Thus the discussion of mathematical beauty from book M of the *Metaphysics*[†] and the mention in the 'Arabic Plotinus' were both accessible right from the earliest period of Islamic philosophy, and both bore Aristotle's imprimatur.

THE BANŪ MŪSĀ, THĀBIT IBN QURRA, IBRĀHĪM IBN SINĀN

Scanty biographical details survive for the three ninth-century fraternal mathematicians and astronomers Jaʿfar Muḥammad (d. 873 CE), Abū al-Qāsim Aḥmad, and al-Ḥasan ibn Mūsā ibn Shākir, known collectively to posterity as the Banū Mūsā (the sons of Mūsā),[4] but they left a lasting legacy in science. The most important mathematical work ascribed to them is the *Book on the Measurement of Plane and Spherical Figures*, which applied the Greek method for determining area and volume to the circle and sphere. This work became well-known in the Christian and Islamicate worlds, being one of the works translated into Latin by Gerald of Cremona (c. 1114–1187).[5]

The Banū Mūsā held privileged positions in the court of the ʿAbbāsid caliphs, and were placed to finance searches for Greek manuscripts in the Byzantine Empire and to support a group of translators.[6] In particular, Muḥammad ibn Mūsā met Thābit ibn Qurra (826–901 CE) working as a money-changer in Ḥarran. Impressed by his linguistic abilities, Muḥammad invited Thābit to Baghdad and arranged for his scientific education. Thābit was a prolific translator of Greek works into Arabic, including Archimedes' two books *On the Sphere and the Cylinder*, the pseudo-Archimedean *Book of Lemmas*, books 5–7 of Apollonius' *Conics*, and Nicomachus' *Introduction to Arithmetic*.[7] He became an eminent mathematician and astronomer in his own right and surpassed his benefactor and his brethren.[8] Thābit was clearly aware of beauty in mathematics, for he was willing to use the term *ḥasana* [حَسَنَة] in his translation of Apollonius.[‡]

Thābit ibn Qurra was the first of his line recruited to mathematics and science. His son Sinān ibn Thābit (c. 880–943 CE) became a physician, mathematician, and astronomer; *his* son Ibrāhīm ibn Sinān (909–46 CE) left behind a substantial body of work on mathem-

[1] Adamson, *Al-Kindī*, p. 6.

[2] Bertolacci, 'On the Arabic Translations of Aristotle's *Metaphysics*', pp. 244–5.

[3] Adamson, 'The Arabic Plotinus', p. 13.

[*] ▸ pp. 193–4

[†] ▸ pp. 89–90

[4] Casulleras, 'Banū Musā'.

[5] al-Dabbagh, 'Banū Mūsā', p. 444; Clagett, *The Arabo-Latin Tradition*, p. 30.

[6] Rashed, *Hist. Arabic Sci. Math.*, vol. 1, p. 6.

[7] Rashed, *Hist. Arabic Sci. Math.*, vol. 1, pp. 120–1; Rosenfeld & Grigorian, 'Thābit ibn Qurra', p. 289.

[8] Rosenfeld & Grigorian, 'Thābit ibn Qurra', p. 288.

[‡] ▸ pp. 113–14

atics and astronomy. Amongst the accomplishments he listed in his autobiography, he wrote that he had demonstrated that previous and contemporary astronomers had erred in their models of the motions of the Sun (in the geocentric cosmos). Ibn Sinān claimed credit for locating the problems and describing how to test alternative hypotheses; his demonstrations, he wrote, were 'by means of propositions on spherical geometry, with very beautiful [*ḥasana* حسنة] methods'.[1]

Ibn Sinān's autobiography makes clear that he valued not only rigour, but clarity and simplicity. Pursuit of these goals was reason enough to continue working on an already-proven result.[2] He specifically discussed achieving these aims in his treatise *On the Measurement of the Parabola*. Ibn Sinān's grandfather, Thābit ibn Qurra, had determined this measurement in twenty propositions, using *reductio ad absurdum*. Al-Māhānī (d. c. 880 CE) did it in only five or six, based upon some arithmetical lemmata, again via *reductio*. Ibn Sinān took Thābit's accomplishment as a challenge, and proved the result directly in three propositions, without requiring arithmetical lemmata, and without using *reductio*.[3]

This episode points to Ibn Sinān valuing brevity, and having doubts, aesthetic or epistemological or paedagogical, about *reductio ad absurdum*. It may thus be the first entry in — or at least a prelude to — a long historical debate over preferring or avoiding proofs by contradiction. The pride he took in avoiding arithmetical lemmata may also indicate a concern about *metabasis eis allo genos*,* which feeds into later discussions about the notion of purity in proof.

AL-FĀRĀBĪ

Abū Naṣr al-Fārābī (c. 870–950/1) was a philosopher of such renown that he became known as the 'Second Teacher',[4] with only Aristotle taking precedence. After his accomplishments as a philosopher, he is remembered as a music theorist, although he had enough skill and interest in mathematics to devise constructions of parabolae and regular polygons[5] and to write commentaries on Euclid's *Elements* and Ptolemy's *Almagest*.[6] In his *Enumeration of the Sciences*, he presented a more comprehensive classification of the mathematical sciences than any previous thinker. He defined the mathematical sciences [*al Faṣl al-Thālith fī 'Ilm al-Taʿālīm*] as those dealing with numbers and magnitudes, the latter being understood as geometrical entities. For him, these sciences went beyond the quadrivium, comprising seven fields: arithmetic, geometry, optics, astronomy, music, weights, and mechanical artifices, each further divided, often into practical and theoretical branches.[7]

Al-Fārābī seems nowhere to have *explicitly* addressed beauty in mathematics. But his aesthetics and ontology left room for it. The

[1] Ibrahīm ibn Sinān, 'Opuscule', p. 12.

[2] Rashed, *Hist. Arabic Sci. Math.*, vol. 1, pp. 462–3.

[3] Ibrahīm ibn Sinān, 'Opuscule', p. 18.

* ▸ p. 91

[4] Walzer, 'al-Fārābī', p. 778.

[5] Rosenfeld, 'Geometry in Islamic Mathematics', p. 2066.

[6] Mahdi, 'al-Fārābī', pp. 523, 525.

[7] Netton, *Al-Fārābī and his School*, p. 39; Bakar, 'Science', pp. 932–3.

discussion of beauty in his book *On the Perfect State* is grounded on a distinction of intelligible and sensible beauty. In the background of this separation sits Plotinus’ thought, which was filtered through the misattributed *Theology of Aristotle* and so seemingly bore the Philosopher’s authority. However, al-Fārābī used the noun *jamāl* [جَمَال] to refer to the concept of beauty, rather than *ḥusn* [حُسْن], which was used in the *Theology of Aristotle*.[1] The term *ḥusn* is broader in scope, and can refer to goodness or grace as well as beauty; the term *jamāl* more narrowly denotes visual beauty.

On the Perfect State opens with a definition: ‘The First Existent is the First Cause of the existence of all the other existents.’[2] The First was the unique existent that was perfect, that, unlike all other things, ‘is free of every kind of deficiency’.[3] The definition of the First is the starting-point of the book as a whole, and also the foundation upon which sits al-Fārābī’s aesthetic theory. Beauty, for al-Fārābī, was based upon the degree of perfection of a thing:

> ‘Beauty and brilliance and splendour mean in the case of every existent that it is in its most excellent state of existence and that it has attained its ultimate perfection. But since the First is in the most excellent state of existence, its beauty surpasses the beauty of every other beautiful existent’.[4]

These sentences imply that beauty was an objective property. But the enjoyment of that beauty could vary, depending on how accurately the object of contemplation is perceived.[5]

Besides the sensory faculties, al-Fārābī held that the soul has a representative and a rational faculty.[6] Once the sensory faculties have gathered their data, the representative and rational faculties acted on these data, combining and abstracting them, leading to aesthetic perception and intellectual understanding. When it comes to intelligible objects, such as those supplied by the active intellect,[7] the mental faculty of representation imitated intelligible objects using the most suitable sensible analogues.[8]

> ‘It thus imitates the intelligibles of utmost perfection, like the First Cause, the immaterial things and the heavens, with the most excellent and most perfect sensibles, like things beautiful to look at’.[9]

Thus the most perfect intelligible objects, which were those that have least descended from the First, were represented by the most beautiful mental images. So the correlation between perfection and beauty was not limited to *within* the class of intelligible objects or *within* the class of sensible objects, but spanned the division. And when the faculty of representation fulfilled this perfect intelligible–beautiful sensible representation, one experienced ‘overwhelming and wonderful pleasure’ and saw ‘wonderful things which can in no way whatever be found among the other existents’.[10]

[1] Puerta Vílchez, *Aesthetics in Arabic Thought*, p. 585.

[2] al-Farabi, *On the Perfect State*, § 1.1 ⟦p. 57⟧.

[3] *Ibid.*, § 1.1 ⟦p. 57⟧.

[4] *Ibid.*, § 1.13 ⟦pp. 83–5⟧.

[5] *Ibid.*, § 1.14.

[6] *Ibid.*, § 10.1 ⟦p. 165⟧.

[7] *Ibid.*, § 14.8 ⟦p. 223⟧.

[8] *Ibid.*, § 14.6 ⟦p. 219⟧.

[9] *Ibid.*, § 14.6 ⟦p. 109⟧.

[10] *Ibid.*, § 14.9 ⟦p. 225⟧.

This theory points to utmost beauty being found within the class of intelligible things, and, given al-Fārābī’s ontology of mathematical objects, is compatible with beauty in mathematics. For al-Fārābī, numbers and geometrical objects such as lines, surfaces, and solids were intelligibles, separate from the sensible objects that may support them.[1] Thus he seems to have come close, without openly stating it, to admitting beauty in mathematical objects.

IKHWĀN AL ṢAFĀ’

The Ikhwān al-Ṣafā’ [إخوان الصفا], a name usually translated ‘Brethren of Purity’, were the authors of an encyclopaedic series of fifty-two epistles [*rasā’il* رسائل], grouped into four sections addressing the mathematical, natural, psychological or rational, and theological sciences.[2] The authorship of the epistles is disputed, but they were composed over a period in the late ninth and first half of the tenth centuries CE.[3] The mathematical epistles address arithmetic, geometry, astronomy, music, and proportion. The authors seem to have viewed geometry as a practical tool to be applied in engineering, architecture, and government, and did not directly discuss any aesthetic application, such as in geometrical art, although this role is arguably sometimes implicit in the text.[4]

The Ikhwān’s fifth epistle, *On Music*, which is the focus here, is a nexus linking their geometry, number theory, aesthetic theory, and cosmology.[5] It opens with a disclaimer: the authors’ intention is not to instruct in the practice of music.[6] Instead, their starting-point was that music is the unique manual craft whose effects are not physical, but rather spiritual: music affected the soul.[7] The reason for the soul taking pleasure in music was that the experience triggered recollection of the delights of the even purer notes and melodies of the world of the celestial spheres,[8] which the Ikhwān described as ‘sweet, delightful, and joyful’ and ‘delightful, pleasing, harmonious tones’.[9]

This effect of music was, for the Ikhwān, an instance of a more general phenomenon: anything that was perceived as beautiful reflected a principle that was innate in the symmetries and proportions of the universe: the soul’s attention

> ‘fixes itself longingly beautiful things because of the affinity between them resulting from the fact that the beauties of this world contain traces of the perfect celestial souls.’[10]

The pleasure experienced in music was dependent on an ‘affinity and resemblance between the substance of the Soul and harmonic numbers’.[11] All pleasure in beauty came about because of ‘beauties of things existing in nature result from the harmony of their colours and the judicious ordering of their constituent parts’,[12] but the beauty of

[1] Rashed, ‘The Philosophy of Mathematics’, p. 168.

[2] Netton, *Muslim Neoplatonists*, p. 2.

[3] Baffioni, ‘Ikhwân al-Safâ’’, § 2.

[4] Puerta Vílchez, *Aesthetics in Arabic Thought*, pp. 170–1.

[5] *Ibid.*, p. 174.

[6] Brethren of Purity, *On Music*, p. 75.

[7] *Ibid.*, pp. 75–84.

[8] *Ibid.*, p. 117.

[9] *Ibid.*, pp. 120, 147–8.

[10] *Ibid.*, p. 169; transliteration deleted.

[11] *Ibid.*, p. 168.

[12] *Ibid.*, p. 169.

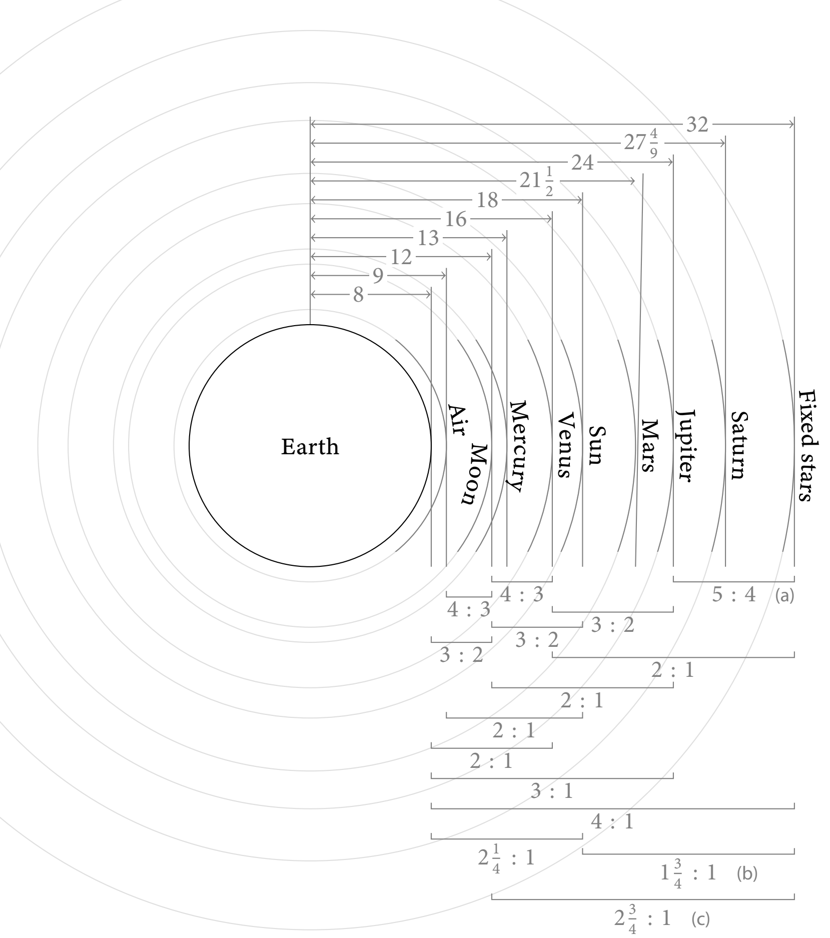

FIGURE 6.3
The numerical structure of the cosmology of the *Ikhwān al-Ṣafā*. The ratios are as precisely as listed in Brethren of Purity, *On Music*, pp. 132–3, where the *diameters* of Earth and the various spheres are given; the relative ratios of radii are same. The ratios labelled (a) and (c) are incorrect and should be 4 : 3 and $1\frac{2}{3}$: 1 respectively. The proportion labelled (b) is an approximation to $1\frac{7}{9}$: 1.

music seems to have been, for the Ikhwān, distinguished as peculiarly numerical. Thus the cosmos, music, number, and beauty were all connected.

For the Ikhwān, the universe was shaped according to a system of numerical concords that clearly owed something to pythagoreanism.[1] In this system, ‘the ideal proportion [...] consists of the ratios [2 :]1, 3 : 2, 4 : 3, 5 : 4, and 9 : 8’,[2] which recalls the harmonic theory that started with Philolaus* (c. 470–c. 385 BCE).

But the fundamental role of 9 : 8, the Philolaan tone, was bound up with a preference for the number 8, which seems not to trace further back. This partiality towards 8 was based on an arithmetic–geometric connection to the cube. The number 8 is the first cubic number (the number 1 presumably being excluded as either not properly a number or not properly cubic).[3] The first perfect number is 6, and ‘the cube is the supreme [*afḍal*] hexahedron’,[4] for it has the greatest equality among its dimensions. The cube has 8 vertices and its faces have a total of 24 = 3 × 8 corners.

The third epistle, *On Astronomy*, says that the cube is also the most ‘equivalent’ shape after the sphere.[5] The sphere stood supreme:

[1] Puerta Vílchez, *Aesthetics in Arabic Thought*, pp. 167–8.

[2] Brethren of Purity, *On Music*, p. 144, insertion in ed.

* ▸ p. 50

[3] *Ibid.*, pp. 131–2.

[4] *Ibid.*, p. 132.

[5] Quot. in Puerta Vílchez, *Aesthetics in Arabic Thought*, p. 173, n. 210.

the celestial bodies were made spherical by the Creator because the sphere was

> ‘the most perfect of all shapes: it is the most ample and the freest from any defect, the swiftest, it has its center at the middle point, all of its parts are equal, a single surface surrounds it, only a surface can touch it at a single point — all these features belong to it alone — and in addition its movement is circular, which is the most perfect of all movements.’[1]

This passage could have been written by any number of Greek philosophers such as Plato,[*] Alcinous,[†] Iamblichus,[‡] Proclus;[§] or Greek mathematicians such as Posidonius,[⁋] Cleomedes,[♠] Pappus.[♥] It makes clear that, for the Ikhwān, the sphere was still seen to have aesthetic primacy among solid shapes.

Numerical concords and the beauty of the sphere lead naturally to the numerical structure the Ikhwān posited for the geocentric cosmos. If the diameter of Earth is supposed to be 8, then the diameter of the sphere of air is 9 and the diameters of the spheres of the Moon, Mercury, Venus, the Sun, Mars, Jupiter, Saturn, and the fixed stars are respectively 12, 13, 16, 18, $21\frac{1}{2}$, 24, $27\frac{4}{9}$, and 32.[2] *On Music* highlights some of the ratios between these dimensions, with particular focus on the ratios in the ‘ideal proportion’ mentioned above (see Figure 6.3).[3]

The preference for 8 and for the circle is also evident in the discussion of calligraphy in *On Music*. The Ikhwān saw all scripts, including Arabic, Hebrew, and Syriac, as well as the Hindu numerals, as being formed from ‘the straight line which is the diameter of a circle and the curved line which is its circumference’.[4] The ‘most excellent of scripts’ was that in which the dimensions of the letters have ‘the most perfect proportion’.[5] *On Music* cites a ‘skilful scribe and geometer’,[6] which doubtless refers to Ibn Muqla (885/6–940/1), as recommending construction of all the letters from some basic measure, and then gives the example of making the Arabic letter ا [*ʾalif*] have a length-to-width ratio of 8 : 1, and using it as the diameter of the circle from which all the Arabic letters are derived.[7][♦]

In a platonic vein, the Ikhwān thought that the practice of mathematics elevated the mind beyond the sensible world, yielding both moral and intellectual improvement.[8] This effect was part of their distinction between practical and theoretical geometry. The benefits of the former are worldly; the benefits of the latter are spiritual:

> ‘the study of sensible geometry aids skilfulness in the arts, whilst the study of intellective geometry and the knowledge of the properties of numbers and figures help in grasping the manner by which the heavenly bodies affect the lower natural entities, and also in understanding how the sounds of music

[1] Quot. in Puerta Vílchez, *Aesthetics in Arabic Thought*, p. 173.

[*] ▸ pp. 61–2

[†] ▸ p. 138

[‡] ▸ pp. 145–8

[§] ▸ pp. 149–58

[⁋] ▸ p. 117

[♠] ▸ p. 117

[♥] ▸ pp. 123–4

[2] Brethren of Purity, *On Music*, pp. 132–3.

[3] *Ibid.*, p. 133.

[4] *Ibid.*, p. 140.

[5] *Ibid.*, p. 141.

[6] *Loc. cit.*

[7] *Ibid.*, pp. 141–2.

[♦] ▸ p. 219

[8] Fakhry, *Hist. Islamic Phil.*, p. 175.

> affect the souls of the listeners. Studying the manner by which these two types influence their effects is a prerequisite to knowing the way by virtue of which the separate [i.e., incorporeal] souls impact the embodied souls in the realm of generation and corruption.'[1]

In spite of the ultimately platonic origin of this notion, the Ikhwān inherited it from neopythagoreanism, and so music was the key science,[2] although the other branches of mathematics can be a means to the end of music. In particular, the tones and melodies produced by the motions of the celestial spheres are audible to those who have undergone mathematical education, like Pythagoras,

> 'who heard the music of the spheres after having been cleansed of the defilement of corporeal appetites and raised to the sublime by constant reflection and by the sciences of arithmetic, geometry, and music.'[3]

On hearing the music of the spheres, it will become evident to any intelligent person that the universe was made by a 'creator', 'a skilled artificer', and a 'benign composer'.[4]

ABŪ'L-WAFĀ'

Abū'l-Wafā' al-Būzjānī (940–77/8 CE) is most noted for the textbooks he wrote on arithmetic as used in business and geometry as used by artisans,[5] which will feature in the discussion of geometrical art in the next chapter.* He also wrote one of the earliest extant treatises dedicated to magic squares, which focuses on constructing magic squares of small orders and building composite squares. It appears to be an elementary introduction, rather than a survey of known methods.[6] In his discussion, he repeatedly referred to the aesthetic value of the *methods* he described.

For instance, as a prefatory remark to a method of constructing a magic square of order 4, he wrote:

> 'It is possible to arrive at the magic arrangement in this square by means of methods without displacement showing regularity and elegance [*niẓām wa-tartīb ḥasan* نظام وترتيب حسن]'[7]

The method he described that showed 'regularity and elegance' was first to place the number 1 in a corner, 2 and 3 adjacent to the opposite corner, and 4 diagonally adjacent to 1; then to place 5 to 8 in reverse order in positions that were horizontally symmetrically opposite to 1 to 4; and finally to place $17 - n$ diagonally two places away from n for $n = 1, \ldots, 8$:

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[1] Brethren of Purity, *On Geometry*, pp. 159–60, insertion in ed.; transliterations omitted.

[2] Hogendijk & Sabra, *The Enterprise of Science in Islam*, p. 133.

[3] Brethren of Purity, *On Music*, p. 148.

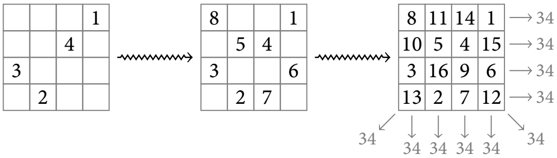
[4] *Loc. cit.*

[5] Youschkevitch, 'Abū'l-Wafā' al-Būzjānī', pp. 40–2.

* ▸ p. 224

[6] Sesiano, *Magic Squares*, p. 13.

[7] Abū'l-Wafā' al-Būzjānī, 'On Magic Arrangement in Squares', p. 217.

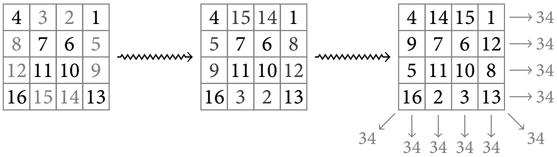


			1
		4	
3			
	2		

8			1
	5	4	
3			6
	2	7	

8	11	14	1
10	5	4	15
3	16	9	6
13	2	7	12

This procedure he contrasted with the first method he described, which used 'displacement' (or rearrangement) of entries: start with a square filled naturally with the numbers 1 to 16, then swap the middle two entries *between* the outer rows and the outer columns, then swap the middle two entries *within* each outer row and outer column:



4	3	2	1
8	7	6	5
12	11	10	9
16	15	14	13

4	15	14	1
5	7	6	8
9	11	10	12
16	3	2	13

4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13

For Abū'l-Wafā', the absence of displacement was clearly not a necessary condition for elegance, since he characterized a method for constructing a magic square of order 6 involving displacement using exactly the same form of words: 'showing regularity and elegance [*niẓām wa-tartīb ḥasan* نظام وترتيب حسن]'.[1] Finally, he closed his discussion of methods for constructing composite magic squares by presenting 'a manner more elegant [*uḥsun* احسن]'[2] than those he had hitherto described.

* * *

There is an interesting postscript to the Arabic tradition of magic squares. Little is known of the life of Manuel Moschopoulos (late 13th–early 14th century), but he was the greatest philologist of his time and a commentator on several classical texts. He also wrote a short treatise on magic squares, which was the first European and first Greek-language work to discuss the subject. There is a coincidence between magic squares he constructed and those in Arabic texts, but a *lack* of parallels in terminology. It thus seems likely that he found the magic squares in an Arabic text but was unable to understand fully the descriptions of their construction.[3]

In his treatise, Moschopoulos made a passing reference to elegance: a magic square of order 8 he constructed (see Figure 6.4) had unspecified 'elegant and interesting [γλαφυρὰ καὶ ἀστεῖα *glaphyra kai asteia*]' properties that a previous one did not enjoy.[4]

1	62	59	8	9	54	51	16
60	7	2	61	52	15	10	53
6	57	64	3	14	49	56	11
63	4	5	58	55	12	13	50
17	46	43	24	25	38	35	32
44	23	18	45	36	31	26	37
22	41	48	19	30	33	40	27
47	20	21	42	39	28	29	34

FIGURE 6.4
A magic square of order 8.

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[1] Abū'l-Wafā' al-Būzjānī, 'On Magic Arrangement in Squares', p. 219.

[2] *Ibid.*, p. 252.

[3] Sesiano, 'Les carrés magiques de Manuel Moschopoulos', pp. 392–3.

[4] Moschopoulos, 'Sur les Carrés Magiques', p. 110.

AL-KŪHĪ

Abū Sahl al-Kūhī (or al-Qūhī; *fl.* c. 970–c. 1000) was regarded by contemporaries as the ‘Master of his age in the art of geometry’,[1] and his mathematical work was focused in this area.[2] He thought that ‘the pleasure that mathematicians take in geometry is greater than the pleasure they take in the other sciences’.[3] This statement does not necessarily indicate *aesthetic* pleasure, but is suggestive of it, especially given that the statement immediately precedes an assertion that philosophers esteemed mathematics more highly than the other sciences in part because of its beauty.[4] Mathematicians’ preference for geometry over other ‘sciences’ doubtless placed it above both the non-mathematical sciences and the other branches of mathematics, such as arithmetic, astronomy, and music.

Beauty and problems

In the introduction to his treatise ‘On the Construction of an Equilateral Pentagon in a Given Square’, al-Kūhī wrote of his motivation for considering the problem:

> ‘Having completed the construction of a regular heptagon in a circle, we set out to investigate another proposition, one more beautiful [*ʾaḥsan* أحسن], deeper [*ʾabʿad* أبعد], more opaque [*ʾaghmaḍ* أغمض] and more difficult [*ʾaṣʿab* أصعب] to find out than the construction of the heptagon, to give pleasure to the King Sharaf al-Dawla. This is the construction of an equilateral pentagon in a known square. This was achieved successfully by his subject Wayjan ibn Rustam,[5] in his reign and with his approval, and that thanks to the good opinion and the high consideration in which the King holds all those who investigate this science. I hope this construction will receive a welcome commensurate with its magnitude and its depth.’[6]

Al-Kūhī had previously given a construction of the regular heptagon that depended on intersecting a parabola and a hyperbola. (Al-Kūhī’s solution to this problem seems to be independent of al-Sijzī’s claimed solution.[7][*]) The quotation above implies that al-Kūhī thought that this construction was beautiful. The new problem, of constructing an equilateral (but not equiangular and thus not regular) pentagon in a square was even more so. And his stated aim for studying a problem that was ‘more beautiful, deeper, more opaque and more difficult’ was to ‘give pleasure’ to the Sharaf al-Dawla (Abu’l-Fawāris Shīrdhīl; c. 960–88/9, r. 983–8/9); this goal suggests that the monarch had an appreciation for these qualities. Certainly, al-Kūhī credited geometrical advances to Sharaf al-Dawla’s patronage of the sciences and arts.[8]

A detailed analysis led al-Kūhī to the following construction (see Figure 6.5). Start from a square *AGID*, with *AG* and *DI* bisected at *B*

[1] Quot. in Berggren, ‘Tenth-Century Mathematics’, p. 178.

[2] Dold-Samplonius, ‘al-Qūhī’, p. 240.

[3] al-Qūhī, ‘On the Division of a Known Angle into Three Equal Parts’, p. 494.

[4] *Loc. cit.*

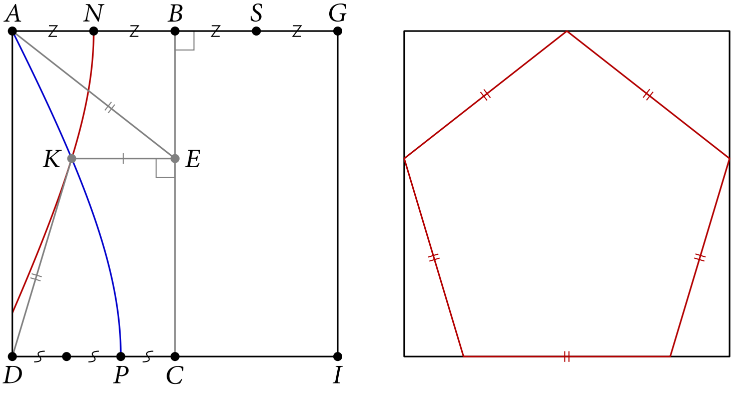
[5] al-Kūhī.

[6] al-Qūhī, ‘On the Construction of an Equilateral Pentagon in a Given Square’, p. 532.

[7] Dold-Samplonius, ‘al-Qūhī’, p. 240.

[*] ▸ pp. 209–10

[8] al-Qūhī, ‘On the Determination of the Side of the Heptagon’, pp. 639–40.



Construction of the necessary segments

The equilateral pentagon

FIGURE 6.5
Al-Kūhi's construction of an equilateral pentagon in a square. The red and blue curves are hyperbolae with latus rectum equal to twice AG and major axes NS and PI respectively. Their intersection K determines the point E. The segments AE and DK have equal length and are both twice the length of KE. A straightforward translation of these segments gives the pentagon.

and C respectively, AB and GS bisected at N and S, respectively, and with P dividing DI so that PI is twice DP. Draw two hyperbolae with latus rectum equal to twice AG, one with major axis NS, one with major axis PI. The perpendicular through their intersection K meets BC at E. The segments AE and DK are equal in length and equal to twice the length of KE. The segments AE are DK are equal to the sides of the desired equilateral pentagon.

Al-Kūhī referred to the greater beauty, depth, and difficulty of the *proposition*, which must indicate the method of construction of the equilateral pentagon, possibly including both the analysis and synthesis, rather than to the bare statement that it *can* be constructed. Given the greater symmetry of the regular heptagon over the irregular but equilateral pentagon inscribed in the square, it is unlikely that the *object* of the construction contributed to giving it greater beauty; if anything, the less-symmetrical pentagon would tend to detract from the beauty of the construction. Thus the difference in beauty must be dependent on the *methods* of construction. Several aspects of the method could contribute: the greater overall simplicity of the construction of the equilateral pentagon; or the use of two conic sections of the same species — hyperbolae — instead of two different species — a parabola and a hyperbola.

Effects of beauty

In the works of al-Kūhī are found what seem to be the earliest extant reflections by a mathematician on the effects of mathematical beauty. Archimedes was honoured in part because of the beauty of his work:

> 'Mathematicians agree in recognizing Archimedes is eminent and holds first place among the Ancients, because they have taken note of how many beautiful and advanced things he discovered, and ⟨how many⟩ difficult and abstruse propositions, in the highly valued demonstrative sciences'.[1]

[1] al-Qūhī, 'On the Construction of the Side of the Regular Heptagon', p. 651.

As already noted, al-Kūhī thought that

> ‘philosophers are more proud of the sciences of the mathematicians than of the sciences of other ⟨scholars⟩, because of their beauty, their truth, their absence of delusive imagination, the proofs they afford, their lack of connections with religions and power, and their disdain for rhetoric and language’.[1]

Observe how al-Kūhi mentioned beauty *first*, prioritizing the aesthetic over the aletheic and epistemic.

The last of al-Kūhī’s reasons for the philosophers’ admiration for mathematics deserves exploration, and he developed it further. Since the mathematical sciences are independent of language,

> ‘they do not disappear with the disappearance of people who learn the tongue, nor change with a change of language, as is the case for the science of grammar. That is why mathematics endure the passage of time much better than other sciences.’[2]

That is, mathematics was admired because it had greater *permanence* than other fields of study. And for al-Kūhī, this permanence seems to have been bound up with the beauty of mathematics: the permanence is valuable, because there endures a thing of beauty. Thus mathematicians might toil without promise of reward to solve a problem,

> ‘confident as they were in what their good faith had promised them: the survival of geometry for all time, its development beyond their life spans, its permanence as a beautiful unchanging memory and a treasure huge and inexhaustible’.[3]

[The connection between the beauty and the durability of mathematics recurs much later, most famously in the work of G. H. Hardy (1877–1947).*]

IBN AL-HAYTHAM

Ibn al-Haytham (965–c. 1040), historically called Alhazen or Alhacen, wrote prolifically on optics, astronomy, and mathematics. His *Optics* was a thoroughgoing reexamination of a subject that he thought had become shrouded in confusion, and he grounded his theories of light and of vision in experimental and mathematical reasoning.[4] In the Islamicate world, it languished in obscurity for three centuries,[5] but both it and his treatise *On Parabolic Burning Mirrors* were translated into Latin;[6] the latter work was cited with admiration for the beauty of its proofs by Roger Bacon† (c. 1219–c. 1292).

The Optics is important in this history because, excepting possibly some writings on calligraphy,[7] it contains the only mediaeval Islamic

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[1] al-Qūhī, ‘On the Division of a Known Angle into Three Equal Parts’, p. 494.

[2] *Loc. cit.*

[3] al-Qūhī, ‘On the Determination of the Side of the Heptagon’, p. 640.

* ▸ p. 508

[4] Sabra, ‘Ibn al-Haytham’, pp. 190–1.

[5] Sabra, intro. to Ibn al-Haytham, *Optics*, vol. ii, pp. lxiv–lxxi.

[6] Sabra, intro. to Ibn al-Haytham, *Optics*, vol. ii, pp. lxxiii–lxxv; Sabra, ‘Ibn al-Haytham’, p. 196.

† ▸ pp. 180–1

[7] Sabra, comm. on Ibn al-Haytham, *Optics*, ch. ii.3, paras 200–31 [vol. ii, p. 97].

analysis of beauty *independent of moral goodness*.[1] Ibn al-Haytham wrote of twenty-two properties that can be perceived in sight:

> 'light, colour, distance, position, solidity, shape, size, separation, continuity, number, motion, rest, roughness, smoothness, transparency, opacity, shadow, darkness, beauty, ugliness, and the similarity and dissimilarity between all the particular properties taken by themselves or between all forms composed of the particular properties.'[2]

Although beauty and ugliness were two of these properties, they were special in that they depend on others. Each of the other properties *produced* a certain kind of beauty.[3] Ugliness was the absence of beauty of every kind.[4] The perception of beauty, therefore, was entangled with the subjective processes of visual perception.

Moreover, visual perception did not comprise pure sensation alone: judgement and inference can supplement it. And these were not a part of sight, but separate faculties.[5] Therefore these faculties also contributed to the perception of beauty. A complete and certain understanding of the form of an object could require both perception and contemplation.[6]

Ibn al-Haytham's choice of aesthetic terminology in *The Optics* merits examination. He preferred not the usual adjective *ḥasan* [حَسَن] for 'beautiful', but the passive term *mustaḥsan* [مُسْتَحْسَن], which emphasizes the subjectivity of judgements of beauty.[7] He also used terms such as 'order' [*tartīb* تَرْتِيب] and 'proportion' [*tanāsub* تَنَاسُب],[8] in which there seems to be a faint echo of Aristotle's criteria for beauty.* But mathematical beauty, although not incompatible with Ibn al-Haytham's account of visual beauty and the role of cognition in supporting it, is not treated directly in *The Optics*. (There is a statement that '[n]umber produces beauty', but in the sense that greater numerosity produced greater beauty, such as regions of the night sky with many stars being more beautiful than those with fewer.[9])

But Ibn al-Haytham did recognize and indeed use mathematical beauty. In the preface to his *Completion of the Conics*, in which he attempted to reconstruct Apollonius' lost eighth book,† he presented various arguments in favour of his work accurately following that of Apollonius. One of these arguments is grounded upon judgements of beauty:

> 'It is inconceivable that the work did not deal with these notions we mentioned and referred to, because they are beautiful notions, the beauty of which is not less than the beauty of what the seven books contain. On the contrary, among them are [some] which exceed in beauty and understanding the propositions that were transmitted. Hence it is most likely that these notions are [the notions] which the eighth book contained. However, he [Apollonius] did not mention them before the

[1] Endress, 'Mathematics and Philosophy', p. 145.

[2] Ibn al-Haytham, *Optics*, ch. II.3, para. 44.

[3] *Ibid.*, ch. II.3, paras 200–22.

[4] *Ibid.*, ch. II.3, para. 232.

[5] *Ibid.*, ch. II.3, paras 16–17.

[6] *Ibid.*, ch. II.4, paras 2–5.

[7] Sabra, comm. on *ibid.*, ch. II.3, paras 200–31 [vol. II, pp. 97–8].

[8] Puerta Vílchez, *Aesthetics in Arabic Thought*, p. 736.

* ▸ pp. 89–90

[9] Ibn al-Haytham, *Optics*, ch. II.3, para. 211.

† ▸ pp. 113–15

> eighth book, since he could dispense with using them in the books which were transmitted.'[1]

Ibn al-Haytham here made a historiographical argument based upon aesthetic appreciation of the mathematics. His own work, he held, contained certain notions that equalled and some that surpassed those in the extant part of the *Conics*. Thus, thought Ibn al-Haytham, his predecessor Apollonius, with his own motivation to pursue beauty, would certainly have reached these notions. Hence he *did* reach them, and must have written them down in the lost eighth book. Implicit in this argument are the premisses that Apollonius would have skill and perseverance to equal Ibn al-Haytham's achievements — which is a reasonable assumption — and that Ibn al-Haytham and Apollonius would have made the same judgements of beauty.

In the quotation above from *Completion of the Conics*, the words translated as 'beauty' are inflections of *ḥasan* [حَسَن], not *mustaḥsan* [مُسْتَحْسَن] which he used in *The Optics*. Together with the implicit premiss that his aesthetic judgements of mathematics and Apollonius' would agree, this choice of terminology suggests that he thought that there were objective judgements of intellectual and in particular of mathematical beauty.

AL-SIJZĪ, AL-JŪD, BEAUTY, AND PRIORITY

Al-Sijzī's (c. 945–c. 1020) main focus was astrology, but he is better known as a geometer for a number of significant papers he wrote. He became involved in a dispute with Abū al-Jūd (*fl.* 10th century CE) over the construction of the regular heptagon.

Al-Jūd wrote three essays on this problem. Al-Sijzī had noticed an error in the first of al-Jūd's essays (composed 968/9 CE), related to dividing a line in a certain ratio, but had been unable to correct it himself. He thus sent it to Ibn Sahl (*fl.* late 10th century CE), who showed that such a division was possible using conic sections. Al-Sijzī made a synthesis based on Ibn Sahl's analysis and presented it as his own solution to the problem of constructing the regular heptagon. In a second essay on the subject, al-Jūd, in a barb seemingly aimed at al-Sijzī, accused an unnamed geometer of plagiarism.[2] Al-Sijzī responded by writing that al-Jūd

> 'rejects as ugly all the propositions that it is not possible for him to understand.'[3]

The phrase 'rejects as ugly' translates a form of the Arabic verb *istaqbaḥa* [اِسْتَقْبَحَ], whose meaning encompasses 'to consider ugly, bad' or 'to abhor'. The combination of aesthetic and epistemological judge-

[1] Ibn al-Haytham, *Completion of the Conics*, § 0.h ⟦p. 624⟧, insertions in ed.

[2] Rashed, *Hist. Arabic Sci. Math.*, vol. 3, p. 302–3; cf. al-Shannī, 'Book on the Discovery of the Deceit of Abū al-Jūd', pp. 678 sqq.

[3] al-Sijzī, quot. in Rashed, *Hist. Arabic Sci. Math.*, vol. 3, p. 303.

ment in this accusation is important. It implicitly linked aesthetic appreciation of results with the ability to understand them, and suggested that al-Jūd was incompetent and disguised his lack of comprehension behind aesthetic dismissal. Al-Sijzī made a similar aesthetically-based criticism of Abū al-Jūd in his treatise on constructing the regular heptagon, accusing him again of 'rejecting as ugly' Archimedes' method and thus of being inadequate to appreciate the work of Archimedes, which was the foundation and the ideal of geometers of the time.[1]

IBN SĪNĀ

Ibn Sinā (980–1037 CE), historically known in Europe as Avicenna, was a philosopher in a new age of mathematics. Since the lives of al-Kindī* and al-Fārābī,† mathematicians in the Islamicate world had completely assimilated and then transcended the accomplishments of their Greek forebears. Mathematicians had turned their gaze to foundational questions that had thitherto lain in the sphere of philosophy.[2] Ibn Sinā would have learned about these accomplishments during his education, which was excellent and wide-ranging. In him were allied a scholarly precocity and a prodigious memory, and he rapidly mastered logic, mathematics, the sciences, and medicine.[3] He came to have a lasting impact on philosophy, and in a sense most subsequent Islamicate philosophy bears the stamp of his influence. Notably, al-Ghazālī's criticism and Ibn Rushd's defence both cast philosophy in the mould of Ibn Sīnā.[4]

Ibn Sinā wrote prolifically, but relatively little on mathematics.[5] He clearly valued the subject: his compendium of philosophy and science, *The Cure* [*Kitāb al-Shifā* كتاب الشفاء] (so called for its aim of rectifying ignorance), includes books on geometry, arithmetic, astronomy, and music.[6] The content of these books reflects standard works: Euclid's *Elements*, Nicomachus' *Introduction to Arithmetic*, and Ptolemy's *Almagest* and *Harmonics*. *The Cure* thus does not include the more advanced geometry of Archimedes or Apollonius or the algebra of al-Khwārizmī, although there is no doubt that Ibn Sinā knew and understood these works.[7]

One of the strongest indications that Ibn Sīnā recognized mathematical beauty is in the account of his life that started as an autobiography and was completed by his pupil Abū 'Ubayd al-Jūzjānī (*fl.* 11th century) based on those parts of his master's life that he witnessed.[8] While describing Ibn Sinā finishing *The Cure* after his move to Iṣfahān, al-Jūzjānī wrote that

> 'In Euclid he presented some geometrical figures, in the Arithmetic some excellent [*ḥasana* حسنة] numerical properties'.[9]

* ▸ pp. 194–7
† ▸ pp. 198

[1] al-Sijzī, 'Sur la construction de l'heptagone dans le cercle et la trisection de l'angle rectiligne', p. 741.

[2] Rashed, 'Avicenna', pp. 249–50.

[3] Anawati, 'Ibn Sīnā', p. 494.

[4] Inati, 'Ibn Sīnā', pp. 242–3.

[5] Gutas, *Avicenna and the Aristotelian Tradition*, p. 460.

[6] *Ibid.*, p. 105.

[7] Rashed, 'Avicenna', pp. 251–2.

[8] Ibn Sina & al-Jūzjānī, *The Life of Ibn Sina*, pp. 44–5.

[9] *Ibid.*, pp. 64–7.

Following al-Fārābī and other neoplatonists, Ibn Sinā conceived of beauty as closely bound to perfection, to goodness,[1] and to pure intellect. He associated all four qualities in the metaphysical part of *The Cure*:

> 'There can be neither beauty nor splendor above the quiddity's being purely intellectual, pure goodness, free from each one of the facets of deficiency, one in every aspect.'[2]

He made the link to beauty even more explicit when he explained that these qualities are flawlessly realized by 'the Necessary Existent — who is ultimate perfection, beauty, and splendor'.[3] His psychological explanation was that the rational soul recognizes that the closer a thing is to the 'First Object of love', the greater its order, harmony, unity, and concord.[4]

For Ibn Sinā, there were four levels of perception, via sense, imagination, estimation, and reason.[5] Aesthetic perception could take place on any of these levels. Via sense, the aesthetic perception was closely related to matter, but the faculties of imagination, estimation, and reason each in turn produced a greater abstraction.[6] There was greater aesthetic value in intellectual perception; the soul aspired to such perception so that it could contemplate 'absolute Truth, absolute Goodness, and true Beauty'.[7]

At the start of the book on metaphysics, Ibn Sinā listed the subject-matter of the mathematical sciences:

> 'either measure abstracted in the mind from matter, or measure apprehended in the mind with matter, or number abstracted from matter, or number in matter.'[8]

Mathematical objects — magnitudes or numbers — had two kinds of existence: in the mind and in physical objects.[9] It has been argued, however, that Ibn Sinā thought that mathematical objects as mental existents were not completely separated from matter.[10]

Ibn Sinā also wrote in the metaphysical book of *The Cure* that

> 'mathematical things do not separate from goodness. This is because they are in themselves possessors of an abundant share of arrangement [*tartīb* الترتيب], order [*niẓām* النظام], and symmetry [*iʿtidāl* الاعتدال]. Thus, each of them is in accordance with what it ought to have. And this is the good of all things.'[11]

There is a clear — and doubtless conscious — echo here of the passage in book M of the *Metaphysics* where Aristotle refuted the notion that the mathematical sciences said nothing of the beautiful or the good, because they especially demonstrate the chief forms of beauty, namely order, symmetry, and definiteness.* But here Ibn Sinā made a more limited statement, that 'mathematical things do not separate from goodness'. This statement is at least in part aesthetic, for, as discussed above, the core of Ibn Sinā's aesthetics was a beauty–perfection–

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[1] Abolghassemi, 'Avicenna on Beauty', p. 49.

[2] Avicenna, *The Metaphysics of* The Healing, § VIII.7.15, ⟦p. 297⟧.

[3] *Ibid.*, § VIII.7.16, ⟦p. 297⟧.

[4] Ibn Sīna, 'Treatise on Love', p. 220.

[5] Avicenna, *Psychology*, p. 38.

[6] Avicenna, *Psychology*, p. 38; Puerta Vílchez, *Aesthetics in Arabic Thought*, pp. 612–4.

[7] Quot. in Puerta Vílchez, *Aesthetics in Arabic Thought*, p. 622.

[8] Avicenna, *The Metaphysics of* The Healing, § I.2.3 ⟦p. 7⟧.

[9] Ardeshir, 'Ibn Sīnā's Philosophy of Mathematics', p. 50.

[10] Zarepour, 'Avicenna's Philosophy of Mathematics', ch. 2.

[11] Avicenna, *The Metaphysics of* The Healing, § VII.3.25 ⟦p. 256⟧.

* ▸ pp. 89–90

goodness–intellect nexus. But Ibn Sinā seems only to have pointed to mathematical *objects* as possessing the qualities of arrangement, order, and symmetry.

AL-NASAWĪ

The mathematician al-Nasawī (*fl.* 1029–44) was lost to history, unmentioned by Arabic biographies, until a study of one of his manuscripts in the nineteenth century.[1] He wrote on both arithmetic and geometry, and in particular commentated upon the pseudo-Archimedean *Book of Lemmas*. This work, though ascribed to the great geometer, is more likely to be a Greek compilation from late antiquity,[2] and was translated into Arabic by Thābit ibn Qurra.[3] It inspired Al-Kūhī to write his *Ornamentation of the Lemmas of Archimedes*, the complete version of which is lost, with only certain results preserved in the commentary on the *Book of Lemmas* by al-Nasawī.

Al-Nasawī's commentary was included in al-Ṭūsī's edition of the Middle Books,[4] a collection of treatises called by that name because they were read by students who had finished Euclid's *Elements* before they tackled Ptolemy's *Almagest*.[5] Al-Ṭūsī signalled his approbation by calling al-Nasawī 'the Competent Scholar'.[6] In his preface, al-Nasawī said of the *Book of Lemmas*:

> 'It contains beautiful [*ḥasana* حسنة] propositions, few in number but with many benefits, on the principles of Geometry; (they are) extremely good and subtle.'[7]

He drew particular attention to the work's fifth proposition, which al-Nasawī considered to be a special case of a more general result, and observed:

> 'Abū Sahl Qūhī [...] presented a proof of this proposition in a more general and more beautiful [*’aḥsan* أحسن] way'.[8]

This statement seems to be the earliest extant description of a *proof* as beautiful.

The fifth proposition is one of three results in the *Book of Lemmas* that concern the *arbēlos*:*

THEOREM A. *'Let AB be the diameter of a semicircle, C any point on AB, and CD perpendicular to it, and let semicircles be described within the first semicircle and having AC, CB as diameters. Then, if two circles be drawn touching CD on different sides and each touching two of the semicircles, the circles so drawn will be equal.'*[9] (See Figure 6.6.)

The more general results, which al-Kūhī proved 'in a more general and beautiful way' did not require the two smaller semicircles to be

[1] Saidan, 'al-Nasawī', p. 614.

[2] Clagett, *The Arabo-Latin Tradition*, pp. 632–3; Coşkun, 'Thābit ibn Qurra's Translation', pp. 54–5; Hogendijk, 'Two Beautiful Geometrical Theorems', pp. 2–3.

[3] Sezgin, *Geschichte des Arabischen Schrifttums*, vol. v, p. 272.

[4] Hogendijk, 'Two Beautiful Geometrical Theorems', p. 5.

[5] Roughan, 'The Little Astronomy and Middle Books', esp. chs 5, 8, 9.

[6] Hogendijk, 'Two Beautiful Geometrical Theorems', p. 9.

[7] al-Nasawī, quot. in *ibid.*, p. 9; typo corrected.

[8] al-Nasawī, quot. in *ibid.*, p. 9.

* ▸ p. 124

[9] ps.-Archimedes, *Book of Lemmas*, Prop. v.

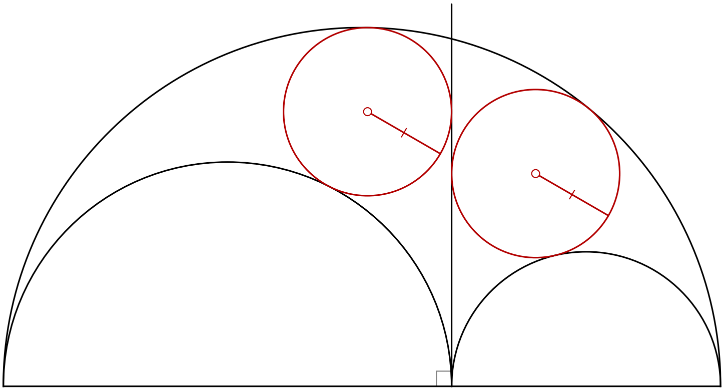

FIGURE 6.6
The fifth proposition from the pseudo-Archimedean *Book of Lemmas*.

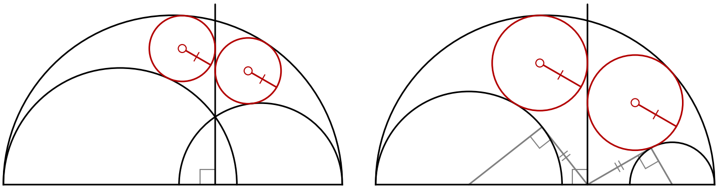

FIGURE 6.7
Al-Kūhī's generalizations of the pseudo-Archimedean result.

mutually tangent; they needed only be tangent to the main semicircle. If the two smaller semicircles crossed, the perpendicular was drawn through their intersection. If they were disjoint, the perpendicular was drawn from the point on the main diameter from which the tangents to the two smaller semicircles were equal. In each case, the two circles that are tangent to the main semicircle, the perpendicular, and one of the smaller semicircles are equal (see Figure 6.7).[1]

It is uncertain what al-Nasawī found more beautiful about the proofs. Al-Kūhī's construction and diagrams do not differ greatly from those in the *Book of Lemmas*, but do have the merit of being more self-contained and not appealing to the pseudo-Archimedean lost commentary on the also-lost work *Treatise on the Right-Angled Triangles*.

Since al-Ṭūsī seemingly thought highly enough of al-Nasawī and his commentary to preserve it, it is a reasonable inference that he would not have disagreed with the al-Kūhi's aesthetic assessments of his own proofs.

AL-GHAZĀLĪ

It is perhaps a historical irony that Abū Ḥāmid al-Ghazālī (c. 1056–1111) is counted among the philosophers. He was a theologian and jurist who tried to point out the errors in *falsafa*. But his opposition was based on thorough study and profound philosophical acumen.[2] In his book *The Incoherence of the Philosophers*, he criticized what he saw as the hubris of placing demonstrative proof above the interpretation of revealed truth, and offered refutations of the arguments supporting various metaphysical and scientific theses.[1]

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[1] Hogendijk, 'Two Beautiful Geometrical Theorems', pp. 13, 14.

[2] Campanini, 'Al-Ghazzālī', p. 258.

An erroneous thread in literature on the history of science holds that al-Ghazālī disparaged mathematics and natural science. On the contrary, in *The Incoherence of the Philosophers*, he defended the pursuit of both,[2] and specifically criticized those who mistook an attack on science for a defence of religion.[3] He marked off sciences like mathematics and logic, which were unobjectionable from a religious perspective, from metaphysics, which could lead to heresy. The difference was that in mathematics, the arguments were truly demonstrative; in metaphysics, they were *conjectural*.[4] And while mathematical demonstrations leave no room for doubt,[5] someone who reads mathematics may be seduced into thinking that all philosophical arguments share its rigour:

> ‘a person who examines mathematics comes to marvel at its subtleties and the compelling nature of its demonstrations, and because of this he takes on a favorable attitude towards the philosophers. He supposes that all their sciences are as clear as this science and that all their demonstrations are just as firm.’[6]

Notice the twofold cause of the error: marvelling at the ‘subtleties’ of mathematics and the ‘compelling nature’ of its arguments. There was thus something *separate* from the rigour of mathematical proof that so impressed the reader. Did al-Ghazālī think that mathematical beauty could play a part? Just possibly, for in his *Alchemy of Happiness* [*Kīmīyā-yi Saʿādat* كيمياى سعادت], he discussed inner intellectual beauty, and thought that anyone with the wit to accept its existence would see its superiority:

> ‘whoever possess intellect does not deny internal beauty and prefers that to the external form. [...] beauty is two: external and internal. The beauty of the inner form is loved just as external beauty; rather, it is more loved by whoever has a modicum of intellect.’[7]

The outer beauty, such as it is, of a written work or a painting or a building revealed the inner beauty of the author or artist or architect.[8] When compared with this inner beauty, the outer beauty was seen as ugliness.[9]

The love of beauty was connected to the pleasure it evokes, and higher pleasure comes from the contemplation of inner beauty.[10] The above quotation from al-Ghazālī’s *Alchemy of Happiness* prefaces an argument for the love of God being the supreme form of beauty and pleasure. While al-Ghazālī skirted around explicitly asserting beauty in mathematics, as part of his argument he pointed to the pleasures of mental endeavour. There was intellectual pleasure in learning, including geometry.[1] A scholar would always choose the pleasures of their chosen field over more mundane enjoyments:

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[1] al-Ghazālī, *Incoherence*, Intro., paras 4–5 ⟦p. 2⟧.

[2] Bakar, ‘Science’, p. 938.

[3] al-Ghazālī, *Incoherence*, Intro., paras 15–16 ⟦pp. 5–6⟧.

[4] al-Ghazālī, ‘The Rescuer from Error’, p. 69.

[5] al-Ghazālī, *Incoherence*, Intro., para. 16 ⟦p. 6⟧; Marmura, ‘Ghazali and Demonstrative Science’, p. 188.

[6] al-Ghazālī, ‘The Rescuer from Error’, p. 69.

[7] al-Ghazzālī, *Alchemy of Happiness*, [vol. 2] pp. 902–3.

[8] Behrens-Abouseif, *Beauty in Arabic Culture*, p. 23.

[9] Leaman, *Islamic Aesthetics*, p. 94.

[10] Behrens-Abouseif, *Beauty in Arabic Culture*, p. 24.

> 'for the scholar who studies the science of mathematics, or the science of geometry, or the science of Religious Law and Doctrine, or that in which he finds a pleasure — when it is not deficient and is perfect — he chooses that above all pleasures.'[2]

And the importance of a field influences the degree of pleasure obtained: 'that person who knows geometry and the form and dimensions of the heavens will be happier than he who knows how to play chess'.[3]

IBN RUSHD

The Andalusian philosopher, jurist, and physician Ibn Rushd (1127–98), who became known in Europe as Averroës, thought that no-one had added 'a word worthy of attention' to Aristotle's logic, natural science, and divine science in the preceding one-and-a-half millennia.[4] Thus he maintained that Aristotelianism should be the approach to philosophical inquiry, and he wrote an extensive set of commentaries on Aristotle's works. Modern scholarship has grouped these works into epitomai or short commentaries; paraphrases or middle commentaries; and literal or long commentaries.[5]

Aristotle's *Metaphysics* was one of the works on which Ibn Rushd composed all three kinds of commentary. But they seem to supply no evidence of Ibn Rushd's response to the passage on beauty in mathematics in book M:[*] the epitome does not touch on this passage; the middle commentary on book M seems not to treat it;[6] and the part of the long commentary dealing with book M is not extant.[7] That he had some response to Aristotle's remarks on beauty in mathematics is not in doubt. Even leaving aside his intimate familiarity with the Aristotelian corpus, it is clear that Ibn Rushd knew book M of the *Metaphysics*, for he described its content and context in the introduction to book Λ in the long commentary.[8]

Ibn Rushd did not address mathematical beauty directly, though nothing stands explicitly opposed to it in his attitude to aesthetics.[9] But there is an indication that he saw beauty in the circle and circular motion, in his commentary on a part of the *Metaphysics* in which Aristotle discussed the circular motion of the heavenly bodies. Aristotle had only indirectly indicated an aesthetic preference for circular motion,[10][†] and had considered desire as a cause for the circular motion. The commentator Alexander of Aphrodisias (*fl.* c. 200 CE) further explained this statement as meaning that the world desired to imitate the 'first god' — his interpretation of the prime mover — but gave no indication that this motive was in any way aesthetic. Rather, the spher-

[1] al-Ghazzālī, *Alchemy of Happiness*, [vol. 2] p. 908.

[2] *Ibid.*, [vol. 2] p. 909.

[3] *Ibid.*, [vol. 1] p. 29.

[4] Ibn Rushd, '*Prooemium Long Comm. Aristotle's Physics*', p. 83.

[5] Ben Ahmed & Pasnau, 'Ibn Rushd', § 1.2.

[*] ▸ pp. 89–91

[6] Aouad, intro. to Averroes, *Middle Comm.* Metaphysics, p. 39 indicates that the only partially-extant Arabic text does not, but the author has been unable to verify whether the (complete) mediaeval Hebrew translation does.

[7] Averroes, *On Aristotle's "Metaphysics"*, *passim*; Aouad, intro. to Averroes, *Middle Comm.* Metaphysics, pp. 17, 39.

[8] Ibn Rushd, *Comm.* Metaphysics Lām, BOU 1398, 1405.

[9] Puerta Vílchez, *Aesthetics in Arabic Thought*, pp. 658–701.

[10] Aristotle, *Metaphysics*, BEK Λ.1072a22 sqq.

[†] ▸ pp. 91–2

ical shape was simply 'the best and most complete [ἄριστόν τε καὶ τελειότατον]'.[1] Ibn Rushd extolled the aesthetic value that underlay this desire:

> 'that since the longing of the celestial bodies is aroused by the intellect and the intellect desires only what is more beautiful than itself, then it follows necessarily that the celestial bodies desire in this motion that which is more beautiful than themselves, and since they are the most excellent and most beautiful sensible bodies, then the beautiful object which they desire is the most excellent being, in particular that which the whole heaven desires in its daily motion.'[2]

Thus, for Ibn Rushd, the circular *motion* of the celestial bodies was more beautiful than the celestial bodies themselves. The aesthetic primacy of the circle, which Pythagoras had once proclaimed,* was still recognized.

◆

[1] Alexander, *Quaestiones*, BRU 62–3.

[2] Ibn Rushd, *Comm.* Metaphysics Lām, BOU 1597–8.

* ▸ p. 44

7

The Artist and the Mathematician

Practice, theory, and Islamic geometrical art

> ‘The artist and the mathematician
> in Arab civilisation have become one.
> And I mean that quite literally.’
>
> — Jacob Bronowski, *The Ascent of Man*, p. 172.

> ‘no other culture used geometric repeat patterns
> in such inventive and imaginative ways.’
>
> — Richard Ettinghausen, Oleg Grabar & Marilyn Jenkins-Madina,
> *Islamic Art and Architecture 650–1250*, p. 61.

Although there is a strong association of geometrical art with Islam, primary sources do not attest to it being essential; figural depictions, for instance, have formed a component of art, albeit a debated one, throughout Islamic history.[1] From the start, there were restrictions that excluded figural depiction from Qurʾanic illumination and religious buildings. But it does not follow that abstract geometrical designs were a necessary consequence of these restrictions, for there was available a wide range of non-geometrical forms on which to draw. Geometrical designs only became predominant from the tenth and eleventh centuries CE, and not everywhere at once: their rise was bound up with the political and religious events of the time.[2]

When the ʿAbbāsid caliphate was at its height in the ninth century CE, it evinced a tendency towards rationalism, and many of the early thinkers of the period embraced Muʿtazilism, the rationalist theology that used the tools of dialectic to defend Islam.[3] But the ʿAbbāsid caliphate declined politically from the start of the tenth century CE, with secessions and the founding of rival caliphates in the west, and the caliphs themselves fell under the secular control first of the Persian Buyids in 955 CE, and then of the Seljuk Turks in 1056 CE. From the mid-ninth century CE, Muʿtazilism was rejected by the ʿAbbāsid caliphs in favour of the anti-rationalist Sunni sect of Ḥanbalism.[4] The move towards religious traditionalism was doubtless bound up with the loss of worldly power.

In the middle of the tenth century CE, almost the whole house of Islam was under the sway of Shīʿite rulers; the major exception

[1] For a survey, see Gruber, ‘Idols and Figural Images in Islam’.

[2] Necipoğlu, *The Topkapı Scroll*, p. 92; Tabbaa, *The Transformation of Islamic Art*, p. 7.

[3] El-Hibri, *The Abbasid Caliphate*, pp. 117, 121–30; van Ess, *Theology and Society*, esp. pt C [vols 3–4].

[4] El-Hibri, *The Abbasid Caliphate*, pp. 125–6, 196.

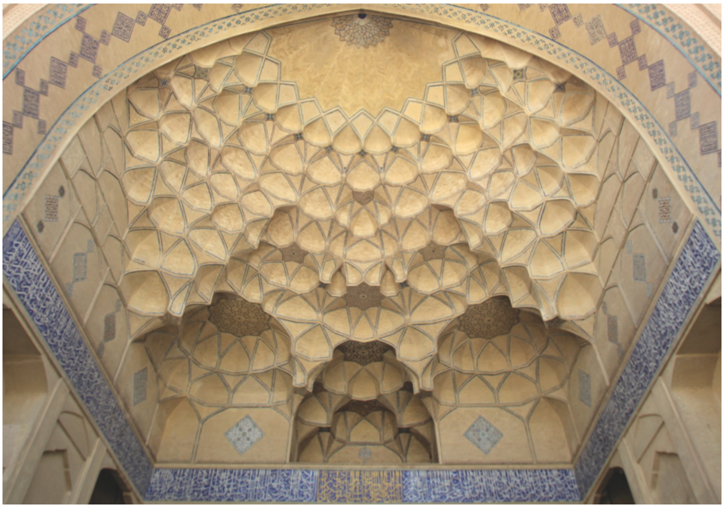

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FIGURE 7.1
The *muqarnas* vaulting in the eastern *ʾīwān* of the Jāmeh Mosque of Isfahan.

was the Umayyad caliphate in al-Andalus. What history came to see as the Sunni revival had its roots in the decision of the caliph al-Qādir (947/8–1031 CE; r. 991–1031 CE), who was politically subject to the Buyid rulers, to put in place a traditionalist orthodoxy that rejected Shīʿism and rationalism. Certain symbolic forms became associated to the ʿAbbāsid caliphate and thence to the Sunni revival, including the *girih* [گره] mode of two-dimensional geometrical patterns, *muqarnas* [مقرنص] vaults, and a geometrically defined script. These forms were an indicator of the spiritual primacy of the ʿAbbāsid caliphs, serving both as a reflection of certain facets of their Sunni theology and a statement of distinction from their political and religious adversaries, the Fāṭimid caliphs.[1]

The origin of the *girih* mode cannot be dated or located conclusively, but indirect evidence points to Baghdad in the ninth or tenth centuries CE.[2] From the beginning, it was based upon a limited stock of star and polygonal configurations, but these formations were combined in ways that excelled previous designs in abstraction, symmetry, and intricacy.[3] Strictly, the word *girih*, which means 'knot', denotes the vertices of the geometrical grids that were the bases for patterns used in architectural plans or in two- and three-dimensional decorations. In any such grid, the vertices had a rotational symmetry of some degree.[4]

A *muqarnas* vault comprises tiered arrangements of repeating units (see Figure 7.1). Each row seems to support the one above it, although most *muqarnas* are decorative and have no structural function. The arrangement of the units is based upon a two-dimensional projection which has a rotational symmetry and can be just as intricate as *girih*.[5]

The abstract geometry of symbolic forms such as *girih*, with their association to the Sunni revival and the ʿAbbāsid caliphate, were adopted more slowly in the Shīʿite Fāṭimid world. There, ornament retained traces of naturalism, and the *girih* mode did not take hold until the second half of the twelfth century.[1] In time, the association of such forms with the ʿAbbāsid caliphate or with its dogma declined,[2] and they became an autonomous style.

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[1] Necipoğlu, *The Topkapı Scroll*, pp. 97, 104; Tabbaa, *The Transformation of Islamic Art*, pp. 8, 72, 133–5.

[2] Necipoğlu, *The Topkapı Scroll*, p. 104.

[3] *Ibid.*, pp. 92–4, 103.

[4] *Ibid.*, p. 9.

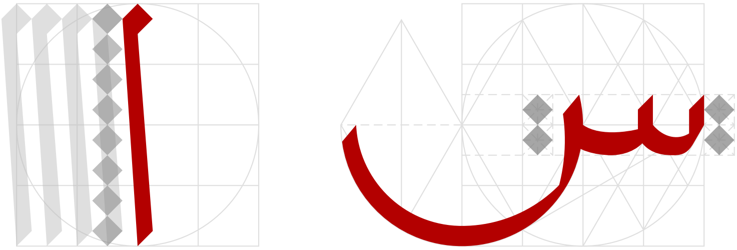
[5] al-Asad, 'The Muqarnas'.

GEOMETRY AND CALLIGRAPHY

There seems to have been a long-standing connection in mediaeval Islamic thought between calligraphy and geometry, at least in the visual sense. Geometry seems to have been illuminated by some of the glory of calligraphy, whose value, beyond the aesthetic, is made clear in the Qur'an.[3] Muḥammad, after all, was the transmitter of the Qur'an; the divinity of the text, not the prophet, was primary. Thus Islam was focused on text and writing to a greater degree than even the other religions of the book.

The emergence of *girih* was approximately contemporaneous with the codification of calligraphy by Ibn Muqla (885/6–940/1 CE), a vizier at the ʿAbbāsid court. Ibn Muqla's exact system can be only tentatively reconstructed from later sources,[4] but it is plausible that he founded his reformed Arabic script on geometry. The proportions were defined using the rhombic dots that a reed pen produced. The *ʾalif* [ا], the first and in some sense the simplest letter of the Arabic alphabet, was to have a height equal to a certain number of dots. Its size defined a circle, which was then used to shape the other letters. (Figure 7.2 shows a later geometrical specification, which is incompatible with Ibn Muqla's requirement that the *ʾalif* be vertical.[5]) Ibn Muqla did not connect geometry and calligraphy *ex nihilo*. The Kufic script [الخط الكوفي], the oldest of the calligraphic Arabic scripts, has a rectilinear variant that lent itself to architectural inscriptions, alongside abstract geometrical designs (see Figure 7.3).

The value assigned to both geometry and calligraphy is summed up in a thought twice quoted with evident approval by Abū Ḥayyān al-Tawḥīdī (c. 930–1023[6]) in his *Epistle on Penmanship*: 'handwriting



Specification of the structure of *ʾalif* [ا].

Specification of *ṣād* [س], based upon the same structure as *ʾalif*.

FIGURE 7.2
An illustration of a geometrical construction of Arabic letter forms (Tabbaa, *The Transformation of Islamic Art*, p. 37, Fig. 6).

[1] Tabbaa, *The Transformation of Islamic Art*, pp. 80–1.

[2] *Ibid.*, p. 128.

[3] Nasr, *Islamic Art and Spirituality*, pp. 21–6.

[4] Blair, *Islamic Calligraphy*, pp. 158–9.

[5] Yaghan, 'Mathematical concepts in Arabic calligraphy', p. 3.

[6] Genequand, 'al-Tawhidi, Abu Hayyan'.

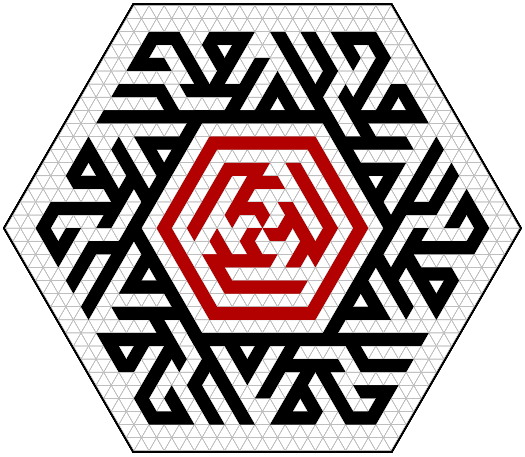


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FIGURE 7.3
Frame 71 from the Topkapı scroll, showing a rotationally symmetric hexagonal panel with Kufic inscriptions of محمد [*Muḥammad*] repeated six times in black and علي [*ʿAlī*] repeated three times in red (Necipoğlu, *The Topkapı Scroll*, pp. 266, 325).

is spiritual geometry by means of a corporeal instrument'.[1] The idea was echoed much later by Aḥmad Ibrāhīmī Ḥusaynī Qummī (b. 1546), who quoted in his treatise *Calligraphers and Painters* (c. 1606) the statement that '[w]riting is the geometry of the soul'.[2] Al-Tawḥīdī attributed the idea to Euclid; Qummī pointed to Plato. These statements by two authors, six centuries apart, and their citation of ancient authorities, suggest that the idea was long-standing and respected. Although calligraphy can clearly be appreciated without any perception of a geometrical structure or an attribution of the expression to the soul, these opinions are evidence of the high status of geometry and calligraphy combined.

ARTISANS AND THEORISTS

Although Islamic geometrical designs are most commonly associated with architecture, they are evident also in ceramics, glassware, textiles, woodworking.[3] Whatever their situation, they were the work of artisans, but there was not a sharp division between the techniques of the artisan and those of the theoretical geometer. The close relationship between theory and practice is evidenced by the placement of application alongside theory in various classifications of knowledge, such as that of al-Fārābī.[4]*

Start with the indirect evidence: the mathematical techniques necessary to create the geometrical patterns found in mediaeval Islamic art and architecture are far from trivial; this alone suggests that theorists would have found them interesting. (Ernest Hankin's (1865–1939) view that only a small amount of geometrical knowledge was needed[5] is accurate only if approximations were accepted in place of exact constructions.) There seems to have been an increase in the precision and complexity of geometrical designs from the thirteenth century.

[1] Rosenthal, 'Abū Ḥaiyān al-Tawḥīdī on Penmanship', pp. 6, 15.

[2] Qāḍī Aḥmad, *Calligraphers and Painters*, p. 52.

[3] See Blair & Bloom, *Cosmophilia*, pp. 107–33.

[4] Necipoğlu, *The Topkapı Scroll*, p. 132.

* ▸ p. 198

[5] See Hankin, *The Drawing of Geometric Patterns*, p. 3.

Departures from regularity that would have previously been tolerated, such as asymmetries and irregular polygons, were to be avoided.[1] This greater precision suggests some influence of the theoretical geometers on the practising artisans.

But such inferences are not the only indication of connection between theory and praxis: there are documented institutional connections. For instance, while his palace at Baghdad was being built, the caliph al-Muʿtadid (853/4 or 860/1–902 CE; r. 892–902 CE) established and financially supported an area for the teaching of both the crafts and the theoretical sciences.[2]

The mathematician Abū'l-Wafā' al-Būzjānī (940–77/8 CE) wrote in his book on geometry useful for craftsmen[*] that he had been present at discussions between artisans and geometers.[3] Al-Khayyāmī (c. 1048–c. 1131), the father of algebra, wrote of attending such a meeting.[4] During the construction of the astronomical observatory in Samarkand, Ghiyāth al-Dīn Jamshīd al-Kāshī (d. 1429) solved a problem about levelling an instrument in front of artisans, scholars, and dignitaries.[5] Caʿfer Efendi (*fl.* early 17th century) wrote of discussions between 'learned men' and architects on the subject of geometry.[6] That these discussions ranged in time and place from tenth-century ʿAbbāsid Baghdad to sixteenth-century Ottoman Constantinople led the historian Alpay Özdural (1944–2003) to posit a tradition of *conversazioni* between the theorists and the practitioners on the subject of geometry: such gatherings would have facilitated an exchange of knowledge — problems, ideas, techniques — between the two groups.[7] The very fact that Abū'l-Wafā' wrote a practical geometry text aimed at and incorporating ideas from artisans, in preference to a presumably more intellectually prestigious theoretical geometry text, points to a collaboration between artisans and geometers,[8] as do references in the anonymous compendium 'On Interlocking Figures' to 'some artisans' and 'some practitioners of geometry'.[9]

Al-Khayyāmī was also involved in tackling problems that arose from geometrical ornamentation.[10] He wrote a treatise offering several solutions to the problem of dividing a line in a particular way. There was a straightforward solution using the theory of conic sections, but al-Khayyāmī was more interested in finding a solution that would be accessible to those, such as artisans, who would not be familiar with the theory.[11] To avoid the use of conics, he offered an algebraic and numerical solution, and geometrical procedures in which one could avoid applying the theory of conics by using 'certain instruments'[†] (which would have fallen foul of Plato's strictures against mechanical methods).[12] It may indeed have been algebraic investigations into a problem raised at a *conversazione* that led al-Khayyāmī to compose his textbook on algebra.[13]

In his book *The Meccan Revelations* [*Kitāb al-Futūḥāt al-Makkiyya* كِتَابُ الفُتُوحَاتِ المَكِّيَّة], the Sufi scholar Ibn ʿArabi (1165–1240) made a number of points that bear on the connection between geometry

[1] Necipoğlu, 'Ornamental Geometries', p. 45.

[2] Necipoğlu, *The Topkapı Scroll*, p. 123; Necipoğlu, 'Ornamental Geometries', p. 61.

[*] ▸ p. 224

[3] Özdural, 'Omar Khayyam, Mathematicians, and *Conversazioni*', p. 55; Necipoğlu, 'Ornamental Geometries', p. 31.

[4] Özdural, 'Omar Khayyam, Mathematicians, and *Conversazioni*', p. 57.

[5] al-Kāshi, undated letter to his father, repr. in Kennedy, 'A Letter of Jamshid al-Kāshī', pp. 198–9.

[6] Caʿfer Efendi, quot. in Özdural, 'Omar Khayyam, Mathematicians, and *Conversazioni*', p. 54, inc. n. 1; Özdural, 'On Interlocking Figures', p. 192.

[7] Özdural, 'Omar Khayyam, Mathematicians, and *Conversazioni*', p. 55; Özdural, 'On Interlocking Figures', p. 192.

[8] Necipoğlu, 'Ornamental Geometries', pp. 33–4.

[9] [Anonymous], 'On Interlocking Figures', fols 187v, 188v; see also Özdural, 'On Interlocking Figures', p. 192.

[10] Özdural, 'Omar Khayyam, Mathematicians, and *Conversazioni*', pp. 59–60.

[11] *Ibid.*, pp. 60–1.

[†] ▸ p. 79

[12] *Ibid.*, p. 61.

[13] *Ibid.*, p. 79.

and art. He discussed how a productive process can span individuals: one person can plan theoretically and another can implement practically. His specific example was how a geometer can communicate knowledge to a carpenter; the resulting artefact is the product of the combined minds.[1] More directly germane is the explanation he offered of al-Bārī, one of the names of God in Islam, which translates as ‘the Innovator’ or ‘the Evolver’:

> ‘As for the Name al-Bārī [الباري], from it comes [God’s] expansion toward intelligent geometers, discoverers, inventors of arts, and makers of unusual shapes. They are all inspired by this Name, which extends to those who create beautiful forms on the balance-scale [*mīzān* مِيزَان].’[2]

The term *balance-scale* is here a term of art referring to divine artistic inspiration.[3] The importance of Ibn ‘Arabi’s explanation is that it serves as a link between geometry, beauty, and art.

Another connection between the theoretical geometer and the practical craftsman is in the person of Abu’l-Faḍl ibn ‘Abd al-Karīm (d. 1202/3), whom Ibn abi Uṣaybiʿah recorded as having been called ‘the geometrician [*al-muhandis*]’ for his famed knowledge of geometry. He was by profession a very skilled carpenter, and also a stonemason. He studied Euclid’s *Elements* to improve his carpentry; he then proceeded to Ptolemy’s *Almagest* and became an authority in geometry. His skill is evident in the doors he made for the Bīmaristān Nūr al-Dīn in Damascus, which have upon them a classic example of *girih*, made up of brass nails that form a pattern of hexagons and pentagrams (see Figure 7.4).[4]

MEANING IN GEOMETRICAL ART

There has been a historiographical tendency to attribute some deep symbolic, religious, or spiritual dimension to Islamic geometrical art, but such work has been criticized as unrigorous and prone to unsubstantiated sweeping generalizations,[5] such as Titus Burckhardt’s (1908–84) statement that

> ‘geometric interlacement doubtless represents the most intellectually satisfying form for it is an extremely direct expression of the ideas of the Divine Unity, underlying the inexhaustible variety of the world.’[6]

In fact, there is no contemporaneous information about any meaning that the *girih* mode was seen to have had when it first emerged.[7] Unwarrantedly assigning religious and spiritual meaning to Islamic geometrical designs is a symptom of what the historian of art Ernst Gombrich (1909–2001) called ‘refusal to accept the autonomy of decorative

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[1] Akkach, *Cosmology and Architecture in Premodern Islam*, p. 47.

[2] Ibn ‘Arabi, quot. in Puerta Vílchez, *Aesthetics in Arabic Thought*, p. 813, ins. ‘[God’s]’ in ed.

[3] *Ibid.*, p. 813, n. 1035.

[4] Ibn Abī Uṣaybiʿah, *A Literary History of Medicine*, § 15.33 ⟦vol. 2-2, pp. 1271–4; trans. vol. 3-2, pp. 1444–7⟧.

[5] See, for example, Necipoğlu, *The Topkapı Scroll*, ch. 5; Tabbaa, *The Transformation of Islamic Art*, p. 75.

[6] Burckhardt, *Art of Islam*, p. 73.

[7] Necipoğlu, *The Topkapı Scroll*, p. 116.

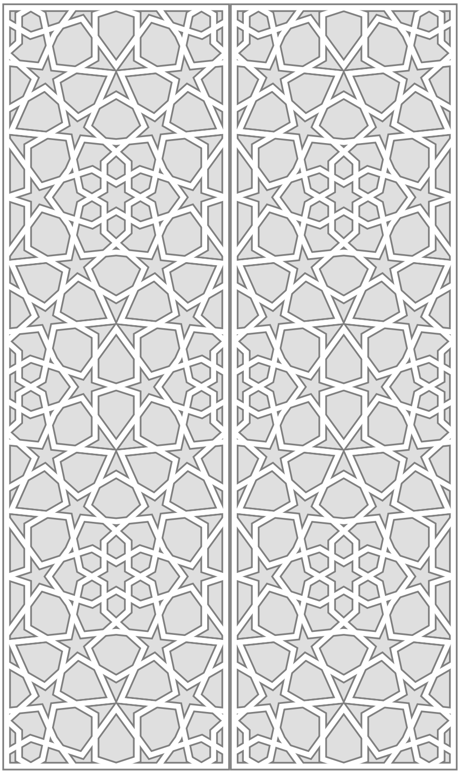


FIGURE 7.4
The *girih* on the doors of the Bīmaristān Nūr al-Dīn, designed by Abu'l-Faḍl ibn ʿAbd al-Karīm.

design'.[1] Perhaps the most that can be said is that the *girih* mode's association to religious architecture could have been self-reinforcing: it would affirm the religious nature of the structure and become an ever-more-strongly accepted indicator of religious significance.[2]

As a matter of experience, it is quite possible to contemplate and find beauty in the geometrical artworks without being aware either of the theoretical principles that underlie their construction or of any notion that they are supposed to have some worth beyond the visual beauty.[3] While there are common characteristics sufficient to identify styles of geometrical art, it seems to have no function beyond visual beautification of the building or object, and thus seems to lack any intentional intellectual content.[4]

Perhaps the most that one can say is that a colourful abstract geometrical design gives primacy to light and to colour, which are the first two properties producing beauty that Ibn al-Haytham (965–c. 1040) listed in the discussion in his *Optics*.[5]* While it is doubtful that Ibn al-Haytham had an influence upon the development of geometrical art, he was certainly part of the same cultural milieu.

[1] Gombrich, *The Sense of Order*, p. 225.

[2] Tabbaa, *The Transformation of Islamic Art*, p. 102.

[3] Leaman, *Islamic Aesthetics*, pp. 138–40.

[4] Grabar, *The Formation of Islamic Art*, p. 179.

[5] Ibn al-Haytham, *Optics*, §§ II.3.202–3; cf. Necipoğlu, *The Topkapı Scroll*, p. 193.

* ▸ pp. 207–9

ABŪ’L-WAFĀʾ AND ARTISANAL GEOMETRY

Abū’l-Wafāʾ al-Būzjānī (940–77/8 CE) wrote the practical *Book on What the Artisan Requires of Geometrical Constructions* [*Kitāb fī mā yaḥtaju ilayhi al-ṣāniʿ min al-aʿmāl al-handasa* كتاب في ما يحتاج إليه الصانع من الأعمال الهندسية], for brevity called *Geometrical Constructions*. This book includes techniques for inscribing regular polygons in circles, constructions of the five regular and two of the archimedean solids,[1] and mechanical methods for the approximate duplication of the cube and the trisection of an angle.[2] It seems to have had wide influence, with its circulation being evidenced by its survival in five Arabic and three Persian manuscripts, the latter seemingly from two different translations, with two commentaries in Arabic and one in Persian.[3]

In the *Geometrical Constructions*, Abū’l-Wafāʾ’s stated aim was to

> ‘substantiate the notions that were discussed in the presence of His Excellency, concerning the geometrical constructions most commonly employed by artisans leaving out all the demonstrations and proofs, in order to make these easy for artisans (*al-ṣunnāʿ*) to use and to familiarize them with their method.’[4]

He wished to describe geometrically correct procedures of practical use, both because artisans often made geometrical errors, and because geometers were sometimes unsuccessful in extracting an applicable method from a correct theoretical solution.[5]

For instance, in the tenth chapter of *Geometrical Constructions*, Abū’l-Wafāʾ wrote that he had participated in a *conversazione* between artisans and geometers, at which the artisans raised the question of how to construct a square with area three times a given square. To the algebraist, this problem meant constructing a square with side length $\sqrt{3}$, and a geometer had no difficulty in giving a theoretically correct solution. But the artisans were not satisfied: they desired to see three squares of side length 1 dissected and re-assembled into a single square.[6]

At the same *conversazione*, Abū’l-Wafāʾ showed the artisans a proof of the pythagorean theorem. The diagram he used (see Figure 7.5 (upper left)) seems to have been admired by the artisans, for it was incorporated into architecture, generally with the right-angled triangles having the ratio $1 : 2$, which gives the outer square side length 3, such as on the western[7] and northern[8] *ʾīwānāt* of the Jāmeh Mosque of Isfahan and the Imam Ibrahim Mosque in Mosul.[9] This diagram could be used to generate an ornamental pattern by doubling the right-angled triangles to form kites, dividing the kites, and then repeating the diagram (see Figure 7.5 (lower left, right)).[10]

[1] Youschkevitch, ‘Abū’l-Wafāʾ al-Būzjānī’, p. 41.

[2] Necipoğlu, *The Topkapı Scroll*, p. 138; Özdural, ‘Omar Khayyam, Mathematicians, and *Conversazioni*’, pp. 61–2.

[3] Necipoğlu, ‘Ornamental Geometries’, p. 32.

[4] Abū’l-Wafāʾ, quot. in *ibid.*, p. 31; some transliterations omitted.

[5] *Ibid.*, p. 25.

[6] *Ibid.*, pp. 24–5.

[7] Özdural, ‘Omar Khayyam, Mathematicians, and *Conversazioni*’, p. 58.

[8] Kheirandish, ‘An Early Tradition in Practical Geometry’, p. 95.

[9] *Loc. cit.*

[10] Özdural, ‘Omar Khayyam, Mathematicians, and *Conversazioni*’, pp. 58–9.

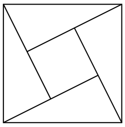
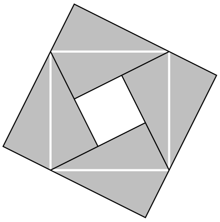
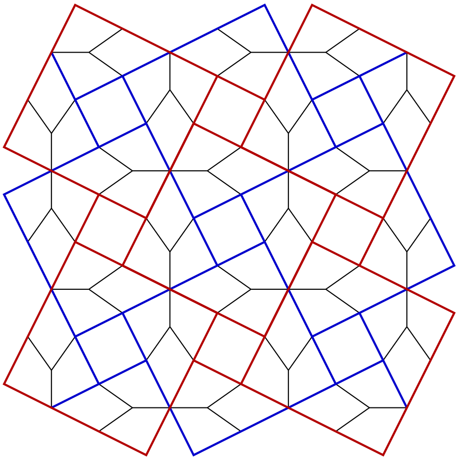


Diagram for proving the pythagorean theorem



Right-angled triangles doubled to produce kites



Tiling made from the resulting shape, with kites divided into smaller kites

FIGURE 7.5
A diagram Abu'l Wafā' used to prove the pythagorean theorem, and the ornamental forms it generates. (The right-angled triangles have the ratio 1 : 2.)

'ON INTERLOCKING FIGURES'

In one Persian compilation of mathematical texts, mainly on practical geometry, the *Geometrical Constructions* of Abū'l-Wafā' is followed by an anonymous compendium 'On Similar and Complementary Interlocking Figures' [*Fī tadākhul al-ashkāl al-mutashābiha aw al-mutawāfiqa* في تداخل الأشكال المتشابهة او المتوافقة], for brevity called 'On Interlocking Figures', which comprises drawings of ornamental geometry with explanatory texts.[1] It is limited to two-dimensional forms, and does not include geometrical patterns for *muqarnas* vaults or *bannā'ī* [بنائى] masonry designs.[2] (The difference between *bannā'ī* and tiling is that in the former case, the pattern is part of the *structure* of the building, not a surface detail.[3])

The authors of 'On Interlocking Figures' are unknown, but various textual hints suggest that it was compiled in the fourteenth century,[4] possibly from notes taken at *conversazioni* between artisans and geometers. *Geometrical Constructions* and 'On Interlocking Figures' seem to be the only mediaeval geometrical texts that are known to have been directly addressed to craftsmen.[5]

'On Interlocking Figures' shows that the *girih* mode comprised related geometrical patterns that were designed to harmoniously link together.[6] It consists of procedures for constructing *girih* patterns, drawing on the more general geometry of Abū'l-Wafā' and others. The procedures include both theoretically correct constructions and simpler mechanical methods.[7] For instance, it presents a mechanical method for constructing the ornamental design shown in Figure 7.6 (right). The key is to find the point *H*; after *H* has been constructed, the remainder of the configuration can be found using elementary techniques such as angle bisection. One could find *H* using the theory of conic sections, but there is a simpler mechanical method: to find *H*, drop a perpendicular from the top of the square to its intersection

---

[1] Özdural, 'On Interlocking Figures', p. 191.

[2] Necipoğlu, *The Topkapı Scroll*, p. 172.

[3] Kana'an, 'Architectural Decoration in Islam', pp. 365–6.

[4] Necipoğlu, 'Ornamental Geometries', pp. 12–24.

[5] Necipoğlu, *The Topkapı Scroll*, p. 139.

[6] *Ibid.*, pp. 133–4.

[7] *Ibid.*, p. 169.

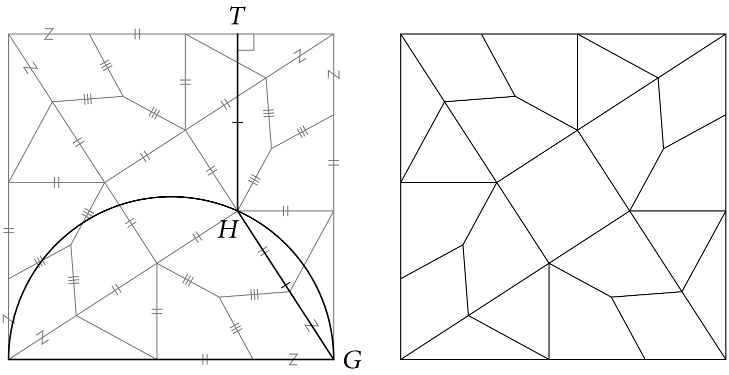


FIGURE 7.6
A mechanical construction of the configuration on the right, from 'On Interlocking Figures' ([Anonymous], 'On Interlocking Figures', 191r). The point H is found by moving the perpendicular until HT and HG have the same length; the rest of the construction uses only elementary geometry.

with a semicircle erected on the bottom, then move the perpendicular so that the distance from the intersection to the top equals the distance from the intersection to G; when this equality is achieved, the intersection is the required point H.

It has been suggested that the geometrical patterns in 'On Interlocking Figures' point to an unwritten conception of beauty linked with proportion and harmony.[1] Exactly one of the constructions in 'On Interlocking Figures' is explicitly praised in aesthetic terms, as being 'extremely elegant [*bi-ghāyat laṭīf ast* بغايت لطيف است]'.[2] The relevant procedure is a way of dissecting a square so as to provide pieces that can be assembled into a choice of stars and polygons; see Figure 7.7. Whether the elegance is seen by an artisan or a geometer or both is not clear; the former might be more likely, since the elegance seems somehow connected to the efficient use of the materials.

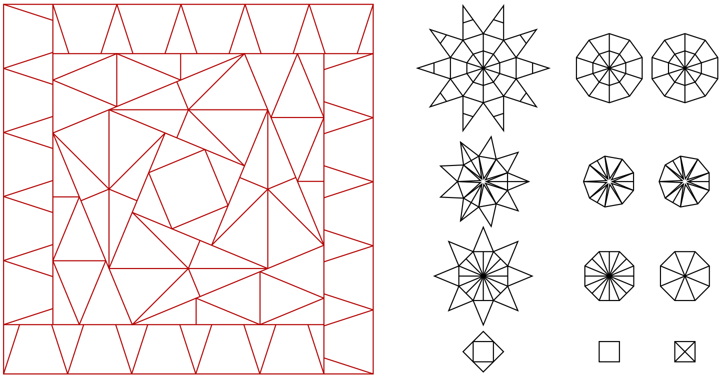


FIGURE 7.7
The dissection of a square prescribed in 'On Interlocking Figures', fol. 180v, p. 190, and shapes that can be constructed from the pieces: either a 10-pointed star or two decagons; either an approximate 9-pointed star or two approximate nonagons; either an 8-pointed star or two octagons; and either a large square or two small ones.

IBN KHALDŪN

The philosopher and historian Ibn Khaldūn's (1332–1406) *magnum opus* was a world history whose first book, the *Muqaddimah* or *Introduction*, is seen in retrospect as a founding text of social science. Although he wrote nothing about the application of geometry to visual art,[1] certain passages indicate a recognition of its role mediated via carpentry.

---

[1] Necipoğlu, 'Ornamental Geometries', pp. 60–1.

[2] [Anonymous], 'On Interlocking Figures', fol. 180v, p. 190.

For Ibn Khaldūn, aesthetically correct carpentry required a knowledge of proportions. It thus did not simply make use of geometry, but depended upon it:

> 'In view of its origin, carpentry needs a good deal of geometry of all kinds. It requires either a general or a specialized knowledge of proportion and measurement, in order to bring the forms (of things) from potentiality into actuality *in the proper manner*, and for the knowledge of proportions one must have recourse to the geometrician.'[2]

It was for this reason, he thought, that the preëminent Greek geometers had also been 'master carpenters': Euclid, Apollonius, Menelaus, and others.[3]

He also made the broader point that the theory of conic sections played an auxiliary role in plastic arts such as sculpture and architecture.[4] In general, he seems to have placed a high aesthetic value on symmetry when he discussed how carpentry can be used to create works that are visually pleasing as well as functional:

> 'one puts these pieces together in certain symmetrical arrangements and nails them together, so that they appear to the eye to be of one piece. They consist of different shapes all symmetrically combined. This is done with all the possible shapes in which wood can be cut, which turn out to be very elegant things.'[5]

He described in general terms the use of geometrical designs to cover walls, emphasizing the importance of symmetry:

> 'The walls are occasionally also covered with pieces of marble, brick, clay, shells (mother-of-pearl), or jet. (The material) may be divided either into identically shaped or differently shaped pieces. These pieces are arranged in whatever symmetrical figures and arrangements are being utilized.'[6]

LATER GEOMETRICAL ART

When the *girih* mode emerged in the tenth and eleventh centuries, it was closely associated with the ʿAbbāsid caliphate, centred on Baghdad, and thus it was not so readily accepted across north Africa and in al-Andalus, which were ruled by the Fāṭimid and Umayyad dynasties. In the Maghreb and al-Andalus, the spread of the geometrical mode came later, when they were unified under the

[1] Puerta Vílchez, *Aesthetics in Arabic Thought*, p. 161.

[2] Ibn Khaldûn, *Muqaddimah*, vol. 2, p. 365, emphasis added.

[3] *Ibid.*, vol. 2, p. 365.

[4] *Ibid.*, vol. 3, p. 132.

[5] *Ibid.*, vol. 2, p. 364.

[6] *Loc. cit.*

Later geometrical art

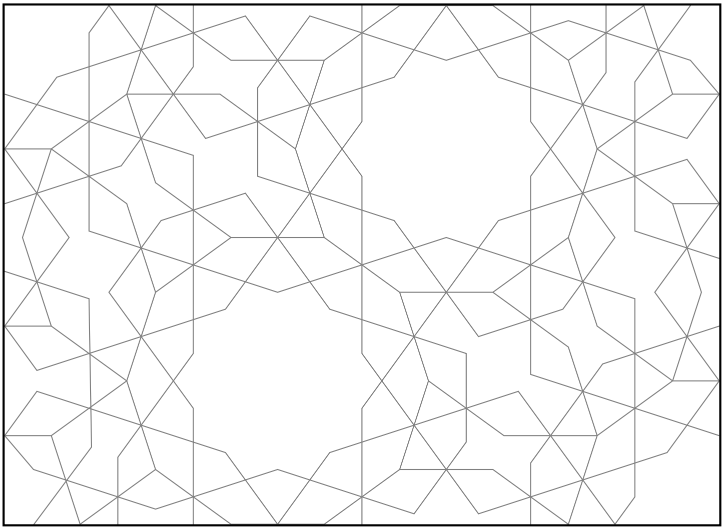


FIGURE 7.8
Frame 52 from the Topkapı scroll (redrawn) (Necipoğlu, *The Topkapı Scroll*, pp. 260, 315).

Almoravid dynasty (1056–1147), which acknowledged the spiritual leadership of the ʿAbbāsid caliphs.[1]

The internal development of geometrical patterns in the Maghreb and al-Andalus reached its apogee in the designs that adorn the Alhambra, the great palace and fortress built by the Naṣrid emirs of Granada (1230–1492). These rulers had reacted against more ascetic artistic standards that had been imposed by the Almohad caliphate (1121–1269), under the influence of al-Ghazālī, and they reintroduced richer, more luxurious designs.[2] The Alhambra represents the culmination of three centuries of development of geometrical planar artwork and *muqarnas* vaulting,[3] and is perhaps the symbol and the exemplar of Islamic geometrical art.

Yet if the greatest product of this artistic mode is in the far west, it is in the east where survives the best documentary evidence of the underlying geometrical techniques. From the compendium ‘On Interlocking Figures’ descend Timurid-Turkmen and Uzbek scrolls that describe intricate *girih*, which, for all the sophistication in their designs, are unlikely to have been much informed by theoretical mathematics.[4] Foremost among these works is the Topkapı scroll, a collection of drawings of geometrical patterns that would have been applicable to architecture in the Timurid-Turkmen sphere or early Safavid Persia. It comprises square geometrical patterns and inscriptions for *bannāʾī* brickwork patterns, tiling patterns of stars and polygons, and projections of *muqarnas* vaults.[5] The kind of designs it contains and its internal organization point to it being compiled in western or central Iran in the late fifteenth or the sixteenth centuries, and not being a copy of an older document.[6]

The Topkapı scroll embodies the collective memory of master builders who refined and extended the mediaeval *girih*, using geometrical imagination, manuals of practical geometry, and doubtless

[1] Necipoğlu, *The Topkapı Scroll*, p. 101.

[2] *Ibid.*, p. 102.

[3] Tabbaa, *The Transformation of Islamic Art*, p. 128.

[4] Necipoğlu, *The Topkapı Scroll*, p. 172.

[5] *Ibid.*, p. 31.

[6] *Ibid.*, pp. 32–4, 37–8.

empirical experience.[1] The innovations around the time the scroll was compiled seem to point to a self-conscious desire to outdo previous designs in terms of intricacy and elaboration (see Figure 7.8).[2]

Further documentary evidence lies in the fragmentary Tashkent scrolls, which comprise a series of drawings of ground plans, geometrical and calligraphic patterns, and two-dimensional projections of *muqarnas* vaults. There is no explanatory text or measurement, which suggests that they were artisanal *aides-mémoires* of established ideal patterns, whose common theme is the symmetrical repetition of a basic unit. They have been dated to the sixteenth century, but the conservatism of architectural practice in the region suggests that they are representative of the standards of the fifteenth century Timurid-Turkmen sphere.[3]

EXALTATION OF GEOMETRICAL ART

The aesthetic experience of geometrical art is attested by various sources. The traveller and poet Ibn Jubayr (1145–1217) wrote that on seeing the pulpit in the Great Mosque of Aleppo: 'the eyes consider the most beautiful sight in the world' because of the interlocking polygons that are joined 'without any interval appearing'.[4]

Caʿfer Efendi (*fl.* early 17th century), the author of a treatise on architecture, recorded the reaction of Sultan Murad III (1546–95; r. 1574–95) to a lectern with a geometrical design:

> 'From top to bottom [it was covered with] the interlocking sides of triangles and quadrangles and the sides of pentagons and hexagons and heptagons, and the patterns were possessed of various forms. That is, looking from one angle, one type of form of circle was seen, and when one looked again at that place from another angle, other types of designs and patterns emerging, other forms appeared. [...] When the late [sultan], out of his delight, turned and examined and inspected it, now from this direction, now from that, he recited extemporaneously this noble verse.
>
> God! God! What are these beautiful forms?
> Like wine, they instantly caused me to lose my head.'[5]

Those who decorated the Alhambra were evidently very conscious of the splendour of the geometrical designs they created. Consider the Hall of the Two Sisters alone, which Jules Goury (1803–34), Owen Jones (1809–74) & Pascual de Gayangos (1809–97) described in their survey as being 'unequalled by any other parts of the palace, for the beauty and symmetry of the ornaments'.[6] It is perhaps difficult to disagree with their assessment (see Figure 7.9). In this Hall, the inscriptions emplaced by the designers sing the praises of its ornament:

---

[1] Necipoğlu, *The Topkapı Scroll*, p. 172.

[2] *Ibid.*, p. 215.

[3] *Ibid.*, pp. 9–14.

[4] Ibn Jubayr, *The Travels of Ibn Jubayr*, p. 283.

[5] Caʿfer Efendi, *Risāle-i Miʿmāriyye*, p. 34; insertions in orig.

[6] Goury, Jones & Gayangos, *Alhambra*, vol. I, description of pl. XV.

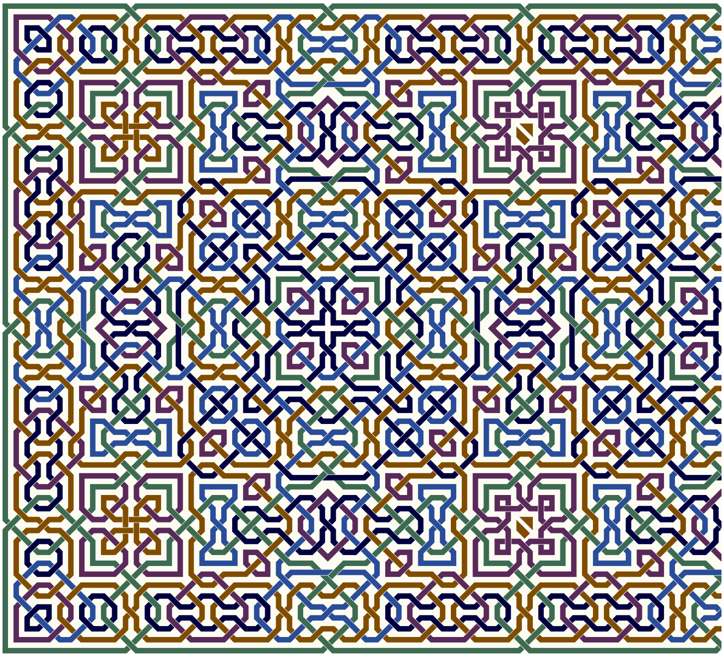


FIGURE 7.9
A mosaic from the Hall of the Two Sisters in the Alhambra (Goury, Jones & Gayangos, *Alhambra*, vol. I, pl. XLII).

> ‘I am the garden, and every morn do I appear decked out in beauty. Look attentively at my elegance, thou wilt reap the benefit of a commentary on decoration.’
>
> ‘Here are columns ornamented with every perfection, and the beauty of which has become proverbial.’[1]

LEGACY

Beauty created in the tradition of Islamic geometrical art has had a lasting effect. Perhaps the most visible influence was on the work of the artist M. C. Escher (1898–1972), who had his existing interest in tessellations and geometry greatly strengthened by a visit to the Alhambra in 1936.[2]*

The very sophistication of mediaeval Islamic geometrical artworks has tempted some to project modern mathematical conceptions back onto their creators. In 2007, Peter J. Lu (1978–) & Paul J. Steinhardt (1952–), who were involved in the search for natural quasi-crystals,† suggested that aperiodic tilings could be found in Islamic geometrical art. From an analysis of numerous *girih* patterns, they concluded that there had emerged before 1200 a new technique for constructing tessellations using five tiles laid so that lines on their surfaces are aligned.[3] This new method, they maintained, allowed the subdivision of large tiles into smaller ones,[4] in a way that is similar to the kind of

[1] Goury, Jones & Gayangos, *Alhambra*, vol. I, description of pl. XV.

[2] Bool et al., *M. C. Escher*, pp. 50, 52–5.

* ▸ pp. 698–700

† ▸ pp. 926–7

[3] Lu & Steinhardt, ‘Decagonal and Quasi-Crystalline Tilings’, p. 1106.

[4] *Ibid.*, p. 1107.

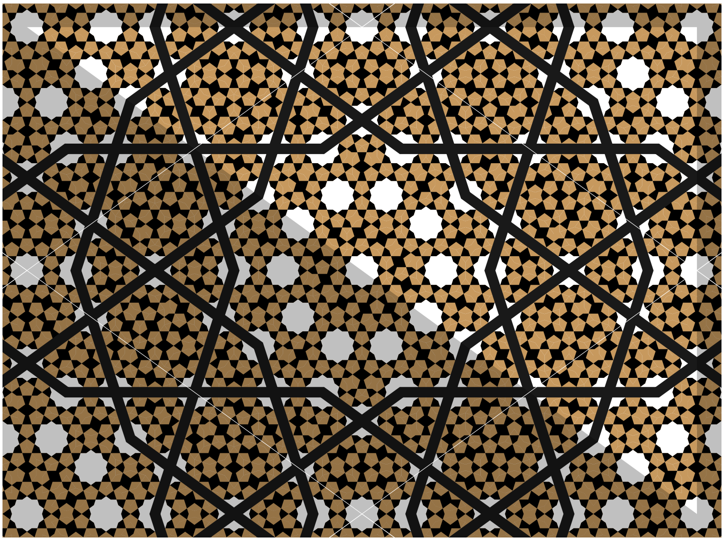


FIGURE 7.10
An extension of a tiling found in a spandrel in the Imamzadeh Darb-i Imam. The unshaded area is an idealized version of the original tiling. The shaded area shows how the original can be extended to a periodic pattern, with the repeating unit outlined in white.

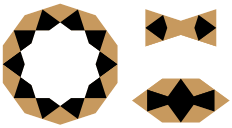


FIGURE 7.11
Tiles that could compose the tiling in Figure 7.10.

self-similarity constructions used to create many kinds of aperiodic tilings.

One of Lu & Steinhardt's key examples was the tiling in a spandrel at the Imamzadeh Darb-i Imam [امامزاده درب امام] in Isfahan. A geometrically idealized version of this spandrel tiling occupies the unshaded area in Figure 7.10. This pattern can be assembled by three tiles, shown in Figure 7.11. It seems likely, as Lu & Steinhardt pointed out, that the tiling was created by subdivision of the large-scale tiling formed by the black lines,[1] and there is an abundance of designs exhibiting such subdivision of tiles.[2]

In particular, as the extension of the tiling (shown shaded in Figure 7.10) makes clear, the tiling of the central area has decagonal symmetry. As Lu & Steinhardt noted, a subdivision and decagonal symmetry can be combined to yield aperiodic tilings akin to Penrose tilings,[3][*] and the spandrel tiling pattern can be mapped almost exactly to a Penrose tiling.[4]

But while the *tools* to create aperiodic tilings were available, it is a stretch to suggest that the *concept* had been formulated. The idea of an aperiodic tiling depends upon tiling (in principle) the entire plane in a way that does not repeat. This in turn implies the idea of a potentially infinite recursive application of a substitution rule and an inflation rule to expand the tiles back to their original size. The visual evidence suggests a single 'substitution', in the sense of a filling-in of areas to create a two-level design. The pattern of the Imamzadeh Darb-i Imam spandrel is a small enough patch that no translational symmetry is evident. And so, whatever ideas were held by the artisans who created it, the pattern can be seen as part of an aperiodic tiling. But it is simpler to imagine a two-level design based on filling in a periodic pattern, such as shown in Figure 7.10.[5]

[1] Lu & Steinhardt, 'Decagonal and Quasi-Crystalline Tilings', p. 1107.

[2] See, for example, Bonner, *Islamic Geometric Patterns*, § 1.24.

[3] Lu & Steinhardt, 'Decagonal and Quasi-Crystalline Tilings', p. 1108.

[*] ▸ p. 689

[4] *Loc. cit.*

[5] Cf. Cromwell, 'The Search for Quasi-Periodicity', pp. 52–4; Makovicky, 'Comment on "Decagonal and Quasi-Crystalline Tilings"'.

Without some further evidence, to posit an awareness of aperiodic tilings in the mediaeval Islamicate world is to lapse into fallacy.[1] Nor is there any need to: the great achievements of Islamic geometrical art do not need to be buttressed by anachronistic hypothetical advances in theory.

◆

[1] Cromwell, 'The Search for Quasi-Periodicity', p. 36; Saltzman, 'Quasi-Periodicity', pp. 586–7.

A Kind of Concord

8

Mathematical beauty in the Renaissance

> ‘Beauty is a kind of concord
> and mutual interplay of the parts of a thing.
> This concord is realized in a particular number, proportion,
> and arrangement demanded by harmony,
> which is the fundamental principle of nature.’
>
> — Leon Battista Alberti, *De re aedificatoria*, bk ix, ch. 5
> (trans. Adam Czerniawski & Ann Czerniawski,
> quot. in Tatarkiewicz, *Hist. Aesthetics*, vol. iii, p. 94).

> ‘Many other artes also there are which beautifie the minde of man:
> but of all other none do more garnishe & beautifie it,
> then those artes which are called Mathematicall.’
>
> — Henry Billingsley, ‘The Translator to the Reader’, n.p.
> In: Euclid of Megara [*sic*], *Elements of Geometrie*.

In the Renaissance, culture — in its broadest sense — began a slow turn away from spiritual to temporal matters, and personal identity from the group to the individual.[1] Art, science, and commerce advanced, first in Italy, then elsewhere in Europe. There developed a philosophy of active participation in political life and a more secular conception of culture, in part shaped by and also shaping a new direct engagement with the heritage of antiquity.[2]

The germ of the revival of mathematics had come earlier, in the twelfth and thirteenth centuries, with translations from the Arabic and Greek and the founding of the first European universities.[*] As the Renaissance progressed, more ancient mathematical texts became available in Latin translation. And with the introduction of printing using moveable type, the ancient texts could circulate more freely. The first mathematical work of any importance to be printed in Europe, and the first to incorporate geometrical diagrams, was the edition of Euclid's *Elements* published in Venice in 1482 by Erhard Ratdolt (1447–before 1529), based on Campanus of Novara's (c. 1220–1296) translation from the Arabic.[3]

Federico Commandino (1509–75), the most prolific translator, published Latin versions of works by Apollonius, Archimedes, Aristarchus, Euclid, Eutocius, Hero, Pappus, Ptolemy, and Serenus.[4] Further translations were made into other languages, either directly from the Greek, like Henry Billingsley's (d. 1606) English translation of Euclid's *Elements*, or from the Latin, like Niccolò Tartaglia's

[1] Burckhardt, *The Civilisation of the Renaissance in Italy*, pt ii.

[2] Baron, *The Crisis of the Early Italian Renaissance*.

[*] ▸ pp. 176–7

[3] Heath, intro. to Euclid, *Elements*, vol. i, p. 97.

[4] Rosen, ‘Commandino, Federico’, p. 364.

(1499/1500–1557) Italian translation of Archimedes and Pierre Forcadel's (c. 1500–c. 1573) French translations of Euclid and Archimedes.

Renaissance mathematicians knew — and declared — that they were recovering and building upon an ancient tradition. For example, one of the most important new developments in mathematics was the growth of algebra out of the abacist schools of practical calculation. But François Viète (1540–1603), who would begin the reformulation of algebraic techniques into a theory of equations, saw himself as recovering a lost mathematical art that the Greeks had known.[*]

Archimedes was the paragon:[1] Regiomontanus (1436–76), for instance, said that he was preëminent among ancient mathematicians, for the breadth and variety of his work;[2] Girolamo Cardano (1501–76) placed him first in a list of twelve outstanding scientific and philosophical minds.[3] Perhaps it was proclaimed most unequivocally by Peter Ramus (1515–72), who gave the apotheosis of Archimedes a divine mandate:

> 'God had decided that there should be in each art something like a unique idea which everybody studying the discipline would propose to himself as a model — as in eloquence, Demosthenes and Cicero, and in medicine Hippocrates and Galen — thus Archimedes in mathematics.'[4]

There also seems to have been aesthetic admiration for Archimedes' mathematics: Bernardino Baldi (1553–1617) wrote in his biographical study of Italian mathematicians that Guidobaldo del Monte (1545–1607) had demonstrated that the work of Archimedes on levers was in accordance with the text *Mechanics*, which was attributed to Aristotle but is now considered spurious. Thus

> 'Archimedes followed entirely in the footsteps of Aristotle as to the principles, but added to them the exquisiteness [esquisitezza] of his proofs.'[5]

Baldi's report almost suggests that the 'exquisiteness' of Archimedes' proofs was to be expected. [Baldi was also sensitive to the aesthetic value of unapplied mathematics, writing that in Archytas' method for duplicating the cube[†] 'the demonstration is beautiful and the intuition ingenious', in spite of it being difficult to apply in practice.[6]]

When Forcadel translated the pseudo-Archimedean[7] mediaeval text *De ponderibus Archimenidis*, he declared an aesthetic motivation:

> 'I have translated and annotated this work so that every one can revel in such a beautiful treatment. […]
>
> […] May God give me the grace to be able with diligence to benefit those who learn and to satisfy those who are studious in the mathematical sciences with many beautiful demonstrations, as I hope to have done in this book.'[8]

Forcadel's translation was based upon Francisco de Mello's (1490–1536) heavily reworked version of the text, which in particular con-

[*] ▸ pp. 272–4

[1] Høyrup, 'Archimedism, not Platonism'.

[2] Regiomontanus, 'Oratio', p. 45.

[3] Cardanus, *De Subtilitate*, p. 607.

[4] Ramus, *Scholarum Mathematicarum*, p. 26; trans. Høyrup, 'Archimedism, not Platonism', p. 99.

[5] Baldi, 'Vite Inedite di Matematici Italiani', p. 439; trans. mod. Drake, Intro. to Drake & Drabkin, *Mechanics in Sixteenth-Century Italy*, p. 15.

[†] ▸ p. 53

[6] Baldi, 'Vite Inedite di Matematici Italiani', p. 367.

[7] Clagett, *The Fate of the Medieval Archimedes* (Part III), p. 1071.

[8] Forcadel, preface to Archimede, *Le Livre des Pois*, pp. 3, 30 [incorrectly numbered 03]; trans. Clagett, *The Fate of the Medieval Archimedes* (Part III), pp. 1122, 1140.

tained changed proofs.[1] It is a matter for speculation whether Forcadel would have praised the book in such glowing aesthetic terms had he known that work was not by Archimedes, nor even by a pseudo-Archimedes of the Middle Ages, but largely by a contemporary. Nevertheless, Forcadel seems to have written as though his aesthetic judgement of the pseudo-Archimedes would be uncontroversial.

If mathematics itself was guided by the model and the spirit of Archimedes, its aesthetic evaluation in the Renaissance was platonic. Scholars who wrote on aesthetics looked primarily to Plato and Vitruvius as their guides.[2] Renaissance platonism was only in a limited way an inheritance from the Middle Ages; of greater influence were the works of Plato himself, available anew to European scholars. Much of this chapter is devoted to this platonic current, most evident in the renewed veneration of the five regular solids. But the sphere of platonism and the regular solids was gradually circumscribed, as other solids began to play a greater role; notably, the archimedean solids were gradually rediscovered; see Figure 8.1. The thought of Johannes Kepler (1571–1630) was the swansong for the regular solids in an aesthetically-driven cosmology: he celebrated them in his earlier work, but dislodged them in favour of conic sections in his later work.[*]

PLATONISM AND POLYHEDRA

Just as mathematics benefitted from a surge in translation and publication, so too did philosophy. For most of the Middle Ages, Europe had access to Plato's thought only through the partial translation of the *Timaeus* by Calcidius (*fl.* c. late 4th/early 5th century CE), supplemented by a number of shorter testimonia.[†] In the later Middle Ages, Henricus Aristippus (1105/10–62 CE) produced less than competent translations of the *Phaedo* and the *Meno*, and William of Moerbeke partially translated the *Parmenides*.[3] In the fifteenth century, there was an explosion in the number of translations, culminating in Marsilio Ficino's (1433–99) publication at the end of the century of the first complete Latin version of the Platonic corpus.[4] Platonism took a strong hold on the philosophy of the time, although there is no simple reason why Plato held such attraction for the Renaissance.[5] Here the concern is limited to the effect on the aesthetics of mathematics.

Calcidius' incomplete version of the *Timaeus* ends before the description of the roles of the fairest triangles[‡] and the regular solids[§] in the cosmos. But the Timaean association of polyhedra with elements was not entirely unknown: Calcidius made passing mentions in his commentary on the *Timaeus*,[6] and Aristotle discussed it in *On the Heavens* in order to criticize it on physical grounds.[7] A new awareness of the Timaean role of polyhedra thus entered European

[1] Clagett, *The Fate of the Medieval Archimedes* (Part III), pp. 1075–7.

[2] Tatarkiewicz, *Hist. Aesthetics*, vol. II, pp. 38–9.

[*] ▸ pp. 279–94

[†] ▸ pp. 172–3

[3] Hankins, *Plato in the Italian Renaissance*, vol. I, p. 4.

[4] *Ibid.*, vol. I, p. 5.

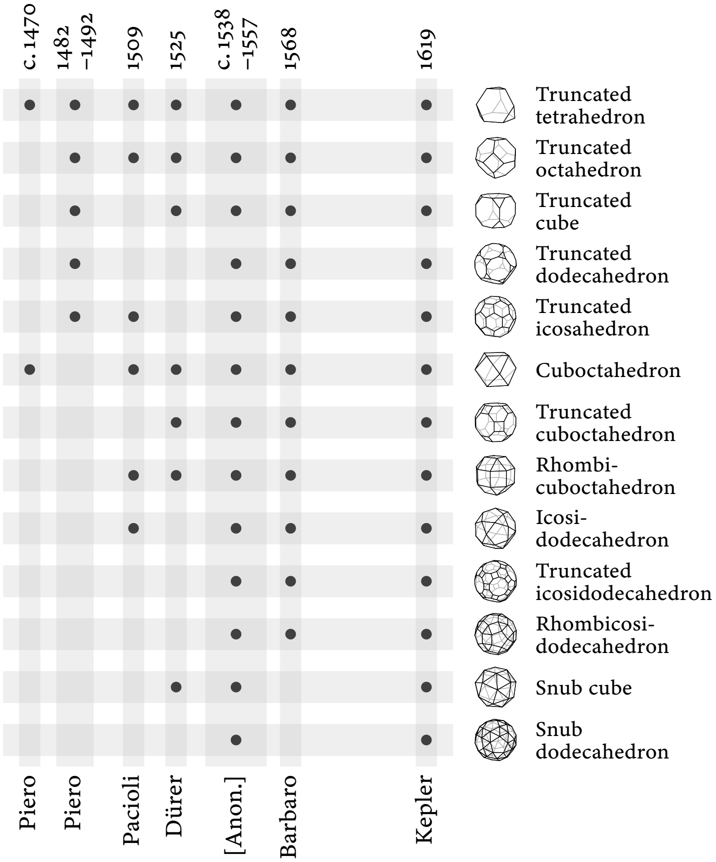
[5] *Ibid.*, vol. I, pp. 7–17.

[‡] ▸ pp. 65–7

[§] ▸ pp. 67–70

[6] Calcidius, *Comm. Timaeus*, §§ 20, 126.

[7] Aristotle, *On the Heavens*, BEK III.1.299a1–300a19, 7.306a1–b2, 8.306b3–7b24.



	Piero c. 1470	Piero 1482–1492	Pacioli 1509	Dürer 1525	[Anon.] c. 1538–1557	Barbaro 1568	Kepler 1619
Truncated tetrahedron	●	●	●	●	●	●	●
Truncated octahedron		●	●	●	●	●	●
Truncated cube		●		●	●	●	●
Truncated dodecahedron		●			●	●	●
Truncated icosahedron		●	●		●	●	●
Cuboctahedron	●		●	●	●	●	●
Truncated cuboctahedron				●	●	●	●
Rhombicuboctahedron			●	●	●	●	●
Icosidodecahedron			●		●	●	●
Truncated icosidodecahedron					●	●	●
Rhombicosidodecahedron					●	●	●
Snub cube				●	●		●
Snub dodecahedron					●		●

FIGURE 8.1
Timeline of the appearance of the archimedean solids in Piero della Francesca's 'Trattato d'abaco' (c. 1470) and 'De quinque corporibus regularibus' (c. 1482–1492), Luca Pacioli's *Divina proportione* (1509), Albrecht Dürer's *Underweysung der Messung* (1525), a set of unpublished printing blocks (c. 1538–1557), Daniele Barbaro's *La Practica della Perspettiva* (1568), and Johannes Kepler's *Harmonice Mundi* (1619) (Field, 'Rediscovering the Archimedean Polyhedra', Tbl. 1; Schreiber, Fischer & Sternath, 'New light on the rediscovery of the Archimedean solids').

thought when *On the Heavens* became accessible through Gerard of Cremona's (c. 1114–1187) translation from an Arabic text, followed by Michael Scot's (before 1200–c. 1235) translation of Ibn Rushd's long commentary, which includes the text itself, and then by William of Moerbeke's (1215/35–c. 1286) translation from the Greek. Many authors came to accept and reiterate Aristotle's physical arguments for rejecting the Timaean theory.[1]

But neither Calcidius nor — more importantly — Aristotle drew attention to the use of the *regular* solids in the *Timaeus*, nor that there are precisely *five*. And the treatises in which Plutarch (before 50–after 120 CE) discussed the regular solids* were not translated into Latin until the middle of the sixteenth century.[2] Thus it was Ficino's translation of the *Timaeus* that introduced Renaissance scholars to Plato's view of the cosmic role and the aesthetic importance of the regular solids, the 'fairest [*kallista*] bodies'.[3] And within a few years of Ficino's publication of the Latin *Timaeus*, Euclid's *Elements* was in print. The thirteen books of the *Elements* end with the construction of the five regular solids; the so-called fourteenth and fifteenth books, long known to be spurious but at that time thought to be Euclid's

[1] Sanders, 'The Regular Polyhedra', pp. 72–7.

* ▸ pp. 135–6

[2] Becchi, 'Humanist Latin Translations of the *Moralia*', Tbl. 1.

[3] Plato, *Timaeus*, STE 53e.

own work, are supplementary treatments of the regular solids.[1] Thus it was natural, although simply incorrect, to see the regular solids as the climax and the *purpose* of the *Elements*, just as Billingsley wrote in his English translation:

> ‘Wherfore they haue euer of the auncient Philoſophers bene had in great eſtimation and admiration, and haue bene thought worthy of much contemplacion, about which they haue beſtowed moſt diligent ſtudy and endeuour to ſearche out the natures & properties of them. They are as it were the ende and perfection of all Geometry, for whoſe ſake is written whatſoeuer is written in Geometry.’[2]

Thus European thought became re-acquainted, in a short space of time, with both the philosophical-aesthetic significance of the regular solids for Plato and with what was seen as their overmastering mathematical importance for Euclid. In hindsight, it seems hardly surprising that they would become a nexus of Renaissance thought.

A few examples will suffice here as illustration. First, the mathematician and philosopher Charles de Bovelles (1471/9–after 1566) wrote a treatise *On mathematical bodies* [*De mathematicis corporibus*] about the five regular solids and the sphere, which he considered fitted his definition: ‘that which is enclosed on every side by equal and similar surfaces or bases’.[3] For Bovelles, this brought the number of ‘regular solids’ to the perfect 6, with the sphere being preëminent.[4]

From the regular solids, he drew symbolic relations to which he ascribed religious significance. There are only three regular polygons that can compose a regular solid. The three-sided equilateral triangle is the only regular polygon that is the face of more than one regular solid, and in fact is the face of three: the tetrahedron, octahedron, and icosahedron. Each regular solid has a number, equal to the sum of the numbers of vertices, edges, faces, plus 1 for the body of the solid. Hence the number of the tetrahedron is $15 = 4 + 6 + 4 + 1$; of the cube, $27 = 8 + 12 + 6 + 1$; of the octahedron, $27 = 6 + 12 + 8 + 1$; of the dodecahedron, $63 = 20 + 30 + 12 + 1$; and of the icosahedron, $63 = 12 + 30 + 20 + 1$.[5] Bovelles seems to have been pleased that the number of the cube should be 3^3, and that the numbers of the cube and octahedron and of the dodecahedron and icosahedron should be equal.[6] (The equality follows the duality of the polyhedra.) The sum of the numbers of the triangle-faced solids is 105; of all the regular solids, 195. Bovelles pointed out that these numbers are also multiples of three.[7] The number three abounds. And thus, in such facts, Bovelles saw a mental guide toward contemplation of the Trinity.[8]

While Bovelles saw religious significance in the regular solids, Jofrancus Offusius (*fl.* mid 16th century), an itinerant astrologer about whom little is known but who was mentioned by Tycho Brahe (1546–1601) and by Kepler,[9] saw them as informing the structure of the geocentric cosmos. But Offusius in no way presaged Kepler’s celebrated theory in which the regular solids determined the orbs of the heliocen-

[1] Heath, notes to Euclid, *Elements*, vol. III, pp. 512–20.

[2] Billingsley, comm. on Euclide of Megara, *Elements of Geometrie*, fol. 339v.

[3] Bovelles, quot. in Sanders, ‘Charles de Bovelles’s Treatise’, p. 529.

[4] *Ibid.*, p. 549.

[5] *Ibid.*, pp. 540, 547.

[6] *Ibid.*, p. 547.

[7] *Ibid.*, pp. 540, 548.

[8] *Ibid.*, pp. 529, 530–1, 541.

[9] Stephenson, *The Music of the Heavens*, p. 47.

tric cosmos.[*] Rather, Offusius, like Bovelles, extracted numbers from the regular solids and used those numbers for the cosmic structure. Notably, he thought that the distance between the centre of the Earth and the Sun was 576 times the diameter of the Earth. The number 576 was the sum of the number of triangles that, according to Offusius's platonism, make up the regular solids. Each equilateral triangular face of the tetrahedron, octahedron, and icosahedron is made up of 6 triangles with sides in proportion $1 : \sqrt{3} : 2$; each square face of the cube is made up of four isosceles right-angled triangles, and each pentagonal face of the dodecahedron is made up of thirty right-angled triangles, for a total of $\big((4+8+20)\cdot 6\big) + (6\cdot 4) + (12\cdot 30) = 576$.[1] [The division of the pentagon into thirty triangles goes back to the middle platonism of antiquity. Plutarch erroneously claimed that the pentagon could be assembled from thirty of Plato's scalene triangles,[2] but in another place pointed out that the faces of the dodecahedron were made of 'something else'.[3] Alcinous also described the regular pentagon as made up of five triangles each divided into six triangles, for a total of thirty.[4] This description suggests a division such as the one shown in Figure 8.2.[5]]

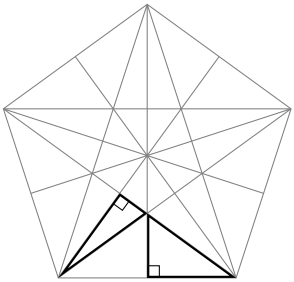


FIGURE 8.2
Division of a pentagon into thirty right-angled triangles, of two types up to similarity.

In the textbook in which he aimed to restore a pristine pre-Platonic geometry,[†] Peter Ramus (1515–72) followed the proof that there were exactly five regular solids by pointing out that what he saw as pythagorean geometry led to the conclusion that God had wanted to adorn the world with regular bodies, had discerned that there were five, and had thus considered how they could be incorporated into the world.[6] But Ramus thought that in this Timaean perspective one should see only an analogy rather than a truth about nature.[7]

NICHOLAS OF CUSA

> ‘ the Cusan, the inventor of geometry's most beautiful secrets ’
>
> — Giordano Bruno, *Cause, Principle and Unity*, p. 97
> (trans. R. de Lucca).

Nicholas of Cusa (c. 1401–1464) is important in the history of mathematical beauty, not because of any direct contribution

[*] ▸ pp. 281–5

[1] Sanders, 'The Regular Polyhedra', p. 217.

[2] Plutarch, *Platonic Questions*, STE V.1.1003d.

[3] Plutarch, *The Obsolescence of Oracles*, STE 428a.

[4] Alcinous, *Handbook of Platonism*, HER 168–9; § 13.2.

[5] Heath, notes to Euclid, *Elements*, Prop. IV.10.

[†] ▸ pp. 268–70

[6] Ramus, *Geometriae*, § XXV.16, ⟦p. 183⟧; see also Sanders, 'The Regular Polyhedra', pp. 198–9.

[7] Ramus, *Scholarum Mathematicarum*, pp. 310–1.

of his own, but because his thought is a bridge between ancient and mediaeval ideas and the Renaissance.[1] He has been seen as the last of the mediaeval thinkers, although he anticipated many later ideas.[2]

Although he entered the University of Padua to study canon law, he spent much time studying mathematics, physics, and astronomy.[3] His mathematical studies strongly influenced his philosophy, and he thought that mathematical arguments could give insight into theological problems: 'If such is the case with regard to things mathematical, then such will be the case more truly with regard to things theological.'[4] He used mathematical analogy and symbolism in his philosophical arguments. He repeatedly drew comparisons in which God was an infinite quantity or what in modern terms would be called the limit of an infinite sequence. For instance, he argued that the infinitely large and infinitely small were both subsumed within the idea of maximality, and since the concept of maximum admits precisely one *absolute*, the infinitely large and infinitely small coincided. And thus 'God Himself can be considered as the Infinite Angle [...] God is like an angle that is both maximal and minimal.'[5] From the same premisses, the most perfect geometric forms — the infinitely large circle and the infinitely large sphere — coincided with their centres.[6] Nicholas was therefore led to liken God to an infinite sphere.[7] Such analogies between the infinite and the divine also illustrated the limitations of reasoning: truth and intellect are like a circle and an inscribed polygon: no matter how many sides a polygon has, it is never equal to the circle.[8]

Nicholas's philosophy was certainly influenced by the platonic tradition, in which he was very well-read, owning over three hundred manuscripts of platonist texts.[9] He also subscribed to the pythagorean notion that 'all things are constituted and understood through the power of numbers'[10] and that number was responsible for the harmony between things.[11]

Nicholas thought that the beauty of the world caused admiration for the 'divine art' with which God shaped it, using arithmetic, geometry, music, and astronomy.[12] In what seems to have been a conscious nod to the phrase 'measure and number and weight' from the book of Wisdom,[13] he wrote that

> 'God, who created all things in number, weight, and measure, arranged the elements in an admirable order. (Number pertains to arithmetic, weight to music, measure to geometry.)'[14]

This specific connection of the mathematical ideas mentioned in the book of Wisdom to the different fields of mathematics seems to have been original to Nicholas. A particular aspect of the beauty of the world was that the motion closest to perfection was circular and the shape closest to perfection was the sphere.[15] And one could identify reasons for this:

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[1] Tatarkiewicz, *Hist. Aesthetics*, vol. III, p. 61.

[2] Dupré & Hudson, 'Nicholas of Cusa', p. 466.

[3] *Loc. cit.*

[4] Nicholas, *De Theologicis Complementis*, p. 3.

[5] *Ibid.*, p. 12.

[6] Hofmann, 'Cusa, Nicholas', p. 514.

[7] Nicholas, *On Learned Ignorance*, WIL 70–3; ch. I.23.

[8] *Ibid.*, WIL 10; ch. I.3.

[9] Moran, 'Nicholas of Cusa', p. 16.

[10] Nicholas, *On Learned Ignorance*, WIL 3; ch. I.1.

[11] Nicholas, *On Learned Ignorance*, WIL 13; ch. I.13; see also Nicholas, *De Mente*, pp. 94–5.

[12] Nicholas, *On Learned Ignorance*, WIL 175–6; ch. II.13.

[13] Wisdom 11:20.

[14] *Ibid.*, WIL 176; ch. II.13.

[15] *Ibid.*, WIL 163; ch. II.12.

> ‘all who have a mind are favorably disposed toward circular figures, which appear to us complete and beautiful because of their uniformity and equality and simplicity. And this is nothing other than the fact that in a circle the form of forms shines forth more clearly than in any other figure. Note how greatly the mind is favorably disposed toward the exemplar of a circle — toward its infinite form and beauty, unto which alone it looks.’[1]

The lasting influence of Nicholas of Cusa was sporadic,[2] but he was admired by Kepler,[*] who gave him the epithet ‘divine’ for the importance of the distinction he made between straight and curved lines and his comparisons which dared ‘to liken a curve to God, a straight line to his creatures’.[3] These comparisons seem to have been Kepler’s interpretation of the difference Nicholas saw between a circle — God — and the approximation offered by an inscribed polygon — human reason.

PIERO DELLA FRANCESCA

Piero della Francesca (c. 1412–1492) should be remembered not just as a great artist, but as a great mathematician. While a two-sided interest in mathematics and art was not uncommon in the Renaissance,[4] Piero’s mathematical treatises were in their time an almost unparalleled synthesis of arithmetic and geometry. Besides exalting him as ‘the greatest geometrician who lived in his times’,[5] the artist and father of art history Giorgio Vasari (1511–74) called him ‘an uncommon master of the problems of regular bodies in both arithmetic and geometry’,[6] and here lay his greatest contribution to pure mathematics.

Vasari reported that Piero studied mathematics in his youth.[7] Certainly he would have learned arithmetic as part of his mercantile apprenticeship. He may have been introduced to geometry in conversation with surveyors or with the town doctor, who would have read the first four books of Euclid’s *Elements* as a student, since geometry was necessary for the astrological component of diagnostic medicine.[8] But he was primarily self-taught, absorbing both the *abaco* calculatory knowledge of the artisan and merchant and the latest theoretical mathematical knowledge of the university,[9] and in particular learning from the scholars and the library at the humanist court of Federico da Montefeltro, Duke of Urbino (1422–82). He obtained manuscript copies — printing using moveable type still being in its infancy — of Latin translations of the *Elements* and *Optics* of Euclid and of the works of Archimedes,[10] and made a copy of the Archimedes manuscript for his own use.[11]

[1] Nicholas, *De Theologicis Complementis*, p. 7; ins. and note markers omitted.

[2] Cranz, ‘Reason and Beyond Reason in Nicholas of Cusa’, p. 20.

[*] ▸ pp. 279–94

[3] Kepler, *Mysterium Cosmographicum* (1596), ch. 11, p. 23; trans. Kepler, *The Secret of the Universe*, ch. 11, p. 93.

[4] Davis, *Piero della Francesca’s Mathematical Treatises*, p. 2.

[5] Vasari, ‘Piero della Francesca’, p. 168.

[6] *Ibid.*, p. 163.

[7] *Ibid.*, p. 164.

[8] Banker, *Piero della Francesca*, p. 10.

[9] *Ibid.*, p. 79.

[10] *Ibid.*, pp. 188–93.

[11] Banker, ‘A manuscript of the works of Archimedes’.

In his treatise ‘De prospectiva pingendi’, Piero held that perspective was vital for painting to fulfil its purpose of accurately depicting a three-dimensional space on a two-dimensional surface.[1] But his interest in mathematics was certainly not limited to that required for perspective, but extended into questions of purely intellectual interest. The five regular solids seem to have held a fascination for him, a fascination absent in earlier authors like Leonardo Pisano[*] (‘Fibonacci’; c. 1170–after 1240).[2] It is unclear whether he found them beautiful, but he certainly did not assign to them a platonic metaphysical significance.[3][†] In his ‘Trattato d’abaco’, he discussed these five solids and the relationships between them. Towards the end of his life, when he seems to have painted less in order to pursue geometry,[4] he made this study the principal topic of ‘De quinque corporibus regularibus’, a work he declared his monument[5] and which has been acclaimed as a work of art in its own right.[6]

‘De quinque corporibus regularibus’ contains some treasures and some remarkable developments in proof. For instance, in the third book, Piero proved a series of results about the relative dimensions of regular solids inscribed in one another. For instance, if an octahedron is inscribed in a dodecahedron, the diagonal of the octahedron (which equals the diameter of the dodecahedron from edge midpoint to edge midpoint), is equal to the sum of length of the edge and the length of the diagonal of the pentagonal face of the dodecahedron.[7] This result is surely a ‘very pretty theorem’,[8] which was accessible to but never discovered by the Greeks. But Piero’s proof would have been utterly alien — and presumably monstrous — to Greek geometry: he considered a regular dodecahedron of edge length 4 and proved that the diameter and the sum both equal $6 + \sqrt{20}$.[9]

In the fourth book of ‘De quinque corporibus regularibus’, Piero also considered non-regular solids, which he had previously touched upon in the ‘Trattato d’abaco’. In both works, he showed that by truncating regular solids, it was possible to obtain new solids inscribable into spheres. An exercise of no special prominence in the ‘Trattato d’abaco’ asks for the edge length of a solid with 4 triangular and 4 hexagonal faces that can be inscribed in a sphere: this solid is a truncated tetrahedron. The next exercise similarly involves a cuboctahedron. These exercises were the first appearance of inscribing such truncated solids into spheres.[10] In ‘De quinque corporibus regularibus’, Piero systematically considered how truncations of the five regular solids yielded the polyhedra now called the truncated tetrahedron, truncated cube, truncated octahedron, truncated dodecahedron, and truncated icosahedron.[11] He paid no special attention to the construction, treating it as obvious,[12] which it doubtless was to one with his ability to think visually and geometrically.[13]

The only account of Archimedes’ study of these solids is in Pappus’ *Collection*. Nothing is known of the whereabouts of the only manuscript of the *Collection* known to have existed at the time, whence all later texts descend.[1] In any case, the text was Greek, which Piero did not read.[2] Furthermore, in 'De quinque corporibus regularibus', Piero did not mention Archimedes in this context.[3] Thus it is likely that Piero did not have any conception of archimedean solids at all. His concern, rather, was simply to investigate which truncations of regular solids were also inscribable in a sphere. Only in retrospect was Piero's work seen as the beginning of the rediscovery of the archimedean solids.[4]

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[1] Banker, *Piero della Francesca*, p. 176.

[*] ▸ pp. 179–80

[2] Davis, *Piero della Francesca’s Mathematical Treatises*, p. 39.

[3] Banker, *Piero della Francesca*, pp. 94, 218.

[†] ▸ pp. 67–70

[4] *Ibid.*, p. 219.

[5] Piero, ‘De quinque corporibus regularibus’, fol. 1v.

[6] Peterson, *Galileo’s Muse*, p. 131.

[7] Piero, ‘De quinque corporibus regularibus’, bk III, § ix; Peterson, ‘The Geometry of Piero della Francesca’, p. 36.

[8] Peterson, *Galileo’s Muse*, p. 134.

[9] Peterson, ‘The Geometry of Piero della Francesca’, p. 36.

[10] Field, ‘Rediscovering the Archimedean Polyhedra’, pp. 248–9.

[11] *Ibid.*, p. 251.

[12] Peterson, ‘The Geometry of Piero della Francesca’, p. 38.

[13] Banker, *Piero della Francesca*, p. xxii.

Piero's study of the regular and archimedean solids seems to have been a first step away from a metaphysical concern towards an intrinsic — possibly purely aesthetic — fascination. The aesthetic aspect is speculative, except that the role of the circumscribing sphere hints at a place for Piero in an aesthetic tradition descending from Pythagoras and extending at least to Kepler.

MARSILIO FICINO

Philosophy owes a great debt to Marsilio Ficino (1433–99): it was he who provided what were for three centuries the definitive Latin translations of Plato, Plotinus, and various platonist thinkers, plus commentaries. These were the key figures of the philosophical tradition with which he identified, and were a vital influence on his own philosophy, for he saw the pursuit of philosophy and the study of its history as intimately related.

Ficino's conception of the history of philosophy was centred on seeing Plato as the sixth sage in — and the culmination of — a far older tradition of *prisca theologia* [ancient theology]; in this idea, he was clearly influenced by Iamblichus.[*] Originally, when Ficino translated *Pimander*, the first book in the Hermetic corpus, he thought that the genealogy started with Hermes Trismegistus,[†] who by this time was thought, on the authority of Augustine, to have lived slightly later than Moses.[5] [Indeed, the Hermetic corpus was then considered so important that Ficino's patron Cosimo de' Medici (1389–1464) ordered him to complete its translation before turning to the Greek philosophers.[6]] The next sages were Orpheus, the legendary pre-Homeric poet and prophet of the eponymous Orphic mysteries, then Aglaophemus, an obscure figure[7] who Iamblichus thought had initiated Pythagoras (c. 570–c. 490 BCE) into the Orphic mysteries.[8] Pythagoras himself was the fourth sage in Ficino's genealogy and the first whose existence, at least, stands on a firm historiographical footing. The penultimate sage was Philolaus (c. 470–c. 385 BCE), before the tradition reached Plato (c. 427–348/347 BCE). But when Ficino wrote *Platonic Theology*, he added the prophet Zoroaster at the head of the tradition and ejected Philolaus to maintain the number of sages. In his commentary to Plotinus' *Enneads*, the persons of the genealogy remained the same, but

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[1] Clagett, *The Fate of the Medieval Archimedes* (Part III), pp. 405–6.

[2] Banker, *Piero della Francesca*, p. 204.

[3] *Ibid.*, p. 193.

[4] Field, 'Rediscovering the Archimedean Polyhedra'.

[*] ▸ pp. 145–8

[†] ▸ p. 145

[5] Augustine, *The City of God*, bk XVIII, § 39.

[6] Yates, *Giordano Bruno*, pp. 12–13.

[7] Schmitt, 'Perrenial [*sic*] Philosophy', p. 508, 510, esp. n. 36.

[8] Iamblichus, *De Vita Pythagorica Liber*, p. 82, ll. 12 sqq. [para. 146].

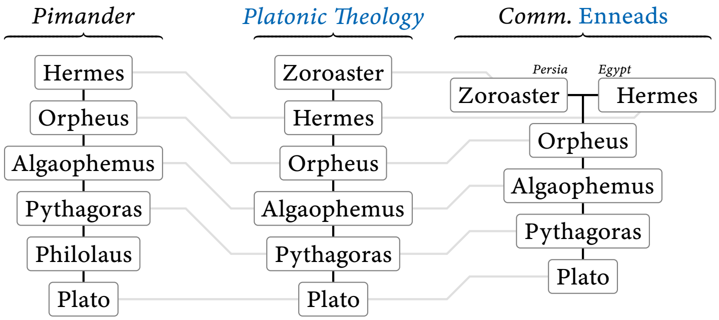

FIGURE 8.3
Ficino's three variant traditions of sages, described in the preface to his translation of *Pimander*, in his *Platonic Theology*, and in his commentary on Plotinus' *Enneads*. Plato is consistently the sixth and last.

Zoroaster and Hermes were made joint progenitors.[1] (See Figure 8.3.) The important features of the tradition for Ficino were its very ancient origin, and the number of sages, both of which served Ficino's goal of enhancing the authority of Plato. Plato was the sixth sage, 6 is a perfect number, and thus Plato's thought was the point where *prisca theologia* had reached its perfection. The desire to hold up six figures doubtless contributed to Ficino excluding other, perhaps more obvious, antecessors of Plato, either historical, such as Socrates (c. 469–399 BCE) or Parmenides (*fl.* early 5th century BCE), or ahistorical, such as Timaeus or Diotima,[2] and also to his not limiting himself to the genealogy Proclus (410/12–485 CE) gave, comprising only Orpheus, Aglaophemus, Pythagoras, and Plato.[3]

For Ficino, the preference for 6 extended to modifying the cosmos: he took the marks that the Timaean demiurge marked on a strip of world-stuff* and re-based them at 6, instead of 1. Using the Λ-shaped scheme, Ficino marked one leg 6–12–24–48 and the other 6–18–54–162, then placed between each adjacent pair their harmonic and their arithmetic means, noting that the interval between the harmonic mean and the smaller number of the pair is a fourth, while the interval between the arithmetic mean and the smaller number is a fifth (see Figure 8.4).[4] Ficino, however, did not proceed to the third step of Plato's demiurge, when the intervals 4/3 are divided further.

Ficino had to modify the system of the *Timaeus* because it did not match his own views or those of his times, which had naturally been shaped by developments in the almost two millennia since Plato had written.[5] In order to interpret platonism as compatible with Christianity, he identified the demiurge, the neoplatonic One–Good, and God.[6] God was thus the first cause of all the harmony in the world.[7] And so Ficino identified the model of the world that existed in the mind of the Timaean demiurge with the divine *Logos* [Λόγος].[8] The beauty of the order of the world pointed to a surpassingly beautiful order in the intellect that so placed that order in the world.[9] This intellect arranged things in the same way as the lesser intellects in the created world were wont to arrange things 'according to number, weight and measure, mode, species and order';[10] a description both platonic and Biblical. Ficino went into detail about how the scrip-

[1] Yates, *Giordano Bruno*, pp. 14–15.

[2] Cf. Allen, 'At Variance', p. 35.

[3] Proclus, *Théologie Platonicienne*, ch. I.5 ⟦vol. I, pp. 25–6⟧.

* ▸ pp. 62–3

[4] Ficino, *Compendium on the* Timaeus, ch. 35.

[5] Cf. Prins, *Echoes of an Invisible World*, p. 47.

[6] Ficino, *Compendium on the* Timaeus, ch. 8.

[7] Prins, *Echoes of an Invisible World*, p. 27.

[8] Ficino, *Compendium on the* Timaeus, ch. 16; see John 1:1.

[9] Ficino, *Comm.* Philebus, p. 204.

[10] *Ibid.*, p. 78.

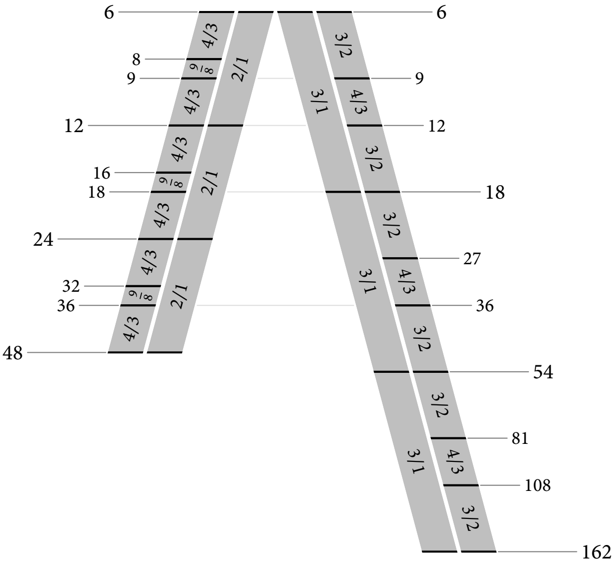

FIGURE 8.4
Ficino's version of the intervals used by the demiurge in constructing the heavens, according to the Plato, *Timaeus*, STE 35b–6c.

tural properties of 'measure and number and weight',[1] by which God ordered the universe, corresponded to 'mathematical images' in the *Timaeus*. The regular solids, which played such an important role in the Timaean system,[*] indicated weight. After mentioning measure, thought Ficino, Plato discussed the forces that were the causes of weights. Lastly, number was indicated by the three means.[2][†]

Ficino saw his own work as descending from the *prisci theologi*, all of whom, he thought, had believed that the created world was harmonious, though they may have differed in the details.[3] He attributed to all pythagoreans and platonists the view that the cosmos was made up of the four Empedoclean elements combined using a musical and geometric ratio, in such a harmonious way that there was never any discord in the heavens.[4] It was in this context that Ficino thought, like his predecessors from Plato onward, that the visible world reflected a superior kind of beauty in the intelligible world.[5]

It was through mathematics that we could come to perceive the harmony of the divine model which was embedded in the created world,[6] because the soul, for Ficino, sat between the intelligible world and the sensible world and so was analogous to mathematics, which was likewise midway between the divine and the natural.[7] The very insertion of mathematical structure into the world-stuff seems to have allowed it to serve as a mediator between the intelligible and sensible realms.[8]

Mathematically, the most significant change Ficino made to the system of the *Timaeus* was to incorporate the perfection of 6 into the demiurge's emplacement of harmony in the world. He cited Philo of Alexandria[‡] (*fl.* early 1st century CE) in order to harmonize the Timaean numerical ideas with the Biblical creation story.[9] He drew the conclusion that Plato was like Moses, who authored the book of

[1] Wisdom 11:20.

[*] ▸ pp. 67–70

[2] Ficino, *Compendium on the* Timaeus, ch. 19.

[†] ▸ pp. 62–5

[3] Prins, *Echoes of an Invisible World*, p. 32.

[4] Ficino, *Compendium on the* Timaeus, ch. 24.

[5] Allen, 'Cosmogony and love', p. 150.

[6] Prins, *Echoes of an Invisible World*, p. 42.

[7] Ficino, *Compendium on the* Timaeus, ch. 28.

[8] Prins, *Echoes of an Invisible World*, p. 80.

[‡] ▸ pp. 130–3

[9] Ficino, *Compendium on the* Timaeus, ch. 12; cf. Prins, *Echoes of an Invisible World*, p. 49.

Genesis, in which the world was created in six days. And Pythagoras was like both, for he linked the number 6 to creation and to marriage, calling it *gamos* [γάμος], for its parts, put side by side, produce the number again, just as a child resembles its parent.[1] Although this justification clearly referred to 6 being the sum of its proper divisors and thus a perfect number, the visual description recalls pythagorean pebble arithmetic.[*]

Although Ficino recognized the importance of both arithmetic and geometry in the *Timaeus*, he did not dwell on geometry, which may be the result of his compendium being unfinished,[2] or may simply indicate that he was more interested in the numerical, musical, and metaphysical perspectives.[3] For subjects outside this focus, Ficino referred readers to other authorities: for mathematics to the writings of Pier Leone da Spoleto (c. 1445–1492); for architecture to the *De Re Aedificatoria Libri Decem* of Leon Battista Alberti (1404–72); for cosmography, to the works of Francesco Berlinghieri (1440–1501).[4]

Even Ficino's discussion of the spherical shape of the cosmos leads back to number via the mediaeval concept of circular number. Ficino seems not to have shared, or at least did not explicitly mention, the historical aesthetic preference for the sphere. Rather, he argued that only the sphere allowed the movement of one mass within another. The shape of the physical world, within which the world-soul moves, reflects the sphericity of the divine world, which turns back to itself through divine understanding and love.[5] But this circularity justifies the role of number in the ordering of the world: specifically, the odd circular number 5 and the even circular number 6,[†] which play roles in the division of the elements.[6]

Even more fundamentally, Ficino offered an account of how numbers came about in the intelligible world, which was logically and ontologically prior to the created sensible world: God acted to create the 'powers' of limit and the unlimited, and so the number 2 arose. The two powers commingled to create a third form; hence 3. The numbers 5, 6, and 7 each arose in turn.[7] Why did Ficino include these numbers in this first generation? Perhaps, by virtue of being circular numbers, the numbers 5 and 6 deserved inclusion, while Philo, whose work Ficino knew and admired, had pointed out that 7 was a number that is neither a divisor nor a multiple of any other number in the decad, and had drawn attention to its beauty.[‡]

REGIOMONTANUS

Johannes von Königsberg (1436–76), better known as Regiomontanus from the Latin calque of his birthplace, was in a sense a man ahead of his time. His knowledge of and ability in the mathematical sciences was perhaps greater than any other fifteenth-century

[1] Ficino, *Compendium on the* Timaeus, ch. 12.

[*] ▸ pp. 46–7

[2] Prins, *Echoes of an Invisible World*, p. 67.

[3] *Ibid.*, p. 75.

[4] Ficino, *Compendium on the* Timaeus, ch. 43.

[5] *Ibid.*, ch. 16.

[†] ▸ pp. 154–5, 168–9

[6] *Ibid.*, ch. 17.

[7] *Loc. cit.*

[‡] ▸ pp. 131–2

European, and the historian of science Noel Swerdlow (1941–2021) once called him ‘the first European to display [...] the intellectual and aesthetic sensibility of modern mathematics’.[1] But his achievements were, if not limited, then certainly structured by his times: rather than seeking new discoveries, he focused on the improvement of mathematical knowledge through his own expositional work and through the recovery and translation of ancient texts. When he worked, university courses in the mathematical sciences were virtually unknown: none of the texts had been printed, in part because of the extra expense entailed in typesetting mathematics and in cutting the necessary diagrams.[2] Most were available only in manuscript, and only in Greek or in very poor Latin translations.[3] By July 1471, he had formulated and announced an intention to publish editions of the most important mathematical works. He died five years later, having succeeded in only a small part of his programme.[4]

The value Regiomontanus placed on the mathematical sciences was made clearest in an inaugural oration that prefaced a set of lectures he delivered at the University of Padua on the work of the astronomer al-Farghānī (d. after 861 CE). In this speech, he praised mathematics on historical, philosophical, utilitarian, and aesthetic grounds. Alongside its exhortations about the importance of the applications and history of mathematics, the oration seems to have had a particular aim of persuading the audience of the beauty and power of mathematics itself.[5] Aesthetic appreciation underlies much of the oration, but he was most explicit when he looked to the history of number theory and algebra:

> ‘Euclid made a much more worthy foundation of numbers in three of his books, the seventh, eighth and ninth, whence Jordanus gathered the ten books of elements of numbers and from this produced his three most beautiful books on given numbers. Diophantus, however, produced thirteen most subtle books [...], in which lie the very flower of all arithmetic, namely the art of assessing and accounting, which today is called algebra after its Arabic name. Indeed, Latin authors treat many fragments of that most beautiful art’.[6]

Aside from the aesthetic value of mathematics, he recognized its practical applications in architecture and perspective,[7] but he considered more important the role of mathematics in the liberal arts and in philosophy.[8] He thought that works in the Aristotelian corpus such as the *Physics*, *Meteorology*, *On the Heavens*, *Metaphysics* could not be completely understood without a solid grounding in mathematics.[9]

But mathematics, thought Regiomontanus, was in fact superior to any of its fields of application, including philosophy. The conclusions of mathematics are *certain*, which is evidenced by the absence from mathematics of ‘schools’ like the Scotists or Thomists of philosophy.[10] The certainty of mathematics gives it *permanence*: Regiomontanus was sure that ‘the discoveries of Archimedes will be no less admired after a thousand centuries’.[1] Regiomontanus’s point was not that the admiration is dependent on the durative nature of Archimedes’ work; rather, the work is lasting and so will remain admirable a centimillennium hence, and therefore the *reason for the admiration* is just as durable. His lecture could therefore be an early indication of a connection between the beauty and permanence of mathematics.[*]

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[1] Swerdlow, ‘Science and Humanism in the Renaissance’, p. 133.

[2] Zinner, *Regiomontanus*, p. 111.

[3] Swerdlow, ‘Science and Humanism in the Renaissance’, pp. 153–4.

[4] Zinner, *Regiomontanus*, pp. 110–15.

[5] Swerdlow, ‘Science and Humanism in the Renaissance’, p. 150; see, for instance, Regiomontanus, ‘Oratio’, p. 52.

[6] Regiomontanus, ‘Oratio’, p. 46; trans. Byrne, ‘A Humanist History of Mathematics?’, pp. 54–5.

[7] Swerdlow, ‘Science and Humanism in the Renaissance’, p. 148.

[8] *Loc. cit.*

[9] Byrne, ‘A Humanist History of Mathematics?’, pp. 57–8.

[10] Regiomontanus, ‘Oratio’, pp. 50–1.

LUCA PACIOLI

Luca Pacioli’s (c. 1445–1517) importance in mathematics generally lies in his encyclopaedic *Summa de arithmetica, geometria, proportioni et proportionalita* (1494), which contains accounts of theoretical and practical arithmetic, a summary of Euclid’s *Elements*, an elementary text on algebra, and an introduction to double-entry bookkeeping.[2] In this work, he described demonstrations and rules and methods as beautiful,[3] and referred to the subject of its second part as ‘beautiful and gentle geometry’.[4]

But perhaps the single most interesting aesthetic judgement in the *Summa* is his statement that Archimedes’ result that the ratio of the circumference of a circle to its diameter is approximately $3\frac{1}{7}$ was a ‘beautiful and subtle [bella e ſotile]’[5] discovery. Pacioli followed Leonardo Pisano’s exposition,[6][†] but mangled the line of Archimedes’ thought even more than Leonardo had. Pacioli ignored or abandoned the idea of finding bounds; in his version, the proof has degenerated to a pair of calculations showing π is approximately $3\frac{1}{7}$. Pacioli admired the beauty of the result while having an almost complete lack of understanding of the proof.

Polyhedra

Another of Pacioli’s books, *Divina proportione*, written in 1496–8 and published in 1509, is relevant to the history of mathematical beauty, but its role is very unlike what golden numberists later claimed.[‡] Its importance lies primarily in the treatment of regular solids and the part Pacioli’s contribution played in the rediscovery of the archimedean solids.

In his discussion of the regular solids in *Divina proportione*, Pacioli made many references to Plato’s *Timaeus*. Perhaps influenced by Plato, and doubtless glad of his imprimatur, Pacioli asserted the aesthetic value of the various solids he undertook to study in *Divina proportione*. He hoped that his patron Ludovico Sforza (1452–1508), the Duke of Milan, to whom he addressed the work, would come to see ‘their most sweet harmony’,[7] which he later emphasized is not musical harmony.[8] In the end, *Divina proportione* was published

[1] Regiomontanus, ‘Oratio’, p. 51; trans. Byrne, ‘A Humanist History of Mathematics?’, p. 59.

[*] ▸ p. 508

[2] Jayawardene, ‘Pacioli’, p. 270.

[3] Pacioli, *Summa*, pt i, fol. 130v, fol. 24r, fols 27v–8r.

[4] *Ibid.*, pt ii (new pagination), fol. 1r.

[5] *Ibid.*, pt ii (new pagination), fol. 31r; trans. C.

[6] Clagett, *The Fate of the Medieval Archimedes (Part iii)*, p. 432.

[†] ▸ pp. 179–80

[‡] ▸ pp. 553–4, 555

[7] Pacioli, *Divina proportione*, ch. ii, fol. 2r; trans. C.

[8] *Ibid.*, ch. ii, fol. 2v.

after Sforza's death. But Pacioli thought that their aesthetic value was grounded in reason: an epigram in the form of the solids addressing the reader reads

> 'The sweet fruit, charming and pleasant,
> Already forced the philosophers to seek
> Our origins, to nourish the intellect.'[1]

Their aesthetic value he linked to that of the sphere from which he saw them as deriving; the sphere itself had no variation but was completely uniform.[2] He seems to have placed special value on the dodecahedron, which he called 'most noble'.[3] He did not explain this judgement explicitly, but later explained how simply and symmetrically the other regular solids can be inscribed in it: the icosahedron is its dual and thus can be inscribed in a dodecahedron so that each of its vertices touches the centre of a face; the octahedron can be inscribed so that each of its vertices touches the midpoint of an edge; the cube and tetrahedron can be inscribed so that each of their vertices touches a vertex (see Figure 8.5).[4] And he thought that Plato had assigned the dodecahedron to the universe because it could encompass all the other regular solids thus.[5]

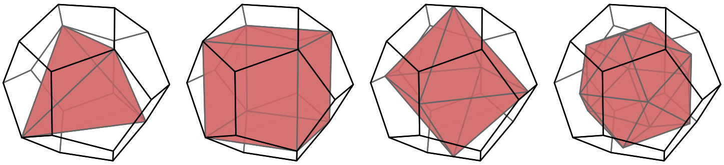

FIGURE 8.5
Pacioli noted how simply the other four regular solids can be inscribed in the dodecahedron.

Pacioli also considered truncating the five regular solids, though he did not distinguish different types of truncation. For example, truncating the cube can yield either the truncated cube or the cuboctahedron. Thus he obtained just five new polyhedra: the truncated tetrahedron, cuboctahedron, truncated octahedron, truncated icosahedron, and icosidodecahedron.[6] Except for the icosidodecahedron, these solids had been known to Piero della Francesca. Pacioli also discussed the rhombicuboctahedron and gave a vague description of how it could be obtained from the cube by cutting along its edges.[7] Finally, he discussed a 72-hedron that he claimed Euclid had used in Proposition XII.17 of the *Elements*: *Given two spheres about the same centre, to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface.* This result clearly requires an ever-higher number of faces as the two spheres approach one another, but Pacioli seems rather to have been concerned with giving a concrete example of a shape that he could compare to a dome.[8]

But in the same treatment, he considered what he called 'elevated' solids, formed by adding a suitably-based pyramid (with equilateral triangular faces) to each face of a solid (see, for example, Figure 8.6). Giving the number of vertices, edges, and faces, he described the

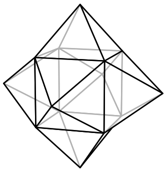

FIGURE 8.6
An elevated cube.

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[1] Pacioli, *Divina proportione*, epigrams, n.fol. trans. C.

[2] *Ibid.*, ch. LV.

[3] *Ibid.*, contents entry for ch. XXX, n.fol.

[4] *Ibid.*, chs XL–XLIII.

[5] *Ibid.*, ch. XLVI.

[6] *Ibid.*, chs XLVIII–LII.

[7] *Ibid.*, ch. LIII.

[8] *Ibid.*, ch. LIV.

elevated tetrahedron, cube, octahedron, dodecahedron, icosahedron, truncated dodecahedron, and rhombicuboctahedron.[1]

This mixture suggests that Pacioli did not have the *concept* of an archimedean solid. This suggestion is reinforced by his statement that a process of truncation could yield ever more solids, by implication having regular faces.[2] This statement further suggests that he had not grasped that only very restricted types of truncations maintain the regularity of faces. Further, since he listed thirteen properties of division in extreme and mean ratio[3] because this number was the count of Jesus and his Apostles,[4] he surely would have expounded similarly on the thirteen archimedean solids had he known of them.[5]

It is reasonable to infer that Pacioli had some special preference for the rhombicuboctahedron, since it was selected for prominent — even spectacular — placement in the famous portrait of Pacioli (Figure 8.7) attributed to Jacopo de' Barbari (d. c. 1516); compare the much more subdued dodecahedron. Presumably this preference stemmed from Pacioli's independent rediscovery of it, but the bare fact of the rediscovery being his own did not distinguish it from, say, the various elevated solids that he discussed, which also seem not to have been considered earlier. Thus he seems to have ranked the archimedean solids that he knew — the rhombicuboctahedron and those he found by truncation — above the elevated regular solids. If so, it seems that his preference was not based solely on rotational and reflectional symmetries, for the symmetries of an elevated solid are the same as those of its original; nor could it have been based simply on convexity or inscription in a sphere, for he discussed also the 72-hedron. Rather, it suggests a combination of both, which fits into the pattern stretching from Piero to Kepler.

Legacy

The beauty of Pacioli's mathematical ideas was recognized in the later Renaissance. When Baldi evaluated Pacioli's works in the biography he wrote, he judged their style most harshly, and thought that the aesthetic value of the underlying ideas was the only thing that made them worthwhile:

> 'His writing is of [such] a barbarous, irregular, rough, and unsuccessful style that it nauseates those who read it, and certainly, if beneath such verbal filth [sordidezza di parole] there were not such beautiful and useful [belle et util] things, his work would not be worthy of seeing the light'.[6]

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[1] Pacioli, *Divina proportione*, chs XLVIII–LIII.

[2] *Ibid.*, ch. LV.

[3] *Ibid.*, chs VII–22.

[4] *Ibid.*, ch. XXIII.

[5] Field, 'Rediscovering the Archimedean Polyhedra', p. 264.

[6] Baldi, 'Vite Inedite di Tre Matematici', p. 424; trans. mod. Saiber, *Measured Words*, p. 101; first ins. in trans.

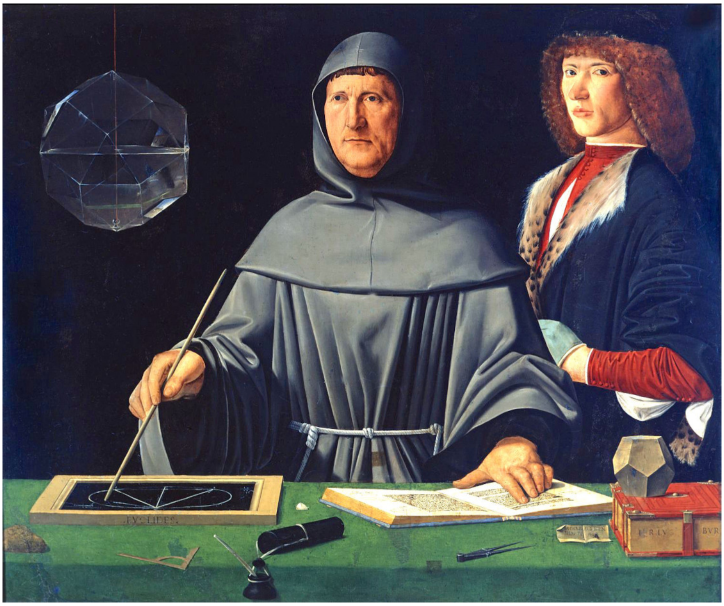

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FIGURE 8.7
Portrait of Luca Pacioli, attributed to Jacopo de' Barbari (d. c. 1516). A glass rhombicuboctahedron hangs behind Pacioli; a small dodecahedron sits on top of the book on the table. Pacioli's slate shows the diagram for Proposition XIII.12 of the *Elements*, which shows that the radius of a circle and the side length of an inscribed equilateral triangle are in the ratio $1 : \sqrt{3}$; this result is used in the *Elements* only in the construction of the regular tetrahedron, but can be used in an alternative construction of the regular dodecahedron given by Pappus (MacKinnon, 'The Portrait of Fra Luca Pacioli', p. 133).

LEONARDO DA VINCI

> ‘ The Great Engineer leaned forward,
> his eyes suddenly filled with fiery enthusiasm.
> "[…] The strange thing about these mathematical fancies is not
> that they generate what we apprehend as beauty,
> but that one cannot predict which geometry is beautiful. […]" ’
>
> — Paul J. McAuley, *Pasquale's Angel*, pt 3, ch. 10.

Leonardo da Vinci (1452–1519) seems to have been influenced towards a deeper interest in mathematics by Luca Pacioli.[1] Famously, he drew the illustrations of polyhedra for Pacioli's *Divina proportione*.[2][*] The skeletal diagrams in particular (see the examples in Figure 8.8), which are beautiful works of art in their own right, were an unparalleled depiction of the forms of these polyhedra.

Leonardo's notebooks contain a sketch of the five regular solids and a quotation of the epigram Pacioli used in *Divina proportione*.[3][†][4] The epigram strongly suggests that Leonardo saw in them the same fascination that Pacioli did. This idea is borne out by sketches in the *Codex Atlanticus*[5] which point to Leonardo's own investigation of polyhedra obtainable by truncating or otherwise symmetrically modifying the regular solids.[6] The beauty that he seems to have seen in the regular solids or polyhedra generally sparked his interest and led to him undertaking a systematic study of mathematics.[7]

His mathematical studies and investigations were wide-ranging, including both arithmetical and geometrical attempts to duplicate the cube.[8] His researches were at least partly informed by aesthetic influence. One of the most spectacular evidences of this influence is an exploration in the *Codex Atlanticus* of numerous configurations,

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[1] Kemp, *Leonardo Da Vinci*, pp. 129–31.

[2] Pacioli, *Divina proportione*, ch. LXX.

[*] ▸ pp. 247–9

[3] Léonard, 'Manuscrit M', fol. 80v.

[†] ▸ p. 248

[4] Kemp, *Leonardo Da Vinci*, pp. 135–6.

[5] Leonardo, *Codex Atlanticus*, fol. 735v.

[6] Field, 'Rediscovering the Archimedean Polyhedra', p. 263.

[7] Kemp, *Leonardo Da Vinci*, pp. 135–6.

[8] Duvernoy, 'Leonardo and Theoretical Mathematics', pp. 40–7.

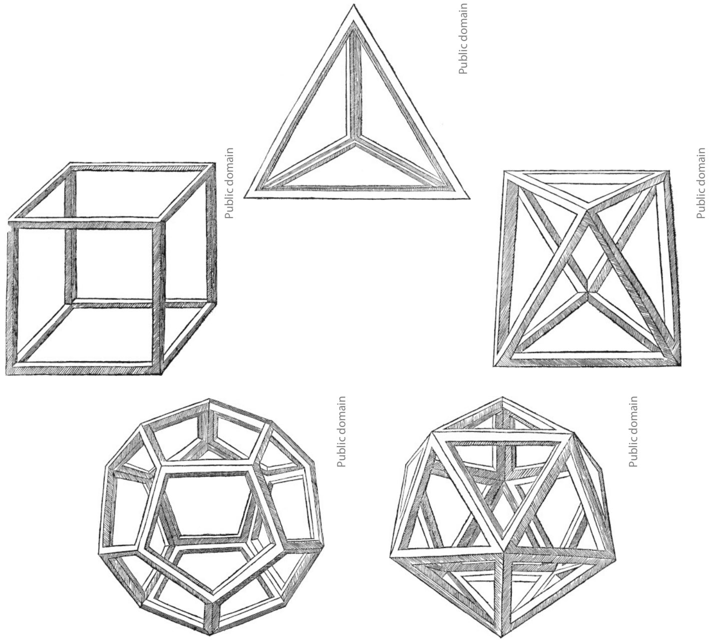
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FIGURE 8.8
The five regular solids as drawn by Leonardo da Vinci for Pacioli's *Divina proportione*, pls II, VIII, XVI, XXII, XXVIII.

mostly symmetric, inscribed in semicircles and circles, and how areas within those configurations relate to each other.[1] This study seems to have been a part of his attempts to square the circle, which at one time he believed he had achieved.[2]

Leonardo had a strong interest in symmetry and he devoted significant effort to the design of symmetrical buildings, including how best to install cupolas and attach chapels.[3] [However, it is a stretch to claim, as the mathematician and physicist Hermann Weyl (1885–1955) did, that Leonardo 'engaged in systematically determining the possible symmetries of a central building and how to attach chapels and niches without destroying the symmetry of the nucleus',[4] and that, in abstract terms, he had essentially characterized the two-dimensional central symmetries as cyclic and dihedral.]

However, since after his death Leonardo's manuscripts were unpublished until Jean Paul Richter's (1847–1937) editions and translations in 1883,[5] his mathematical and scientific work long remained unknown and unstudied.[6] He is thus almost a dead end in the present story: none of the scholars and artists below were influenced by him in terms of aesthetic value in mathematics, save via his illustrations in *Divina proportione*.

[1] Leonardo, *Codex Atlanticus*, fols 455r, 471v.

[2] Duvernoy, 'Leonardo and Theoretical Mathematics', pp. 47–8.

[3] Reynolds, 'The Octagon in Leonardo's Drawings'; Xavier, 'Leonardo's Representational Technique'.

[4] Weyl, *Symmetry*, p. 66.

[5] Kemp, Preface to Leonardo, *Notebooks*.

[6] See, for example, Verga, *Bibliografia Vinciana*.

ALBRECHT DÜRER

The artist Albrecht Dürer (1471–1528) held a platonic view of beauty as governed by geometry, which he at least partly developed on learning of the new developments in art theory regarding perspective and proportion.[1] Lamenting the loss of the books of the ancient artists, he wrote that he would seek beauty 'according to measure, number and weight'.[2] This same formulation is found in Pacioli's *Divina proportione*,[3] where it was explicitly quoted from the Bible.[4][*] Dürer passed most of his life in Nuremberg, but twice visited Italy and may have studied under Pacioli;[5] perhaps his use of the phrase points to Pacioli. But the phrase is suggestive of platonism, and Dürer, though lacking a classical humanist education, learned much of such matters from his close friend Willibald Pirckheimer (1470–1530), who descended from a learned patrician family and who became a great scholar and translator.[6] Whatever lay behind Dürer's use of the expression, he maintained that artistic genius was ultimately founded on mathematics.[7]

Dürer was a natural geometer, understood proofs, and grasped the notion of infinity,[8] though his visual imagination seems to have been at least partly responsible for his erroneous conclusion that the shape obtained by cutting a cone at an angle less than its slope would be ovoid rather than elliptical (see Figure 8.9).[9]

Dürer's 1525 treatise *Underweysung der Messung* [*Instruction on Measure*] was aimed at anyone who used geometry. It ranges widely through geometrical objects and constructions, including epicycloids and helices, and it seems to have been the earliest publication of nets of polyhedra. (A net is a diagram of an 'unfolded' polyhedron.) In book IV of *Underweysung der Messung*, Dürer first showed a net, plan, and elevation of each of the five regular solids.[10] Then he stated that he would offer, to anyone who wished to make 'even more beautiful [hübscher] bodies',[11] the nets of various other polyhedra inscribable in spheres. He first presented a net, plan, and elevation of a 128-hedron that formed an approximation of a globe (see Figure 8.10 (left)).[12] These diagrams he followed by nets of seven of the archimedean solids:[†] a truncated octahedron, truncated cube, cuboctahedron, truncated octahedron, rhombicuboctahedron, snub cube, truncated cuboctahedron.[13] Then he gave nets of two polyhedra of his own invention: one formed by twice truncating the vertices of a cube so that the square faces become dodecagonal (Figure 8.10 (mid-left)), one formed by adding a hexagonal cone to the upper and lower faces of a hexagonal prism (Figure 8.10 (mid-right)).[14]

What governed his choice of these polyhedra? Dürer explicitly emphasized that each of the solids he presented could be inscribed in a sphere.[15] This property seems to have been more important than all the faces being regular polygons, as indicated by the isosceles triangular faces of the twice-truncated cube in Figure 8.10 (mid-left).[16] Further, although a hexagonal prism *without* the addition of

[1] Klibansky, Panofsky & Saxl, *Saturn and Melancholy*, p. 364.

[2] Conway, *Literary Remains*, pp. 177–8, 204; trans. Doorly, 'Durer's *Melencolia I*', p. 265.

[3] Pacioli, *Divina proportione*, ch. 11.

[4] See Wisdom 11:20.

[*] ▸ pp. 160–1

[5] Panofsky, *The Life and Art of Albrecht Dürer*, p. 252.

[6] Ebneth, 'Pir(c)kheimer, Willibald'; Panofsky, *The Life and Art of Albrecht Dürer*, p. 7.

[7] Klibansky, Panofsky & Saxl, *Saturn and Melancholy*, p. 341.

[8] Panofsky, *The Life and Art of Albrecht Dürer*, p. 254.

[9] Panofsky, *The Life and Art of Albrecht Dürer*, p. 255; Herz-Fischler, 'Dürer's Paradox'.

[10] Dürer, *Underweysung der Messung*, Figs 4.29–33.

[11] *Ibid.*, Fig. 4.34.

[12] *Loc. cit.*

[†] ▸ pp. 109–12

[13] *Ibid.*, Figs 4.35–41.

[14] *Ibid.*, Figs 4.42–3 (fig. 43 is incorrectly labelled '34').

[15] Schreiber, 'A New Hypothesis on Dürer's Enigmatic Polyhedron', p. 373.

[16] Dürer, *Underweysung der Messung*, Fig. 4.42.

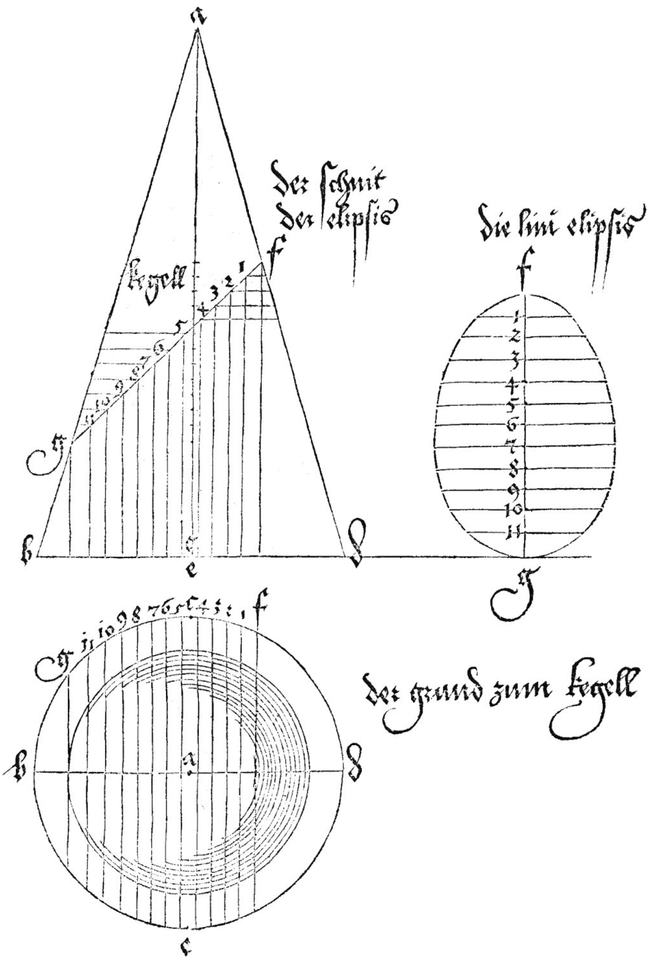

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FIGURE 8.9
Dürer's erroneous reasoning that cutting a cone at an angle less than its slope produced an ovoid (Dürer, *Underweysung der Messung*, Fig. 1.34).

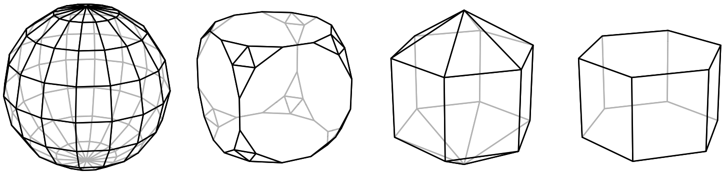
128-hedron
Twice-truncated cube
Hexagonal prism augmented by cones on the hexagonal faces
Hexagonal prism

FIGURE 8.10
Four polyhedra inscribable in spheres. Dürer gave nets for each of these except for the hexagonal prism in *Underweysung der Messung*, Figs 4.34, 42–3 (fig. 4.43 is incorrectly labelled '34').

the hexagonal cones (see Figure 8.10 (right)) would have this same property, it seems to be not as 'sphere-like' as the polyhedron devised by Dürer. Thus the existence of a circumscribed sphere (which is a precise mathematical concept) and 'closeness' to that sphere (which is not) seem to have been in Dürer's eyes criteria for beauty.[1]

Considering polyhedra naturally brings us to Dürer's celebrated 1514 engraving *Melencolia I* (see Figure 8.11), which can be interpreted in connection with mathematical beauty. Such a view must perforce

[1] Schreiber, 'A New Hypothesis on Dürer's Enigmatic Polyhedron', pp. 370–1.

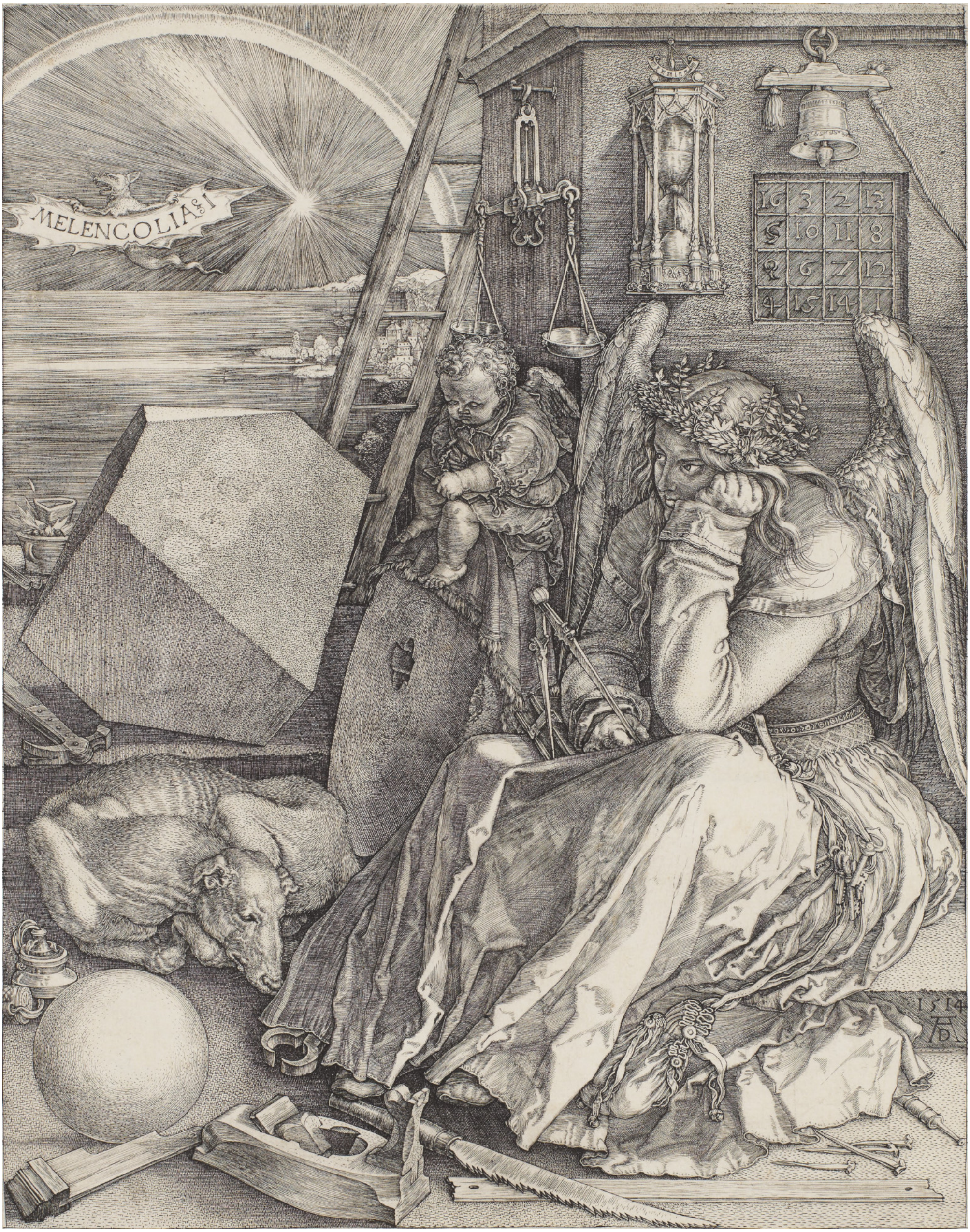
MELENCOLIA I
16 3 2 13
5 10 11 8
9 6 7 12
4 15 14 1
1514

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FIGURE 8.11
Albrecht Dürer, *Melencolia I* (1514).

be tentative, given the sheer range of interpretations that have been assigned to *Melencolia I*: more ink has been expended on this engraving than on any other of his works and indeed possibly on any other work of visual art.[1]

[1] In particular, *weigh* Schuster, *Melencolia I*.

The female figure is an anthropomorphic image of the abstract notion of melancholy,[1] yet she holds a compass and is surrounded by the tools and artefacts of geometry and art and science. Why?

In the astrological tradition, the planet Saturn was the bringer of melancholy.[2] The mediaeval association of the seven liberal arts to the seven planets assigned astronomy to Saturn, just as Dante (1265–1321) described.[*] But a modification was introduced where instead geometry was given to Saturn.[3] One example of such a system appeared in Michael Scot's (before 1200–c. 1235) *Liber Introductorius*:[4]

Trivium	grammar	dialectic	rhetoric	
	Moon	Mercury	Venus	
Quadrivium	arithmetic	music	astronomy	geometry
	Sun	Mars	Jupiter	Saturn

A definitive reason for this change cannot be given, but the god Saturn had been an agricultural deity and so had been responsible for the measurement of land: geo-metry.[5]

Thus through Saturn were geometry and melancholy connected. But the idea that geometry and melancholy were linked goes back to antiquity. A case history of a sufferer, attributed to the Greek physician Rufus of Ephesus (*fl.* late 1st/early 2nd centuries CE), says that '[t]he reason for his illness was the constant contemplation of geometrical sciences'.[6] The mediaeval philosophers Ramon Llull (c. 1232–c. 1315) and Henry of Ghent (c. 1217–1293) both argued for a psychological connection between geometry and melancholy, though from different directions. Llull thought that melancholic temperaments inclined to mathematics; Henry thought that geometrical minds, conscious of their inability to transcend what can be spatially visualized, became melancholic.[7] It is doubtful whether Dürer knew of their arguments, but he was part of an intellectual world in which geometry and melancholy were akin.

And the mathematical allusions to melancholy go beyond geometry. The magic square in *Melencolia I*, whose rows, columns, diagonals, and various other symmetrical combinations of four entries sum to 34 (see Figure 8.12), was a talisman meant to bring the influence of sanguine Jupiter to counteract that of melancholic Saturn.[8] It is likely that Dürer learned of this particular square from Luca Pacioli, who included it, with the association to Jupiter, in his unpublished treatise *De Viribus Quantitatis* [*On the Power of Quantity*], a miscellany of recreational mathematics and conjuring tricks.[9] [Not counting reflections and rotations, there are 880 magic squares containing the numbers 1 to 16, so coincidence is improbable.[10] In *De Viribus Quantitatis*, Pacioli treated magic squares as a mathematical curiosity and only mentioned their magical significance without exploring it.[11] In *De Occulta Philosophia*, Heinrich Cornelius Agrippa (1486–1535) associated to Jupiter a magic square that can be obtained from Dürer's by a rotation and a column swap,[12] but his book was published eighteen

[1] Klibansky, Panofsky & Saxl, *Saturn and Melancholy*, p. 304.

[2] *Ibid.*, esp. pt II.

[*] ▸ pp. 184–5

[3] *Ibid.*, pp. 333–4.

[4] Thorndike, *Michael Scot*, pp. 54–5.

[5] Klibansky, Panofsky & Saxl, *Saturn and Melancholy*, pp. 333–4.

[6] Rufus, *On Melancholy*, fr. 68, § 3 ⟦p. 69⟧.

[7] Klibansky, Panofsky & Saxl, *Saturn and Melancholy*, pp. 337–8; Panofsky, *The Life and Art of Albrecht Dürer*, p. 168.

[8] Klibansky, Panofsky & Saxl, *Saturn and Melancholy*, pp. 325–7.

[9] Pacioli, *De Viribus Quantitatis*, ch. LXXII, fol. 118v.

[10] OEIS, A006052.

[11] Pacioli, *De Viribus Quantitatis*, ch. LXXII, fols 118r sqq.

[12] Agrippa, *De Occulta Philosophia*, bk II, ch. XXII, p. cxlix [page mislabelled clxix].

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

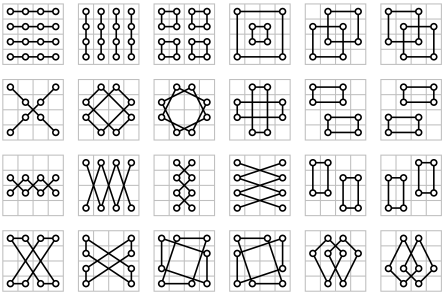

FIGURE 8.12
Dürer’s magic square and combinations of four entries that sum to 34.

years after *Melencolia I* and the original manuscript version from 1510 lacked the planetary squares.[1]]

Erwin Panofsky (1892–1968), a historian of art who made an especial study of Dürer, eloquently described the attitude of the image of melancholy:

> ‘Winged, yet cowering on the ground […] equipped with the tools of art and science, yet brooding in idleness, she gives the impression of a creative being reduced to despair by an awareness of insurmountable barriers which separate her from a higher realm of thought.’[2]

What is the ‘higher realm of thought’, the attempt to reach which may be symbolized by the ladder? The compass she holds, the straightedge at her side, the sphere in front of her, the magic square behind: these suggest mathematics. But *the beautiful* is another possibility. The historian of art Patrick Doorly pointed out that many of the objects that feature in *Melencolia I* seem to allude to Plato’s *Greater Hippias*, the dialogue in which Plato first considered the problem of what is meant by ‘the beautiful’.[3]* In this dialogue, Socrates asks Hippias to help him answer a man who asked him what the beautiful is. Socrates becomes frustrated when Hippias offers *examples* of the beautiful, imagining the man responding to these inadequate answers:

> ‘“[…] For what I am asking is this, man: what is absolute beauty? and I cannot make you hear what I say any more than if you were a stone sitting beside me, and a millstone at that, having neither ears nor brain.”’[4]

And lo, beside the wingèd Melancholy is a millstone with a putto sitting on it. A millstone seems out of place among the tools of geometry and craft, so its presence must be significant. The putto may be writing, but he seems too young and his eyes seem closed: this could be Dürer’s reference to the insensibility of Socrates’ millstone.[5] Socrates proceeds to suggest that the beautiful is the useful, noting that beautiful eyes are those useful for seeing, not those that only seem beautiful but cannot see.[6] Melancholy’s bright eyes contrast the dull, perhaps blind, eyes of the putto.[7] Socrates gives further examples, mentioning useful animals and implements.[8] On the floor around Melancholy are

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[1] Klibansky, Panofsky & Saxl, *Saturn and Melancholy*, p. 327, n. 147.

[2] Panofsky, *The Life and Art of Albrecht Dürer*, p. 168.

[3] Doorly, ‘Durer’s *Melencolia I*’, p. 255.

* ▸ p. 60

[4] Plato, *Greater Hippias*, STE 292d.

[5] Doorly, ‘Durer’s *Melencolia I*’, p. 262.

[6] Plato, *Greater Hippias*, STE 295c.

[7] Doorly, ‘Durer’s *Melencolia I*’, pp. 266–7.

[8] Plato, *Greater Hippias*, STE 295c–e.

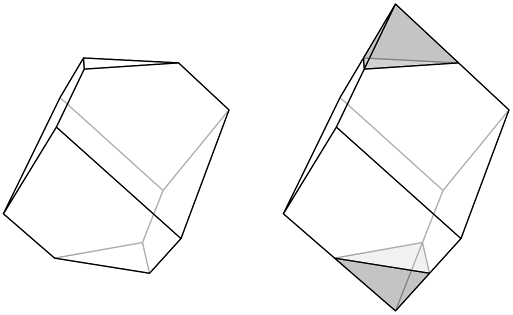


FIGURE 8.13
Dürer’s truncated rhombohedron as it appears in *Melencolia I*, and the restored non-truncated rhombohedron.

a plane, saw, hammer, moulding board, melting pot, tongs, bellows,[1] and a lean hunting dog.[2] These items are just some of the parallels between *Melencolia I* and the *Greater Hippias*.[3] The investigation in this dialogue does not succeed in explaining ‘the beautiful’, and *Melencolia I* can thus be interpreted as revealing the dejection that resulted from this failure and from Dürer’s own frustrated search for the beautiful. The polyhedron in the engraving symbolizes this failure and frustration. It is a rhombohedron truncated at two opposite vertices; see Figure 8.13. (It has sometimes been thought a truncated cube, but that would require an improbable degree of perspective incompetence on the part of Dürer.) Doorly thought that it was supposed to be a ‘botched’ dodecahedron, and so to represent the failure — by Plato and Dürer alike — to understand ‘the beautiful’.[4]

But the representation of this failure is not the whole story, for *Melencolia I* contains another important reference to Plato’s thought, specifically to the conversation with Diotima that Socrates relates in the *Symposium*, where Diotima describes ascending to the ‘highest beauty […] as on *the rungs of a ladder*’.[5][*]

The geometry of the rhombohedron is also important. Though there have been many attempts to recreate its precise shape by determining the angles that perspective has distorted,[6] sketches by Dürer that seem to have been an exploration of ideas for the polyhedron’s faces show a circumcircle.[7] These sketches support the hypothesis (which was advanced before they came to light[8]) that Dürer chose a polyhedron that could be inscribed in a sphere, like the solids he would later exhibit in *Underweysung der Messung*.

The regular dodecahedron would have been the natural choice to represent mathematical success. It is one of the regular solids, which Dürer certainly admired, and is the last-constructed regular solid in the *Elements*.[9] It was distinguished among the regular solids as the most noble by Pacioli,[†] whose work Dürer knew. It served as a symbol of mathematics in the portrait of Luca Pacioli (see Figure 8.7), attributed to Jacopo de’ Barbari, whom Dürer certainly met;[10] in an engraving of Diogenes by Jacopo Caraglio (after a drawing by Parmigianino which does not show a dodecahedron, or a polyhedron of any kind); in Nicolas Neufchâtel’s (c. 1527–after 1567) painting *The Nurem-*

[1] Panofsky, *The Life and Art of Albrecht Dürer*, pp. 155–6.

[2] Doorly, ‘Durer’s *Melencolia I*’, p. 267.

[3] *Ibid.*, p. 270.

[4] *Ibid.*, pp. 269–70.

[5] Plato, *Symposium*, STE 211c, emphasis added.

[*] ▸ p. 77

[6] For a brief survey, see Weitzel, ‘A further hypothesis’, pp. 11–12; see also Futamura, Frantz & Crannell, ‘The cross ratio’.

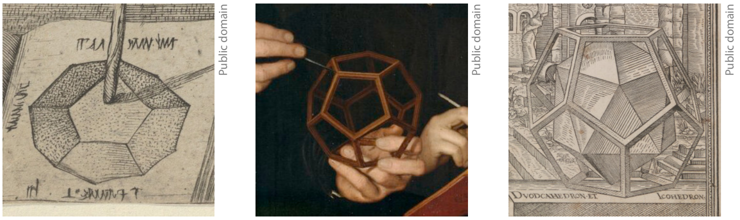
[7] Weitzel, ‘A further hypothesis’, esp. Fig. 1.

[8] Schreiber, ‘A New Hypothesis on Dürer’s Enigmatic Polyhedron’.

[9] Euclid, *Elements*, Prop. XIII.17.

[†] ▸ p. 248

[10] Panofsky, *The Life and Art of Albrecht Dürer*, p. 35.



Jacopo Caraglio, engraving of Diogenes (1526–7)

Nicolas Neufchâtel, *The Nuremberg Writing Master Johann Neudörfer* (1561)

Lorenz Stoer, cover page from *Geometria et Perspectiva* (1567)

FIGURE 8.14
Details from some sixteenth-century artworks that use the dodecahedron as a symbol for geometry or mathematics.

berg Writing Master Johann Neudörfer with a Pupil [*Der Nürnberger Schreibmeister Johann Neudörfer mit einem Schüler*] (1561); and on the cover page for Lorenz Stoer's book *Geometria et Perspectiva* (see Figure 8.14). But Dürer wanted to depict the failure to achieve this perfection. Thus he chose a solid that superficially resembles a regular dodecahedron, with its most prominent faces being pentagonal, and which is indeed beautiful, being highly symmetrical and inscribable in a sphere, but which is manifestly *not* a regular dodecahedron. It is recognizable as both a plausible attempt and an undeniable failure to construct a dodecahedron. It is beautiful but not absolutely so.

Thus, though Dürer declared that he would seek beauty 'according to measure, number and weight', his truncated rhombohedron represents the failure to find in this world — and the futility of here seeking — the perfect beauty of mathematics.

OTHER ARTISTS AND POLYHEDRA

Wenzel Jamnitzer (c. 1507–1585) was a goldsmith in Nuremberg. He was almost certainly aware of Dürer's work since one of his instruments to aid perspective drawing appears to be a refinement of one of Dürer's.[1] He published in 1568 *Perspectiva Corporum Regularium*, a collection of perspective drawings of polyhedra.

Some of the engravings in this book show polyhedra that are akin to the archimedean solids, but Jamnitzer seems to have assigned little importance to regularity of the polygonal faces. Thus, for instance, a polyhedron that superficially resembles the rhombicuboctahedron (see Figure 8.15 (left, bottom-left)) actually has only 6 square faces, with 12 oblong faces.

Jamnitzer seems to have had little concern for a mathematical investigation of polyhedra.[2] Rather, he engaged in a *tour de force* of drawing ever more sophisticated transformations of regular solids: truncations, elevations (in Paciolian and non-Paciolian veins), and other manipulations, *but always applied symmetrically* (see Figures 8.15 and 8.16). Thus, although some of his engravings resemble star polyhedra,[*] it is unlikely that he made the leap to conceive of polyhedra where

[1] Andersen, *The Geometry of an Art*, p. 225.

[2] Field, 'Rediscovering the Archimedean Polyhedra', pp. 258–9, esp. n. 31.

[*] ▸ pp. 289–90

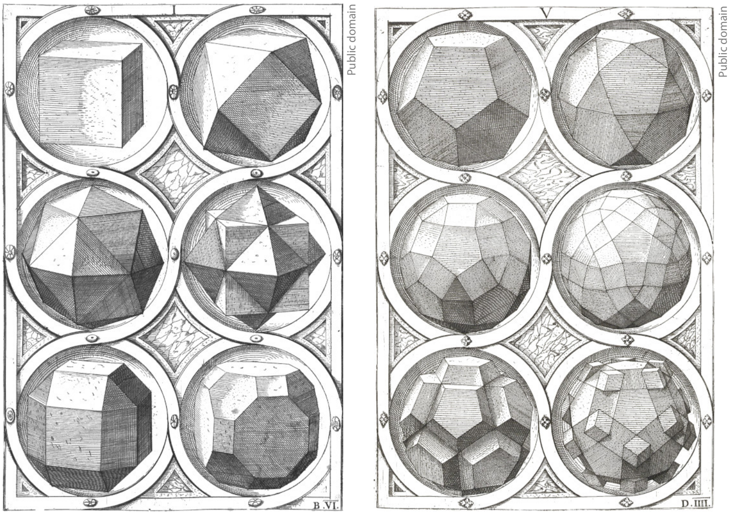


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FIGURE 8.15
Illustrations from Jamnitzer's *Perspectiva Corporum Regularium*, fols 17r, 27r.
Left: A cube (top left) and various polyhedra derived from it. Note that in the bottommost two polyhedra some faces are clearly irregular; thus they are not truly a rhombicuboctahedron and a truncated cuboctahedron.
Right: A regular dodecahedron (top left) and various polyhedra derived from it. Note that on the top-right is an icosidodecahedron and on the middle-left is a rhombicosidodecahedron.

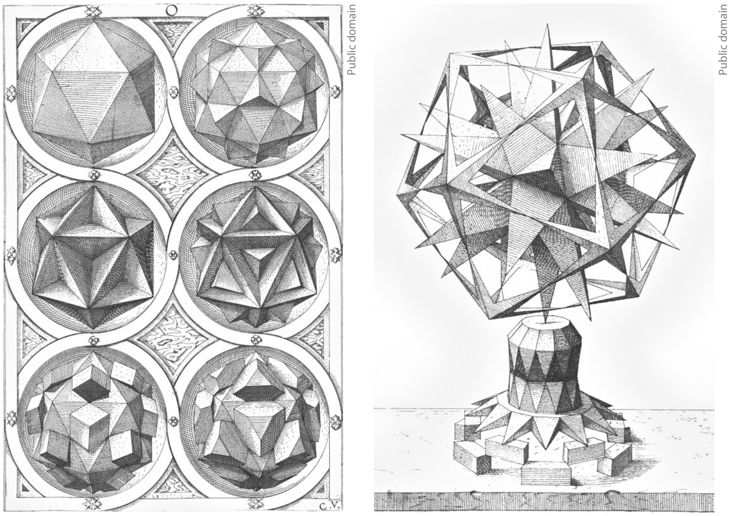

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FIGURE 8.16
Illustrations from Jamnitzer's *Perspectiva Corporum Regularium*, fols 22r, 39r.
Left: A regular icosahedron (top left) and various polyhedra derived from it. Note that the polyhedron below the icosahedron resembles a great dodecahedron.
Right: An elevated icosahedron, resembling a great stellated dodecahedron, within an icosahedral frame.

the faces can pass through each other. Rather, he likely considered the 'great dodecahedron' (Figure 8.16 (left, middle left)) as a solid obtained by removing tetrahedra from each face of an icosahedron, with the shape of tetrahedra chosen so that the faces of adjacent cavities aligned. Similarly, he likely thought of the 'great stellated dodecahedron' (Figure 8.16 (right)) as a particular elevated icosahedron.

A set of printing blocks from the mid-sixteenth century shows that *someone* at that time knew all thirteen archimedean solids. The blocks include nets of all of the regular solids, and truncations and other transformations by which one can obtain the archimedean solids (see, for example, Figure 8.17).[1] The artist who drew the diagrams is unknown,[2] but three of the blocks bear the signature of Hieronymus Andreae (d. 1557).[3] Andreae engraved for Dürer and published a

[1] Schreiber, Fischer & Sternath, 'New light on the rediscovery of the Archimedean solids', pp. 462–5.

[2] *Ibid.*, p. 459.

[3] *Loc. cit.*

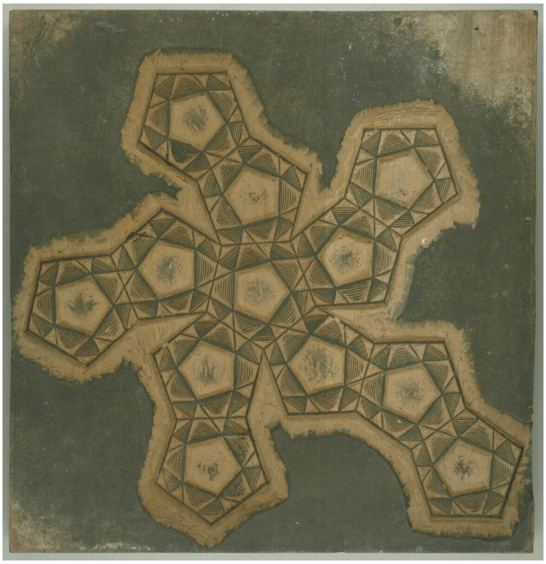
Albertina HO2006/723. [Public domain]

FIGURE 8.17
A printing block that shows the net of a dodecahedron and how to obtain from it a snub dodecahedron. This block is from a collection drawn by an anonymous artist and engraved by Hieronymus Andreae, which indicates that all thirteen archimedean solids were known in the mid-sixteenth century.

posthumous second edition of his *Underweysung der Messung*, edited by Dürer's widow Agnes (1475–1539), in 1538. This second edition contained two archimedean solids not in the first edition. If Andreae had been aware, at the time he published the second edition, of the further discoveries of archimedean solids, he would likely have seen to it that they were at least mentioned. Since they were not, the printing blocks are likely from after c. 1538 and must be from no later than 1557, when Andreae died.[1] But the state of the printing blocks shows that they were only used for printing tests, then corrected; they were never used for printing an edition.[2] Presumably, therefore, they had no contemporaneous influence. In particular, it is impossible to tell from the blocks whether it was known that there were no more archimedean solids to be found.

[What appears to be a print from the block shown in Figure 8.16 has sometimes been claimed to have been drawn by Leonardo for Pacioli's *Divina proportione*.[3] This attribution seems to trace back to the 1948 book *Dürers Gestaltlehre der Mathematik und der bildenden Künste* by the mathematician and historian of mathematics Max Steck (1907–71), where the image is attributed to the 1509 Venice edition of *Divina proportione*.[4] It does not appear there.[5]]

The architect Daniele Barbaro (1514–70) was certainly aware of Dürer's work,[6] and made the closest approach yet to a systematic investigation of the archimedean solids. But his work was mathematically flawed. Although he enumerated all the archimedean solids save the snub cube and snub dodecahedron — which is to say, all those than can be obtained by truncating the regular solids — he also claimed that there was a solid, formed by truncating the vertices of the cuboctahedron, in which some of the vertices were surrounded by two

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[1] Schreiber, Fischer & Sternath, 'New light on the rediscovery of the Archimedean solids', p. 466.

[2] *Ibid.*, p. 459.

[3] See, for example, *ibid.*, p. 462 & Fig. 7.

[4] Steck, *Dürers Gestaltlehre der Mathematik und der bildenden Künste*, Tafel XVIII.

[5] Pacioli, *Divina proportione*, *passim.*

[6] Field, 'Rediscovering the Archimedean Polyhedra', p. 269.

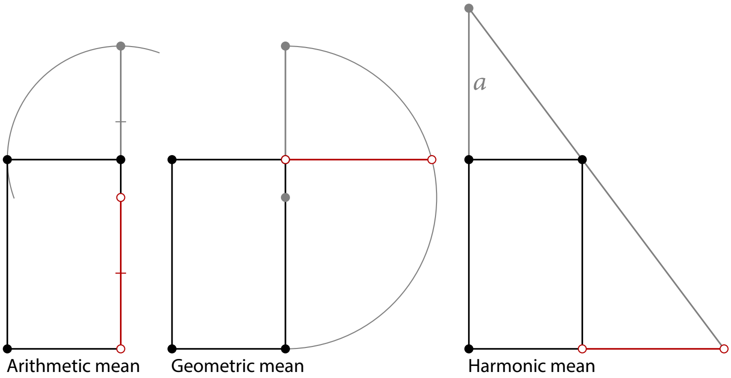

FIGURE 8.18
Palladio's methods for finding the three means of the sides of a rectangle.

regular hexagons and two equilateral triangles. Such a configuration would be flat, and so no such solid exists.[1]

ANDREA PALLADIO

The architect Andrea Palladio (1508–80) owned that Vitruvius* was his master and guide,[2] and followed him in writing a treatise that would have great influence in the development of architecture: *I Qvattro Libri dell'Architettvra* [*The Four Books of Architecture*] (1570). His characterization of beauty clearly echoes Vitruvius': 'Beauty will result from the beautiful form, and from the correspondence of the whole to the parts, of the parts to each other, and of those to the whole.'[3]

As a component in achieving this, Palladio specified the proportions for the shapes of rooms. He wrote that there are seven 'most beautiful and proportionate styles of rooms', which are either circular (which choice is rare), or square, or rectangular with width-to-length proportions of $1 : \sqrt{2}$ (the length being the diagonal of the square of the width), 4 : 3 ('a square and a third'), 3 : 2 ('a square and a half'), 5 : 3 ('a square and two thirds'), 2 : 1 ('two squares').[4] The inclusion of $1 : \sqrt{2}$ is significant, for it represents a departure from the small number ratios of antiquity and specifically from ratios associated to musical concords.

Palladio also specified the heights that rooms should have. If a room has a flat ceiling, its height should equal its width. If a room has a vaulted ceiling, the situation is more complicated. When the room is square, the height of the ceiling should be 4/3 that of the width of the room. When the room is oblong, the architect can choose to use the arithmetic, geometric, or harmonic mean for the height. Palladio did not use this terminology, but gave geometric constructions for the means (see Figure 8.18).[5] For the arithmetic mean: extend the longer side by the length of the shorter and bisect it. For the geometric, draw

[1] Field, 'Rediscovering the Archimedean Polyhedra', p. 271; see Barbaro, *La Pratica della Perspettiva*, ch. XVI, p. 91.

* ▸ pp. 129–30

[2] Palladio, *I Qvattro Libri*, p. 5.

[3] *Ibid.*, p. 6.

[4] *Ibid.*, ch. XXI, p. 52; trans. C.

[5] *Ibid.*, ch. XXIII, pp. 53–4.

a semicircle with diameter the extended side, and take the extension of the shorter side to meet the semicircle. For the harmonic, extend the longer side by the arithmetic mean a and draw the line passing through the new endpoint and the opposite end of the nearer shorter side; take the extension of the further shorter side to its intersection with this line.

Palladio noted that the geometric mean, at least, cannot be found 'in numbers', which refers to the fact that the geometric mean may be incommensurable with the sides of the original rectangle.[1] Nevertheless, his statement reinforces the point that the classical requirement of whole number ratios had, for Palladio, lost its importance. An empirical study of the proportions of the rooms shown in the various plans in *I Qvattro Libri* further confirms this point, suggesting that ratios based on $\sqrt{3}$ may have also played a role.[2] But the significance of this conclusion must be tempered by Palladio's observation that certain heights for vaulted ceilings are not derived from any scheme, being left to the architect's own judgement.[3]

NICCOLÒ TARTAGLIA

Niccolò Tartaglia (1499/1500–1557) was largely self-taught, being unable to afford a tutor. He started to learn mathematics early, and progressed quickly. He made his living as a teacher of mathematics, first in Verona, then in Venice. He was responsible for the first printed edition of Archimedes, publishing William of Moerbeke's Latin translation in 1543.[4]

Tartaglia, like many of his mathematical contemporaries in northern Italy, was interested in the practical applications of mathematics,[5] but he valued the theoretical training that was a necessary prerequisite. The frontispiece of his *La Nova Scientia* (see Figure 8.19) is an allegory showing personifications of the sciences, together with Tartaglia himself, in a walled enclosure whose gate is guarded by Euclid. Directly opposite the gate, steps lead to a smaller, higher enclosure, wherein the personification of philosophy is enthroned. Near the foot of the steps stands Aristotle, more concerned with the physical sciences; at the summit stands Plato, grasping a banner. NEMO HUC GEOMETRIE EXPERS INGREDIATUR; no-one ignorant of geometry may enter, it says, alluding to the sign that tradition placed at the door to Plato's Academy.[6] *

Tartaglia's work *Quesiti et Inventioni Diverse*, as its title indicates, treats of many subjects, including ballistics and other uses of mathematics in war. The seventh book of *Quesiti et Inventioni Diverse*, on the principles of the pseudo-Aristotelian text *Mechanics*, and the eighth book, on weights and levers, takes the form of dialogues between Tartaglia and Diego Hurtado de Mendoza (1503/4–1575), who was

[1] Palladio, *I Qvattro Libri*, ch. XXIII, pp. 53; trans. C.

[2] Mitrović, 'Palladio's Theory of Proportions'.

[3] Palladio, *I Qvattro Libri*, ch. XXIII, pp. 54.

[4] Masotti, 'Tartaglia', p. 259.

[5] Drake, intro. to Drake & Drabkin, *Mechanics in Sixteenth-Century Italy*, p. 13.

[6] Tartaglia, *La Nova Scientia*, Frontispiece.

* ▸ p. 59

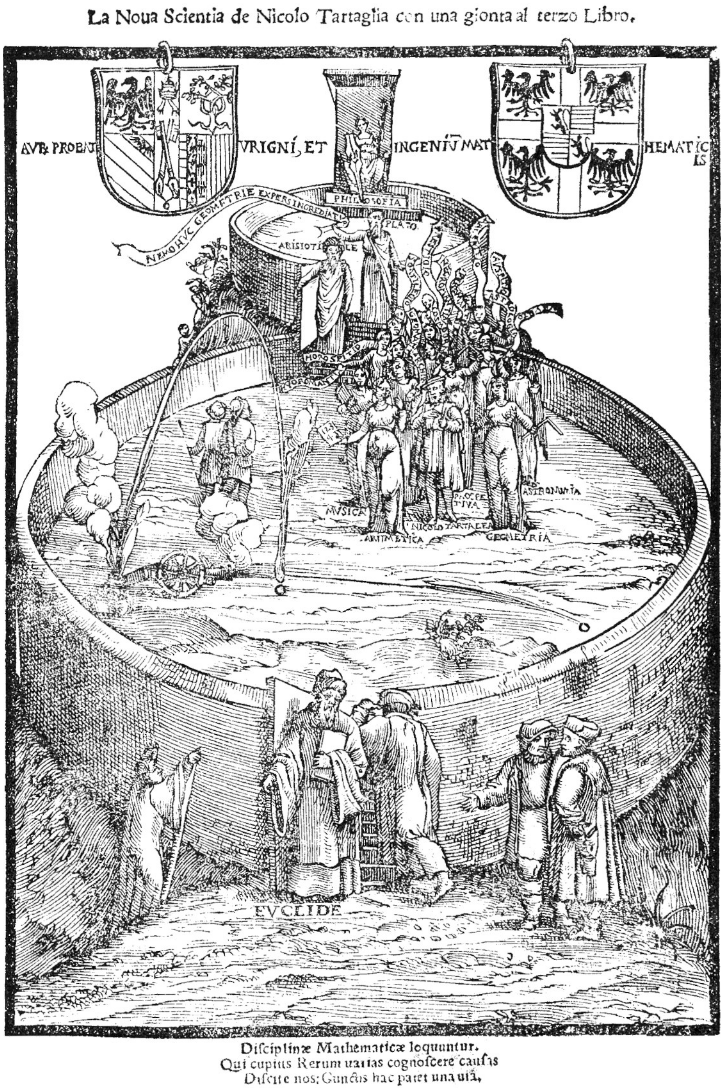
La Noua Scientia de Nicolo Tartaglia con una gionta al terzo Libro.
AVR PROBAT
VRIGNI, ET
INGENIV MAT
HEMATIC IS
NEMO HVC GEOMETRIE EXPERS INGREDIATVR
PHILOSOFIA
PLATO
ARISTOTELE
ASTRONOMIA
PROSPETTIVA
MVSICA
NICOLO TARTALEA
ARITHMETICA
GEOMETRIA
EVCLIDE
Diſciplinæ Mathematicæ loquuntur.
Qui cupitis Rerum uarias cognoſcere cauſas
Diſcite nos; Cunctis hac patet una uiã.

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FIGURE 8.19
The frontispiece of Niccolò Tartaglia's *La Nova Scientia*. Tartaglia is portrayed at the front of the group of the personifications of the sciences. Euclid guards the entrance; Aristotle and Plato stand on the steps leading to the philosophy's own enclosure.

ambassador to the Republic of Venice from the Holy Roman Emperor Charles v (1500–58; r. 1519–56), and who was a scholar who amassed Greek books and manuscripts.

In this dialogue, the character of Mendoza repeatedly gives aesthetic assessments in response to some explanation by Tartaglia's character. For instance, Tartaglia's character states one of the fundamental results from the theory of levers:

> 'If the arms of the balance are proportional to the weights imposed on them, in such a way that the heavier weight is on the shorter arm, then those bodies or weights will be equally heavy positionally.'[1]

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[1] Tartaglia, *Quesiti et Inventioni Diverse*, fol. 93r; trans. Tartaglia, *Various Questions (Extracts)*, pp. 132–3.

After hearing the proof, Mendoza's character says: 'This is a very beautiful [assai bella] proposition.'[1] In another place, he compliments a proof as 'a beautiful demonstration.'[2]

Publishing such assessments[3] of course indicates that Tartaglia recognized mathematical beauty. But he put these remarks into the mouth of his interlocutor, a character named for a real and eminent person, which suggests either that Mendoza himself had made such assessments, or else that Tartaglia thought that Mendoza would not object to his literary self making them.

GIROLAMO CARDANO

Tartaglia's name is forever bound up with that of Girolamo Cardano (1501–76) and the story of their infamous and vitriolic dispute over the solution of the cubic equation. Scipione del Ferro (1465–1526), who was a professor at the University of Bologna, found a way of solving the cubic equation $x^3 + cx = d$. Since the position of a mathematician at the time was dependent on outdoing others in solving problems in public contests, he kept the solution secret, only disclosing it before his death to his student Antonio Maria del Fiore (*fl.* early 16th century) and to his son-in-law and successor to his chair at Bologna, Annibale dalla Nave (1500–58). Tartaglia, challenged by del Fiore, found the same solution to this cubic equation and so won their contest. Cardano pressed Tartaglia to reveal his method; Tartaglia yielded only when Cardano took an oath never to publish what Tartaglia told him. But in time dalla Nave gave Cardano access to del Ferro's papers and thus to the solution that Tartaglia had re-discovered. Considering that his oath did not forbid him from publishing what he had gleaned from *del Ferro*'s work, Cardano included the solution in his *Ars Magna*. Tartaglia thought this act a betrayal and excoriated Cardano publicly and at length.[4]

Cardano's *Ars Magna* (*Great Art*) is one of the seminal texts of algebra, devoted to cubic and quartic equations. In this book, Cardano wrote that del Ferro's solution of the equation $x^3 + cx = d$ was 'a very beautiful and admirable [pulchram & admirabilem] accomplishment.'[5] Del Ferro's technique was to calculate numbers

$$u = \sqrt[3]{\sqrt{\left(\frac{c}{3}\right)^3 + \left(\frac{d}{2}\right)^2} + \frac{d}{2}}, \quad v = \sqrt[3]{\sqrt{\left(\frac{c}{3}\right)^3 + \left(\frac{d}{2}\right)^2} - \frac{d}{2}};$$

the solution is then $u - v$. In essence, this method works because the numbers u and v have the properties that $u^3 - v^3 = d$ and $u^3v^3 = (c/3)^3$, and so

$$(u-v)^3 + c(u-v) = u^3 - 3u^2v + 3uv^2 - v^3 + 3uv(u-v) = d.$$

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[1] Tartaglia, *Quesiti et Inventioni Diverse*, fol. 93v; trans. C.

[2] *Ibid.*

[3] For other examples, see *ibid.*, fols 89r, 90v, 97r.

[4] Katz & Parshall, *Taming the Unknown*, pp. 215–19.

[5] Cardanus, *Ars Magna*, ch. I, p. 222; trans. C.

The modern notation here is very different from what Cardano used, and Cardano's demonstration of the result uses the language of geometry and has recourse to Euclid's *Elements*.[1] One can speculate that Cardano's perception of beauty here arose from the choice of u and v and their very neat interaction with each other and the equation.

Elsewhere in the *Ars Magna*, though not in the first edition,[2] Cardano made an aesthetic comparison between two solutions of the following example problem:

> 'Find three quantities the first plus the second of which is one and one-half times the first and third and the first plus the third of which is one and one-half times the second and third.'[3]

In modern terminology, this problem asks for a solution to these simultaneous equations

$$x + y = 1\tfrac{1}{2}(x + z), \tag{C1}$$

$$x + z = 1\tfrac{1}{2}(y + z). \tag{C2}$$

Cardano gave two solutions to this problem, proceeding along slightly different lines. Here they are given in abbreviated form using modern notation and terminology:

First solution. Subtract z from (C2) to see that

$$x = 1\tfrac{1}{2}y + \tfrac{1}{2}z. \tag{C3}$$

From (C1) and (C3), $x + y = 2\tfrac{1}{4}y + 2\tfrac{1}{4}z$; subtract y to get

$$x = 1\tfrac{1}{4}y + 2\tfrac{1}{4}z. \tag{C4}$$

It follows from (C3) and (C4) that $1\tfrac{1}{2}y + \tfrac{1}{2}z = 1\tfrac{1}{4}y + 2\tfrac{1}{4}z$. Subtracting $1\tfrac{1}{4}y$ and $\tfrac{1}{2}z$ yields $\tfrac{1}{4}y = 1\tfrac{3}{4}z$, and hence $y = 7z$. Assume that $z = 1$, so that $y = 7$ and (C2) implies $x + 1 = 1\tfrac{1}{2}(7 + 1)$, so that $x = 11$.

Second solution. Assume that $z = 1$. From (C1) and (C2) one obtains

$$y = \tfrac{1}{2}x + 1\tfrac{1}{2}, \tag{C5}$$

$$x = 1\tfrac{1}{2}y + \tfrac{1}{2}. \tag{C6}$$

Double (C5) to obtain $2y = x + 3$ and replace the x on the right-hand side using (C6), yielding $2y = 1\tfrac{1}{2}y + 3\tfrac{1}{2}$. Thus $\tfrac{1}{2}y = 3\tfrac{1}{2}$ and so $y = 7$. From (C1) and (C2) one obtains $x = \tfrac{3}{4}x + 2\tfrac{3}{4}$. Hence $\tfrac{1}{4}x = 2\tfrac{3}{4}$ and $x = 11$.

Cardano said that this latter solution 'is more beautiful [pulchrior] when we are working with three quantities'.[4] Presumably Cardano found it more beautiful because making the assumption that $z = 1$ at the *beginning* yields a solution of greater simplicity. In making this assumption, Cardano implicitly utilized the degree of freedom in a system of equations where there are more unknowns than equations.

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[1] Cardanus, *Ars Magna*, ch. xi, pp. 249–50.

[2] Cardano, *Ars Magna* (trans.), ch. ix, p. 75, n. 13.

[3] *Ibid.*, ch. ix, p. 242.

[4] Cardanus, *Ars Magna*, ch. ix, p. 242; trans. C.

Cardano was something of a polymath. His doctorate was in medicine, and as a physician he achieved no less success and recognition than as a mathematician.[1] His mathematical skills were not limited to algebra: he developed the first theory of probability and applied it to his own advantage in gambling. He also wrote *De Subtilitate*, which he intended to be a complete compendium of natural philosophy. It was certainly one of his most widely-read works,[2] and parts of it are important in the history of mathematical beauty.

As already discussed, Cardano made aesthetic judgements in his mathematical work, and, although he nowhere gave a philosophical account of mathematical beauty, his discussion of sensory beauty in *De Subtilitate* at least suggests that he thought that the appreciation of beauty was connected with cognition and involved qualities such as symmetry which can be found also in mathematics:

> ‘every sense especially enjoys what it recognises. […] So what is beauty? A thing perfectly recognised by vision — we cannot love what is not recognised. Vision recognises what is made up simply in the proportion double, triple, quadruple, one and a half times, one and a third times, […]. So in the case of columns arranged on this basis, or trees, or parts of the face, when vision immediately grasps the equality and symmetry in them, it is delighted; delight resides in recognition, just as grief resides in failure to recognise.’[3]

In this passage, Cardano specifically assigned beauty to objects of sight. But imagination of things could also trigger aesthetic responses, and in this situation the response was involuntary

> ‘if you do imagine something beautiful, you are not free to withhold your love. So when a beautiful form is accepted into the power of imagination, we are snatched away into love even against our will. This is why scholarly people are more powerfully in love, because of the strength of their imagining faculty.’[4]

Aesthetic judgements of mathematics appear in book XVI of *De Subtilitate*, which is entitled ‘On the Sciences’. This book begins by presenting sixty useful properties of figures,[5] but the discussion concludes with:

> ‘These are the sixty properties, outstanding in distinction and beauty and regard [nobilitate, & pulchritudine, & admiratione præſtantiores], of the geometrical figures both surface and solid. Yet it does not escape me that properties exist that are practically numberless, but cannot be compared with these for elegance, either because a proof of them has not yet been found, or because they cannot be understood by the criterion of their names alone, or because they are not related to equality

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[1] Gliozzi, ‘Cardano’, pp. 64–5.

[2] Henry & Forrester, intro. to Cardano, *De Subtilitate* (trans.), p. xiii.

[3] Cardanus, *De Subtilitate*, p. 571; trans. Cardano, *De Subtilitate* (trans.), p. 705.

[4] Cardanus, *De Subtilitate*, p. 572; trans. Cardano, *De Subtilitate* (trans.), p. 707.

[5] Cardanus, *De Subtilitate*, p. 592.

but, so to speak, are wanderers; equality is an aim of geometry.’[1]

Cardano’s figures include the circle, sphere, cylinder, cone, ellipse, parabola, hyperbola, spiral, a generic polygon, square, hexagon, heptagon, paraboloid, hyperboloid, and spheroid. The properties he enumerated are largely results about these objects.[2] Some of Cardano’s properties seem to lapse into triviality: for example, the property that a circle contains a point from which all the lines to the circumference have equal length.[3] But others are important results from Euclid, Archimedes, Apollonius, and Ptolemy. Here is a sample:

- If two perpendiculars are dropped from a parabola onto the axis, then the squares on them are to each other as the corresponding segments of the axis (see Figure 8.20).[4] This result, the first property of a parabola that Cardano discussed, is Proposition 1.20 of Apollonius’ *Conics*. If one treats conics from first principles, as the figures formed by taking a section through a cone, such a result certainly seems surprising and perhaps thus beautiful. Today, if one uses analytic geometry and defines a parabola by an equation such as $y^2 = ax$, then the result seems too trivial to be beautiful.

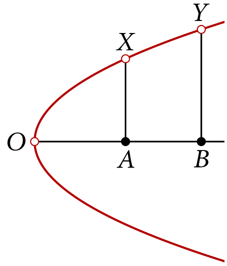

FIGURE 8.20
$AX^2 : BY^2 :: OA : OB$

- Let A_i be the point at the end of the i-th turn of an archimedean spiral. Consider the line through the A_i and the perpendicular to this line through the origin O. Suppose that the tangent to the spiral at A_i intersects the perpendicular at B_i. Then OB_i equals the circumference of the circle with centre O and radius OA_i (see Figure 8.21).[5] This result effectively comprises Propositions 18 and 19 of Archimedes’ *On Spirals*.

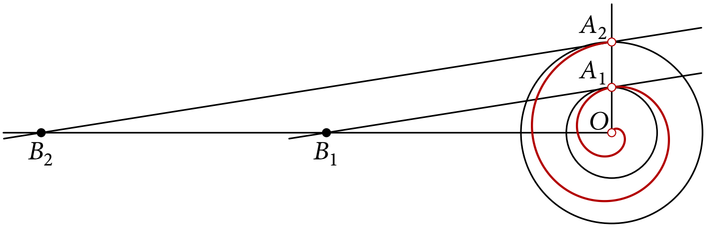

FIGURE 8.21
Diagram illustrating Propositions 18 and 19 of Archimedes’ *On Spirals*.

- A cylinder has three times the volume of a cone with the same height and base and one and a half times the volume of a sphere with great circle equal to the base and diameter equal to the height;[6] these are reformulations of some of Archimedes’ results from his two books *On the Sphere and the Cylinder*.*

- The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the rectangles contained by pairs of opposite sides;[7] this is the result known as Ptolemy’s theorem.[8]†

There is a curious feature here that calls for analysis: these properties that Cardano called beautiful are all properties of figures; that

[1] Cardanus, *De Subtilitate*, p. 598; trans. Cardano, *De Subtilitate* (trans.), p. 790.

[2] Cardanus, *De Subtilitate*, pp. 593–8.

[3] *Ibid.*, p. 593.

[4] *Ibid.*, p. 596.

[5] *Loc. cit.*

[6] *Ibid.*, p. 598.

* ▸ pp. 106–8

[7] *Ibid.*, p. 593.

[8] Ptolemy, *Almagest*, HEI 36–7.

† ▸ p. 642

is, they are geometrical results. In writing *De Subtilitate*, Cardano praised algebra for its usefulness in solving problems and refuting incorrect solutions,[1] but he chose only to exhibit beautiful properties from geometry, despite having judged as beautiful results such as del Ferro's solution of cubic equations.

The most probable explanation is that *Ars Magna* was a specialist mathematical text, while *De Subtilitate* was a general philosophical text. Both the scope of and intended audience for *De Subtilitate* are wider. Thus Cardano chose examples of beautiful and useful mathematics from the comfortable, established, classical genre of geometry, rather than the unsettled frontier of algebra. Furthermore, given his characterization of beauty in terms of vision in *De Subtilitate*, he may have tended to locate beauty at least partly in the geometrical configuration, rather than in the abstracted result. And thus he preferred to list properties *of* rather than propositions *about* mathematical objects.

PETER RAMUS AND HENRY SAVILE

The logician and philosopher Peter Ramus (1515–72) and the mathematician and classical scholar Henry Savile (1549–1622) held almost opposite views on the aesthetic value of Greek mathematics, views that both influenced and were influenced by their understanding of the history of mathematics.

Ramus was not well-educated in mathematics,[2] although he assiduously worked through the works of Euclid, Archimedes, and Ptolemy,[3] and thought that mathematics should be one of four divisions — along with physics, moral philosophy, and dialectical philosophy — of the university curriculum.[4] His turn to mathematics came after he was forbidden from teaching philosophy.

His first mathematical work was an abbreviated Latin edition of the *Elements*, containing only the propositions, without any proofs or diagrams.[5] The separability of the proofs from the propositions had become accepted in the Renaissance, for the proofs were then supposed to be by Theon of Alexandria (*fl.* late 4th century CE).[6] Most manuscripts of the Greek text state in their titles that they are from an edition or from lectures of Theon, who noted in his commentary on Ptolemy that he had supplied the second part of the proof of Proposition VI.33. Only in 1808 did François Peyrard (1759/60–1822) discover that a manuscript from the Vatican Library[7] neither bore a such title nor contained this interpolation, and thus established that its descent was independent of Theon.[8] When in 1505 Bartolomeo Zamberti (1473–after 1539) published his first Latin translation of the *Elements* based on a Greek manuscript from the Theonine tradition, his title attributed the proofs to Theon,[9] and this belief took hold (though not without dissent) in the sixteenth century.

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[1] Cardanus, *De Subtilitate*, p. 602.

[2] Ong, *Ramus*, p. 27.

[3] *Ibid.*, p. 33.

[4] *Ibid.*, p. 198.

[5] Goulding, *Defending Hypatia*, p. 27.

[6] *Ibid.*, pp. 154–9.

[7] Vat. gr. 190 (pt 1, pt 2).

[8] Heath, intro. to Euclid, *Elements*, pp. 46–7, 103.

[9] Heath, intro. to *ibid.*, p. 98.

When Ramus published his edition of the *Elements*, he had a high opinion of Greek mathematics.[1] But he changed his mind, probably during the period 1551–5. Sometime during his studies, his frustrating struggle to understand a particular demonstration in the *Elements* made him lose his temper:

> 'I threw away my drawing-board and ruler, and burst out in rage against mathematics, because it tortures so cruelly those who love it and are eager for it.'[2]

Ramus decided that the fault lay not in him, but in Euclid. Euclid may have been a diligent collector of mathematical truths, but, for Ramus, he had been an incompetent arranger of them. In contrast to his view in 1545, Ramus had come to think that proofs were indispensable to the *Elements*. But this indispensability prompted another criticism of Euclid, for it was Euclid's shortcomings as an expositor that had required Theon to add explanations. Ramus sought a *natural* mathematics, where the propositions were so ordered as to be completely clear and absolutely convincing.[3]

Ramus claimed to detect differences between the version of the *Elements* to which Proclus' *Commentary on Euclid* applied and the known (Theonine) text.[4] Combining these perceived differences with a mistaken chronology that had Proclus flourishing in the second century CE — before Theon lived — led Ramus to think that by comparing the two versions one might discern the scope of Theon's interventions.[5] His conclusion was that Theon had not made enormous changes: removal of analyses, small textual changes, rearrangement of definitions and postulates.[6] But Theon had been in no way special. Ramus thought that all the 'elementators' had gradually shaped the text: Euclid was no different, except as an empty name that had become attached to the *Elements*.[7]

Ramus thought that behind the accretion of the years there was a primaeval, untaught, natural mathematics — a divine gift.[8] Ramus blamed Plato for its loss: Plato could have used his authority to restore this mathematics. Instead, thought Ramus, he had decreed that mathematics should be studied only by philosophers, that philosophy would be 'cheapened' if mathematics were accessible to others. Hence Plato's prohibition on seeking mechanical solutions to geometrical problems: only theoretical solutions with proofs, inaccessible to non-philosophers, were acceptable.[9]

When Ramus wrote, the Euclid of the *Elements* (*fl.* c. 300 BCE) was still thought to be Plato's contemporary Euclid of Megara (*fl.* c. 400 BCE). Although the first source specifically to call Euclid of Megara the author of the *Elements* was fourteenth-century, the confusion is of ancient origin: Valerius Maximus (*fl.* early 1st century CE), thinking of the tale of the duplication of the cube, said that Plato sent those who came to consult him on the problem to Euclid the geometer,[10] who thus must have lived in Plato's time. By the Renaissance, the identification of the two Euclids had become accepted.[11] The historical

[1] Goulding, *Defending Hypatia*, pp. 28–9.

[2] Ramus, 'De sua professione Oratio', p. 521; trans. mod. Goulding, *Defending Hypatia*, p. 30.

[3] Goulding, *Defending Hypatia*, p. 32.

[4] *Ibid.*, pp. 161, 169.

[5] *Ibid.*, p. 169.

[6] *Ibid.*, p. 172.

[7] *Ibid.*, p. 170.

[8] *Ibid.*, p. 40.

[9] *Ibid.*, pp. 46–7.

[10] Valerius Maximus, *Memorable Deeds and Sayings*, § 8.12.ext.1 ⟦p. 295⟧.

[11] Heath, intro. to Euclid, *Elements*, pp. 3–4.

Euclid of Megara was a pupil of Socrates and present at his death,[1] after which, according to Diogenes Laertius, Plato and some other socratic philosophers withdrew to Megara to join him.[2] Thus the confusion of the Megaran and the geometer served to link closely the *Elements* and its author to Plato.

But Ramus did not accept this account, for the *Elements* says nothing about the duplication of the cube, and, had its author been a contemporary of Plato, he would not have ignored such an important problem. He thus rejected Valerius Maximus' account and concluded that the Euclid of the *Elements* was later than Plato. Thus he saw the *Elements* as the product of at least a generation of corruption of a pre-Platonic perfect mathematics.[3]

Before Plato and Euclid, maintained Ramus, Pythagoras had written the first '*Elements*', which had then been improved by Hippocrates (whom Proclus had said was the first to write an '*Elements*'[4]). This tradition, free from the philosophical influence of Plato, was the authentic mathematics that Ramus sought. In particular, he interpreted (incorrectly) Proclus as claiming that Hippocrates was the first to use *reductio ad absurdum* proofs and so thought that pre-Hippocratic geometers like Thales, Pythagoras, Oenopides, Anaxagoras, and Theodorus had direct proofs where the *Elements* used *reductio*.[5] Ramus disdained reductio proof and so saw Hippocrates' innovation as introducing a serious flaw into geometry.[6]

There is an aesthetic undertone throughout these arguments of Ramus, and he implicitly acknowledged the beauty of the unsullied first mathematics he sought: in his *Scholarum Mathematicarum*, Ramus recounted Aristippus' claim that mathematics says nothing of the beautiful or good* and wrote that such nonsense, long refuted by Aristotle, would get him expelled from a good university.[7]

Henry Savile had greater ability in mathematics than Ramus and absorbed the works of Euclid, Archimedes, Ptolemy, and Copernicus. He had what seems to have been a well-founded[8] desire to improve the teaching of mathematics at Oxford, where literary humanism was in the ascendant. When Savile was appointed a lecturer in mathematics, he championed its place in a university education.

In promoting the study of mathematics, Savile had to deal with concerns that it was a craft for astrologers or tradesmen. He therefore portrayed it as a part of humanist education.[9] He criticized Ramus' focus on practical applications, which were not the proper role for mathematics: 'Immortal gods! That mathematics — until now immune from all thoughts of worldly advancement — should be reduced to a mere mechanical skill, as though thrust into some lowly mill!'[10] In this deliberate and perhaps exaggerated platonic guise, he said that he had 'a sort of disgust for external things'.[11] Mathematics, which did not involve such things, would be enjoyable for the student. In his

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[1] Plato, *Phaedo*, STE 59c.

[2] Diogenes Laertius, 'Plato', MEI 3.6.

[3] Goulding, *Defending Hypatia*, pp. 133–8.

[4] Proclus, *Commentary on Euclid*, FRI 66.

[5] Ramus, *Scholarum Mathematicarum*, pp. 95–6.

[6] Goulding, *Defending Hypatia*, p. 66.

* ▸ p. 90

[7] Ramus, *Scholarum Mathematicarum*, p. 73.

[8] Fauvel & Goulding, 'Renaissance Oxford', p. 51; Goulding, *Defending Hypatia*, p. 90.

[9] Fauvel & Goulding, 'Renaissance Oxford', p. 65.

[10] Savile, MS 29, fols 5v–6r, quot. and trans. Goulding, *Defending Hypatia*, p. 109.

[11] Savile, MS 29, fol. 11r, quot. and trans. *ibid.*, p. 107.

own lectures, he gave an account of his own education in geometry, doubtless much distorted for persuasive purposes:[1]

> 'Who would not be delighted by so noble, so pleasing, so agreeable a variety of the most pleasant of things? I was greatly moved by the certain and established passages from first principles to intermediate results, and from them to advanced theorems, set out in a straightforward order of progression. It was pleasant to embark on triangles. Debates on quadrilaterals pleased me greatly. I delighted first to describe circles, and then to unite them with the aforementioned figures. Then, before I had grown tired of circles, the sweet harmony of proportion seized me, in which I should always wish to linger — but I did not wish to die having mastered so little of geometry. And so, not irrationally, I applied myself to irrationals.'[2]

Savile may have structured his account to match Plato's programme of education for the guardians.* Even if his students did not notice the parallel, they would have understood that the pursuit of mathematics is worthwhile for the intellectual delight it brings, independently of practical utility.[3]

At some point during his education, Savile came across Ramus's *Prooemium Mathematicum*,[4] and his lectures show clear borrowing from Ramus's work.[5] But, unlike Ramus, Savile saw Greek mathematics as a triumph. The Greeks had introduced proof — the single most important step in the development of mathematics — and system. Unlike Ramus, Savile saw proof as a good thing.[6] He did not believe in trying to return to some pristine primaeval mathematics. Greater sophistication, such as Hippocrates had introduced, was an admirable thing. His was a Whig history of mathematics as progress, and he wanted to bring to Oxford the same kind of mathematical progress as Greek Alexandria had seen.[7]

In his 1570 lectures, Savile maintained, like Ramus, that Proclus flourished in the second century CE and commentated on a pre-Theonine text of the *Elements*. He saw in Proclus less evidence of difference in the texts, and concluded that Theon's editing had been minimal and that the extant text was almost wholly Euclidean.[8] When he returned to the subject in later life, Johann Jakob Fries (1546–1611) had published in 1592 the correct chronology of platonism, including Proclus.[9] [Savile had also come to accept that the Euclid of the *Elements* was not Euclid of Megara.[10]] Savile therefore had to seek a new argument for his anti-Theonine position. He therefore made an emotive argument, recounting the story of the bloody death of the scholar Hypatia (daughter of Theon; c. 370–415 CE) and the dismemberment and desecration of her corpse,[11] attempting to make a parallel between the horror at the rending of her body and of Ramus' attempt to dissect the *Elements*.[12]

Savile assessed Euclid's *Elements* as near-immaculate:

[1] Goulding, *Defending Hypatia*, p. 111.

[2] Savile, MS 29, fol. 11v, quot. and trans. *ibid.*, p. 110.

* ▸ pp. 75–6

[3] *Ibid.*, p. 111.

[4] Fauvel & Goulding, 'Renaissance Oxford', p. 77.

[5] *Ibid.*, p. 64.

[6] Goulding, *Defending Hypatia*, pp. 105–6.

[7] *Ibid.*, pp. 106–7.

[8] *Ibid.*, p. 173.

[9] Frisius, *Bibliotheca Philosophorum Classicorum*, fol. 41r; Goulding, *Defending Hypatia*, pp. 174–5.

[10] Goulding, *Defending Hypatia*, pp. 138–40.

[11] Savile, *Prælectiones*, pp. 14 sqq.

[12] Goulding, *Defending Hypatia*, p. 177.

> ‘On the most beautiful body of geometry there are two blemishes [nævi] and no more, so far as I know. I have spent long hours poring over the writings of ancients and moderns, as you shall soon see, in order to wipe away and erase those blemishes.’[1]

Given the context of his remark, one blemish was doubtless the parallel postulate, which still seemed out of place as an axiom, and the other was probably the theory of proportion.[2]

FRANÇOIS VIÈTE

In the seventeenth century, René Descartes* (1596–1650) wrote that in his time ‘a sort of arithmetic called “algebra” is flourishing, and this is achieving for numbers what the ancients did for figures’.[3] A few pages later, he discoursed upon a lost ‘true mathematics’, hints of which he thought could be discerned in the works of Pappus and Diophantus. This science, he thought, could be algebra, if only algebra were

> ‘divested of the multiplicity of numbers and incomprehensible figures which overwhelm it and instead possessed that abundance of clarity and simplicity which I believe the true mathematics ought to have.’[4]

Leave aside the question of whether Descartes’s lament here is aesthetic or epistemic: by the time he wrote, the first steps in this reform of algebra had already been taken by François Viète (1540–1603).

Viète was arguably an amateur mathematician. His training and much of his career was in law, though he also served as a cryptanalyst for Henry IV of France.[5] His mathematical legacy was not the result of his professional activities. His most important achievement was his development of algebra towards an autonomous and unified theory of equations, as opposed to a disparate collection of techniques. A component of this work was his extension of symbolic notation: he was the first to use letters for both known and unknown quantities. He saw himself as fulfilling the goal that Descartes would later enunciate: recovering a type of mathematics that the ancients had known, and that had become encumbered by the detritus of the intervening age. In the dedication to the collection of treatises he called *The Analytic Art*, he described both the end and the means of this pursuit, using aesthetic language that has a particular pungency:

> ‘See that the art that I offer is new, or so old and by the barbarians so defiled and polluted that I had to put it into an entirely new form, and, having banished all the pseudo-technical coinages, lest it retain any of the dirt and reek as of old,

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[1] Savile, *Prælectiones*, p. 140; trans. Goulding, *Defending Hypatia*, p. 177.

[2] Goulding, *Defending Hypatia*, p. 177; Hilbert, *Lectures on the Foundations of Geometry*, p. 210; Stäckel & Engel, *Die Theorie der Parallellinien*, p. 22.

* ▸ pp. 303–5

[3] Descartes, *Rules for the Direction of the Mind*, p. 17.

[4] *Ibid.*, p. 19.

[5] Busard, ‘Viète’, p. 18.

> to work out and publish a new vocabulary, to which ears are little accustomed, and thus it will be hardly avoidable that at the very threshold many will be offended and frightened away. And yet underneath the Algebra or Almucabala which they lauded and called the great art, all Mathematicians recognized that incomparable gold lay hidden, though they used to find very little.’[1]

It is clear that Viète appreciated and aimed for mathematics that was aesthetically pleasing. But the one occasion where he described his own work as beautiful serves better as a striking illustration of how novelty can engender beauty and the passage of time can breed *ennui*. Viète described as ‘elegant and very beautiful [elegans & perpulchræ]’[2] the result quoted here in the original Latin and using his original notation:

> ‘Si A quadrato-cubus $\overline{-B-D-G-H-K}$ in A quad. quad.
> $\overline{+B \text{ in } D+B \text{ in } G+B \text{ in } H+B \text{ in } K+D \text{ in } G+D \text{ in } H+D}$
> $\overline{\text{in } K+G \text{ in } H+G \text{ in } K+H \text{ in } K}$ in A cubum $\overline{-B \text{ in } D \text{ in } G}$
> $\overline{-B \text{ in } D \text{ in } H-B \text{ in } D \text{ in } K-B \text{ in } G \text{ in } H-B \text{ in } G \text{ in } K}$
> $\overline{-B \text{ in } H \text{ in } K-D \text{ in } G \text{ in } H-D \text{ in } G \text{ in } K-D \text{ in } H \text{ in } K}$
> $\overline{-G \text{ in } H \text{ in } K}$ in A quad. $\overline{+B \text{ in } D \text{ in } G \text{ in } H+B \text{ in } D \text{ in } G}$
> $\overline{\text{in } K+B \text{ in } D \text{ in } H \text{ in } K+B \text{ in } G \text{ in } H \text{ in } K+D \text{ in } G \text{ in } H}$
> $\overline{\text{in } K}$ in A, æquetur B in D in G in H in K: A explicabilis est de qualibet illarum quinque B, D, G, H, K.
>
> 1QC – 15QQ + 85C – 225Q + 274N, *æquatur* 120. *Fit* 1N 1, 2, 3, 4, *vel* 5.’[3]

Translated into English and into modern algebraic notation, Viète’s result says that *If, for an unknown a and known b, d, g, h, and k,*

$$\left.\begin{aligned} a^5 - a^4&(b+d+g+h+k)\\ &+a^3(bd+bg+bh+bk+dg\\ &\qquad+dh+dk+gh+gk+hk)\\ &-a^2(bdg+bdh+bdk+bgh+bgk\\ &\qquad+bhk+dgh+dgk+dhk+ghk)\\ &+a(bdgh+bdgk+bdhk+bghk+dghk)\\ &\qquad=bdghk,\end{aligned}\right\}\quad(\mathrm{V})$$

then a can be b, d, g, h, or k. For example, if the known terms b, d, g, h, k are respectively 1, 2, 3, 4, 5 then the equation (V) is $a^5 - 15a^4 + 85a^3 - 225a^2 + 274a = 120$, and its solutions are $a = 1, 2, 3, 4, 5$. A mathematician today, working with this modern notation, would presumably notice that if one sets $a = b$ and multiplies out the brackets on the left hand side of (V), then all terms except $bdghk$ cancel; by symmetry, the same reasoning applies for d, g, h, k. Thus, to modern eyes, the result collapses into near-triviality and seems to deserve little if any aesthetic praise.

[1] Vietæ, ‘Inclytæ Principi Melvsindi Catharinæ Parthenæensi’, n.p. trans. mod. Katz & Parshall, *Taming the Unknown*, pp. 236–7.

[2] Vietæ, ‘De Emendatione Æqvationvm (II)’, p. 158; trans. C.

[3] *Ibid.*, p. 158.

Through Viète's *Opera*, there are scattered other references to aesthetic value. None stand out in the context of the sixteenth century; rather, they indicate that his taste was compatible with other writers. Notably, he thought that the archimedean spiral provided an elegant way to tackle problems related to the circle, and the first result he gave on this subject was among the beautiful properties listed by Cardano.[1]*

Before taking leave of Viète, it is worth noting one of his discoveries that has earned lasting aesthetic appreciation. Going beyond Archimedes' inscription of a 96-gon in a circle to approximate π, Viète considered inscribing 4, 8, 16, and in general 2^n-gons in the unit circle, and, relating the side lengths of successive polygons, derived an expression that is today stated as

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots ;$$

'a remarkable formula, one which even today arouses our admiration for its beauty'.[2]

NICOLAUS COPERNICUS

The revolution in astronomy that Nicolaus Copernicus (1473–1543) initiated grew out of a mathematical aesthetic distaste for the Ptolemaic system. Like the heliocentric system that he went on to develop, the Ptolemaic system cleaved to the aesthetic perfection of the circle (and circular motion) and the sphere. But Copernicus had objections, both to the system as a whole and to some of the devices it used. Copernicus may have been influenced towards this view by his teacher, Domenico Maria da Novara (1454–1504), who had also criticized the Ptolemaic system as too awkward to be a real description of the mathematical order of nature.[3]

In his prefatory letter to *De revolutionibus orbium cœlestium*, Copernicus wrote that an orderly creator must have shaped the universe in a mathematically neat way. This aesthetic principle entailed the primacy of the circle and sphere:

> 'the universe is spherical. The reason is either that, of all forms, the sphere is the most perfect, needing no joint and being a complete whole, which can be neither increased nor diminished; or that it is the most capacious of figures, best suited to enclose and retain all things'.[4]

The principle also entailed rejection of devices like the equant,† which violated the axiom of uniformity of motion, which seems to detract from the symmetry of the circle. Eccentrics did not violate the axiom of uniformity of motion, and could be used to model accurately the apparent motions of the planets, but in the Ptolemaic system they did not illuminate the structure or symmetry of the universe:[1]

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[1] Vietæ, *Variorum de Rebvs Mathematicis Responsorvm Liber viii*, p. 355.

* ▸ p. 267

[2] Maor, *To Infinity and Beyond*, p. 10.

[3] Kuhn, *The Copernican Revolution*, p. 129.

[4] Copernicus, *On the Revolutions*, ch. i.1, p. 8.

† ▸ p. 118

> 'like some one taking from various places hands, feet, a head, and other pieces, very well depicted, it may be, but not for the representation of a single person; since these fragments would not belong to one another at all, a monster rather than a man would be put together from them.'[2]

Copernicus clearly found such an unsymmetrical system repugnant.

Frustrated by the problems of the Ptolemaic model, Copernicus scoured the philosophical literature for some other system, and came across[3] a pseudo-Plutarchian account of Philolaus' cosmology, which had a mobile Earth moving in an 'oblique circle' around a fire.[4] [In Copernicus's time, the work in question was attributed to Plutarch; it is a recension by an unknown author of Aëtius' (*fl.* c. late 1st century CE) book *On the Opinions of the Philosophers*.] A neoplatonist symbolic identification of God and the Sun was present in Renaissance literature and art and may have eased Copernicus's acceptance of heliocentrism.[5]

The Copernican system did not displace previous systems because it had an advantage in accuracy. Ephemerides calculated using the Ptolemaic system show errors in the position of Mars that reach approximately the same magnitude as later ones calculated using the Copernican system.[6] Rather, the Copernican system triumphed because it had a greater aesthetic appeal. Copernicus saw in his heliocentric system 'a marvelous symmetry of the universe, and an established harmonious linkage between the motion of the spheres and their size, such as can be found in no other way'.[7] His system had greater coherence. It used only 'minor epicycles' to remove small differences between the model and the phenomena; it did not resort to large epicycles to explain the brute fact of apparent retrograde motion. Thus the Copernican orbits were closer to *being* circles than their geocentric counterparts.[8] Further, the Copernican system did not require an artificial linkage between circles to keep Mercury and Venus close to the Sun, and it connected the relative sizes of the planetary orbits.[9]

The aesthetic superiority of the system was also sufficient to overcome its greatest weakness: a moving Earth implies parallax in the stars. Since none had ever been observed, the sphere of fixed stars had to be vastly greater in the Copernican universe than in the Ptolemaic.[10]

[The lack of measurable parallax was simply because the distances to the stars are beyond what would then have been reasonable conjecture. Given the apparent diameter of the stars, the distance required would have entailed the sphere of stars being truly colossal relative to the Sun. Galileo weakened this objection when he noted that stars appear smaller through the telescope than with the naked eye.[11] The first accurate measure of the parallax of a star came only in the nineteenth century: in 1838, Friedrich Bessel (1784–1846) announced that 61 Cygni had a parallax of 0.314″, corresponding to a distance of 11.4 ly; the accepted modern values are 0.292″ and 10.9 ly.[1]]

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[1] Copernicus, *On the Revolutions*, p. 4.

[2] *Loc. cit.*

[3] *Ibid.*, pp. 4–5.

[4] ps.-Plutarch, *Peri tōn Areskontōn tois Philosophois Physikōn Dogmatōn*, STE III.13.896a; trans. ps.-Plutarch, *Sentiments Concerning Nature*, p. 156; see also DK, 44A21.

[5] Kuhn, *The Copernican Revolution*, pp. 130–1.

[6] Gingerich, '"Crisis" versus Aesthetic', pp. 86, 94.

[7] Copernicus, *On the Revolutions*, ch. I.10, p. 22.

[8] Field, *Kepler's Geometrical Cosmology*, p. 36.

[9] Kuhn, *The Copernican Revolution*, pp. 169–73; Martens, *Kepler's Philosophy*, pp. 65–6; Gingerich, '"Crisis" versus Aesthetic', pp. 89–90.

[10] Stephenson, *The Music of the Heavens*, pp. 71–4.

[11] Martens, *Kepler's Philosophy*, p. 28.

Responses

Georg Joachim Rheticus (1514–74) made a visit to Copernicus in 1539 in order to learn about the new cosmology, and probably in part to persuade him to have his work printed in Nuremberg by Johannes Petreius (1496/7–1550). He ended up studying under Copernicus in Frauenburg for some years.[2] He wrote an account of his teacher's theory in his *Narratio Prima*, and thought that additional evidence in favour of the Copernican system was that it had six circles centred on the Sun, in contrast to the Ptolemaic system's seven centred on the Earth:

> 'the number six is honored beyond all others in the sacred prophecies of God and by the Pythagoreans and the other philosophers. What is more agreeable to God's handiwork than that this first and most perfect work should be summed up in the first and most perfect number?'[3]

◆

The 1566 Basel reprinting of *De revolutionibus orbium cœlestium* came with a testimonial from Erasmus Reinhold (1511–53), praising Copernicus and his work.[4] But Reinhold did not see the importance of Copernicus's system as lying in the heliocentric cosmology, but rather in the technical mathematics of his model.[5] On the title page of his copy of *De revolutionibus orbium cœlestium*, Reinhold wrote 'Axioma Astronomicum: Motus coelestis aequalis est et circularis vel ex aequalibus et circularibus compositus';[6] the motto is a paraphrase of the title of the sixth chapter of the first book of *De revolutionibus*, which translates as 'The motion of the heavenly bodies is uniform, eternal, and circular or compounded of circular motions'.[7] This statement points to what Reinhold thought important in the book: the geometrical model.[8] In particular, Reinhold seems to have appreciated that Copernicus had dispensed with Ptolemy's equant.[9]

◆

Tycho Brahe (1546–1601) was not convinced by Copernicus's system. For him, the arguments against a moving Earth were too strong. Nevertheless, he could not ignore the greater mathematical harmony of Copernicus's theory, and came to abandon the Ptolemaic system for his own hybrid Tychonic system. In this cosmology, the circles of the Moon and Sun and the sphere of fixed stars were centred on

[1] Fricke, 'Bessel', pp. 100–1.

[2] Gingerich, *The Book Nobody Read*, ch. 3.

[3] Rheticus, *Narratio Prima*, p. 147.

[4] Gingerich, 'The role of Erasmus Reinhold', p. 43.

[5] *Ibid.*, pp. 55–6.

[6] Reinhold, quot. in *ibid.*, p. 58.

[7] Copernicus, *On the Revolutions*, p. 10.

[8] Gingerich, 'The role of Erasmus Reinhold', p. 58.

[9] Gingerich, 'From Copernicus to Kepler', p. 515.

the Earth and those of the other five planets on the Sun.[1] This system Copernicus would doubtless have rejected had it occurred to him, for the geometric centre of the universe was not the centre of most of the motions it contained.

GALILEO GALILEI

> ‘Philosophy is written in this grand book
> — I mean the universe — […].
> It is written in the language of mathematics,
> and its characters are triangles, circles,
> and other geometrical figures’
>
> — Galileo, *The Assayer*, pp. 183–4,
> (trans. Stillman Drake).

Galileo Galilei (1564–1642) made important contributions to the ongoing mathematization of science and to observational astronomy. Through experiment and thought-experiment, he undermined the older aristotelian physics and introduced the principle of relativity. On hearing in 1609 of the invention of the telescope, he constructed his own, set about building improved versions, and turned his instruments upon the heavens in January 1610. He discovered mountains on the Moon, countless new stars, and four satellites of Jupiter. Together with the supernova of 1604, which seemed at odds with the aristotelian incorruptibility of the skies, his observations weakened the hold of the old astronomy and cemented his own acceptance of the Copernican system. Later, his personal conflict with religious authority became a symbol of the change in world-view of which he was a part.

Galileo certainly recognized mathematical beauty. Not only is there the word of his son Vincenzio Galilei (1606–49), whose biographical notes of Galileo say that ‘through the mathematical sciences he knew too well the beauty of truth’,[2] but Galileo himself described results and proofs as beautiful. In his dialogue *Two New Sciences*, the participants discuss the result:

THEOREM P. ‘*When a projectile is carried in motion compounded from equable horizontal and from naturally accelerated downward* [*motions*], *it describes a semiparabolic line in its movement.*’[3] $\boxed{\text{P}}$

The character Sagredi refers to this result as ‘this first beautiful proposition’.[4] Elsewhere in the same dialogue, he calls a proof leading to the quadrature of the parabola a ‘beautiful and ingenious demonstration’.[5] This proof uses the method of exhaustion, which Euclid and Archimedes had used and which in the sixteenth century saw new development.[6]

Sagredi’s interlocutor Salviati presents a proof of the following result:

[1] Kuhn, *The Copernican Revolution*, pp. 200–2.

[2] Galilei, ‘Notizie Raccolte’, p. 596.

[3] Galileo, *Le Opere*, p. 269; trans. Galileo, *Two New Sciences*, p. 217, insertion in trans.

[4] Galileo, *Le Opere*, p. 269; trans. C.

[5] Galileo, *Le Opere*, p. 184; trans. Galileo, *Two New Sciences*, p. 141.

[6] Cf. Mueller, *Philosophy of Mathematics*, p. 246.

THEOREM C. *'The surfaces of equal cylinders, excluding [those of] their bases, are in the ratio of the square roots of their lengths.'*[1]

Sagredi then gives his judgement:

> 'SAGR. The proof appears to me so beautiful [tanto bella] that even if it had no power to persuade me of that original purpose for which it was adduced (though indeed for me it has), the brief time devoted to hearing it was well spent in any case.'[2]

This is quite a profound statement: a beautiful proof can be enjoyed even if it fails to convince. Thus for Galileo, the beauty of a proof seems to have stood independent of its power to persuade or explain. The proof of the theorem, in a more concise form than Galileo offered, is as follows:[3]

Proof of C. Let C_{AB} and C_{CD} be two cylinders of equal volume with heights AB and CD. Let E be the geometric mean of these heights. Cut the first cylinder at F to obtain C_{AF}, where $AF = CD$. (See Figure 8.22.) The bases of cylinders of equal volume are inversely proportional to their heights, so the ratio of the base of C_{CD} to the base of C_{AB} is $AB : CD$. Circles are to one another as the squares on the diameters, so the squares on the diameters of the bases also have the same ratio $AB : CD$. But $AB : CD :: \text{sq}(AB) : \text{sq}(E)$. So the squares on the diameters of the circles have the ratio $\text{sq}(AB) : \text{sq}(E)$. Hence the diameters of the circles have the ratio $AB : E$, and so do the surfaces of equal-height cylinders based on them. Hence the surfaces of C_{CD} and C_{AF} have the ratio $AB : E$. Since the ratio of the surface of C_{AF} and the surface of C_{AB} is $AF : AB$, the ratio of the surface of C_{CD} and the surface of C_{AB} will be $AF : E$. So the ratio of the surface of C_{AB} and the surface of C_{CD} is $E : CD$, or $AB : E$, which is the square root of $AB : CD$. $\boxed{\text{C}}$

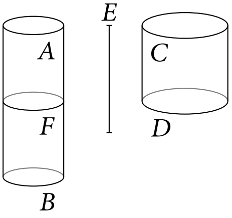

FIGURE 8.22
Cylinders in Galileo's proof.

In one respect, Galileo's aesthetic judgement held him back: he could not accept Kepler's theory of elliptical planetary motion.* He was too strongly attached to the pythagorean–platonic conception of the circle as perfect and of the highest beauty. In principle, he thought that human value judgements should not be imposed upon the world:

* ▸ pp. 285–7

> 'we must not ask nature to accommodate herself to what we may think the best arrangement and disposition, but must adapt our intellect to what she has produced'.[4]

But in practice he could not detach himself from his aesthetic preferences. Early in the *Dialogue Concerning the Two Chief World Systems*, it is clear that Galileo accepted the privileged position of the circle from aesthetic and mechanical perspectives. In the arguments he advanced for circular motion, he appealed to the perfection of the circle and the imperfection of the straight line, and held that straight-line motion was unnatural in a perfectly ordered universe.[5] Only

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[1] Galileo, *Le Opere*, p. 98; trans. Galileo, *Two New Sciences*, p. 59, insertion in trans.

[2] Galileo, *Le Opere*, p. 99; trans. mod. Galileo, *Two New Sciences*, p. 60.

[3] Galileo, *Le Opere*, p. 98.

[4] Galileo, letter to Cesi, dated 30 Jun. 1612, repr. in Galileo, *Carteggio*, p. 344; trans. Panofsky, *Galileo as a Critic of the Arts*, p. 12.

[5] Galileo, *Dialogue Concerning the Two Chief World Systems*, pp. 18–20.

circular motion was uniform and can be perpetually maintained, for in circular motion the tendency to move away and towards a point were in balance, and thus there was neither acceleration nor deceleration.[1] Although more than two decades had elapsed since Kepler had published his *Astronomia Nova*, Galileo could not bring himself to abandon the perfect order he saw in the circle-based universe for Kepler's ellipses, which he thought of as distorted circles.[2]

JOHANNES KEPLER

Johannes Kepler's (1571–1630) thought is clearly part of the platonic tradition that saw the universe as shaped according to mathematical beauty. He was probably influenced towards this tradition as a student at University of Tübingen by Philip Melanchthon (1497–1560).[3] For the structure of the cosmos, Kepler looked, as previous platonist thinkers had, to the five regular solids and the circle and sphere. He never abandoned the judgement that these forms were beautiful, but he was the first to break the implication from the beauty of the sphere and circle to the motions of the planets being circular or circle-based. In retrospect, this was a giant intellectual step.

Platonism and the beauty of the universe

Kepler was a platonist, but a nuanced one.[4] He called Plato and Pythagoras 'our true masters',[5] and was heavily influenced by the *Timaeus*, which he thought 'a kind of commentary on the first chapter of Genesis, or the first book of Moses, converting it to the Pythagorean philosophy'.[6] (By 'Pythagorean philosophy', Kepler here meant heliocentric cosmology.[7]) He shared with Plato the idea that the creator of the universe was a geometer,[8] but he rejected the polyhedral particles that Plato described in the *Timaeus*. Kepler wanted not only a mathematical model of the cosmos, but a model that could be checked against observation. Plato's use of polyhedra described insensible properties and was immune from empirical checking.[9]

Kepler's platonism was also informed by later writers in the tradition. He agreed with Proclus' ideas that the mind comes into being containing mathematical ideas and that the structure of the universe is revealed through mathematics.[10] He may also have been influenced in this view by Nicholas of Cusa,* whom he called 'divine' for likening the sphere to God.[11] Again, he did not treat this platonic heritage uncritically. He rejected Robert Fludd's (1574–1637) metaphysics[12] and Nicholas of Cusa's unbounded universe.[13] Beneath his philosophical refutation of the latter seems to have lain a kind of revulsion at the notion of infinite space:

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[1] Galileo, *Dialogue Concerning the Two Chief World Systems*, pp. 31–2.

[2] Panofsky, *Galileo as a Critic of the Arts*, p. 25.

[3] Martens, *Kepler's Philosophy*, pp. 11–12.

[4] Field, 'Le platonisme de Johannes Kepler'.

[5] Kepler, letter to Galileo, dated 7 Oct. 1597, repr. in Kepler, *Briefe 1590–1599*, p. 145.

[6] Kepler, *The Harmony of the World*, bk IV, ch. i, p. 301.

[7] Field, 'Le platonisme de Johannes Kepler', p. 11.

[8] Kepler, *Mysterium Cosmographicum* (1596), ch. II, p. 26; Kepler, *The Harmony of the World*, p. 407.

[9] Field, *Kepler's Geometrical Cosmology*, p. 16.

[10] Martens, *Kepler's Philosophy*, pp. 34–5.

* ▸ pp. 238–40

[11] Kepler, *Mysterium Cosmographicum* (1596), ch. II, p. 23.

[12] Martens, *Kepler's Philosophy*, p. 35.

[13] Koyré, *From the Closed World*, ch. III.

> ‘This very cogitation carries with it I don’t know what secret, hidden horror; indeed one finds oneself wandering in this immensity, to which are denied limits and center and therefore also all determinate places.’[1]

Such ‘horror’ made it impossible, for Kepler, that the universe should be infinite, since he thought that it was absolutely necessary that God should have created the most beautiful world. In both editions of *Mysterium Cosmographicum* [*The Secret of the Universe*] (1596, 1621), Kepler stated that the universe is so shaped because its creator, being most perfect, must produce a work of the greatest beauty, for the *best* [optimus] must be the *most beautiful* [pulcherrimum]. This last doctrine he ascribed to Cicero, who, he said, had followed Plato’s *Timaeus*;[2] this points to him having read Cicero’s translation.[*] In order to create a best and most beautiful universe, and so that humans could understand these ideas, the creator made measure and quantity.[3] He expressed similar ideas in *Harmonice Mundi* [*The Harmony of the World*] (1619), where he linked this beauty to geometry:

> ‘For geometry, the part of which that looks in this direction was embraced in the two previous books, is coeternal with God, and by shining forth in the divine mind supplied patterns to God, as was said in the preamble to this Book, for the furnishing of the world, so that it should become best and most beautiful and above all most like to the Creator.’[4]

The structure of the heavens, and thus their geometry, he explicitly held to be superior in beauty to commonplace things: ‘the heavens, the first of God’s works, were laid out much more beautifully than the remaining small and common things’.[5] And the geometry is absolutely fundamental to the beauty: an oft-quoted (but little-cited) position of Kepler was that he saw geometry as the archetype of the beauty of the world: ‘Geometria archetypus pulchritudinis mundi’.[6]

Kepler seems to have held to the idea of a universal mathematical harmony throughout his working life, as indicated by his correspondence[7] and the notes to the second edition of *Mysterium Cosmographicum*, whose tone is ‘rather that of pleasure in his later success than one of explanation or apology for his earlier failures’.[8] The geometrical beauty of the universe was indeed a motivation for his work:

> ‘I certainly know that I owe it this duty, that as I have attested it as true in my deepest soul, and as I contemplate its beauty with incredible and ravishing delight, I should also publicly defend it to my readers with all the force at my command.’[9]

A particular instance of this harmony, which points back to Plato, is that Kepler thought that the diameter of the orbit of Saturn would be the geometric mean of the diameter of the Sun and of the radius of the sphere of fixed stars.[10] [Archimedes attributed to Aristarchus an analogous position about the orbit of the Earth.[11]] Kepler published

[1] Kepler, *De Stella Nova*, p. 253; trans. Koyré, *From the Closed World*, p. 61.

[2] Kepler, *Mysterium Cosmographicum* (1596), ch. 11, pp. 23–4.

[*] ▸ pp. 127–9

[3] *Ibid.*, ch. 11, p. 24.

[4] Kepler, *The Harmony of the World*, bk III, ch. i, p. 146.

[5] Kepler, *Mysterium Cosmographicum* (1621), p. 15; trans. Field, *Kepler’s Geometrical Cosmology*, p. 96.

[6] Kepler, ‘Tertius Interveniens’, p. 204.

[7] Field, *Kepler’s Geometrical Cosmology*, pp. 97–8.

[8] *Ibid.*, p. 74.

[9] Kepler, *Epitome Astronomiae Copernicanae*, p. 8; trans. Burtt, *Metaphysical Foundations*, p. 47.

[10] Field, *Kepler’s Geometrical Cosmology*, p. 27.

[11] Dijksterhuis, *Archimedes*, pp. 362–3.

two calculations of the size of the sphere of the fixed stars made on this assumption, in *De Stella Nova* and *Epitome Astronomiae Copernicanae*. In the relevant passage of the former work, he justified his assumption by appealing to the beauty and rationality of the world.[1] Although Kepler did not cite Plato in his argument, it is reasonable to conclude that one of his premisses was Plato's distinction of the geometric mean as the 'fairest of bonds'.[*]

Regular solids and Mysterium Cosmographicum

Kepler's first book was *Mysterium Cosmographicum*, a paean[2] to Copernicus' theory. It was the first work to emphasize the systematic aspect: Copernicus himself had treated the movements of all the planets separately, rather than as components of a unitary whole.[3]

In this book, Kepler tackled two fundamental questions: 'Why are there six planets?' and 'Why do the five intervals between them have their given sizes?'.[4] The 'why' in these questions was nothing less than an inquiry into the motivation of the Creator in so shaping the universe; and the answers are grounded in geometrical beauty. It is perhaps no surprise that *Mysterium Cosmographicum* has been called 'the most daring Neoplatonic document since *Timaeus* itself'.[5]

Kepler considered the questions in terms of the orbs of the planets. An *orb* was a hollow sphere, whose inner and outer surfaces bound a volume containing the orbit of the planet. He scorned explanations for the sizes of the orbs in terms of the properties of numbers.[6] He particularly rejected symbolical explanations such as Rheticus's claim that there were six planets because of the 'sanctity' of the number 6,[†] for it was a fallacy to seek reasons for the design of the world in numbers that only subsequently gained some dignity.[7]

In July 1595, Kepler had the idea of explaining the orbs using the regular solids. On seeing that the ratio of the incircle and circumcircle of an equilateral triangle reminded him of that of the orbs of Jupiter and Saturn, he first tried to fit regular polygons between the orbs of the other planets. Some gaps required a single polygon; others needed nested polygons of different types. This first idea he rejected, partly because the process would not let him reach the Sun itself, but mainly because it did not explain the reason for there being six planets.[8] Then he asked himself why plane figures should lie between the three-dimensional orbs, recalled the five regular solids, and found the solution:

> '*The Earth is the orb which is the measure of all: Construct a dodecahedron around it: The orb surrounding that will be Mars. Around Mars construct a tetrahedron: The orb surrounding that will be Jupiter. Around Jupiter construct a cube: The orb surrounding that will be Saturn. Now construct an icosahedron inside the Earth: The orb inscribed within that will be Venus.*

[1] Kepler, *De Stella Nova*, p. 234.

[*] ▸ pp. 64–5

[2] Stephenson, *The Music of the Heavens*, p. ix.

[3] Field, *Kepler's Geometrical Cosmology*, p. 32.

[4] *Loc. cit.*

[5] Peterson, *Galileo's Muse*, p. 177.

[6] Field, *Kepler's Geometrical Cosmology*, p. 46.

[†] ▸ p. 276

[7] Kepler, *Mysterium Cosmographicum* (1596), p. 10; Field, 'Kepler's rejection of numerology', p. 274.

[8] Kepler, *Mysterium Cosmographicum* (1596), pp. 11–12; Field, *Kepler's Geometrical Cosmology*, p. 48.

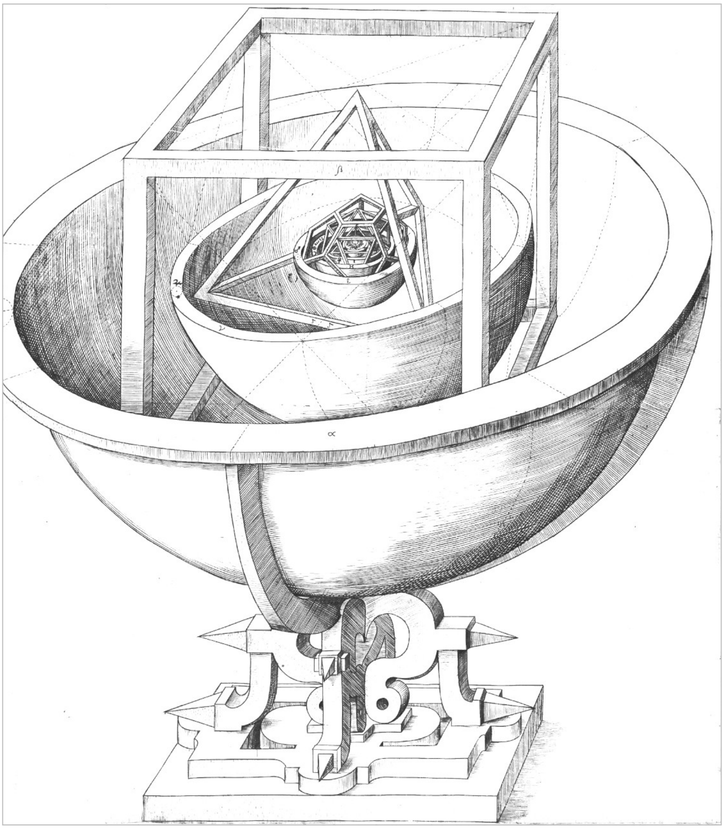

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FIGURE 8.23
Kepler's theory of the separation of the orbs of the planets by the five regular solids, as illustrated in the second edition of *Mysterium Cosmographicum*. (Kepler, *Mysterium Cosmographicum* (1621), pl. following p. 26; clipping of left-hand side in orig.)

> *Inside Venus inscribe an octahedron: The orb inscribed within that will be Mercury.* There you have the explanation of the number of the planets.'[1]

This arrangement is depicted in a famous illustration in *Mysterium Cosmographicum*; see Figure 8.23.

In explaining the reasons for the universe being shaped thus, Kepler said that the mathematical concepts of measure and quantity, introduced by the Creator for the design and understandability of the best and most beautiful world, lead to the fundamental distinction of straight and curved,[2] both of which were thus installed in the universe. The curved was emplaced in the sphere of fixed stars that circumscribed the world.[3] The straight is found best exemplified, in three dimensions, by looking for a class of bodies that are most regular, in having equal sides and angles: the regular solids.[4] (Much later in life, Kepler mentioned that 'nothing can be more beautiful in geometry than equality'.[5])

Kepler seems to have sought to present his work as part of an ancient tradition, emphasizing what he saw as the pythagoreanism of his system.[6] His dedicatory poem outlined the questions he wished to answer and declared: 'Pythagoras reveals all this to you by five figures',[1] and he coined the term *Copernicopythagoraea* to describe his view of the cosmos.[2]

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[1] Kepler, *Mysterium Cosmographicum* (1596), p. 13, emphasis in orig.; trans. mod. Kepler, *The Secret of the Universe*, p. 69.

[2] Kepler, *Mysterium Cosmographicum* (1596), ch. II, p. 24.

[3] *Ibid.*, ch. II, pp. 24–5.

[4] *Ibid.*, ch. II, p. 25.

[5] Kepler, *Epitome Astronomiae Copernicanae*, pt III.iv, p. 209.

[6] Regier, 'Johannes Kepler and the Pythagoreans', p. 349.

It is worth emphasizing that the illustration in Figure 8.23 was not an astronomical diagram. In particular, the small circle within the thickness of each orb cannot have indicated an epicycle, since the deferent (the larger circle on which an epicycle was posited to move) would not have been concentric with the orb.[3] In any case, Kepler did not posit the physical existence of these orbs.[4] Each orb, for Kepler, was simply a volume within which lay the orbit of the planet. More generally, Kepler maintained a distinction between astronomical and geometrical hypotheses. The former concerned the actual orbit of a planet; the latter concerned the mathematical construction of that orbit. Two geometrical hypotheses could be very different and yet yield the same orbit. For astronomy, there was no difference between them. In particular, a geometrical hypothesis did not entail any ontological commitment about the mechanism of a planet's movement. A choice between astronomically-equivalent geometrical hypotheses could not be made on an empirical basis.[5]

Kepler argued that order of the five regular solids between the various orbs was determined *a priori*. He first classed the five solids as *primary* or *secondary*. In the first class were the tetrahedron, the cube, and the dodecahedron; in the latter were the octahedron and the icosahedron. He gave seven features that led to this distinction:[6]

1) The primary solids had different polygons as faces; the secondary solids all had triangular faces.
2) Each of the primary solids had a face of its own; the secondary solids had the same face as the tetrahedron.
3) In the primary solids, three faces met at each vertex; in the second solids, more than three faces met at each vertex.
4) The primary solids were not derived from any others; the secondary solids could be obtained by taking the centres of the faces of primary solids as vertices of new solids.
5) The primary solids had their highest rotational symmetry about an axis through the centre of a face; the secondary solids had their highest rotational symmetry about an axis through a pair of opposite vertices. (Kepler's language here is unclear; this interpretation is due to the historian of science J. V. Field (1943–).[7])
6) Primary solids most naturally sat on a face; secondary solids were most naturally suspended by a vertex.
7) There were three primary solids and two secondary solids; 3 was a perfect number (in a non-Euclidean sense) and 2 an imperfect one. Also, the internal plane angles at the vertices of the primary solids were acute for the tetrahedron, right for the cube, and obtuse for the dodecahedron, covering all three possibilities. In the secondary solids, the internal angles were obtuse. In the octahedron, the plane angles of the faces were acute, the angle between

[1] Kepler, *Mysterium Cosmographicum* (1596), 'LECTOR AMICE SALVE' [p. 4], l. 6; trans. Kepler, *The Secret of the Universe*, p. 49.

[2] Kepler, *Mysterium Cosmographicum* (1596), p. 7.

[3] Field, *Kepler's Geometrical Cosmology*, p. 41.

[4] Field, 'Kepler's Rejection of Solid Celestial Spheres'.

[5] Martens, *Kepler's Philosophy*, pp. 60–1.

[6] Kepler, *Mysterium Cosmographicum* (1596), ch. III, p. 29; paraphrased following Field, *Kepler's Geometrical Cosmology*, pp. 54–5.

[7] Field, *Kepler's Geometrical Cosmology*, p. 55.

two opposite edges at a vertex was right, and the angle between two adjacent faces was obtuse.

Note that features 1 and 2 seem hardly distinguishable. Feature 4 alluded to the octahedron and icosahedron being duals of the cube and dodecahedron; the cube and dodecahedron can be obtained from the octahedron and icosahedron by the same process of taking centres of faces as vertices. Feature 6 stands out as purely a matter of taste, with no mathematical aspect. Kepler expanded on this point:

> 'For if you roll the latter [secondary] onto their base, or stand the former [primary] on a vertex, in either case the onlooker will avert his eyes at the awkwardness of the spectacle.'[1]

This list of criteria is a strong indicator of the importance of aesthetic considerations to Kepler. What Kepler saw as the awkwardness or harmony of a situation informed his inference of the truth.

This division between the three primary and two secondary solids explained why there were two planets inside the Earth's orbit and three outside; the three primary solids took the more perfect *active* role of containing the Earth, while the two secondary solids took the less perfect *passive* role of being contained by it. This distinction was the first stage in the determination of the order of the solids. The overall order was determined by the degree of perfection of the solids, with the most perfect outermost.

For Kepler, the cube was the most perfect of the solids; he gave nine reasons for this:[2]

1) The cube was generated by its face; the other solids were either cut out from the cube or formed by adding to the cube.
2) The cube was the only solid that could be cut up into equal bodies of the same shape.
3) The cube was the only solid that exhibited all dimensions however it is turned: all its parts were either mutually perpendicular or parallel.
4) It had the same number of faces as the three dimensions have limits, namely six; and twice that number of edges, namely twelve.
5) All of its angles, whether between edges or faces, were right.
6) It was the only body exemplifying the criterion for the perfection of 3 quoted by Simplicius from Ptolemy, that no more than three straight lines through a point could be mutually perpendicular.
7) It was the simplest of all polyhedra and is used as the measure of volume.
8) Inscribing a right angle in a circle was a fundamental step towards proving many other results.
9) Humans, the most perfect animals, were like the cube in having six surfaces: upper, lower, back, front, left, right.

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[1] Kepler, *Mysterium Cosmographicum* (1596), ch. III, p. 29; trans. Martens, *Kepler's Philosophy*, p. 43, insertions in trans.

[2] Kepler, *Mysterium Cosmographicum* (1596), ch. V, pp. 31–2; paraphrased following Field, *Kepler's Geometrical Cosmology*, pp. 57–9.

These properties, excepting the last, are more mathematical than the primary/secondary distinction.[1] But the idea that they together constituted or imply 'perfection' is *not* mathematical.

Following Kepler's discussion of the perfection of the cube, there follows a similar treatment of the tetrahedron, an invitation to the reader to treat of the dodecahedron, and shorter treatments of the octahedron and icosahedron.[2] In total, these treatments set up the hierarchy of perfection that Kepler required to establish the order in which the regular solids were placed between the orbs. It was a point of some pride to Kepler that he first of all assigned the polyhedra according only to the relative distances of the planets, and only *afterwards* worked out their hierarchy and saw how it fit with his hypotheses.[3] This admission was presumably a rhetorical device intended to disarm the suspicion that he devised the criteria for his hierarchy to fit his hypotheses,[4] but it could also have been a defence against a charge of having modified the hypotheses to fit his model.[5]

Kepler was delighted that Copernicus' *a posteriori* reasoning from the observable phenomena yielded the same conclusions as his own *a priori* reasoning from the idea that the created world is the best and most beautiful.[6] This convergence, for Kepler, pointed to a superiority of the Copernican theory over Ptolemaic cosmologies, and formed an argument for its truth.[7] The systems of the ancients failed to *explain* features such as the number of planets or the sizes of their orbits. In contrast, 'since a most beautiful order is found in Copernicus, the cause must also necessarily be found in it'.[8] In Kepler's view, there were six planets *because* God shaped the cosmos around the five regular solids.

Kepler also came to believe that moons around other planets would make his polyhedral system even more accurate, for the orbs of the planets would have to be thickened to accommodate the moons, which would fill in some excess space, just as accounting for Earth's Moon did.[9] But his system was primarily grounded in the aesthetic value of the regular solids and their mathematical properties, and only secondarily on the positions of the orbits as deduced from the phenomena. Although the latter condition is the most important one from a modern perspective, Kepler's theory does fit the evidence well and its problems stem from its philosophical basis.[10]

Kepler apparently saw the aesthetic aspect of a theory as explaining what aristotelians would have termed its final and formal causes, namely its teleological purpose and its arrangement or shape. Thus if a theory was empirically adequate and aesthetically superior, he would have held it to be explanatorily successful.[11]

Ellipses and the Astronomia Nova

When Tycho Brahe died, Kepler succeeded him as court mathematician to the Holy Roman Emperor Rudolph II (1552–

[1] Field, *Kepler's Geometrical Cosmology*, p. 59.

[2] Kepler, *Mysterium Cosmographicum* (1596), chs VI–VIII, pp. 32–4; Field, *Kepler's Geometrical Cosmology*, pp. 59–60.

[3] Stephenson, *The Music of the Heavens*, p. 79.

[4] *Loc. cit.*

[5] Martens, *Kepler's Philosophy*, pp. 44–5.

[6] Kepler, *Mysterium Cosmographicum* (1596), ch. II, p. 25.

[7] Rothman, *The Pursuit of Harmony*, pp. 47–8.

[8] Kepler, *Mysterium Cosmographicum* (1596), ch. I, p. 15; trans. Rothman, *The Pursuit of Harmony*, p. 47.

[9] Field, *Kepler's Geometrical Cosmology*, pp. 80, 89.

[10] *Ibid.*, p. 52.

[11] Martens, *Kepler's Philosophy*, p. 44.

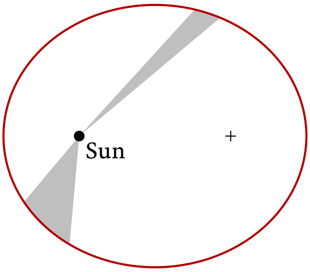

FIGURE 8.24
Kepler's first and second laws:

1) The orbit of a planet is an ellipse with the Sun at one focus.
2) A line joining the Sun to the planet sweeps out equal areas (such as those shown in grey) in equal times.

1612) and gained access to his predecessor's observational records. These records allowed him to mount seriously an attack on the problem of the motion of Mars.[1] His *Astronomia Nova* [*New Astronomy*] is an account of the investigations that led to the first two of the three laws that bear his name. In modern terms, these two laws are:[2]

1) The orbit of a planet is an ellipse with the Sun at one focus.
2) A line from the Sun to a planet sweeps out equal areas in equal times.

(See Figure 8.24.)

Kepler wanted to eliminate Copernicus' epicycles in favour of Ptolemy's equants. There were both aesthetic and metaphysical reasons for this. Aesthetically, he preferred the perfect circular motion of the equant to the not-quite-circular motion of the epicycles:

> 'Copernicus recast the Ptolemaic equant [...]. Consequently, there is something wanting in the motion or the revolution, so it is not perfectly circular as it is in Ptolemy. But it makes no difference that is differs very little from a circle. Is the perfection now undone if it differs a hair's breadth with regard to geometrical beauty?'[3]

It may have been Proclus' criticisms of the epicycles* that planted the seeds of Kepler's distaste for them.[4]

Metaphysically, his rejection of epicycles was based on his view that it was implausible that there should be a causal connection between a planet and an empty point in space.[5] This view also manifested in his rejection of the mean Sun in favour of the true Sun. The *mean Sun* was the point in empty space that was the centre of Earth's circular orbit in the Copernican system. Kepler had philosophical reasons for preferring the true Sun, especially an analogy between the Christian doctrine of the Trinity and the spherical cosmos: the Father was represented by the Sun, the Son by the sphere of fixed stars, and the Holy Spirit by the 'celestial air' ['auram coelestem'] that filled it.[6] But the switch to the true Sun also yielded immediate aesthetic advantages in the symmetry of observed planetary positions and movements.[7]

Kepler's intention was still to use circular orbits, with equants based on the true Sun, but he failed to construct such a system for the orbit of Mars that agreed with observation.[8] He was thus led to consider what had been thitherto unthinkable, or at least unthought: that

[1] Caspar, *Kepler*, §§ 111.2–3.

[2] Murray & Dermott, *Solar System Dynamics*, § 1.3.

[3] Kepler, 'Consideratio Hypotheseos circa aequantem', p. 512; trans. Voelkel, *The Composition of Kepler's* Astronomia Nova, p. 108.

* ▸ p. 155

[4] Field, *Kepler's Geometrical Cosmology*, p. 169.

[5] Kepler, *New Astronomy*, pp. 127–8.

[6] Kepler, *Mysterium Cosmographicum* (1596), ch. 11, p. 24–5.

[7] Martens, *Kepler's Philosophy*, p. 74.

[8] *Ibid.*, pp. 76–9.

the orbits might be based on *non-circular* motions.[1] His first efforts involved ovals — in the specific sense of egg-shaped, non-elliptical, curves — and he seems to have been aware of Dürer's flawed analysis of the ellipse,[*] for there is a one-word marginal note 'Dürer' at the relevant part of the *New Astronomy*.[2] After considering how to explain the libration of planets, which he solved by means of a 'beautiful geometrical demonstration',[3] he realized that he could modify this explanation slightly to use an ellipse. And once he had made the leap to an elliptical orbit, he saw that his libration model and the distance model were unified and seems to have gained faith that he had hit upon the right answer.[4] (These developments took place over the period 1602–5,[5] of which time Kepler wrote: 'I was straining and gnashing my teeth then'.[6]) In introducing elliptical motion to astronomy, Kepler made a giant intellectual step. Before his *New Astronomy*, there was no suspicion — there was *no reason* to suspect — that celestial bodies had anything but circular motion.[7]

Kepler had unchained his astronomical thought from the aesthetic primacy of the circle, but he had not abandoned it. The circle was still the most beautiful figure: it had 'mental beauty and perfection'.[8] But this beauty was mental, and the celestial motions were not. Even under the animistic hypothesis that the planets have intellects and so *desire* circular motion, translating the mental conception of a circle into motion is an imperfect process:

> 'Even assuming that we were to endow the planets with intelligences, these intelligences would still be unable to achieve what they want, that is to say, the absolute perfection of the circle; for, if it were a question of only the beauty of the circle, the circle would not only be correctly perceived by the mind but also adorn all bodies, and above all the celestial ones; the most beautiful quantity [would adorn] that which has quantity. But since, in order to produce movement, there would also be necessary, in addition to the mind, the natural and animal faculties, these would follow their own inclinations'.[9]

The acceptance of the Keplerian ellipse was slow. Some astronomers found them too strange and cleaved to circle-based systems. Galileo was one;[†] Kepler's correspondent David Fabricius (1564–1617), also an astronomer, was another:

> 'You have deprived the movements of their circularity which, upon careful consideration, seems absurd to me [...]. Celestial bodies are perfectly round, that is clear from ☉ and ☽. Therefore there can be no doubt that all movements of all celestial bodies take place in perfect circles, and not in ellipses.'[10]

[1] Martens, *Kepler's Philosophy*, p. 88.

[*] ▸ p. 252

[2] Kepler, *New Astronomy*, ch. 46, p. 465.

[3] *Ibid.*, p. 554.

[4] Martens, *Kepler's Philosophy*, pp. 96–7.

[5] Whiteside, 'Keplerian Planetary Eggs', pp. 8, 12.

[6] Kepler, *New Astronomy*, ch. 45, p. 458.

[7] Martens, *Kepler's Philosophy*, p. 21.

[8] Kepler, *Epitome Astronomiae Copernicanae*, § iv.ii.ii, p. 295.

[9] Kepler, *Epitome Astronomiae Copernicanae*, § iv.iii.i ⟦p. 331⟧; trans. Panofsky, *Galileo as a Critic of the Arts*, 30–1; ins. in trans.

[†] ▸ pp. 278–9

[10] Fabricius, letter to Kepler, dated 20 Jan. 1608, repr. in Kepler, *Briefe 1604–1607*; trans. mod. Panofsky, *Galileo as a Critic of the Arts*, p. 29.

But, in time, Kepler's ellipses became accepted as the *ne plus ultra* of mathematical beauty in systems that lack a deep theoretical foundation: the astronomer and philosopher John Herschel (1792–1871) said that Kepler's laws

> 'constitute undoubtedly the most important and beautiful system of geometrical relations which have ever been discovered by a mere inductive process, independent of any consideration of a theoretical kind.'[1]

Non-regular solids, geometric harmony, and Harmonice Mundi

In writing *Harmonice Mundi*, Kepler built upon the same foundations as *Mysterium Cosmographicum* (1596) an edifice that both subsumed and extensively modified the polyhedral structure of the earlier work. But the observational evidence meant that he no longer thought that the regular solids served as the sole formal cause of the structure of the universe. The tetrahedron still fit exactly between the perihelion orbs of Jupiter and Mars. But there was too little space to fit the cube and octahedron between the orbs of Jupiter and Saturn and of Mercury and Venus: with their vertices touching the outer orbs, their faces would cut into the inner orbs. Conversely, there was too much space for the dodecahedron and icosahedron between the orbs of Earth and Mars and of Venus and Earth: with their vertices touching the outer orbs, there would be a gap between their faces and the inner orbs.[2] There were other formal causes that influenced the structure of the universe, and these lie in the harmonies in the motions of the planets. But Kepler saw the ratios expressing these harmonies as also deriving from polyhedra. In the course of his investigation, he also expanded the range of polyhedra he examined from the five regular solids to include star polyhedra, rhombic polyhedra, and the archimedean solids.

Geometric harmony

Kepler's intention in book v of *Harmonice Mundi* was to develop a system of harmonies that would explain why the planets had been assigned their eccentricities and periods.[3] But harmonic theories in the pythagorean tradition, derived from the tetractys, were too rigid. They did not concur with what the sense of hearing finds consonant.[4] Kepler thus strove to devise a mathematical theory that would nevertheless 'satisfy the judgement of the ears'.[5] And so, while the harmonies were numerical ratios, Kepler thought that they had to be grounded in geometry.[6] Most of the ratios depended ultimately on the geometric properties of the regular solids associated with the spaces between the planets. This idea, for Kepler, was a solid foundation on which to build: it agreed well with observation and was aesthetically

[1] Herschel, *A Preliminary Discourse on the Study of Natural Philosophy*, § 296.

[2] Kepler, *The Harmony of the World*, bk v, § xvii.

[3] Kepler, *The Harmony of the World*, ch. v.iv–viii; Martens, *Kepler's Philosophy*, pp. 123–36.

[4] Kepler, *The Harmony of the World*, bk III, p. 137.

[5] *Loc. cit.*

[6] Stephenson, *The Music of the Heavens*, p. 3; Rothman, *The Pursuit of Harmony*, pp. 21–2.

and philosophically satisfying.[1] And this basis maintained his view of the Creator as platonic rather than pythagorean.[2]

Among what Kepler called a 'diverse and manifold'[3] relationship, it suffices to examine a sample: He considered the 'marriage' of each regular solid with its dual: the cube with the octahedron, the dodecahedron with the icosahedron, the self-dual tetrahedron standing alone. To each 'marriage' he assigned harmonic ratios closest to the ratios of the circumscribed and inscribed spheres: $1 : 2$ and $3 : 5$ to the cube–octahedron; $3 : 4$, $5 : 8$, $4 : 5$, and $5 : 6$ to the dodecahedron–icosahedron; $1 : 8$ and $1 : 4$ to the tetrahedron. The ratio of the two spheres of the tetrahedron was in fact $1 : 3$, but Kepler had assigned $1 : 2$ and $1 : 3$ to the cube and so decided that their squares should be assigned to the tetrahedron — except that the square of $1 : 3$ is $1 : 9$, which is not harmonic and so was replaced by $1 : 8$.[4]

The foundation of this reasoning is clear: his desire to force numerical harmonies into a geometric frame.[5] And the elaborate construction was ultimately less than successful, as the historian of science Owen Gingerich (1930–2023) acutely observed:

> 'All too many correspondences are approximate, as even Kepler admits. [...] Kepler gives no fewer than fifty propositions to justify every deviation, and to argue for an intricate set of interlocking harmonies and tonal intervals. But surely *any* intonation could be hammered into such a frame. From anyone else, the carefully-crafted excuses and scales would be considered the edifice of a madman.'[6]

In *Harmonice Mundi*, Kepler superficially seems to have been just as enthusiastic as he had been in *Mysterium Cosmographicum* about the beauty the creator emplaced in the world. Yet a slight uneasiness can be detected, as if Kepler were uncomfortable about the less exact correspondence between the geometric archetypes of beauty and the world.[7] Seen in this light, the complex system of harmonies he deduced might suggest a kind of desperation: a man whose intellectual honesty has created nagging doubts about his fundamental belief in God as an aesthetician and mathematician who was guided by geometric beauty in shaping the world.

Star polyhedra

By his discovery of star polyhedra, Kepler partly undermined the primacy he assigned to the regular solids. He discovered two star polyhedra, each with twelve star pentagonal faces: one has five faces meeting at each vertex; one has three faces meeting at each vertex. These are now known respectively as the small and great stellated dodecahedra, and are two of the four regular star polyhedra (see Figure 8.25). The faces cut through each other, in defiance of physical realization. The very concept of such polyhedra was original to Kepler.[8] The first detailed description was in book II of *Harmonice*

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[1] Field, *Kepler's Geometrical Cosmology*, p. 176.

[2] *Ibid.*, p. 99.

[3] Kepler, *The Harmony of the World*, ch. v.ii, p. 399.

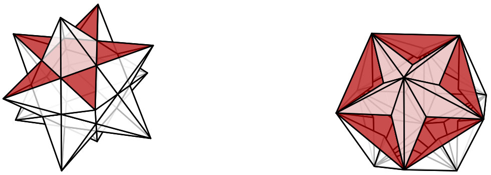
[4] *Ibid.*, ch. v.ii.

[5] Martens, *Kepler's Philosophy*, p. 121.

[6] Gingerich, 'Kepler, Galilei, and the Harmony of the World', p. 60.

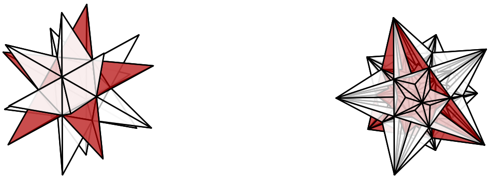
[7] Martens, *Kepler's Philosophy*, pp. 140–1.

[8] Field, *Kepler's Geometrical Cosmology*, p. 107.



Small stellated dodecahedron
12{5/2}; 5/2.5/2.5/2.5/2.5/2

Great dodecahedron
12{5}; 5.5.5.5.5



Great stellated dodecahedron
12{5/2}; 5/2.5/2.5/2

Great icosahedron
20{3}; 3.3.3.3.3

FIGURE 8.25
The four regular star polyhedra, with one face of each polyhedron shaded to show how it intersects other faces and passes through the interior.

Mundi[1] (1619), but he knew of one (though probably not both) by 1599.[2] Kepler's account of star polyhedra was short and terse, for book II of *Harmonice Mundi* was focused on applicable mathematics.[3]

Though Kepler had discovered these two new polyhedra, he maintained a distinction between them and the five regular solids. He did not discuss them alongside the regular solids in *Harmonice Mundi*, but in the following section, lumping them in with 'semisolid' partial figures with star octagonal and star decagonal faces.[4] Further, the description he gave, passages in his correspondence, and a draft of his unfinished treatise on geometry, all make it clear that he viewed the new polyhedra as 'augmented', derived from the dodecahedron and the icosahedron by erecting suitable pyramids on their faces, just as the star pentagon was derived from the pentagon be erecting a suitable triangle on each of its edges.[5]

This conception of the star polyhedra seems to be why Kepler saw no need to modify his polyhedral system to find places for the new star polyhedra. Only the twelve-vertex star polyhedron finds a derivative role, fitting between the orbs of Venus and Mars,[6] with Kepler unconcerned to find a role for the twenty-vertex star polyhedron.

Archimedean solids

With Kepler, the rediscovery of the archimedean solids* reached its slightly anticlimactic dénouement. Piero, Pacioli, Dürer, Jamnitzer, and Barbaro had investigated solids that were in some sense 'regular', but less so than the five regular solids, and Dürer and Barbaro between them had discussed all the archimedean solids save the snub dodecahedron. An anonymous artist had made drawings of them all some time between c. 1538 and 1557, although it is unknown whether

[1] Kepler, *The Harmony of the World*, bk II, § xxvi, pp. 116–17.

[2] Field, 'Kepler's star polyhedra', p. 109.

[3] *Ibid.*, p. 110.

[4] Kepler, *The Harmony of the World*, bk II, §§ xxvi–xxvii.

[5] Field, 'Kepler's star polyhedra', p. 111.

[6] Kepler, *The Harmony of the World*, bk V, ch. iii, p. 407.

* ▸ pp. 109–12

these drawings resulted from a systematic and exhaustive investigation. Even if they were not, it seems likely that the *concept* of the archimedean solids would shortly have been independently rediscovered.

But Commandino, working directly from some of the manuscripts, had translated from Greek into Latin the surviving portion of Pappus' *Collection*; his translation was published posthumously in 1588.[1] Kepler knew the *Collection*, for he explicitly mentioned it in *Harmonice Mundi*.[2] He did not refer to it in his treatment of the archimedean solids, though his familiarity with Pappus' account is indicated by his use of the name 'Archimedean'.[3] He almost certainly worked from Commandino's translation rather than a Greek manuscript.[4]

So, although Kepler filled in the last gap in the list, he had an advantage: Kepler knew from Pappus the incomplete definition of the class of solids Archimedes had considered, and he knew the list of thirteen solids and the numbers of their faces. Kepler's achievement was to work out a correct complete definition and to construct the actual solids.

The terminology Kepler used was rather different from today's, and was slightly more complicated because of his concurrent treatment of tessellations and polyhedra. Limited to polyhedra, his terminology was as follows:

perfect: at each vertex, faces meet in the same way.[5]

most perfect: perfect, and all the faces are congruent.[6]

completely regular: most perfect, and all the faces are regular. (The condition 'most perfect' is implicit, since Kepler pointed out that all vertices lie on the same sphere.)[7]

semiregular: all the faces are non-square rhombi,[8] and all vertices lie on at most two concentric spheres.[9] (Given Kepler's classification of these solids,[*] he must have required that only acute or only obtuse rhombic angles meet at each vertex, otherwise a distorted cube, such as the untruncated version of Dürer's solid,[†] would also qualify.)

perfect of an inferior degree: all the faces are regular, all vertices lie on the same spherical surface, and at each vertex faces meet in the same way.[10]

imperfect: perfect of an inferior degree and the largest face only occurs once or twice.[11]

Thus, for Kepler, prisms and anti-prisms were imperfect and hence implicitly excluded from the class of solids that were perfect of an inferior degree. He found the remaining solids that were perfect of an inferior degree by systematically enumerating the possible configurations of faces at vertices, and so constructed thirteen solids, to which he gave their modern names (see Figure 8.26). These, he declared, constituted the archimedean solids.[12] Kepler's list should

[1] Treweek, 'Pappus of Alexandria', pp. 228–30.

[2] Kepler, *The Harmony of the World*, bk I, §§ xlvi–xlvii, p. 85–9.

[3] *Ibid.*, bk II, § xxviii, p. 118.

[4] *Ibid.*, bk XII, p. 118, n. 34.

[5] *Ibid.*, bk II, § ii, p. 99.

[6] *Ibid.*, bk II, § vi, p. 100.

[7] *Ibid.*, bk II, § vii, p. 100.

[8] *Ibid.*, bk I, § iii.

[9] *Ibid.*, bk II, § vii, p. 100–1.

[*] ▸ pp. 293

[†] ▸ p. 258

[10] *Ibid.*, bk II, § ix, p. 101.

[11] *Ibid.*, bk II, § x, p. 101.

[12] *Ibid.*, bk II, § xxviii, p. 118.

have included the pseudo-rhombicuboctahedron, which is by his own definition also perfect of an inferior degree. Kepler's enumeration of vertex configurations was sound; his mistake was to assume that each vertex configuration would give rise to at most one solid.

But notice the form of Kepler's classification: he did not think that the solids that were perfect of an inferior degree consisted of the two infinite classes of prisms and anti-prisms and the thirteen (as he thought it) individual solids. He *excluded* the prisms and anti-prisms and relegated them to the class of *imperfect* solids. Why should they not have stood with the thirteen archimedean solids? Because they were 'discus-shaped, like a plane, not globe-shaped, like a sphere'.[1] Thus Kepler's classification, like that of Archimedes before him, was in the end tinged by a preference for the sphere. Whether Kepler saw that Archimedes must have excluded the prisms and anti-prisms and followed him, or whether he decided on their ouster in advance is scarcely relevant: at minimum, he *accepted* the preference for the sphere and so placed himself in an aesthetic tradition that descended from Pythagoras.

Kepler's appreciation for these solids that echo the sphere is of a mathematical nature. Most of those who came before him — Pacioli being the exception — were artists first, and they were doubtless influenced by the visual beauty of the solids and the technical skill needed to render them. But Kepler's diagrams (see Figure 8.26) — especially beside Leonardo's or Jamnitzer's illustrations — are crude, with no regard for perspective and with little direct visual beauty; they serve merely to convey the shape of the solids.

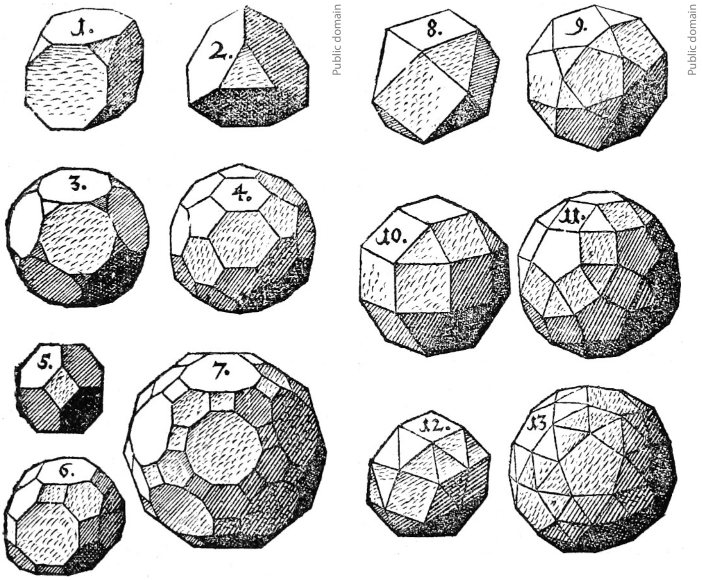


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FIGURE 8.26
Kepler's diagrams of the thirteen archimedean solids.

1: Truncated cube
2: Truncated tetrahedron
3: Truncated dodecahedron
4: Truncated icosahedron
5: Truncated octahedron
6: Truncated cuboctahedron
7: Truncated icosidodecahedron
8: Cuboctahedron
9: Icosidodecahedron
10: Rhombicuboctahedron
11: Rhombicosidodecahedron
12: Snub cube
13: Snub dodecahedron

(Kepler, *The Harmony of the World*, bk ii, § xxviii, pp. 122, 120)

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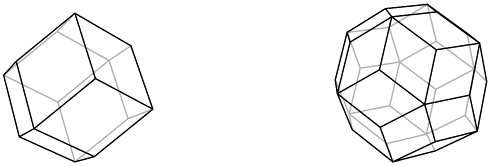
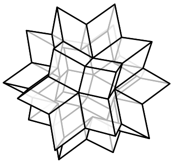
[1] Kepler, *The Harmony of the World*, bk ii, § xiii, p. 102;cf. § cii.



Rhombic dodecahedron
12 faces;
6 vertices with 4 incident faces,
8 vertices with 3 incident faces

Rhombic triacontahedron
30 faces;
20 vertices with 3 incident faces,
12 vertices with 5 incident faces



60 faces;
20 vertices with 3 incident faces,
30 vertices with 4 incident faces
12 vertices with 5 incident faces

FIGURE 8.27
The three rhombic solids other than the cube that are semiregular in Kepler's sense. The rhombic dodecahedron and rhombic triacontahedron are the only such solids that are convex.

Rhombic solids

When Kepler heard in March 1610 about Galileo's discovery of four 'new planets', his first concern was how his system of polyhedra could be preserved undamaged, for the five regular solids implied only six planets. This led him to the hypothesis that perhaps these new 'planets' might be smaller bodies orbiting Venus, Mars, Jupiter, and Saturn, with the body orbiting Mercury having remained unobserved due to the proximity of the Sun.[1] Based on the partial information he had at that point, Kepler was considering ideas that would preserve his system and give it a greater unity by granting every planet a moon. When he received Galileo's *Sidereus Nunicius*, on which he wrote the commentary 'Dissertatio cum Nuncio Sidereo', he learned that all four of the new planets were moons of Jupiter. His geometrical explanation for there being four moons was that there were three solids with rhombic faces that are semiregular in his sense: the cube (the square being a kind of rhombus), a twelve-faced solid that was derived from the cuboctahedron, and a thirty-faced solid that was derived from the icosidodecahedron.[2] The latter two solids must be the rhombic dodecahedron, which is the dual of the cuboctahedron, and the rhombic triacontahedron, which is the dual of the icosidodecahedron (see Figure 8.27). Kepler wrote that he had considered the problem of whether there were solids, 'similar to the five regular solids and the thirteen Archimedean solids', with rhombic faces, on inspecting how opposite-facing honeycombs built by bees join in an array of rhombi.[3] He thus discovered the rhombic dodecahedron and triacontahedron,[4] and later proved that these were the only semiregular rhombic solids other than the cube.[5]

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[1] Field, *Kepler's Geometrical Cosmology*, pp. 77–8.

[2] Kepler, 'Dissertatio cum Nuncio Sidereo', p. 309.

[3] Kepler, *The Six-Cornered Snowflake*, pp. 9–11.

[4] *Ibid.*, p. 11.

[5] Kepler, *The Harmony of the World*, bk II, § xxvii.

[Kepler actually showed that there are only two *convex* rhombic solids except the cube. There is also the non-convex rhombic hexecontahedron, which has sixty faces and 62 vertices: 20 vertices with 3 incident faces, 30 vertices with 4 incident faces, and 12 vertices with 5 incident faces; see Figure 8.27.]

At the time Kepler put forward this hypothesis, he did not know the dimensions of the orbits of the Jovian moons. Later, in the *Epitome Astronomiae Copernicanae*, he argued that the three rhombic solids fit the spaces between their orbs, just as the five regular solids fit the orbs of the planets.[1] However, his account is rather vague about whether the orbits fit with the insphere (which touches the solid at the centres of its faces), the inner circumsphere (the circumsphere of the regular solid whose edges are the *short* diagonals of the rhombic faces), or the outer circumsphere (the circumsphere of the regular solid whose edges are the *long* diagonals of the rhombic faces), and in any case there is at best only a very rough correspondence between the posited orbs and the observed orbits.[2]

Nevertheless, the fact that Kepler looked to rhombic solids is an indicator of his aesthetic preference. First, he thought that the beauty of the cosmos required some mathematical explanation of the number of moons. Second, he desired a class of solids that would share at least some of the properties of the regular solids and thus presumably partake in some of their beauty. Most importantly, in *The Six-Cornered Snowflake* (1611) he explicitly noted the 'perfection, beauty, or dignity of the rhombic figure'.[3] (The context is three-dimensional, so by 'figure' Kepler meant a *solid* figure.)

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[1] Field, *Kepler's Geometrical Cosmology*, pp. 79–80.

[2] *Ibid.*, pp. 215–19.

[3] Kepler, *The Six-Cornered Snowflake*, p. 19; trans. mod. C.

The Most Beautiful Order

9

Aesthetics and early modern mathematics

‘definitions in the first place;
secondly, postulates;
and thirdly, axioms, […]
in the fourth place, there follow propositions
with their demonstrations […].
And this is the most beautiful order of doctrine,
which even Aristotle taught,
and which outside of mathematics, especially pure mathematics,
is never found.’

— Josephus Blancanus,
A Treatise on the Nature of Mathematics, ch. 4, p. 206
(trans. Gyula Klima).

‘we find there the most beautiful order we can desire,
surpassing anything we had expected’

— Gottfried Wilhelm Leibniz,
‘On what is Independent of Sense and of Matter’, p. 552
(letter to Sophia Charlotte of Hanover, dated 1702,
trans. Leroy E. Loemker).

It is impossible to draw a line and declare that *here* begins modern mathematics, but there is a period where that beginning undoubtedly lies: the first half of the seventeenth century. René Descartes published *La Geometrie* in 1637, marking the beginning of analytic geometry and taking the first steps on a road that would lead to Leibniz’s and Newton’s development of the calculus.[1] Pierre de Fermat, who independently invented analytic geometry but whose work was not published until after his death, founded modern number theory.

For the historian of mathematical beauty, the seventeenth century is important because the new developments were accompanied by a wealth of references to beauty seen in mathematics. Judgements of beauty are found both in published works and in the extensive epistolary record of Fermat, Descartes, and others. The letters they exchanged were part of the ‘learned commerce’ (*commercium litterarium*) of the ‘Republic of Letters’, whose cosmopolitan citizenry comprised scholars who were linked — and perhaps at the time were self-identified — by their correspondence.[2] The origins of the Repub-

[1] Cf. Rouse Ball, *A Short Account of the History of Mathematics*, p. 218; Gray, ‘Algebraic and analytic geometry’, p. 847.

[2] Bots & Waquet, *La République des lettres*; Waquet, ‘Qu’est-ce que la République des Lettres?’, see also.

lic lie farther back in time, but it became an accepted idea during the seventeenth century, when the development of state postal services made it easier to exchange letters.[1]

Through their letters, these scholars created and maintained networks of relationships that formed a structure through which knowledge could be freely disseminated, for, unlike printed books, personal correspondence was free from the strictures of censorship.[2] Nicolas Fabri de Peiresc (1580–1637) developed a scientific network of correspondents spanning England, France, Italy, and the Netherlands; on his death, Marin Mersenne (1588–1648) replaced him as its coördinator.[3] Although his mathematical ability was quite limited, and he carefully managed his correspondence in order to enhance his own reputation, Mersenne was an intellectual clearing-house for many of the leading mathematicians of his time.[4]

In the second half of the seventeenth century, the character of the Republic of Letters began to change. Its members were drawn into national academies like the Royal Society of London and the Royal Academy of Sciences in Paris, which had been founded to advance knowledge and so to bring glory to the monarchs who patronized them.[5] Scholarly journals were established and began to take over from the correspondence networks part of the role of disseminating knowledge.[6] But the flow of letters was stimulated rather than diminished,[7] and those that were exchanged in the mathematical part of the Republic continued to contain many judgements, aesthetic and otherwise, that mathematicians made of their own and others' work.

In mathematics, the later seventeenth century was dominated by the work of Leibniz and Newton, and the correspondence between them and others was immensely important in the disputes over their development of the calculus, in which aesthetic judgements had a part to play.*

In the eighteenth century, standing on the shoulders of these and other giants, mathematicians took analytic geometry, the calculus, number theory, and algebra to new heights, and it is clear that aesthetic value played a role in their motivation and judgements. Jean-Étienne Montucla's (1725–99) *Histoire des Mathématiques*, the second edition of which was published at the end of the eighteenth century, is a natural point at which to look back and reflect on this period.

CYCLOID

An enduring locus of mathematical beauty in the seventeenth century was the application of the new mathematics to curves like the cycloid and the catenary and their rectification and quadrature. The cycloid is so important in this history as to merit its own introduction here: it drew attention from many noteworthy

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[1] See, for example Goodman, *The Republic of Letters*, esp. ch. 1.

[2] Kronick, 'The Commerce of Letters', p. 30.

[3] Pearl, 'The Role of Personal Correspondence', pp. 107–8; Crombie, 'Mersenne, Marin', p. 316.

[4] Grosslight, 'Small Skills, Big Networks'.

[5] Goodman, *The Republic of Letters*, pp. 19–21.

[6] Kronick, 'The Commerce of Letters', p. 31.

[7] *Ibid.*, p. 32.

* ▸ pp. 324–6

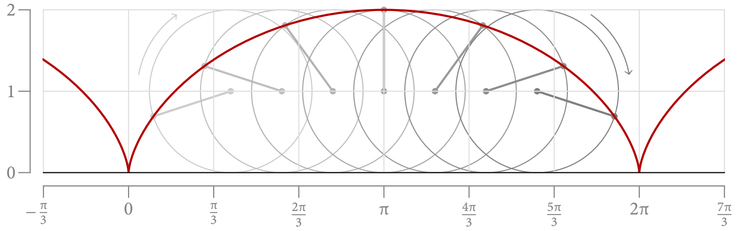

FIGURE 9.1
Generation of a cycloid as the path followed by a point on the circumference of a circle rolling along a straight line.

mathematicians, and was (and indeed *is*) widely agreed to be a beautiful object with many fascinating beautiful properties. Notably, in the second edition of his *Histoire des Mathématiques* (1799–1802), Montucla[*] bestowed on the cycloid the epithet of the 'Helen of the geometers' ('Hélène des géomètres'[1]): like Helen of Troy, it was beautiful and a cause of conflict. [In the passage where he called it by this name, Montucla discussed the controversies of which it was the focus, and elsewhere made it clear he found it beautiful; later authors interpreted the epithet as a reference to the beauty of the cycloid as well as the feuding over it.[2]]

A cycloid is the path followed by a point on the circumference of a circle rolling along a straight line; see Figure 9.1.[3] The resulting curve is defined parametrically by

$$x = r(\vartheta - \sin\vartheta) \qquad y = r(1 - \cos\vartheta),$$

where r is the radius of the circle. Scaling the radius of the circle thus corresponds simply to an equal scaling of the cycloid; it is thus natural to speak collectively of the family of curves as *the* cycloid. It was first used by Charles de Bovelles (1471/9–after 1566), but Galileo[†] (1564–1642) was the earliest to study it seriously, and it was he who coined the term *cycloid*.[4] As this chapter will discuss, the cycloid arose in different guises during the seventeenth century, including as the solutions to the tautochrone[‡] and brachistochrone problems.[§]

PIERRE DE FERMAT[5]

Although Pierre de Fermat (1607–65) was professionally a jurist, as a mathematician he can be credited with both the co-foundation of analytic geometry and the foundation of modern number theory; the popular historian of mathematics E. T. Bell (1883–1960) thus proclaimed him 'The Prince of Amateurs'.[6] His accomplishments were judged, both by himself and others, in terms of their beauty. Fermat is perhaps the earliest mathematician in whose work aesthetic judgements were liberally applied to results, methods, proofs, objects, and theories.

Mersenne opened a correspondence with Fermat in 1636, having heard about Fermat's work from his former colleague Pierre de Car-

[*] ▸ pp. 346–51

[1] Montucla, *Histoire* New ed., vol. 2, p. 55.

[2] Cajori, *A History of Mathematics*, § 'Descartes to Newton'.

[3] For mathematical (non-historical) information about the cycloid, see Simmons, *Calculus Gems*, chs B.21–B.23.

[†] ▸ pp. 277–9

[4] Martin, 'The Helen of Geometry', pp. 17–18.

[‡] ▸ p. 309

[§] ▸ pp. 331–3

[5] The third volume of Fermat's *Œuvres* contains French translations of Latin works and passages in the first two volumes. Where relevant, both the original and translation are cited here.

[6] Bell, *Men of Mathematics*, ch. 4.

cavi (c. 1600–1684).[1] Their exchanges presaged Fermat's entry into the scientific community, and early the next year Fermat sent his treatise 'Introduction to Plane and Solid Loci' ['Ad Locos Planos et Solidus Isagoge'] to Mersenne and other Parisian correspondents. About the same time, Mersenne received from René Descartes the galley proofs of *La Geometrie*. Both works contained the same profound insight of connecting algebraic equations with two unknowns to lines or curves in the plane,[2] which is seen as a step of the utmost importance in the development of mathematics.[3] This coincidence was one of the contributions to what became a bitter quarrel between Fermat and Descartes, less over priority than over the correctness, rigour, and contribution of the two disputants' mathematics.[4]

Analytic geometry

Fermat's early research career was dominated by his work on plane loci, which he called 'the most beautiful theory in all geometry' at the start of his restoration of Apollonius'[*] (c. 260–c. 190 BCE) two books on *Plane Loci*.[5] Apollonius' work had long been lost, but was known through the testimony of Pappus, who gave eight theorems from each book as exemplars of the classes of loci of which Apollonius treated.[6] Fermat set himself to reconstruct Apollonius' work, not as a philologist reassembling a lost text but as a mathematician seeking to recreate the lost results. His familiarity with François Viète's[†] (1540–1603) algebra and its application to geometry led him to a reconstruction that replaced Apollonius' geometrical analyses with algebraic ones. He wrote of an early version of this work that it lacked the 'most difficult and most beautiful question', which he had been unable to solve;[7] he probably referred to Proposition II.5 of his restoration.[8] Fermat sent this result to Gilles de Roberval (1602–75) in 1637, calling it 'one of the most beautiful in Geometry':[9]

PROPOSITION F. *Given a list of points $P_1, \ldots, P_n$ and an area, to find a circle such that, from any point Q on its circumference, the sums of the squares on the lines $QP_1, \ldots QP_n$ is equal to the given area.* $\boxed{\text{F}}$

Fermat stated the proposition and gave the construction of the circle for five points A, G, F, H, E (see Figure 9.2). Draw the line AE and project orthogonally the points G, F, H onto AE at B, C, D. Choose O on AE so that AO is $1/5$ (since there are five points) of the sum of the segments on one side of A, including AA: thus $AO = \frac{1}{5}(AB + AC + AD + AE)$. Erect an orthogonal ON, and let I be such that OI is $1/5$ of the sum of the lengths of the perpendiculars GB, FC, HD. Draw the lines AI, GI, FI, HI, EI. The sum of their squares must be less than the given area; subtract the sum from the area and let Z be the difference. Take $1/5$ of Z and construct the equivalent square M^2. The desired circle has centre I and radius M.

In the projection of the points onto the line through O and the consideration of distances between O and the feet of the perpendicu-

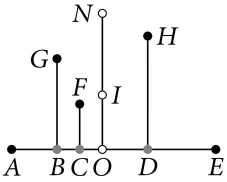

FIGURE 9.2
Fermat's construction for Proposition F.

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[1] Mahoney, *Fermat*, pp. 52–3.

[2] *Ibid.*, p. 72.

[3] Cf. Mill, *Sir William Hamilton's Philosophy*, p. 478; Boyer, *Hist. Analytic Geometry*, ch. v.

[4] Mahoney, *Fermat*, ch. IV; Clarke, *Descartes*, pp. 168–75; cf. Gaukroger, *Descartes*, p. 323.

[*] ▸ pp. 113–15

[5] Fermat, 'Apollonii Pergæi Libri Duo', p. 3; Fermat, *Œuvres*, vol. III, p. 3.

[6] Pappus, *Collection 7*, §§ 21–6.

[†] ▸ pp. 272–4

[7] Fermat, letter to Mersenne, dated 26 Apr. 1636, repr. in Fermat, *Œuvres*, vol. II, p. 5; trans. C.

[8] Mahoney, *Fermat*, p. 95.

[9] Fermat, letter to Roberval, dated c. Feb. 1637, repr. in Fermat, *Œuvres*, vol. II, p. 100; vol. III, p. 300; trans. C.

lars, one sees a hint of a coördinate system.[1] Fermat's proof has been characterized as 'algebraic reasoning in geometric garb'.[2] His search for a satisfactory treatment of this result led to him studying in 'Introduction to Plane and Solid Loci' the general question of how to apply algebra to problems of plane loci. Having developed his new techniques, Fermat acknowledged that the reasoning in his restoration of Apollonius' work could have been much more elegant.[3]

In May 1637, Mersenne communicated this result to Descartes, who said — perhaps condescendingly[4] — it was 'very beautiful' and 'delighted him', especially because it could be proved easily using the new methods from *La Geometrie*.[5] (At this time, the controversy between Fermat and Descartes had not yet erupted; Descartes did not see Fermat as a rival.[6]) Thus for both Descartes and Fermat, this result's beauty came from its setting within the algebra–geometry nexus. But their perspectives seem to have been different. Descartes found the result especially beautiful as an *exemplar* of the kind of result that succumbed to the methods of *La Geometrie*. But Fermat's judgement seems to have been a mirror image of Descartes's: his work on proving this result pointed the way to the broader theory of analytic geometry he developed in 'Introduction to Plane and Solid Loci', and it was when he was almost ready to circulate this treatise in 1637 that he called the result 'one of the most beautiful in Geometry'. Both Fermat and Descartes described the *result* as beautiful, but for both the cause of the beauty was external to the result. To a modern reader, the result seems to have lost at least some of its beauty. In the time of Fermat and Descartes, the new geometry was a new and exciting departure and any result that showed its power reflected its glory. Today, with analytic geometry having acquired the respectable patina of age, something more is required from a result.

Besides this particular result, Fermat's work on loci on planes and surfaces contains several results that he called 'most beautiful'.[7]

Number theory

Fermat directed his attention at different times to analytic geometry, the method of maxima and minima, and the quadrature of curves. Number theory was the one subject on which he worked throughout his life.[8] The works of Diophantus (*fl.* c. 250 CE) fascinated and inspired Fermat, as evidenced by the forty-eight notes he made in the narrow margins of his copy of Claude-Gaspar Bachet's (1581–1638) edition and translation, and which were later published by his eldest son Clément-Samuel (1634–after 1675). But Diophantus' methods in the *Arithmetica* aimed at rational solutions. Fermat sought methods for integer solutions, and in the course of his researches, effectively created modern number theory as separate from the study of continuous magnitudes.[9]

[1] Mahoney, *Fermat*, p. 108.

[2] *Ibid.*, p. 105.

[3] Fermat, 'Ad Locos Planos et Solidus', p. 103; Fermat, *Œuvres*, vol. III, p. 96.

[4] Mahoney, *Fermat*, p. 174.

[5] Descartes, letter to Mersenne, dated 25 May 1637, repr. in Descartes, *Œuvres*, vol. I, p. 377; trans. C.

[6] Mahoney, *Fermat*, p. 174.

[7] See Fermat, 'Apollonii Pergæi Libri Duo', p. 26; Fermat, 'Ad Locos Planos et Solidus', pp. 93, 102; Fermat, 'Isagoge ad Locos ad Superficiem', pp. 113, 115; Fermat, *Œuvres*, vol. III, pp. 25, 87, 95, 104, 106.

[8] Mahoney, *Fermat*, p. xiv.

[9] *Ibid.*, pp. 284, 286.

One reason for concluding that number theory was the subject that held the most interest for Fermat was that he wrote to correspondents more about number-theoretic results than about results of any other kind.[1] He may also have considered it the most beautiful subject, for in his forty-second note on Diophantus (to Question VI.9 of the *Arithmetica*), he wrote:

> 'Can there be, among whole numbers, a square besides 25 which, increased by 2, makes a cube? This seems at first glance difficult to approach, but I can prove, by rigorous demonstration, that 25 is indeed the only square that is two units less than a cube. Among fractional numbers, Bachet's method supplies an infinity of such squares, but *the theory of whole numbers, which is very beautiful and very subtle*, has not been known until now, neither by Bachet, nor by any author whose works I have read.'[2]

Here Fermat claimed to have proved that $x^2 + 2 = y^3$ has only one solution in the integers: $x = 5$, $y = 3$. It is arguable that the phrase 'the theory of whole numbers, which is very beautiful and very subtle' applied only to the theory of integer solutions of equations, but for Fermat that would be a distinction without a difference.

In his eighteenth note (to Bachet's commentary on Question IV.31 of the *Arithmetica*), Fermat stated his celebrated polygonal number theorem:

FERMAT'S POLYGONAL NUMBER THEOREM. *Every natural number is the sum of at most n n-gonal numbers.* ☐

He claimed to have a proof, but did not supply it. He described this result as 'a most beautiful and wholly general proposition [...] this marvellous proposition',[3] and, together with the result *In both the integers and the rationals, any number of the form* $8k - 1$ *is the sum of four squares only,* as 'most beautiful'.[4]

Descartes agreed that the polygonal number theorem was 'without doubt one of the most beautiful that can be found concerning numbers' and he imagined that the proof would be so difficult that he did not dare seek it. He seems to have thought that the result was due to André Jumeau (*fl.* early 17th century), the prior of Sainte-Croix; since his dispute with Fermat had begun by the time he gave his opinion in July 1638, one must speculate that he would have been less generous in his praise had he known the result's true provenance.[5]

Again, in his forty-sixth note (to Proposition 9 of *On Polygonal Numbers*), Fermat noted the following theorem, here given in modern notation:[6]

THEOREM T. *For* $n \in \mathbb{N}$, *let* $T_n^{(1)} = \sum_{i=1}^{n} i$ *be the n-th triangular number. For* $d \in \mathbb{N}$ *with* $d \geqslant 2$, *let*

$$T_n^{(d)} = \sum_{i=1}^{n} T_i^{(d-1)}.$$

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[1] Mahoney, *Fermat*, pp. 285–6.

[2] Fermat, 'Observations', pp. 333–4; Fermat, *Œuvres*, vol. III, p. 269; trans. C., emphasis added.

[3] Fermat, 'Observations', p. 305; Fermat, *Œuvres*, vol. III, p. 252; trans. C.

[4] Fermat, letter to Mersenne for Ste-Croix, dated Sep.–Oct. 1656, repr. in Fermat, *Œuvres*, vol. II, p. 65; vol. III, p. 287; trans. C.

[5] Descartes, letter to Mersenne, dated 27 Jul. 1638, repr. in Descartes, *Œuvres*, vol. II, p. 256; trans. C.

[6] Fermat, 'Observations', p. 341; Fermat, *Œuvres*, vol. III, p. 273.

Then

$$n(n+1) = 2T_n^{(1)}, \quad nT_n^{(1)} = 3T_n^{(2)}, \quad nT_n^{(2)} = 4T_n^{(3)}, \quad \dots$$

and in general $nT_n^{(d)} = (d+2)T_n^{(d+1)}$. $\boxed{\mathrm{T}}$

He described this result in his note as 'very beautiful and marvellous proposition [...]. I believe that there is no theorem on numbers that is more beautiful or general'.[1] And again in correspondence, he called it 'very beautiful'[2] and 'most beautiful'.[3]

But Fermat did not see beauty only in those number-theoretic propositions that involved figurate numbers. In correspondence, he gave three propositions, which he proved not without effort, which he called the foundations of the discovery of perfect numbers, which he hoped to build upon, and which he called 'very beautiful propositions':[4]

- *If* n *is composite, then* $2^n - 1$ *is composite.*
- *If* p *is an odd prime, then* $2^p - 2$ *is divisible by* $2p$.
- *If* p *is an odd prime, then* $2^p - 1$ *is only divisible by numbers of the form* $2pk + 1$.

Fermat also seems to have counted magic squares and analogous configurations as part of number theory, and wrote that: 'I know hardly anything more beautiful in arithmetic than these numbers that some call *planetary* and others *magic*'.[5] (The term 'planetary' is derived from certain treatises linking the magic squares to planets used in talismans.[6]) This statement seems to indicate that Fermat saw beauty in magic squares; that is, in the *objects*, not the results or the theory. He also claimed to have found a rule to find magic cubes (see Figure 9.3), in which any orthogonal row or diagonal parallel to a face has the same sum, and also determined how many different ways each such cube can be arranged, which he called 'one of the most beautiful things in arithmetic'.[7]

Besides theories as a whole, results, and objects, Fermat judged beautiful both conjectures and problems. He wrote to Mersenne that

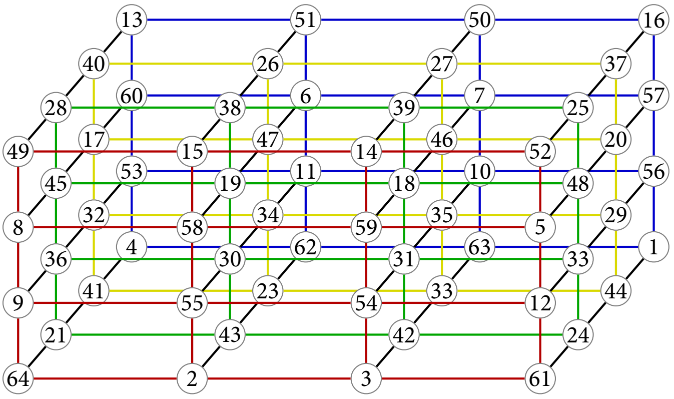

FIGURE 9.3
Fermat's magic cube. In each of the 12 square layers parallel to a side of the cube, each row, column, or diagonal (there are 72 in total) sums to 130 (Fermat, letter to Mersenne, dated 1 Apr. 1640, repr. in Fermat, *Œuvres*, vol. II, pp. 190–1).

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[1] Fermat, 'Observations', p. 341; Fermat, *Œuvres*, vol. III, p. 273.

[2] Fermat, letter to Roberval, dated 4 Nov. 1636, repr. in Fermat, *Œuvres*, vol. II, p. 84; trans. C.

[3] Fermat, letter to Mersenne, dated Sep.–Oct. 1636, repr. in *ibid.*, vol. II, p. 70; trans. C.

[4] Fermat, letter to Mersenne, dated c. Jun. 1640, repr. in *ibid.*, vol. II, p. 198; trans. C.

[5] Fermat, letter to Mersenne, dated 1 Apr. 1640, repr. in *ibid.*, vol. II, p. 194, emphases in orig.; trans. C.

[6] Comes, 'The Transmission of Azarquiel's Magic Squares', pp. 160–2.

[7] Fermat, letter to Mersenne, dated 1 Apr. 1640, repr. in Fermat, *Œuvres*, vol. II, p. 190; trans. C.

he had solved the problem that was perhaps the most beautiful in all of arithmetic: of finding, for any arithmetic progression, the sum of the squares, cubes, bisquares, or higher powers of its terms.[1] There is little to hint at why he found this problem so beautiful.

It is possible to say something more about Fermat's famous conjecture that numbers of the form $F_n = 2^{2^n} + 1$ are all prime. He based his hypothesis on the limited numerical evidence he had available (for $0 \leqslant n \leqslant 4$), and asked for a proof of this 'very beautiful' statement. It seems that part of the reason Fermat found it beautiful was that it would provide an easy way to find a prime larger than any number.[2] The conjecture has famously proved to be spectacularly unsuccessful: Leonhard Euler* (1707–83) showed that it fails for $n = 5$:[3]

$$F_5 = 2^{2^5} + 1 = 4\,294\,967\,297 = 641 \times 6\,700\,417.$$

And for over three hundred values of n, including $5 \leqslant n \leqslant 32$, the numbers F_n are known to be composite; the only values of n for which F_n is known to be prime are the five values Fermat knew: $n = 0, 1, 2, 3, 4$.[4]

Fermat wrote nothing about the beauty of his famous two-squares theorem when he first stated it in the form *Every prime number of the form* $4n + 1$ *is once the hypotenuse of a right-angled triangle*[5] in his seventh note on the *Arithmetica*. Nor did he mention its beauty in the letter to Mersenne in which he announced it.[6] This omission does not mean that he did not find it beautiful: it is more likely to be simply a historical irony that he did not mention in writing his judgement of *this* result, which seems to attract near-universal aesthetic approbation.[7]

This discussion of number theory would be incomplete without mentioning Fermat's celebrated 'last theorem', which states that *For* $n > 2$, *the equation* $x^n + y^n = z^n$ *has no solution in the natural numbers.* It originated in the second note he made on Diophantus (to Question ii.8 of the *Arithmetica*), where he infamously claimed to have a 'truly miraculous [mirabilem sane] proof' of the result, but that the margin in which he wrote was too small to contain it.[8] Fermat offered no aesthetic judgement of the 'theorem', but Euler would later call it 'very beautiful'.†

Elegance

Fermat explicitly sought elegance in his solutions: in the appendix to *Introduction to Plane and Solid Loci*, he told the reader that they could

> 'Forget, therefore, Viète's *climactic parapleroses* by which he reduces quartic equations to quadratics by means of cubic equations of a squared root. For, as has been shown, quartic and cubic equations can henceforth be solved with the same

[1] Fermat, letter to Mersenne, dated Sep.–Oct. 1636, repr. in Fermat, *Œuvres*, vol. ii, p. 69; vol. iii, p. 291.

[2] Fermat, letter to Digby, dated Jun. 1658, repr. in *ibid.*, vol. ii, p. 405; vol. iii, p. 316.

* ▸ pp. 334–41

[3] Euler, 'Observationes de theoremate'.

[4] OEIS, A019434.

[5] Fermat, 'Observations', pp. 293–4; Fermat, *Œuvres*, vol. iii, p. 243.

[6] Fermat, letter to Mersenne, dated 25 Dec. 1640, repr. in Fermat, *Œuvres*, vol. ii, pp. 212–17.

[7] See, for instance, Bell, *Men of Mathematics*, ch. 4, ch. 14; Hardy, *Apology*, §§ 12–13.

[8] Fermat, 'Observations', p. 291; Fermat, *Œuvres*, vol. iii, p. 241; trans. C.

† ▸ p. 336

> elegance, ease, and brevity; and it is not possible, I think, to imagine a more elegant solution.’[1]

Viète would have proceeded to reduce a quartic equation like $x^4+bx = c$ to $x^2+xy = (b/2y)-y^2/2$ where y is a solution of $(y^2)^3+4c(y^2) = b$. Fermat's approach was to reduce the equation to a geometric problem. Starting from an equation $x^4 = cx + d$, he added terms $-2b^2x^2 + b^4$ (for some b) to obtain

$$x^4 - 2b^2x^2 + b^4 = cx + d - 2b^2x^2 + b^4.$$

The left-hand side of this equation equals $(x^2 - b^2)^2$. He then let $2b^2 = n^2$ and let y be such that

$$x^2 - b^2 = ny \quad \text{and} \quad b^4 + d + cx - n^2x^2 = n^2y^2.$$

These two equations define, respectively, a parabola and a circle, and so the quartic equation has been reduced to tracing these curves.

What did Fermat mean by elegance? Certainly it meant something distinct from beauty: although he judged results, theories, and objects beautiful, he seems never to have judged any of them as elegant. Rather, elegance was a value he saw in solutions, methods, constructions, proofs.[2] There also seems to have been only one occasion when Fermat wrote of a proof method being beautiful: he wrote to Pascal that he hoped that one of his methods would seem beautiful.[3] (And another of his correspondents, Bernard Frenicle de Bessy (d. 1675), judged Fermat's methods to be ‘very beautiful’.[4]) Elegance thus seems to have been a property of reasoning processes, not of static things like results. It seems to involve an avoidance of *unnecessary* complexity.[5] And complexity seems to relate to the intricacy of the formulae or statements involved, not to the concepts. Thus, for Fermat, introducing geometric concepts like conic sections into the solving of equations could yield greater elegance.

RENÉ DESCARTES

If one ranks mathematician–philosophers by their combined influence on and contribution to both subjects, René Descartes (1596–1650) probably places first. It is thus disappointing for the historian that he did not bring his philosophical acumen to bear on mathematical beauty. No work of his was dedicated to aesthetics, although he did make occasional remarks that relate to the subject.[6] Notably, Mersenne asked him for his explanation of why certain sounds are more pleasant than others and of the cause of beauty. In his reply, Descartes noted that beauty seemed most applicable to the sense of sight, and wrote that neither beauty nor pleasure signified anything but our judgement of an object. Thus, given the variety of

[1] Fermat, ‘Ad Locos Planos et Solidus’, p. 107; Fermat, *Œuvres*, vol. III, p. 99; trans. mod. Mahoney, *Fermat*, p. 128.

[2] For example, see, respectively, Fermat, ‘Observations’, p. 301; Fermat, ‘Ad Locos Planos et Solidus’, p. 104; Fermat, ‘De Contactibus Sphæricis’, p. 52; Fermat, ‘De Linearum Curvarum’, p. 214; Fermat, *Œuvres*, vol. III, pp. 249, 96, 49, 183.

[3] Fermat, letter to Pascal, dated 25 Sep. 1654, repr. in Fermat, *Œuvres*, vol. II, p. 313.

[4] Frenicle de Bessy, letter to Fermat, dated 2 Aug. 1641, repr. in *ibid.*, vol. II, p. 227.

[5] Cf. Mahoney, *Fermat*, pp. 131–5.

[6] Tatarkiewicz, *Hist. Aesthetics*, vol. III, pp. 161–3.

judgements we reach, it was pointless to seek any kind of criterion for beauty.[1] Elsewhere, he wrote that 'we call beautiful or ugly that which is thus represented by our exterior senses'.[2] Strictly, he did not limit the application of the term *beautiful* to sense-objects. And in practice he used it to describe mathematics, like some of Fermat's results,[*] and from his remarks one can deduce something of the kind of mathematics he found beautiful.

For Descartes, a result could be beautiful but simple, though he did not think such a result worth boasting about. When he heard from Mersenne that Roberval had proven that the area enclosed by the cycloid[†] is three times that of the generating circle,[3] Descartes admitted that the cycloid was something he had not thought about, and acknowledged the result as 'quite beautiful', but was distinctly unimpressed: he wrote 'I do not see what there is to make such a fuss about, having found such an easy thing' and proceeded to give a proof.[4] [This episode contributed to the fractious relationship between Descartes and Roberval.[5]]

He congratulated an unknown correspondent for the 'beautiful rule that they had found to solve solid problems with the hyperbola',[6] saying that it might be possible to find a shorter rule, but not a more beautiful one. This statement suggests that, for Descartes, brevity did not constitute an overmastering contribution to beauty. But perhaps one should not rely too much on what Descartes wrote here: he seems to have been writing to a social superior, and did not restrain himself from fulsome praise and excessively complimentary language.

Descartes thought Florimond de Beaune's (1601–52) technique of deducing properties of curves by examining their tangents to be so beautiful that he preferred it to Archimedes' quadrature of the parabola. He gave some explanation of why he found it beautiful: Archimedes had studied a given line; de Beaune studied a line not yet given.[7] Here Descartes stepped away from individual results: he called a general *method* beautiful. Similarly, he said the problem of finding tangents to curved lines was the most beautiful and most useful that he knew: here, he called beautiful a problem asking for a general method.[8]

Franciscus van Schooten (1615–60) reported that Descartes said of Fermat's work:

> 'he has invented a number of beautiful particular things [...]. But for my part, I have always studied to consider very general things, in order to be able to deduce rules, which are always of use elsewhere.'[9]

The implied criticism — that Fermat's successes related to solving isolated problems rather than developing systematic theories — was not unfair.[10] It fits with Descartes's opinion of Proposition F on page 299, where the beauty seems to relate to its demonstration of the power of Descartes's own theory. But his praise of Fermat's polygonal

[1] Descartes, letter to Mersenne, dated 18 Mar. 1630, repr. in Descartes, *Œuvres*, vol. I, pp. 132–3.

[2] Descartes, *Les Passions de l'Ame*, Art. LXXXV, p. 391; trans. C.

[*] ▸ pp. 299–300

[†] ▸ pp. 296–7

[3] Mersenne, letter to Descartes, dated 28 Apr. 1638, repr. in Descartes, *Œuvres*, vol. II, pp. 116–17.

[4] Descartes, letter to Mersenne, dated 27 May 1638, repr. in *ibid.*, vol. I, p. 377; trans. C.

[5] Jullien, 'Descartes-Roberval', esp. p. 367.

[6] Descartes, letter to ***, dated c. Oct. 1637, repr. in Descartes, *Œuvres*, vol. I, p. 459; trans. C.

[7] Descartes, letter to de Beaune, dated 20 Feb. 1639, repr. in *ibid.*, vol. II, pp. 513–14.

[8] Descartes, letter to Mersenne, dated c. Jan. 1638, repr. in *ibid.*, vol. I, p. 492.

[9] van Schooten, letter to Huygens, dated 19 Jul. 1658, repr. in Huygens, *Œuvres Complètes*, vol. 2, pp. 221–2; trans. C.

[10] Mahoney, *Fermat*, p. 282.

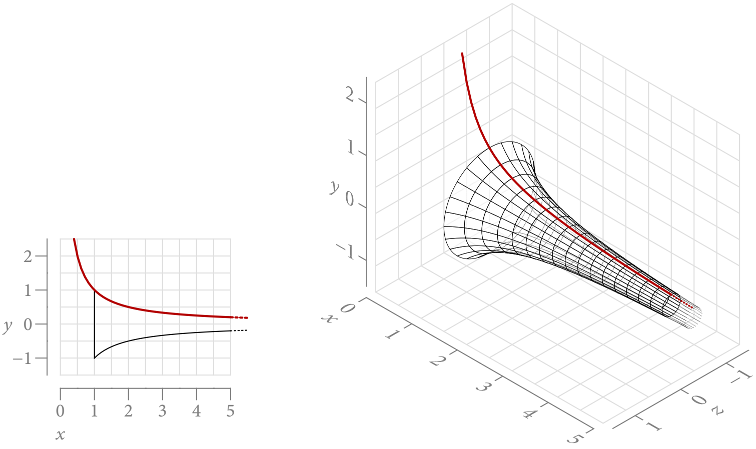

FIGURE 9.4
Torricelli's solid, which has infinite length, infinite surface area, but finite volume, here formed by rotating the positive branch of the hyperbola $y = 1/x$ about the x axis and truncating at the plane $x = 1$.

number theorem[*] suggests that he was not entirely immune to the charms of individual gems.

Evangelista Torricelli's (1608–47) solid is defined (in modern terms) by rotating a hyperbola about one of its asymptotes and truncating it at a plane normal to that asymptote (see Figure 9.4). The resulting solid is of infinite length, and indeed infinite surface area, but has finite volume. The discovery of such a counter-intuitive solid caused philosophical disturbance, for it seemed to violate the distinction between finite and infinite.[1] Torricelli, foreseeing the scrutiny to which his work would be subjected, took the precaution of pre-empting some criticisms by supplying two different proofs, one by 'indivisibles', one by exhaustion.[2] Descartes seems not to have been provoked to any such philosophical objections, and found Torricelli's demonstration of the existence of an infinitely long solid of finite volume to be beautiful.[3]

As regards proof techniques, there is just a hint of aesthetic disdain in Descartes's complaint that *reductio ad absurdum* is 'the least valued & the least ingenious' of the tools of mathematics.[4] But one cannot read too much into this remark, for its context was Descartes's attempted demolition of Fermat's work as a rival to *La Geometrie*, in which he contrasts his own use of *a priori* argument.

BLAISE PASCAL

> ‘I have always seen with great joy
> the beautiful things with which you enrich geometry.’
>
> — Claude Mylon, letter to Pascal, dated 27 Dec. 1658,
> repr. in Pascal, *Œuvres*, vol. IX, p. 153
> (trans. C).

Blaise Pascal (1623–62) seems to have been the earliest mathematician to repudiate explicitly a notion of mathematical beauty: among the fragments that were posthumously published as his *Pensées* is found the following:

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[*] ▸ p. 300

[1] Mancosu & Vailati, 'Torricelli's Infinitely Long Solid', esp. pp. 60 sqq.

[2] *Ibid.*, pp. 55–7.

[3] Descartes, letter to Mersenne, dated 2 Nov. 1646, repr. in Descartes, *Œuvres*, vol. IV, p. 553.

[4] Descartes, letter to Mersenne, dated Jan. 1638, repr. in *ibid.*, vol. I, p. 490; trans. C.

> 'Just as we talk of poetic beauty, so we should also talk of mathematical beauty and medicinal beauty. But we do not talk like that for the very good reason that we know what the object of mathematics is, namely proof; and what the object of medicine is, namely cure; but we do not know what constitutes the attraction which is the object of poetry.'[1]

But in a letter to Fermat, Pascal called mathematics 'the most beautiful occupation in the world'.[2] It is unclear how to reconcile these statements. Strictly, there is no contradiction between mathematics *qua* occupation being beautiful but mathematics *qua* subject-matter not being beautiful, but this explanation seems hardly satisfactory. Perhaps Pascal just changed his mind. Or, perhaps, given that the *Pensées* were his notes for a book of apologetics that he intended to publish, he felt he had to take a more nuanced position than he would admit to in private correspondence.

In any case, Pascal seems not to have been immune to beauty in mathematics. In correspondence during 1654, he and Fermat developed probability theory by examining in particular the problem of how to divide the stakes when a game of chance is interrupted. In one of Pascal's letters, he stated a result addressing a special case of this question, and then gave a number of arithmetic propositions that lead up to this result, which have 'quite beautiful properties'.[3] But after this slip, he did not mention which properties he found beautiful. Several years later, writing to Carcavi, he described certain problems relating to conoids as 'certainly admirably beautiful'.[4]

But it was not simply in private correspondence that he used such language. In his 1658 essay on the history of the cycloid, he referred to beautiful questions,[5] discoveries,[6], and methods.[7] In particular, he praised Christopher Wren's (1632–1723) work on the cycloid, which was in response to problems set by Pascal himself:

> 'among all the writings that have been received of this sort, none is more beautiful than that which has been sent by Mr Wren; for besides the beautiful way that he gives to measure the area of the Cycloid, he has given the comparison of the curved line itself and of its parts with the straight line. His proposition is that the line of the Cycloid is quadruple its diameter, of which he has sent the statement without proof.'[8]

But Pascal's essay was not an impartial account[9] and perhaps he used more affective language in an attempt to persuade his readers. [It was Wren's rectification of the cycloid that established his reputation as a mathematician.[10] His proof was published later by John Wallis (1616–1703).[11]]

Thus, for all Pascal's importance as a mathematician and philosopher, and despite his *Œuvres* filling fourteen volumes, his attitude to

[1] Pascal, *Pensées*, § 586.

[2] Pascal, letter to Fermat, dated 10 Aug. 1660, repr. in Pascal, *Œuvres*, vol. x, pp. 4–5; trans. C., emphasis added.

[3] Pascal, letter to Fermat, dated 29 Jul. 1654, repr. in *ibid.*, vol. iii, vol. 385; trans. C.

[4] Pascal, letter to Carcavi, dated Feb. 1659, repr. in *ibid.*, vol. ix, p. 182; trans. C.

[5] Pascal, 'Histoire de la Roulette', p. 195.

[6] *Ibid.*, p. 196.

[7] *Ibid.*, p. 205.

[8] *Ibid.*, p. 204; trans. C.

[9] Taton, 'Pascal, Blaise', p. 336.

[10] Bennett, *The mathematical science of Christopher Wren*, p. 23.

[11] Wallis, 'Tractatus Duo', pp. 533–9.

mathematical beauty remains obscure. In publications, only in the essay on the history of the cycloid did he use aesthetic language to describe mathematics. Otherwise, he only used it in private correspondence, and then far less commonly than Fermat or Descartes. (In particular, the author has been unable to locate any instance where Pascal wrote of *elegance* in mathematics.) Perhaps there was a distinction between his public and private positions, or perhaps his attitude is simply inaccessible.

CHRISTIAAN HUYGENS

The mathematician, astronomer, physicist, horologist Christiaan Huygens (1629–95) was the greatest natural philosopher of the generation before Newton and Leibniz.[1] The best testament to his genius is the tendency of historians to wonder why he did not develop the calculus or Newtonian physics: is this question raised about any other scientist?[2] Huygens consistently prioritized mathematics in his study of nature.[3] As a mathematician, he was broadly conservative, and the importance of his work mostly derives from his refinement and new applications of known methods.[4] He remained sceptical about Leibniz's differential calculus because of Leibniz's secrecy,[5] although towards the end of his life he expressed his aesthetic admiration of it:

> 'I admire more and more the beauty of geometry in these advances that are being made in it these days, in which you have such a big part, sir, not only by your marvellous calculus.'[6]

In his correspondence, Huygens often judged mathematics on its beauty, variously mentioning beautiful problems, results, methods, and theories.[7] Proofs he would describe as elegant, and he made a single remark that suggests he saw a link between elegance and brevity in proofs.[8]

Number theory

Claude Mylon (c. 1618–c. 1660) wrote that Fermat was interested in what Mylon described as Huygens's 'beautiful printed treatises'[9] on numbers. Carcavi gave to Mylon two 'beautiful problems on numbers', which the context suggests were from Fermat, to show to Huygens.[10] It is not known what these problems were: Mylon's letter to Huygens seems to be lost,[11] while the only extant letter from Fermat to Carcavi with a proximate date does not propose any problems on numbers.[12] Nevertheless, it is known that Huygens agreed with Mylon's assessment, writing that these problems of Fermat were 'quite beautiful in the genre'.[13] And when Fermat died, Huygens wrote to Carcavi: ‘I have been extremely saddened by the death of Mr de Fermat, of whom I always expected the beautiful things that he could supply’.[1] These opinions strongly suggest that there was broad agreement among Fermat, Huygens, Carcavi, and Mylon about what problems, at least concerning numbers, could be considered beautiful.

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[1] Cf. Blåsjö, *Transcendental Curves*, p. 69.

[2] Yoder, *Unrolling Time*, p. 177.

[3] *Ibid.*, p. 170.

[4] Bos, 'Huygens, Christiaan', p. 599.

[5] *Loc. cit.*

[6] Huygens, letter to Leibniz, dated 17 Sep. 1693, repr. in Huygens, *Œuvres Complètes*, vol. 10, p. 511.

[7] *Ibid.*, vol. 9, p. 181.

[8] Huygens, letter to Mylon, dated 13 May 1651, repr. in *ibid.*, vol. 1, p. 141.

[9] Mylon, letter to Huygens, dated 15 Apr. 1656, repr. in *ibid.*, vol. 1, p. 400; trans. C.

[10] Carcavi, letter to Huygens, dated 20 May 1656, repr. in *ibid.*, vol. 1, p. 418; trans. C.

[11] *Ibid.*, vol. 1, p. 426, n. 1.

[12] Fermat, letter to Carcavi, dated 1656, repr. in Fermat, *Œuvres*, vol. 2, pp. 315–20.

[13] Huygens, letter to Mylon, dated 1 Jun. 1656, repr. in Huygens, *Œuvres Complètes*, vol. 1, p. 426; trans. C.

Solid geometry

Huygens wrote to Carcavi that Theorem P below, due to Pascal, was ‘very beautiful’.[2] Some definitions are necessary: A *triline* [‘triligne’] is a plane figure bounded by two straight lines perpendicular to one another and a curved line; the two straight lines are called the *axis* and the *base*; the *arm* on the axis or base is the perpendicular to that line passing through the centre of gravity of the triline.

THEOREM P. *‘If a triline is rotated, first about its base, and then about its axis, and thus forms two solids, one around the base and the other around the axis, the distance between the axis and the centre of gravity of the solid around the base is to the distance between the base and the centre of gravity of the solid around the axis as the arm of the triline on the axis to the arm of the triline on the base.*

From which it appears that, if one knows the centre of gravity of the triline and of one of the solids, that of the other is also known.’[3] $\boxed{\text{P}}$

Cycloid

Mersenne sent to Huygens some of Torricelli’s work on the cycloid in 1647,[4], but Huygens only became interested in it when he received a Latin edition of Pascal’s ‘Histoire de la Roulette’ in early 1659.[5] Like Pascal, Huygens was greatly impressed by Wren’s rectification of the cycloid, calling it ‘wonderfully pleasing’[6] and ‘beautiful’.[7] Wren had not supplied a proof, so Huygens’s praise applied solely to the result: *The arc length of the cycloid is four times the diameter of its generating circle.*

Huygens wrote to Wallis that ‘Wren’s discovery [...] is the best that has been found about this curve. For I distinguish the difficult from the elegant.’[8] This difference suggests that Huygens evaluated ‘best’ by aesthetic value, not by technical difficulty. And since his opinion applied solely to the result, its elegance was presumably related to the simple integer relationship between the arc length and the diameter of the generating circle.

His admiration for Wren’s result may have been a trigger for a desire to exhibit his own mathematical prowess.[9] He was moved to verify Wren’s result, and in so doing proved a more general result than Wren had announced, though he only wrote a rough draft of the exposition.[10]

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[1] Huygens, letter to Carcavi, dated 26 Mar. 1665, repr. in Huygens, *Œuvres Complètes*, vol. 5, p. 278; trans. C.

[2] Huygens, letter to Carcavi, dated 22 May 1658, repr. in *ibid.*, vol. 2, p. 412; also n. 4.

[3] Pascal, ‘Traité des Trilignes Rectangles et de leurs Onglets’, ‘Consequence’ on p. 35; trans. C.

[4] Mersenne, letter to Huygens, dated 8 Jan. 1647, repr. in Huygens, *Œuvres Complètes*, vol. 1, p. 52.

[5] Yoder, *Unrolling Time*, p. 78.

[6] Huygens, letter to Wallis, dated 31 Jan. 1659, repr. in Huygens, *Œuvres Complètes*, vol. 2, p. 330; trans. C.

[7] Huygens, letter to van Schooten, dated 7 Feb. 1659, repr. in *ibid.*, vol. 2, p. 343 [ltr 582]; trans. C.

[8] Huygens, letter to Wallis, dated 31 Jan. 1659, repr. in *ibid.*, vol. 2, p. 330; trans. C.

[9] Yoder, *Unrolling Time*, p. 122.

[10] Huygens, *Œuvres Complètes*, vol. 14, pp. 363–7; see also Sloth, ‘Chr. Huygens’ Rectification of the Cycloid’.

Huygens also proved that the evolute (the loci of the centres of curvature) of a cycloid* is an equal cycloid;[1] he further proved that the inverted cycloid is a *tautochrone*: a body starting from rest and freely accelerated by uniform gravity along the curve reaches the lowest point of the curve in the same time, independently of its starting point.[2] (Later, Montucla would consider this result beautiful.[3]) The time taken is equal to $\pi/2$ multiplied by the time required for a body to fall through the diameter of the generating circle: if r is the radius of the circle generating the cycloid and g the acceleration due to gravity, the time to fall through the diameter is $2\sqrt{r/\mathrm{g}}$, while the time of descent along the cycloid is $\pi\sqrt{r/\mathrm{g}}$. Huygens had thus proved that a pendulum moving (frictionlessly) in a cycloid (see Figure 9.5) would oscillate at an equal frequency regardless of its swing length. (However, friction prevents the practical use of cycloid pendula in clock escapements.)

Catenary

Huygens was also greatly interested in the *catenary*, namely the curve formed by a uniform chain hanging between two points of support (see Figure 9.6). It had long been thought to be a parabola, but Huygens proved this claim false by considering the hanging chain as a set of equal weights equally spaced along a parabola.[4] Although other authors also proved that the catenary is not a parabola, none were able to present its actual shape.[5] Curiously, it was only in the May 1690 issue of the *Acta Eruditorum* that the problem of finding the shape of the catenary curve was posed by Jakob I Bernoulli (1655–1705).[6] Huygens approved of the question on aesthetic grounds:

> 'We must be grateful to Mr Bernoulli for having thought of this problem of the catenary, which is beautiful in that it has for object a curved line that is most presented to our eyes, and which is one of those that nature seems trace it itself.'[7]

Thus, for Huygens, the beauty of the problem seems to have rested upon the ubiquity of the physical object that (approximately) embodied the curve, and not upon the mathematics itself.

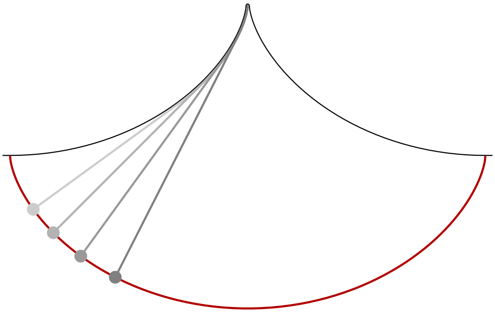

FIGURE 9.5
A cycloid pendulum, in which, if there were no friction, the frequency would be the same regardless of the length of the swing.

* ▸ pp. 296–7

[1] Phillips, 'Brachistochrone, Tautochrone, Cycloid', p. 507.

[2] Whitman, 'Some Historical Notes on the Cycloid', p. 315.

[3] Montucla, *Histoire*, vol. 2, p. 385; Montucla, *Histoire New ed.*, vol. 2, p. 419.

[4] Huygens, letter to Mersenne, dated 1646, repr. in Huygens, *Œuvres Complètes*, vol. 1, pp. 34–40; see also Bukowski, 'Christiaan Huygens', pp. 4–7.

[5] Bukowski, 'Christiaan Huygens', pp. 8–9.

[6] Bernoulli, 'Analysis Problematis Antehac Propositi', p. 424.

[7] Huygens, letter to ?, dated 1691, repr. in Huygens, *Œuvres Complètes*, vol. 10, p. 217; trans. C.

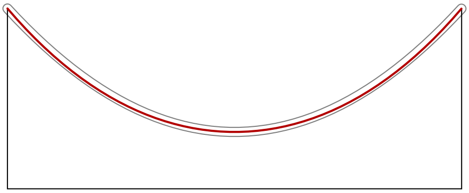

FIGURE 9.6
A catenary, shown in red, with a best-fit parabola, shown in grey outline.

Huygens, Leibniz, and Johann I Bernoulli each offered a solution to Jakob I Bernoulli's problem. Leibniz's solution was a euclidean construction of the curve,[1] and Huygens wrote to Leibniz that it was 'certainly [...] very beautiful'.[2] Johann I Bernoulli thought Leibniz's solution 'most elegant'.[3] In modern terms, the equation of a catenary is

$$y = a\cosh\frac{x}{a} = \frac{a}{2}\left(\mathrm{e}^{x/a} + \mathrm{e}^{-x/a}\right),$$

where a is a constant. Leibniz noted 'the wonderful and elegant harmony of the curve of the chain with logarithms'.[4]

Huygens also thought that if there were a method to discover, from the equation of a curve, whether it is possible to reduce the problem of its quadrature to that of the circle or hyperbola, then such a method would be beautiful. While investigating the catenary, he wrongly thought that Leibniz and Bernoulli (the context suggests Johann I) had such a method, which led him down the wrong path.[5]

Beauty and utility

In August 1662, Huygens's brother Lodewijk (1631–99) sent him three problems, asking for numbers satisfying certain properties. For instance, the second problem asked for three numbers that form a harmonic mean and are the lengths of the sides of a right-angled triangle.[6] Huygens was unimpressed: 'the questions that you have sent me do not merit attention, not being at all beautiful or useful for anything'.[7] The denial of beauty and of utility separately suggests that Huygens saw them as separate motivations for mathematical research.

On the other hand, he seems to have linked beauty and utility in justifying the pursuit of squaring the circle:

> 'The research on the Quadrature of the Circle has made known to Geometers so many beautiful things, that so they are not deprived of such a useful exercise, I take the position of defending against Mr Gregory the possibility of succeeding in it.'[8]

Taken literally, this statement suggests that Huygens found useful the pursuit of beauty, and, more importantly, that this pursuit motivated him to analyse James Gregory's (1638–75) arguments and locate an error.[9]

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[1] Leibniz, 'Solutio Problematis Funicularis'.

[2] Huygens, letter to Leibniz, dated 1 Sep. 1691, repr. in Huygens, *Œuvres Complètes*, vol. 10, p. 129; trans. C.

[3] Bernoulli, 'Positionum de Seriebus Infinitis (4)', p. 113; trans. C.

[4] Leibniz, 'De Solutionibus Problematis Catenarii', pp. 255–6; trans. Blåsjö, *Transcendental Curves*, p. 137.

[5] Huygens, letter to Leibniz, dated 4 Sep. 1691, repr. in Huygens, *Œuvres Complètes*, vol. 10, pp. 140–1; cf. Huygens, letter to Leibniz, dated 16 Nov. 1691, repr. in Huygens, *Œuvres Complètes*, vol. 10, p. 189.

[6] Lodewijk Huygens, letter to Huygens, dated Aug. 1662, repr. in Huygens, *Œuvres Complètes*, vol. 4, p. 211.

[7] Huygens, letter to Lodewijk Huygens, dated 31 Aug. 1662, repr. in *ibid.*, vol. 4, p. 215; trans. C.

[8] Huygens, letter to Gallois, dated 2 Nov. 1668, repr. in *ibid.*, vol. 6, p. 272; trans. C.

[9] Huygens, letter to Gallois, dated 2 Nov. 1668, repr. in *ibid.*, vol. 6, p. 273.

Towards the end of his life, he wrote that 'in research into numbers, the most useful would be to stop at Theorems in which there are beautiful ones and which can serve in encounters',[1] which seems to indicate that beauty and utility (or, at least, applicability) are not connected, save that both are desirable properties. And he described some of Jakob I Bernoulli's work on elastic curves as not being of any use, but which could serve as 'very beautiful and subtle exercises, when no employment is found for more fruitful mathematics'.[2]

ISAAC BARROW

Isaac Barrow (1630–77), in his oration on being installed as the first Lucasian Professor of Mathematics at the University of Cambridge, showed that he partly shared Plato's view on the role of mathematics in moral formation and made an even more direct link to beauty than Plato had. Much of the oration was devoted to praising Henry Lucas (c. 1610–1663) for his bequest of a library of four thousand volumes and an income to support the professorship.[3] Barrow admitted that he had found little reward in his previous position as Professor of Greek, and that nothing could have pleased him more than to be appointed to the Lucasian chair.[4]

Barrow held that mathematics was valuable for without it one could understand neither the philosophy of the great thinkers of antiquity past nor the modern scholars of natural science.[5] The study of mathematics was also useful because it trained one to think diligently and correctly, not rashly or falsely.[6] But it also had a moral effect which Barrow had clearly taken from Plato and which he linked directly to the experience of beauty in the ideas of mathematics:

> 'I shall neither be the only Person, nor the first, who affirms it; that while the Mind is abstracted and elevated from sensible Matter, distinctly views pure Forms, conceives the Beauty of Ideas, and investigates the Harmony of Proportions; the Manners themselves are sensibly corrected and improved, the Affections composed and rectified, the Fancy calmed and settled, and the Understanding raised and excited to more divine Contemplation. All which I might defend by the Authority, and confirm by the Suffrages of the greatest Philosophers.'[7]

[1] Huygens, letter to Leibniz, dated 16 Nov. 1691, repr. in Huygens, *Œuvres Complètes*, vol. 10, p. 190; trans. C.

[2] Huygens, letter to Leibniz, dated 24 Aug. 1694, repr. in *ibid.*, vol. 10, p. 665; trans. C.

[3] Barrow, 'The Prefatory Oration', p. vii.

[4] *Ibid.*, pp. xxii, xxiv.

[5] *Ibid.*, pp. xxvi sqq.

[6] *Ibid.*, p. xxxi.

[7] *Loc. cit.*

ISAAC NEWTON

> ‘“Does that mean I am Wise?” “No.
> You are not Wise but erudite. […]”
> […]
> “Enoch Root — is he Wise?” “Yes.”
> “Leibniz?” “Erudite.”
> “Newton?” “Hard to say.”
> “It seems as though Newton just knows things.
> Finished knowledge appears in his head —
> no one else can make out how he did it […]” ’
>
> — Neal Stephenson, *The Baroque Cycle*, bk 7.

Tradition views Isaac Newton (1642–1726) as one of the three greatest mathematicians in history, alongside Archimedes and Gauss. His reputation is reflected in the inscription — a line from Lucretius — on his statue, sculpted by Louis-François Roubiliac (1702–62), in the ante-Chapel at Trinity College: ‘qui genus humanum ingenio superavit’.[1]

Newton was concerned with the methods of mathematics from a perspective of aesthetics, visualization, and compatibility with ancient tradition.[2] It was on these grounds that he favoured classical geometric methods over cartesian algebraic ones. Algebra was, at best, a tool for discovery and not for exposition.[3] And calculation was alien to geometry:

> ‘anyone who examines the constructions of problems by the straight line and circle devised by the first geometers will readily perceive that geometry was contrived as a means of escaping the tediousness of calculation by the ready drawing of lines. Consequently these two sciences ought not to be confused. The Ancients so assiduously distinguished them one from the other that they never introduced arithmetical terms into geometry; while recent people, by confusing both, have lost the simplicity in which all elegance of geometry consists.’[4]

Note the link between elegance and simplicity. He emphasized this point later in the same text: ‘Geometrical speculations possess elegance only in so far as they have simplicity’.[5]

Newton’s views on the superiority of the ancient methods is most easily seen in his approach to the *Pappus problem*. This problem is related to, but more general than, the three- and four-line loci results that Apollonius had found beautiful.* In modern terms, the givens of the Pappus problem are lines $L_1, \ldots, L_n$, angles $\varphi_1, \ldots, \varphi_n$, and a constant k. The problem is to find the points P satisfying the following conditions. For each i, let d_i be the length of the segment from P to L_i making an angle φ_i with L_i. If $n = 2m$, then $\prod_{i=1}^{m} d_i = k \prod_{i=m+1}^{n} d_i$; if $n = 2m - 1$, then $\prod_{i=1}^{m} d_i = k \prod_{i=m+1}^{n-1} d_i$ (see Figure 9.7).[6] [In Apollonius’ loci, each angle φ_i is $\pi/2$, so that d_i is simply the distance from P to L_i.]

[1] Lucretius, *De Rerum Natura*, bk III, l. 1043.

[2] Guicciardini, *Isaac Newton*, pp. xiv, 231.

[3] Guicciardini, ‘A brief introduction’, p. 399.

[4] Newton, ‘Lucasian Lectures on Algebra’, p. 429.

[5] *Ibid.*, p. 433.

* ▸ pp. 113–15

[6] Pappus, *Collection* 7, §§ 38–40; Grosholz, *Cartesian Method*, p. 29.

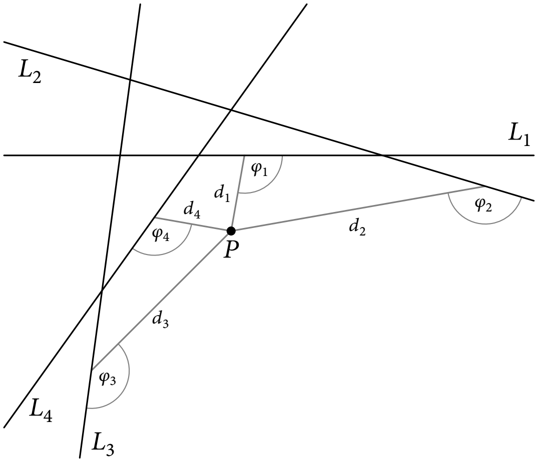

FIGURE 9.7
An instance of the Pappus problem.

Descartes had put the Pappus problem at the centre of *La Geometrie*, so as to demonstrate the power of his method: the Greeks had neither solved it systematically nor generalized it.[1] He solved it for three and four lines by introducing coördinates and using similar triangles to show that the locus is a curve consisting of points satisfying a second-degree equation in two unknowns.[2] Having made this analysis, Descartes turned to synthesis and constructed, using Apollonius' theory, a conic section depending on the second-degree equation. He then proved that its equation coincided with the equation of the locus.[3]

Newton read Pappus' *Collection* and was convinced from Pappus' account of Euclid's lost work *Porisms*[4] that it had contained the key to the ancient geometrical method of analysis. Newton hoped that he could reconstruct this geometrical analysis and so prove wrong Descartes, who had proposed algebra as a tool that could replace euclidean methods.[5]

Newton's preference for geometry over algebra was at least partly aesthetic. His approbation of Antonio Hugo de Omerique's (1634–1705) book *Analysis Geometrica* (1698) makes this clear:

> 'I have lookt into De Omerique's Analysis Geometrica & find it a judicious & valuable piece answering to y^{e} Title. For therein is laid a foundation for restoring the Analysis of the Ancients w^{ch} is more simple more ingenious & more fit for a Geometer then the Algebra of the Moderns. For it leads him more easily & readily to the composition of Problems & the Composition w^{ch} it leads him to is usually more simple & elegant then that w^{ch} is forct from Algebra.'[6]

Newton also preferred the way in which a geometrical analysis could yield a synthesis simply by inverting the 'sequence of argument'.[7] With algebra, the analysis might be easier, but an elegant synthesis more difficult:

[1] Grosholz, *Cartesian Method*, pp. 29–31.

[2] Descartes, *La Geometrie*, pp. 382–4, 397–9.

[3] *Ibid.*, pp. 400–7.

[4] Pappus, *Collection* 7, §§ 13–20.

[5] Guicciardini, *Isaac Newton*, pp. 81–2.

[6] Newton, New College MS 361(2), fol. 14v Iliffe & Mandelbrote, 'Newton Project', THEM00097; repr. inNewton, *Correspondence*, vol. VI, pp. 413–13.

[7] *Ibid.*, p. 102.

> 'Through algebra you easily arrive at equations, but always to pass therefrom to the elegant constructions and demonstrations which usually result by means of the method of porisms is not so easy'.[1]

But although it might be *easier*, he still thought that algebraic analysis of locus problems was less elegant than geometrical analysis, even before the question of synthesis was addressed.[2]

For Newton, the simplicity of an equation was not relevant to whether the curve it described could be used in problems. Algebra was foreign to geometrical simplicity. The equation of a parabola $y = ax^2$ might have been simpler than that of a circle $x^2 + y^2 = r^2$, but the circle was prior because of its simpler construction.[3] From Newton's perspective, the conchoid could be described more simply than any curve except the circle,[4] because it had a straightforward mechanical construction (see Figure 3.4 (left) on page 98), but its equation is of the form $(x - b)(x^2 + y^2) - a^2x^2 = 0$.[5] For Newton, a curve arose from some mechanical action, not from an equation: '*geometry* is founded on mechanical practice'.[6]

Newton was probably influenced towards an aesthetic preference for geometry over Cartesian symbolic analysis by reading Huygens's *Horologium Oscillatorium*, which described physical phenomena in a way compatible with the certainty that Newton thought the ancient geometers had achieved, and which Newton held up as a model of mathematical method.[7] According to Henry Pemberton (1694–1771),

> 'Sir ISAAC NEWTON has several times particularly recommended to me *Huygens*'s stile and manner: He thought him the most elegant of any mathematical writer of modern times, and the most just imitator of the antients. Of their taste, and form of demonstration Sir ISAAC always professed himself a great admirer: I have heard him even censure himself for not following them yet more closely than he did; and speak with regret of his mistake at the beginning of his mathematical studies, in applying himself to the works of *Des Cartes* and other algebraic writers, before he had considered the elements of *Euclide* with that attention which so excellent a writer deserves.'[8]

Since Pemberton supervised the third edition of the *Principia Mathematica*, in the preface of which Newton called him 'a man greatly skilled in these matters',[9] it is reasonable to credit him with an accurate account of Newton's views.

In his attempted restoration of the ancient geometrical analysis, Newton developed many results in projective geometry, but his greatest success was in giving a geometrical solution to the Pappus problem for three or four lines.[10] Working geometrically, he showed that the locus would be a conic section passing through four intersections of the given lines and another constructible point. Thus he reduced the

[1] Newton, 'First Essays at a Treatise on Geometry', p. 261.

[2] Guicciardini, *Isaac Newton*, p. 106.

[3] Newton, 'Lucasian Lectures on Algebra', p. 425.

[4] *Ibid.*, p. 429.

[5] See also Guicciardini, *Isaac Newton*, p. 65.

[6] Newton, *Principia*, 'Author's Preface'.

[7] Guicciardini, *Isaac Newton*, p. 59.

[8] Pemberton, *A View of Sir Isaac Newton's Philosophy*, Preface, n.p. emphases in orig.

[9] Newton, *Principia*, p. 400.

[10] *Ibid.*, § I.5, Corol. 2 to Lem. 19.

problem to constructing a conic section through five given points.[1] Pappus had described how to construct an ellipse through five given points in *Collection*,[2] so Newton thought his solution was evidence that ancient geometers could have solved Pappus' problem.[3]

Thus Newton also considered his solution to have been a victory over Cartesian algebra. And the ancient method, he maintained, was more elegant than the Cartesian.[4] But the victory was not even pyrrhic: first, his solution uses projective geometry and is thoroughly seventeenth-century; second, unlike the algebraic solution, it does not generalize to more than four lines.[5] Also, part of his objection to the Cartesian method was purely expositional: lacking algebraic notation, the ancients could only have expressed such an approach in words, which would 'be so tedious and entangled as to provoke nausea, nor might it be understood'.[6]

Newton held to his preference for geometry when writing the *Principia*. Although in later life he claimed to have first used fluxions to discover the results and then to have re-cast them in a synthetic geometrical form, as a matter of historical fact the *Principia* was never based upon fluxions in the sense of an algebraic technique.[7] It was founded rather upon 'the calculus' as an informal system, not the kind of symbolic manipulations that Newton and Leibniz had found.[8]

Finally, it is worth noting that Newton also seems to have had a preference for arguments that avoided numerical calculation: in the first edition of the *Principia Mathematica*, he elided calculations that were 'too complicated and encumbered by approximations and not exact enough'.[9]

GOTTFRIED WILHELM LEIBNIZ

> ‘ Un bel air de Musiqve qvi sera tousjours chanté,
> est comme un beau theoreme de Geometrie rencontré. ’
>
> — Leibniz, 'Relation de l'état de la république des lettres', p. 569.

Gottfried Wilhelm Leibniz (1646–1716) started as a prodigy and ended as one of the last great polymaths. He read voraciously as a child, and, though his degree and doctorate were in law, his wide-ranging interests were already evident.[10] In 1672, he came to Paris for what became a four-year stay. He was then something of a tyro when it came to the higher mathematics, which caused him embarrassment during a visit to London in 1673, when he proudly claimed a new discovery only to learn that it was already known.[11] Guided in part by Huygens, who became his mathematical mentor, he devoted a year of study to repair the deficiencies in his knowledge, and, by the time he left Paris, he had formulated the core of the differential calculus.[12] Many years later, he would be delighted by his old master admitting to having sufficiently overcome initial reservations

[1] Guicciardini, *Isaac Newton*, pp. 91–2.

[2] Pappus, *Collection*, bk VIII, Prop. 13.

[3] Guicciardini, *Isaac Newton*, pp. 92–3.

[4] Newton, 'Researches into the 'Solid Locus'', p. 277.

[5] Guicciardini, 'A brief introduction', p. 408.

[6] Newton, 'Researches into the 'Solid Locus'', p. 277.

[7] Hall, *Philosophers at War*, p. 28.

[8] *Ibid.*, p. 30.

[9] Newton, *Principia*, bk III, Scholium to Prop. 35, n. a.

[10] Aiton et al., 'Leibniz, Gottfried Wilhelm', p. 149.

[11] Hofmann, *Leibniz in Paris*, pp. 26–7; Aiton, *Leibniz*, p. 44.

[12] Aiton, *Leibniz*, pp. 49, 57–9.

to recognize the same beauty that he himself saw in the calculus:[*] 'I am thrilled [ravi] to see that [...] you have remarked that there is much beauty in our differential calculus'.[1] [That Leibniz *concurred* in Huygens's appreciation is made even clearer by a draft he began writing at the end of Huygens's letter, which reads 'you have now *discovered* that there is much beauty in the differential calculus'.[2]]

Leibniz's work at Paris included his first great mathematical achievement: the 'arithmetic quadrature' of the circle through his alternating series for π/4:[3]

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots. \qquad \text{(L)}$$

He found this series using his *transmutation theorem*, which converted an integral giving an area under a curve to a sum involving an easier-to-compute integral. He seems to have been very impressed by the significance of (L), noting in one of the figures in the paper where he published it in 1682: 'GOD rejoices in the odd number [Numero DEUS impare gaudet]'.[4] The result is a particular case of what is now called the Taylor series for the arctangent function,

$$\arctan x = \int_0^x \frac{\mathrm{d}z}{1+z^2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}, \qquad \text{(T)}$$

which James Gregory (1638–75) had discovered independently in 1671, without noticing (or at least without pointing out) the special case of (L).[5]

After discovering (L) in 1673–4, Leibniz communicated the result to Huygens, who called it 'very beautiful and very pleasing [fort belle et fort heureuse]'.[6] Leibniz himself wrote to two different correspondents that there was no simpler or more elegant [simpliciorem atque elegantiorem] way and that there was no simpler or more beautiful [simpliciorem et pulchriorem] way of expressing the area of a circle using rational numbers.[7] Newton welcomed Leibniz's work on convergent series as 'very elegant [[p]erelegans]', and thought that 'it would have sufficiently revealed the genius of its author, even if he had written nothing else', but cautioned that (L) converged too slowly for practical computation.[8]

[The series (L) seems to have had lasting appeal. More than a half-century later, Christian Goldbach wrote to Leonhard Euler:[†]

> 'If one looks at the facility with which the approximation proceeds, I daresay nothing more elegant has ever been invented for the quadrature of the circle than Leibniz's series.'[9]

At the end of the eighteenth century, Montucla judged it elegant in his history of circle quadrature.[‡] Richard Courant (1888–1972) & Herbert

[*] ▸ p. 307

[1] Leibniz, letter to Huygens, dated 1 Oct. 1693, repr. in Leibniz, *Sämtliche Schriften*, vol. iii.5, p. 645–6 [ltr 191]; trans. mod. O'Hara, *Leibniz's Correspondence*, p. 436.

[2] Leibniz, MS draft repr. in Leibniz, *Sämtliche Schriften*, vol. iii.5, p. 635, emphasis added; trans. mod. O'Hara, *Leibniz's Correspondence*, p. 435.

[3] Aiton, *Leibniz*, pp. 49–52.

[4] Leibnitio, 'De vera proportione Circuli', pl. opp. p. 42; trans. C.

[5] Gregory, letter to Collins, dated 15 Feb. 1671, repr. in Newton, *Correspondence*, vol. i, p. 62 [ltr 24].

[6] Huygens, letter to Leibniz, dated 6 Nov. 1674, repr. in Leibniz, *Sämtliche Schriften*, vol. iii.1, p. 170; trans. C.

[7] Leibniz, letter to Magliabechi, dated Dec. 1679; Leibniz, letter to [Kochański], dated Jul.–Aug. 1680, repr. in *ibid.*, vol. iii.3, pp. 37, 243 [ltrs 2, 91].

[8] Newton, letter to Oldenburg, dated 24 Oct. 1676, repr. in Newton, *Correspondence*, vol. 2, pp. 110, 120–1 [trans. pp. 130, 139; ltr 188]; Iliffe & Mandelbrote, 'Newton Project', NATP00196.

[†] ▸ pp. 334–41

[9] Goldbach, letter to Euler, dated 31 Jul. 1730, repr. in Euler, *Opera Omnia*, vol. iv.A.4.i, p. 118 [ltr 8; trans. vol. iv.A.4.ii, p. 613].

[‡] ▸ pp. 347–8

Robbins (1915–2001) called it ‘one of the most beautiful mathematical discoveries of the seventeenth century’.[1] Atle Selberg (1917–2007) credited it for sparking his interest in mathematics:

> ‘I had accidentally opened a book and come across Leibniz’s series for $\pi/4$ [...]. [...] school mathematics had always bored me but this seemed such a very strange and beautiful relationship that I determined I would read that book in order to find out how this formula came about.’[2]

Calvin Clawson (1941–2012) said it was ‘a beautiful infinite series’.[3]]

Aesthetic motivation

Leibniz declared that aesthetic concerns were one of his motivations for pursuing mathematics, alongside practical utility and importance internal to mathematics. In response to a request to engage with a particular problem that he found banal, he wrote:

> ‘I do not tackle any mathematical problem, unless it is either particularly elegant [valde elegans], useful in matters mechanical or physical, or finally, shows us a new method for solving an infinity of other problems.’[4]

Fifteen years later, he reiterated these motivations in the context of a particular problem posed by Jakob i Bernoulli (1655–1705). The classical isoperimetric problem asked for the shape in the plane with a given length that encloses the maximum area; the answer is a circle. Jakob challenged his brother Johann i Bernoulli (1667–1748) and other mathematicians to solve a modified problem that asks for the curve with a given length, on a given base, such that a certain related curve on the same base encloses the greatest area, and offered a prize of 50 thalers.[5]

Leibniz wrote to Johann about the problem, encouraging him to solve it. But Leibniz saw little intrinsic reason for tackling the problem, and seems to have thought that Jakob held the same view and had thus offered the prize:

> ‘He seems to have thought he had to do this, because the problems he proposes are not in themselves elegant [elegantiae], nor does it appear to have utility, except that it might perhaps advance the *ars inveniendi*.’[6]

Theory of beauty

Leibniz’s contribution to aesthetics has received relatively scant attention from scholars.[7] It is true that he did not develop a systematic aesthetic theory,[8] but aesthetic concepts (most frequently beauty[9]) and motives are embedded throughout his philosophy, so

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[1] Courant & Robbins, *What is Mathematics?*, p. 441.

[2] Selberg, ‘Reflections Around the Ramanujan Centenary’, p. 203.

[3] Clawson, *Mathematical Sorcery*, p. 130.

[4] Leibniz, letter to Pfautz, dated 28 Apr. 1682, repr. in Leibniz, *Sämtliche Schriften*, vol. iii.3, p. 597 [ltr 345]; trans. mod. O’Hara, *Leibniz’s Correspondence*, p. 286.

[5] Bernoulli, ‘Solutio sex problematum fraternorum’, p. 775.

[6] Leibniz, letter to Joh. Bernoulli, dated 28 May 1697, repr. in Leibniz, *Sämtliche Schriften*, vol. iii.7, p. 416–17; trans. mod. O’Hara, *Leibniz’s Correspondence*, p. 631.

[7] Brown, ‘Leibniz and Aesthetic’, p. 70; Portales, ‘Leibniz’s Aesthetics’, pp. 10, 13.

[8] Portales, ‘Leibniz’s Aesthetics’, p. 10.

[9] *Ibid.*, p. 12.

much so that he has been called the 'grandfather of German aesthetics'.[1]

There seems to have been no significant changes in Leibniz's ideas about aesthetics across his writings,[2] so it is justifiable to draw on many works to understand how he viewed beauty in mathematics.

First, he offered an explicit definition of beauty: 'We seek beautiful things because they are pleasant, for I define beauty as that, the contemplation of which is pleasant.'[3] But this recourse to pleasure did not mean the experience was wholly subjective, for he wrote elsewhere that '[p]leasure is a knowledge or feeling of perfection'.[4] (This seems to be the only explicit statement he made that clearly grounds beauty in perfection, but this point is in the background at many points in his writing.[5]) Whatever form it took, perfection was connected with cognition: 'perfection [...] is the degree of positive reality, or what comes to the same thing, the degree of affirmative intelligibility'.[6] The perfection of an object was manifested in its capacity to unite different properties,[7] and 'unity in plurality is nothing but harmony [...] there flows from this harmony the order from which beauty arises'.[8] [Harmony is characterized using variations on the phrase 'unity in plurality' in many different writings of Leibniz.[9]]

The pleasure — whether caused by beauty or by something else — is both a feeling and a cognitive state. Beauty is therefore both subjective and objective: the feeling of pleasure is subjective, but the structural harmony of the percept is objective. Leibniz thus followed a middle course between ancient thinkers who identified beauty with concepts like symmetry and harmony, and those who thought it was completely subjective. He anticipated aesthetic rationality, which saw beauty as a relation between subject and object.[10] The rationality he saw fits with comments he made on Shaftesbury's *Characteristicks*, in which he reserved particular praise for 'the treatise which is unjustly called a *Rhapsody*'.[11] By this phrase he referred to 'The Moralists', subtitled 'A Philosophical Rhapsody', which posited a new platonic theory of beauty in which the mind of the Creator played an essential role.*

Nature had beauty because it had laws, and it was beautiful because these laws could be stated mathematically.[12] An especial aspect of this mathematical structure was the symmetry in the laws of nature, which created harmony and so beauty.[13] But it was necessary to distinguish the harmony and beauty from the symmetry in nature itself. Complete symmetry could not exist either in phenomena or in the reality behind them; Buridan's ass would always be confronted by an asymmetry in its two choices.[14]

The application of mathematics to solve worldly problems was also a source of beauty for Leibniz, and this beauty seemingly reflected back on the mathematics itself. Huygens had published a new curve which he thought could be used to create a pendulum clock that would

[1] Beiser, *Diotima's Children*, pp. 31–2.

[2] Portales, 'Leibniz's Aesthetics', p. 15.

[3] Leibniz, 'Elements of Natural Law', p. 137.

[4] Leibniz, 'Felicity', § 4.

[5] Portales, 'Leibniz's Aesthetics', p. 28.

[6] Leibniz, letter to Wolff, dated winter 1714–15, extr. in Leibniz, *Philosophical Essays*, p. 230.

[7] Beiser, *Diotima's Children*, p. 35.

[8] Leibniz, 'On Wisdom', p. 426.

[9] Portales, 'Leibniz's Aesthetics', p. 34.

[10] Beiser, *Diotima's Children*, pp. 35–6.

[11] Leibniz, 'Remarks on the Three Volumes', p. 633, emphasis in orig.

* ▸ pp. 353–6

[12] Leibniz, 'Initia et Specimina', pp. 118–20; Breger, 'Die mathematisch-physikalische Schönheit bei Leibniz', p. 108.

[13] Breger, 'Die mathematisch-physikalische Schönheit bei Leibniz', p. 110.

[14] Leibniz, 'Essais de Théodicée', pp. 129–30.

not be affected by the motion of a ship and so would solve the problem of finding longitude at sea. Leibniz wrote in response:

> 'You do much honour to Geometry when you find the most beautiful uses of lines that it can furnish. And this new curve, which you give only in a riddle, will be a beautiful proof of this just as your use of the cycloid has been at other times.'[1]

[The 'riddle' was 'a a b b c d e e e e e f i i i i i l l l m m m m n o r r s s t t u u x', which consists of the letters of 'flexilis ambitum cum linea deſerit orbem' ('a flexible circuit with a line leaves the disc').[2]]

Beauty in theorems

Leibniz thought that the pleasure of a beautiful theorem is accessible only to those who can see the inner harmony.[3] But more generally, beauty was sometimes only recognized by the cognoscenti: for example, the movements of the planets seemed chaotic to the untrained mind.[4]

In a short note concerning the beauty of theorems, Leibniz wrote:

> 'Theorems are not intelligible except by their signs or characters. Images are a species of characters. When characters can be similar to things, all the better. The beauty of theorems consists in the beautiful arrangement of their characters.'[5]

This statement does not mean that the beauty of theorems lay only in their text or notation or diagrams. The 'characters' of a theorem could include its mental representation. It is not a stretch to allow that the theorem had to be represented *somehow*, on paper or in the mind, for its beauty to be contemplated,[6] which would fit with remarks Leibniz made about discovering results:

> 'Because they themselves cannot be painted or heard, we paint or hear their representations even if they are not similar and observe some perceptible beauties in them that will produce the effect in us that a theorem or a property of the intellectual thing itself is understood.'[7]

To illustrate the 'beautiful arrangement of characters', Leibniz gave as an example a theorem that he had discovered concerning Berthet's curve. This curve is defined with reference to an arc *AC* centred at *B*. Its defining property is that for any point *E* on the curve, the length of the segment *DE* of the line *BE* equals the arc length *AD* (see Figure 9.8 (left)).

Leibniz found a 'nice theorem' ('Theoreme joli') that yields the tangents to Berthet's curve: for a point *E* on the curve, let *D* be the corresponding point on the generating arc (that is, the intersection of the arc and the line through *E* and the arc's centre); extend the segment of the tangent of the arc at *D* on the same side as *A*; let *F* be

[1] Leibniz, letter to Huygens, dated 11 Oct. 1693, repr. in Huygens, *Œuvres Complètes*, vol. 10, pp. 539–40; trans. C.

[2] Huygens, letter to Editors of *Acta Eruditorum*, dated Sep. 1693, repr. in *ibid.*, vol. 10, p. 515, esp. n. b.

[3] Breger, 'Die mathematisch-physikalische Schönheit bei Leibniz', p. 105.

[4] *Ibid.*, p. 106.

[5] Leibniz, 'De la Beauté des Théorèmes', p. 439; trans. C.

[6] Portales, 'Leibniz's Aesthetics', p. 196.

[7] Leibniz, 'De Characteribus et Compendis', pp. 433–4; trans. Knobloch, 'Leibniz between *ars characteristica* and *ars inveniendi*', p. 292.

on this segment such that FD is to DE as EB is to BD (see Figure 9.8 (left) again).[1] This theorem has a 'beautiful arrangement of characters' in that the proportion $FD : DE :: EB : BD$ that defines F on the tangent through D is easily remembered by the mnemonic that the path $FD \cdot DE \cdot EB \cdot BD$ can be drawn without raising one's pen (see Figure 9.8 (right)).

In diametrical opposition to Newton,[*] Leibniz considered algebra to be one of the most beautiful disciplines,[2] and in particular to be the most beautiful part of 'the art of combinations'.[3] His own investigations into determinants led him to a 'very beautiful' theorem giving the solution to a system of rational equations from the coefficients, and this beauty became even clearer when one wrote the coefficients as numbers, for this made clear the harmony.[4]

For Leibniz, infinite series seemed to be a particular locus of aesthetic value. For instance, besides his praise for the $\pi/4$ series, he wrote that the sine series

$$\begin{aligned} \sin x &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots \\ &= \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \end{aligned}$$

had 'a certain unique elegance [elegantiam quandam singularem]'.[5]

Beauty in methods

In a note first sent to François Pinsson[6] (1612–91) and subsequently published in the *Mémoires pour l'Histoire Des Sciences & des beaux Arts*, known as the *Journal de Trévoux* from its place of publication, Leibniz wrote about beauty in methods:

> 'One of the issues of the *Journal de Trévoux* contains some method of Mr. Jacques Bernoulli, and mixes with it reflections on the differential calculus [...]. The author of these reflections seems to find the way through infinity and infinity of infinity not sure enough, and too far removed from the method of the Ancients. But he will have the goodness to consider that if the discoveries are considerable, the novelty of the method rather increases its beauty. With regard to the sureness of the way, [...] in place of the infinite or of the infinitely small, one

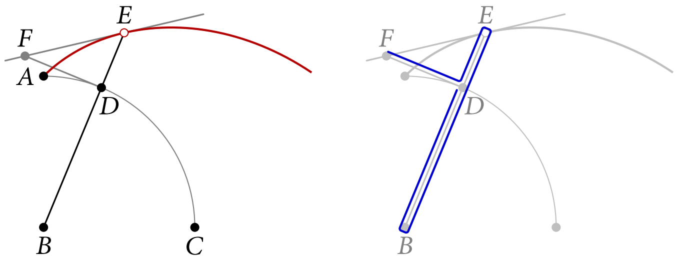

FIGURE 9.8
Left: Leibniz proved that the tangent to Berthet's curve at E can be found by letting F be on the tangent of the generating arc at D such that $FD : DE :: EB : BD$.
Right: Leibniz's mnemonic for the proportion defining F.

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[1] Leibniz, 'De la Beauté des Théorèmes', pp. 439–40.

[*] ▸ pp. 312–15

[2] Breger, 'Die mathematisch-physikalische Schönheit bei Leibniz', p. 110.

[3] Leibniz, letter to Tschirnhaus, dated May 1678, repr. in Leibniz, *Philosophical Papers and Letters*, p. 193.

[4] Breger, 'Die mathematisch-physikalische Schönheit bei Leibniz', p. 110.

[5] Leibniz, letter to Oldenburg, dated 13 May 1676, repr. in Leibniz, *Sämtliche Schriften*, vol. III.1, p. 375 [ltr 80]; trans. C.; cf. letter, dated 17 Aug. 1676, pp. 578–9 [ltr 89].

[6] Leibniz, letter to Pinsson, dated 29 Aug. 1701, repr. in *ibid.*, vol. I.20, pp. 493–4.

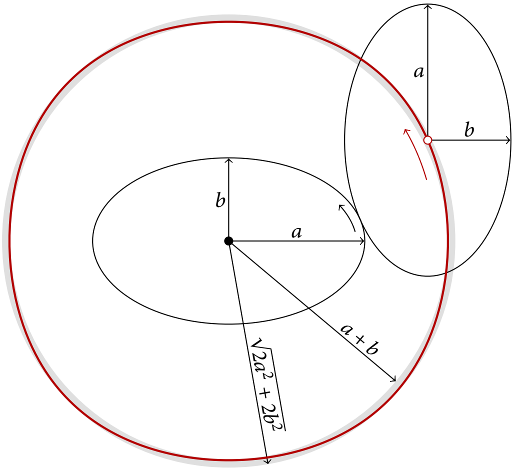

FIGURE 9.9
The path traced by the centre of one ellipse moving around a congruent ellipse, and the circles that inscribe and circumscribe this path.

> can take quantities as great or as small as one needs so that the error be less than the given error. Thus it differs only from Archimedes' style in its expressions which in our method are more direct and more true to the art of discovery.'[1]

Leibniz here said that a method's *novelty* increases its beauty. Taken literally, this means that a method would lose beauty over time, reverting to whatever base or intrinsic beauty it had. But this seems to be contradicted by his view that Viète's method for solving equations in rational numbers was very beautiful.[2] Leibniz wrote this in 1675, but Viète's method was published in his *Introduction to the Analytic Art* in 1591, and thus its novelty must surely have worn off. Perhaps it is best to interpret Leibniz as simply meaning that the new method was more beautiful than the old.

If this is the case, then Leibniz's conclusion that the new method differs only from Archimedes' in its 'expressions' (presumably, notation and text) should probably be interpreted only with regard to rigour, not with regard to beauty, for it seems unlikely that the greater beauty of the new method depends solely on its textual presentation.

Leibniz also thought beautiful[3] a method that Johann I Bernoulli had discussed in correspondence[4] for comparing elliptical and circular arcs. If an ellipse is moved around another congruent but orthogonally placed ellipse, maintaining their relative orientation, then the centre of the moving ellipse traces out a path whose length is twice that of the length of the original ellipse. This path lies between two circles of similar radii, facilitating the comparison of arcs of circles and ellipses (see Figure 9.9).[5]

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[1] Leibniz, 'Mémoire de Mr. G. G. Leibniz Touchant son Sentiment sur le Calcul Différentiel', trans. C.

[2] Leibniz, 'De resolutionibus aequationum cubicarum triradicalium.', p. 143.

[3] Leibniz, letter to Joh. Bernoulli, dated 1 Feb. 1707, repr. in Leibniz, *Mathematische Schriften*, vol. 3.2, p. 811.

[4] Joh. Bernoulli, letter to Leibniz, dated 15 Jan. 1707, repr. in *ibid.*, vol. 3.2, pp. 803–10.

[5] Blåsjö, *Transcendental Curves*, p. 125.

Leibniz wrote that his discovery of the formula (L) via the transmutation theorem was 'a completely general and beautiful method'.[1] This assessment, together with evidence from his *Nachlass*, shows that, for Leibniz, generality implied beauty,[2] which fits with his view of 'unity in plurality' being harmony and thus giving rise to beauty, for *ceteris paribus* a general theorem or method would bring great plurality into unity.

Symmetrical reasoning both has greater harmony and thus beauty, but also pays dividends in accuracy, for it can make it easier to spot errors via violation of the symmetry.[3]

But for Leibniz the beauty of a method did not mean that it was simple or easy: he described a method he recalled hearing about for summing series as 'quite difficult and beautiful [satis difficilis et pulchra]'.[4]

◆

Using binary ('dyadic') notation reveals a beautiful order and harmony and more beautiful properties of numbers. The dyadic system is 'the most beautiful symbol of the continuous creation of things from nothing, and of their dependence on God',[5] so there is nothing more beautiful and suitable to lead us to God.[6]

◆

Leibniz's search for mathematical beauty was one of the motivations for his *analysis situs* programme, which involved both mathematical and philosophical study of the foundations and practice of geometry.[7] Such a programme was necessary because algebra itself did not always yield the shortest proofs or most beautiful constructions,[8] though sometimes it did lead to greater elegance.[9] He hoped that his programme would yield beautiful constructions as a result of the kind of calculations analytic geometry and the infinitesimal calculus had made available.[10]

◆

There are points where Leibniz's aesthetic standards might seem to diverge from modern ones. He once set up a system of three equations of motion, one for each dimension, specifically motivated by beauty and harmony, even though only two were necessary ['pour la beauté et pour l'harmonie, neantmoins deux en pourroient suffire pour la necessité'].[11] A reader holding to modern notions of simplicity would doubtless balk at the redundancy.

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[1] Leibniz, *De quadratura arithmetica*, p. 90; trans. Knobloch, 'Generality in Leibniz's mathematics', p. 94.

[2] Knobloch, 'Generality in Leibniz's mathematics', pp. 94–5.

[3] Breger, 'Symmetry in Leibnizean Physics', p. 16; Breger, 'Die mathematisch-physikalische Schönheit bei Leibniz', p. 110.

[4] Leibniz, letter to Oldenburg, dated 30 Mar. 1675, repr. in Leibniz, *Sämtliche Schriften*, vol. III.1, p. 211 [ltr 46]; trans. C.

[5] Leibniz, letter to Schulenburg, dated 29 Mar. 1698, quot. in Leibniz, *Mathematische Schriften*, vol. 7, p. 239; trans. Leibniz, *The Shorter Leibniz Texts*, p. 39.

[6] Breger, 'Die mathematisch-physikalische Schönheit bei Leibniz', p. 110.

[7] De Risi, 'Analysis Situs'.

[8] Leibniz, letter to Huygens, dated 8 Sep. 1679, repr. in Leibniz, *Philosophical Papers and Letters*, pp. 248–9.

[9] Leibniz, letter to Tschirnhaus, dated May 1678, repr. in *ibid.*, p. 193.

[10] Breger, 'Die mathematisch-physikalische Schönheit bei Leibniz', p. 111.

[11] Leibniz, 'Essay de Dynamique', p. 228; see Breger, 'Die mathematisch-physikalische Schönheit bei Leibniz', p. 108.

Geometrical calculus

Leibniz was not satisfied with the application of algebra to geometry, because it did not yield proofs that used beautiful geometrical constructions:

> 'I search for almost nothing more in Geometry than the art of finding right away beautiful constructions. I see more and more that Algebra is not the natural way to reach them.'[1]

> 'But after all the progress that I have made in these matters, I am not yet happy with the Algebra, in that it gives neither the shorter routes nor the more beautiful constructions of Geometry.'[2]

Thus Leibniz wanted another calculus — a properly geometrical calculus — that would directly express *situation*.[3] This new calculus, which Leibniz started to develop, would permit rigorous description of geometric configurations and would eliminate any need to appeal to geometric figures.[4] The problem with using algebra in geometry was that the former was the characteristic of *magnitude*, not *situation*. To apply algebra to geometry entailed translating relations of situation to relations of magnitude, requiring one to set up coördinate systems and to use two or three magnitudes to represent a single geometrical point.[5] But his new geometrical calculus would avoid this, and so would help in finding beautiful constructions:

> '*firstly*, I can express perfectly by this calculus the entire nature or definition of the figure (which Algebra can never do, for saying that $x^2 + y^2$ equals a^2 is the equation of a circle, it is necessary to explain by diagram what is x and y, that is to say what are the straight lines, the one perpendicular to the other, and the one starting at the centre, the other at the circumference of the diagram). And I can do this for all figures, for they are all explained by spherics, circular plans, and straight lines that I have made in them. For the points of other curves can be found by straight lines and circles. Now all machines only make certain figures, which I can describe by these characters; and I can explain the changing of the situation that they can make, that is to say their movement. *Secondly*, when one can explain perfectly the definition of something, one can also find all its properties. This characteristic will serve a lot to find beautiful constructions, because the calculus and the construction are found together but I do not say that one can still find by it the most absolutely beautiful.'[6]

Sound and vision

Leibniz held that pleasures that closely approached intellectual pleasure could be found in sight and hearing. These pleas-

[1] Leibniz, letter to Galloys, dated Dec. 1678, repr. in Leibniz, *Mathematische Schriften*, vol. 1, p. 183; trans. C.

[2] Leibniz, letter to Huygens, dated 8 Sep. 1679, repr. in Huygens, *Œuvres Complètes*, vol. 8, p. 216; trans. C.

[3] Leibniz, letter to Huygens, dated 8 Sep. 1679, repr. in *ibid.*, vol. 8, p. 216.

[4] Couturat, *La Logique de Leibniz*, p. 390.

[5] *Ibid.*, pp. 398–9.

[6] Leibniz, letter to Huygens, dated Dec. 1679, repr. in Huygens, *Œuvres Complètes*, vol. 8, pp. 249–50, emphases in orig.; trans. C.

ures were symmetry in sight and music in hearing.[1] In this context, by 'symmetrical' Leibniz probably meant 'well-proportioned' and not any kind of exact mathematical symmetry.[2] But both the beauty of music and beauty in sight were closely connected with beauty in mathematics, being based on an unconscious act of counting:

> 'Music charms us, although its beauty consists only in the agreement of numbers and in the counting, which we do not perceive but which the soul nevertheless continues to carry out, of the beats or vibrations of sounding bodies which coincide at certain intervals. The pleasures which the eye finds in proportions are of the same nature, and those caused by other senses amount to something similar, although we may not be able to explain them so distinctly.'[3]

This statement seems pythagorean, but a new element has appeared: *counting*. The beauty was perceived at least partly through the unconscious mental act of counting, not just in a static contemplation of the numbers.

THE CALCULUS CONTROVERSY

Newton famously wrote: 'If I have seen further it is by standing on ye sholders of Giants.'[4] The quarrel between Newton and Leibniz over priority in the discovery of the differential calculus only developed because preceding giants had already found parts and seen disjointed glimpses, and so the field was ready for a mathematician of Newton's or Leibniz's stature to realize the grand structure.[5] What follows is not a complete account of the dispute, but a mere outline that gives context to occasions on which aesthetic concerns arose.

Starting in late 1664, Newton assimilated and began to surpass the most advanced mathematics of his time. By the middle of 1665, he had grasped that integration and differentiation were mutually inverse and had obtained standard methods for differentiation. By the end of 1666, he knew all the essential techniques of the calculus.[6] Knowledge of his discoveries filtered out to other mathematicians, but remained unpublished.[7]

When Leibniz set himself to master the latest mathematics after his visit to London in 1673,* he maintained a correspondence with Henry Oldenburg (c. 1618–1677), from whom he received intelligence about new scientific developments in England and Scotland.[8] In June and October of 1676, Newton sent two letters containing some of his work to Oldenburg for forwarding to Leibniz.[9]

It was not until June 1677 that Leibniz received what he called the 'truly beautiful [sane pulcherrimis]' second letter, and quickly sent back an acknowledgement with his preliminary thoughts, praising

[1] Leibniz, 'Felicity', § 8a.

[2] Breger, 'Symmetry in Leibnizean Physics', p. 14.

[3] Leibniz, 'Principes de la Nature et de la Grace', § 17; trans. Leibniz, 'The Principles of Nature and of Grace', § 17; cf. Leibniz, letter to Goldbach, dated 17 Apr. 1712, repr. in Leibnitius, 'Epistolæ Tres', p. 438.

[4] Newton, letter to Hooke, dated 5 Feb. 1676, repr. in Newton, *Correspondence*, vol. i, p. 416 [ltr 154].

[5] Boyer, *Hist. Calculus*, pp. 187–8; Hall, *Philosophers at War*, p. 7.

[6] Hall, *Philosophers at War*, pp. 12–15.

[7] *Ibid.*, pp. 16–17.

* ▸ p. 315

[8] Aiton, *Leibniz*, pp. 49–66.

[9] Newton, letter to Oldenburg, dated 24 Oct. 1676, repr. in Newton, *Correspondence*, vol. ii, pp. 20–32, 110–29 [ltrs 165, 188]; trans. 32–41, 130–49.

Newton's results variously as 'truly elegant [elegantia sane]' and 'most beautiful [[p]ulcherrimae]'; noting in particular Newton's 'most beautiful and most useful [pulcherrima et utilissima]' remarks on series and the 'most elegant and least expected [[e]legantissima et minime expectata]' way in which he had deduced the arctangent series.[1]* Newton had mentioned the problem of constructing a curve through any given number of points. Euclid could draw a circle passing through three points;[2] Apollonius had the tools necessary to describe a conic section through five given points, but did not do so explicitly. Newton said that of the general problem: 'it ranks among the most beautiful [pulcherrimis] of all that I could wish to solve';[3] his solution later appeared in the *Principia*.[4] Leibniz in response described the problem as 'very elegant [perelegans]'.[5]

This aesthetic agreement is characteristic of the relationship that existed between Newton and Leibniz at the time: one of cordiality and mutual respect, although tempered by the care not to reveal too much without a fair return.[6] In this second letter, Newton concealed in an anagram the explicit statement of the inverse relationship of differentiation and integration, but openly wrote enough for Leibniz to realize that Newton was familiar with a set of methods similar to his own, independently developed, calculus.[7]

Leibniz published his first, very terse, paper on the calculus in October 1684. It introduced the d■ notation and gave, without proof, some of the rules for differentiation.[8] Johann i Bernoulli accurately characterized the paper as 'an enigma rather than an explication'.[9] In his inaugural oration at the University of Basel, he lamented that Leibniz had deigned initially 'to show to the mathematical community the beauty of his invention only through a fog curtain'.[10]

Leibniz's second paper appeared in 1686, and from here the development of the calculus on a Leibnizian basis was taken up by Johann and Jakob i Bernoulli, Guillaume de l'Hospital (1661–1704), Pierre Varignon (1654–1722), and others.[11]

In 1703, George Cheyne (1671?–1743) published an expository book *On the Inverse Method of Fluxions* [*Fluxionum Methodos Inversa*], in which he declared that all that had been achieved in the calculus were 'repetitions, or not difficult corollaries', of what Newton had published or sent to friends.[12] Johann Bernoulli wrote to Leibniz, seemingly impressed by the book's mathematical content, but quoted and deplored Cheyne's assertion, which reduced him, Leibniz, and others to 'Newton's apes [Newtoni Simiae]'.[13] Leibniz thought Cheyne's book trivial: 'it offers no new beautiful [pulchram] Series, no elegant [elegans] Theorems'.[14]

Cheyne's unsatisfactory account moved Newton to publish some of his own work in *De Quadratura Curvarum* (1704).[15] Leibniz wrote in the *Acta Eruditorum* an unsigned review, which was approving if slightly lukewarm.[16] He referred to himself, perhaps injudiciously, as

[1] Leibniz, letter to Oldenburg, dated 21 Jun. 1677, repr. in Leibniz, *Sämtliche Schriften*, vol. iii.2, p. 167, 173, 175 [ltr 54].

[*] ▸ p. 316

[2] Euclid, *Elements*, Prop. iv.5.

[3] Newton, letter to Oldenburg, dated 24 Oct. 1676, repr. in Newton, *Correspondence*, vol. ii, p. 119 [ltr 188]; trans. p. 137.

[4] Newton, *Principia*, bk iii, Lem. 5.

[5] Leibniz, letter to Oldenburg, dated 21 Jun. 1677, repr. in Leibniz, *Sämtliche Schriften*, vol. iii.2, p. 174 [ltr 54].

[6] Aiton, *Leibniz*, pp. 63–4.

[7] Hall, *Philosophers at War*, pp. 16–17.

[8] Boyer, *Hist. Calculus*, pp. 207–8.

[9] Bernoulli, ms autobio. repr. in Wolf, *Biographien zur Kulturgeschichte der Schweiz*, vol. 2, p. 72; trans. C.

[10] Bernoulli, 'De fato novæ analyseos', p. 24; trans. mod. Roero, 'Leibniz, First Three Papers', p. 53.

[11] Boyer, *Hist. Calculus*, pp. 238 sqq.

[12] Cheyne, *Fluxionum Methodos Inversa*, pp. 59–60; trans. C.

[13] Joh. Bernoulli, letter to Leibniz, dated 29 Sep. 1703, repr. in Leibniz, *Sämtliche Schriften*, vol. iii.9, pp. 344–5 [ltr 108]; trans. C.

[14] Leibniz, letter to Joh. Bernoulli, dated 22 Nov. 1703, repr. in *ibid.*, vol. iii.9, pp. 384 [ltr 124]; trans. C.

[15] Hall, *Philosophers at War*, pp. 135–6.

[16] *Ibid.*, pp. 138–9.

the inventor of the differential calculus, signifying his own particular version.[1] He continued:

> 'Instead of the Leibnizian differences, [...] Mr. Newton employs and has always employed, fluxions [...]. He has made elegant use of these both in his *Mathematical Principles of Nature* and in all publications since'.[2]

It was John Keill (1671–1721) who made what amounted to the first open accusation of plagiarism, in a paper published in the *Philosophical Transactions of the Royal Society*. In this paper, dated 1708 but actually printed in 1710, Keill said that Leibniz had taken Newton's fluxions, changed the name and notation, and published the work as his own.[3] Leibniz protested to the Royal Society, whose president at the time was Newton. To justify himself, Keill sent to Newton the review Leibniz had written of *De Quadratura Curvarum*. Leibniz had probably not intended to express any disparaging sentiment, but Newton interpreted the passage quoted above as an assertion that Leibniz had developed the calculus first, and that Newton had substituted his own fluxions for Leibniz's differences.[4]

So erupted the open feud between Newton and Leibniz. The first and main exchange of fire by the two principals comprised the *Commercium Epistolicum*, a collection of letters with annotations issued by the Royal Society on Newton's behalf in early 1713,[5] and the 'Charta Volans', the 'flying paper' published later the same year, which was Leibniz's counter.[6]

In the 'Charta Volans', Leibniz cited 'the judgement of a leading mathematician most skilled in these matters and free from bias',[7] who was later revealed to be Johann I Bernoulli.[8] He quoted the 'leading mathematician' (with some interpolations, omissions, and paraphrases[9]) disparaging (not entirely fairly) Newton's notation, and in particular the use of *o* to denote the increment in a quantity.[10]

In 1714, Newton drafted a never-published[11] reply to the 'Charta Volans' in which, referring to himself in the third person, he defended the *o* notation on aesthetic grounds:

> 'whereas the great Mathematician represents that Mr Newton uses the Letter *o* in the vulgar manner wch destroys the advantages of the Differential method: he uses it, and has used it ever since the writing of his Analysis, in such a manner as makes his method more beautiful more Geometrical & more advantageous then the Differential, & [...] much more universal.'[12]

He reiterated its aesthetic superiority of the method: it was 'more convenient more elegant & more suitable to Geometry then by the differential notation & as universal.'[13]

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[1] [Leibniz], Review of *Tractatus Duo*, p. 34.

[2] [Leibniz], Review of *Tractatus Duo*, p. 35; trans. Newton, *Correspondence*, vol. V, p. 116.

[3] Hall, *Philosophers at War*, p. 145.

[4] *Ibid.*, pp. 139–41.

[5] *Ibid.*, pp. 179–92.

[6] *Ibid.*, pp. 199–201.

[7] Leibniz, 'Charta Volans', p. 16 [trans. p. 18].

[8] Hall & Tilling, notes to *ibid.*, pp. 20–1, nn. (1), (3).

[9] Hall & Tilling, notes to *ibid.*, pp. 20–1, nn. (4)–(8).

[10] Joh. Bernoulli, letter to Leibniz, dated 7 Jun. 1713, repr. in Leibniz, *Mathematische Schriften*, vol. III.2, p. 911; extr. in Newton, *Correspondence*, vol. VI, p. 2 [trans. p. 4].

[11] Hall & Tilling, notes to Newton, 'Draft response to *Charta Volans*', p. 90, n. (1).

[12] *Ibid.*, p. 85.

[13] *Ibid.*, p. 86.

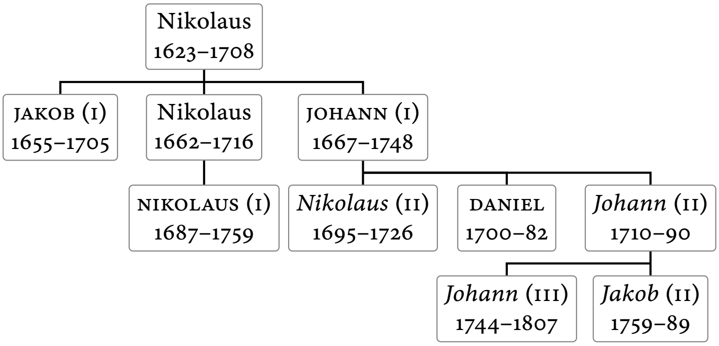

FIGURE 9.10
Part of the Bernoulli family tree, illustrating the descent of the mathematicians in the family. Family members whose names are in small capitals are mentioned in the text. Other mathematicians' names are in italics.

THE BERNOULLIS

The role of the Bernoulli family in mathematics could be compared to that of the Bach family in music: in three generations, the Bernoulli family produced eight mathematicians of the first rank (see Figure 9.10). But whereas the genius of Johann Sebastian overshadows all the other Bachs, at least three of the Bernoullis stand out above the others: Jakob I (1655–1705), his younger brother and student Johann I (1667–1748), both of whom were instrumental in the development of analysis, and Johann's son Daniel (1700–82), who was a pioneer in fluid mechanics.

Here the focus will be on the first-generation mathematical siblings, Jakob I and Johann I. Jakob was the first of the family to show mathematical excellence, and was attracted to the subject to such a degree as to study it against his father's will. He mastered all the latest techniques, including the infinitesimal calculus.[1] When Johann attended university, he secretly studied mathematics under Jakob.[2]

In time, competitiveness between the two brothers developed into bad blood,[3] which was sometimes manifested in aesthetic judgements. Writing to Pierre Varignon (1654–1722), and showing possibly exaggerated humility, Johann said:

> 'I expected from my brother some extraordinarily ingenious method, in which the artifice, the beauty and the singular gracefulness would be such, that mine would dare to appear with it; this is what I dreaded most'.[4]

But Johann disparaged Jakob's work: it was entirely opposite to this expectation: no beauty or gracefulness, just complexity, perplexity, obscurity, and verbosity.[5]

The works and correspondence of Jakob and Johann indicate that they both saw beauty and elegance in a range of mathematics, and in loci including theorems, proofs, properties, methods, problems, and solutions. Both men seem to have preferred terms cognate with *elegant* when writing in Latin and with *beautiful* when writing in French, and it seems likely that they did not distinguish two separate aesthetic categories. It is perhaps noteworthy that Johann, writing

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[1] Hofmann, 'Bernoulli, Jakob (Jacques) I', pp. 46–7.

[2] Fleckenstein, 'Bernoulli, Johann (Jean) I', p. 52.

[3] Hofmann, 'Bernoulli, Jakob (Jacques) I', p. 48.

[4] Joh. Bernoulli, letter to Varignon, dated 5 Jul. 1701, repr. in Bernoulli, *Der Briefwechsel*, vol. 2, p. 302; trans. C.

[5] Joh. Bernoulli, letter to Varignon, dated 5 Jul. 1701, repr. in *ibid.*, vol. 2, p. 302; trans. C.

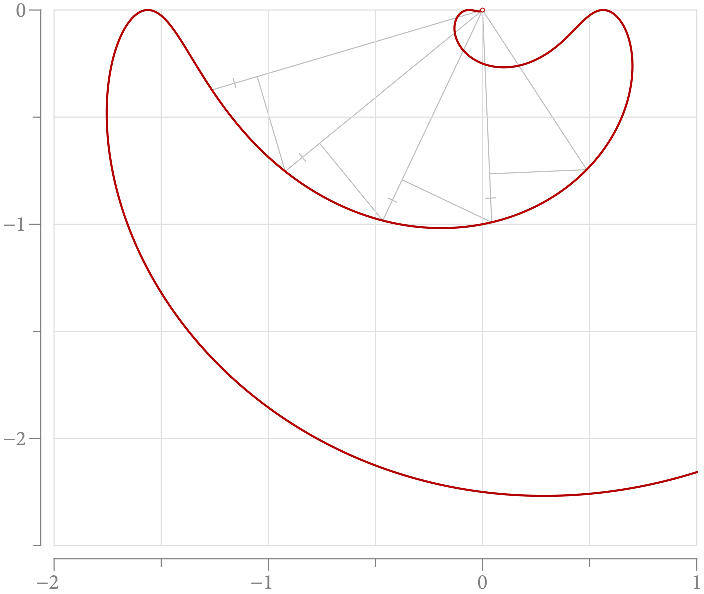

FIGURE 9.11
A paracentric isochrone. A point accelerated by gravity moves along this curve so that its distance from the start increases linearly with time.

in French, reported, with a precise citation, that 'the incomparable Mr. LEIBNIZ' had described Johann's solution to the brachistochrone problem as having 'such a singular beauty [une beauté ſi singulière]',[1] but following the citation shows that Leibniz, writing in Latin, had there called it 'very elegant [perelegans]'.[2]

Jakob I Bernoulli

Like other mathematicians of his time, including his younger brother Johann, Jakob admired the cycloid, that 'most noble [nobilissima] curve'[3] and was pleased to discover 'another very beautiful property'[4] of it, even though it was a rather technical one.

But he also thought highly of the mathematics of other curves. For instance, he described as 'most elegant'[5] the problem, posed by Leibniz,[6] of finding a paracentric isochrone: a curve along which a weight accelerated by gravity will descend so that its distance from the origin increases at a constant rate (see Figure 9.11). And he praised l'Hospital's 'full and elegant [plene & eleganter] solution' to the problem, proposed by Johann, of constructing the shortest path on a conoid between two given points.[7]

Jakob also argued on partly aesthetic grounds for the admissibility of certain methods. For instance, he argued against a limitation Descartes had placed on techniques for constructing equations. In modern terms, constructing an equation $f(x) = 0$ comprises finding two curves $g(x, y) = 0$ and $h(x, y) = 0$, both of certain accepted types and as simple as possible, such that the solutions of the original equation lie among the x-coördinates of the intersections of g and h. For instance, a construction of a cubic equation might lead to a

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[1] Bernoulli, 'Remarques sur […] problems sur les isoperimetres', p. 266; trans. C.

[2] Leibniz, 'Communicatio suae pariter duarumque', p. 334; trans. C.

[3] Bernoulli, 'Oratio de Historia Cycloidis', p. 265.

[4] Bernoulli, 'Sur le Probléme des Isoperimetres', p. 816; trans. C.

[5] Bernoulli, 'Solutio problematis Leibnitiani', p. 601.

[6] Leibniz, 'De linea isochrona', p. 237.

[7] Bernoulli, 'Solutio sex problematum fraternorum', p. 801.

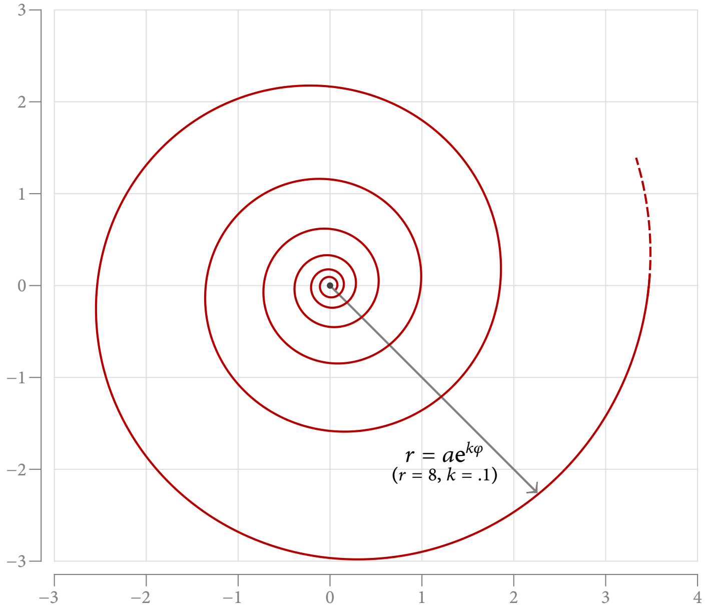

FIGURE 9.12
Part of a logarithmic spiral.

parabola g and a circle h.[1] It seems that Descartes thought — or at least seventeenth-century mathematicians interpreted him as having thought — that in constructing higher-order equations, one of the curves should be a circle.[2] Jakob thought that Descartes had misstepped in imposing this restriction, for it excluded more aesthetically pleasing possibilities:

> ‘Since therefore these constructions, so elegant [elegantes] and so easy, are nevertheless eliminated from Geometry by DESCARTES, it is certainly clear that even the method that he prescribes for constructing equations beyond the quintic should rather be rejected’.[3]

Jakob admired greatly the properties of the logarithmic spiral (see Figure 9.12) and declared the pleasure he took in contemplating it: ‘on account of its so singular and so astonishing properties, this marvellous spiral pleases me so highly, that I can hardly satisfy my desire to contemplate it’.[4] Jakob’s admiration was such that he declared that he would be happy to have his grave marked with the logarithmic spiral, in conscious imitation of the cylinder and sphere on Archimedes’ tomb:[*] ‘I gladly would will this spiral upon my tomb with the inscription *Eadem numero mutata resurget*’.[5] The Latin phrase loses something in the translation, but could be rendered as ‘Though much changed; it will arise again the same’. It refers to the fact that a logarithmic spiral, if scaled, is still congruent to itself under a rotation. In the end, Jakob’s epitaph in Basel Minster bore *Eadem mutata resurgo* (see

[1] Bos, ‘Arguments on Motivation’, pp. 343–4.

[2] *Ibid.*, p. 345.

[3] Bernoulli, ‘Animadversio in geometriam Cartesianam & Constructio quorundam Problematum Hypersolidorum’, p. 351; trans. C.

[4] Bernoulli, ‘Lineæ Cycloidales’, p. 502; trans. C.

[*] ▸ p. 106

[5] *Ibid.*, p. 502; trans. C.

Figure 9.13), which points more closely to the spiral being a metaphor for the Christian doctrine of resurrection:[1] 'Though changed; I arise again the same'. But the spiral that accompanied it was, through some mistake, made archimedean rather than logarithmic.

Johann i Bernoulli

In a draft autobiography, Johann recalled the aesthetic impulse in his early mathematical studies: 'the singular pleasure that I felt in this beautiful and divine science made me progress with an incredible rapidity'.[2]

Johann saw beauty and elegance in subjects from geometry to algebra to what is now known as combinatorics. A few illustrative examples will suffice. He thought that one of the most beautiful properties of Pascal's triangle was that its rows give the binomial coefficients.[3] In a discussion of the division of angles in a letter to Guillaume de L'Hospital (1661–1704), Johann gave an example of a geometrical lemma that he considered 'very beautiful and very useful'[4] in such work: *If a circular arc between A and C is divided into any number of equal parts by the chords* $AB_1, AB_2, \ldots, AB_n$, *then* $AB_1 : AB_2 :: AB_k : (AB_{k-1} + AB_{k+1})$ *for* $k = 2, \ldots, n-1$. Finally, he seems to have found the whole theory of the calculus to be beautiful.[5]

Johann commented on a series that appeared in Pierre Rémond de Montmort's (1678–1719) book *Essay on the Analysis of Games of Chance* [*Essay d'Analyse sur les Jeux de Hazard*], where it was given in the form[6]

$$\left.\begin{aligned}&\frac{1}{1}-\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}-\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\\&\qquad-\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\&c.\end{aligned}\right\}\quad\text{(D)}$$

Bernoulli called the series 'very beautiful and very curious [tres belle & tres curieuſe]'.[7] In Montmort's book, the series appeared in a discussion of probabilities involved in the game of chance called 'Treize' or 'Rencontre'. In this game, two players have cards labelled $1, 2, \ldots, n$. At each turn, each player selects one of their own cards. If at any point the cards they select have the same label, the first player wins. If they never select cards with the same label, the second player wins. The question is to find the probabilities of winning for each player. Montmort gave the solution without proof, while the first proofs were due to Nikolaus i Bernoulli (1687–1759).[8]

Later, Leonhard Euler, unaware of Montmort's work, told Pierre-Louis Maupertuis (1698–1759) that finding this probability was a 'beautiful problem' and that he had been unable to rest until he had solved it.[9]

In modern terms, the probability of the second player winning is the proportion of the permutations of an n-set that are derangements

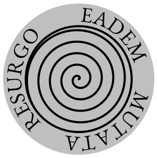

FIGURE 9.13
The spiral on Jakob's epitaph.

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[1] Cf. Fontenelle, 'Éloge de Bernoulli', p. 120.

[2] Bernoulli, MS autobio. repr. in Wolf, *Biographien zur Kulturgeschichte der Schweiz*, vol. 2, p. 72; trans. C.

[3] Bernoulli, letter to de Montmort, dated 17 May 1710, repr. in Bernoulli, *Opera Omnia*, p. 460.

[4] Bernoulli, letter to l'Hôpital, dated 22 Jan. 1701, repr. in Bernoulli, *Der Briefwechsel*, vol. 1 , p. 371; trans. C.

[5] Bernoulli, 'De fato novæ analyseos', p. 24.

[6] Montmort, *Essay d'Analyse sur les Jeux de Hazard*, p. 59.

[7] Joh. Bernoulli, letter to Montmort, dated 17 May 1710, repr. in; trans. C. Bernoulli, *Opera Omnia*, vol. 1, p. 459.

[8] Hald, *Hist. Probability and Statistics*, §§ 19.2–3.

[9] Euler, letter to Maupertuis, dated 19 Aug. 1747, repr. in Euler, *Opera Omnia*, vol. IV.A.6, p. 203 [ltr 98]; trans. C.

(permutations that leave fixed no element). The limit as n tends to infinity of this proportion is the sum of the series (D), namely $1 - e^{-1}$.

* * *

Jean-Étienne Montucla* (1725–99) credited Johann with having had a part in the solution of the most beautiful problems that concerned mathematicians of the time and of having proposed several himself.[1] The most beautiful and influential instance of his involvement was certainly the brachistochrone problem: to find the path between two points along which a body starting from rest at the higher point and freely accelerated by uniform gravity would descend in minimum time to the lower. Johann challenged other mathematicians to solve this problem in 1696,[2] declaring that it was 'a problem whose usefulness linked with beauty will be seen by all who successfully apply themselves to it'.[3] L'Hospital concurred, saying that it seemed to him very beautiful.[4] Leibniz also agreed, writing to Johann that

> 'The problem is surely the most beautiful, and it draws me reluctantly and resistingly to it by its beauty, like the apple did Eve. For it is a grave and harmful temptation to me, impaired in strength and burdened by a mass of other things; so that I do not readily dare more things which require more intense labour of meditation.'[5]

Leibniz reiterated that it was 'exceptionally beautiful [auß der maßen schohn]' to his correspondent Rudolf Christian von Bodenhausen (1640–98).[6] Leibniz was so taken by the problem that he solved it immediately,[7] though his solution was limited to giving a construction of the curve via the quadrature of the circle. Bernoulli considered that Leibniz, in the course of solving the problem, had discovered a 'very beautiful property' of the brachistochrone: the horizontal coördinates of the curve have the same proportions as the corresponding segments of what one retrospectively recognizes as the generating circle of the cycloid.[8]

Leibniz preferred the solution offered by Johann himself, who had 'progressed further and beautifully found it to be the same as the cycloid'.[9]

Johann expected his readers to be 'petrified with astonishment'[10] by the fact that the cycloid, the tautochrone,† and the brachistochrone coincide (see Figure 9.14). Bernoulli felt this surprise himself, and the aesthetic satisfaction the result gave him is evident in his interpretation of the result as evidence of some deep design in nature:

> 'Before I conclude, I cannot refrain from again expressing the amazement which I experienced over the unexpected identity of *Huygens's tautochrone* and our *brachistochrone*. Furthermore I think it is noteworthy that this identity is found only under the hypothesis of Galileo so that even from this we may

---

* ▸ pp. 346–51

[1] Montucla, *Histoire*, vol. 2, p. 356.

[2] Bernoulli, 'On the Brachistochrone Problem', pp. 645 sqq.

[3] *Ibid.*, p. 646.

[4] l'Hôpital, letter to Joh. Bernoulli, dated 30 Nov. 1696, repr. in Bernoulli, *Der Briefwechsel*, vol. 1, p. 325.

[5] Leibniz, letter to Joh. Bernoulli, dated 26 Jun. 1696, repr. in Leibniz, *Sämtliche Schriften*, vol. III.6, p. 799; trans. Blåsjö, *Transcendental Curves*, p. 184.

[6] Leibniz, letter to von Bodenhausen, dated 28 Jun. 1696, repr. in Leibniz, *Sämtliche Schriften*, vol. III.6, p. 805.

[7] Bernoulli, 'On the Brachistochrone Problem', p. 649.

[8] Bernoulli, 'Sur le Probléme des Isoperimetres', p. 198; trans. C.

[9] Leibniz, letter to Joh. Bernoulli, dated 10 Aug. 1696, repr. in Leibniz, *Sämtliche Schriften*, vol. III.7, p. 71; trans. Blåsjö, *Transcendental Curves*, p. 185.

[10] Bernoulli, 'On the Brachistochrone Problem', p. 649.

† ▸ p. 309

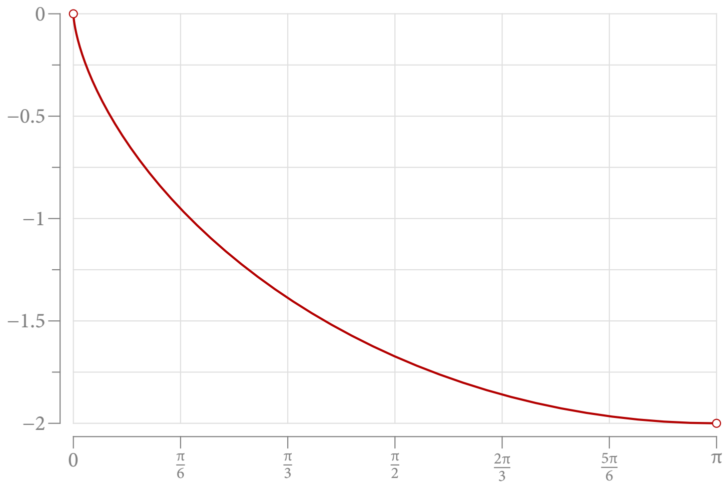

FIGURE 9.14
The solution to the brachistochrone problem; a segment of the cycloid.

> conjecture that nature wanted it to be thus. For, as nature is accustomed to proceed always in the simplest fashion, so here she accomplishes two different services through one and the same curve, while under every other hypothesis two curves would be necessary the one for oscillations of equal duration the other for quickest descent. If, for example, the velocity of a falling body varied not as the square root but as the cube root of the height [fallen through], then the *brachistochrone* would be algebraic, the *tautochrone* on the other hand transcendental; but if the velocity varied as the height [fallen through] then the curves would be algebraic, the one a circle, the other a straight line.'[1]

Johann said that he found both indirect and direct solutions.[2] In the indirect solution, he imagined a point descending along some frictionless path accelerated by uniform gravity. It is straightforward to see that its velocity v is given by $v = \sqrt{2gy}$, where y is the vertical distance it has descended. Then consider the point descending through thin layers, indicated by the horizontal lines in the figure; the velocity depends on the layer the point crosses. This brings to mind light passing through layers of material of different densities, so that in passing through the layer of depth y, its velocity is $v_y = \sqrt{2gy}$ and it travels at some angle α_y (see Figure 9.15). Light always travels along the fastest course between two points, so its path is the brachistochrone. But its bending in passing between layers is governed by Snell's law, so that $(\sin \alpha_y)/v_y$ is a constant. Finally, considering the tangent to the curve gives $\tan(\pi/4 - \alpha) = y'$. Putting all these equations together yields a differential equation whose solution is the cycloid.[3] For Johann, this indirect solution to the brachistochrone problem showed 'a marvellous identity' and 'a beautiful affinity' between the curve of motion and the beam of light.[4]

[1] Bernoulli, 'On the Brachistochrone Problem', p. 654, insertion in ed.; see also La Paz, notes to Bernoulli, 'On the Brachistochrone Problem', p. 654, n. 2, regarding the admissibility of Bernoulli's alternative hypotheses.

[2] Bernoulli, 'On the Brachistochrone Problem', pp. 649–50.

[3] Bernoulli, 'On the Brachistochrone Problem', pp. 650–4; the account here also draws on Polya, *Mathematics and Plausible Reasoning*, [vol. I] § IX.4.

[4] Joh. Bernoulli, letter to l'Hôpital, dated 30 Mar. 1697, repr. in Bernoulli, *Der Briefwechsel*, vol. 1, p. 347; trans. C.

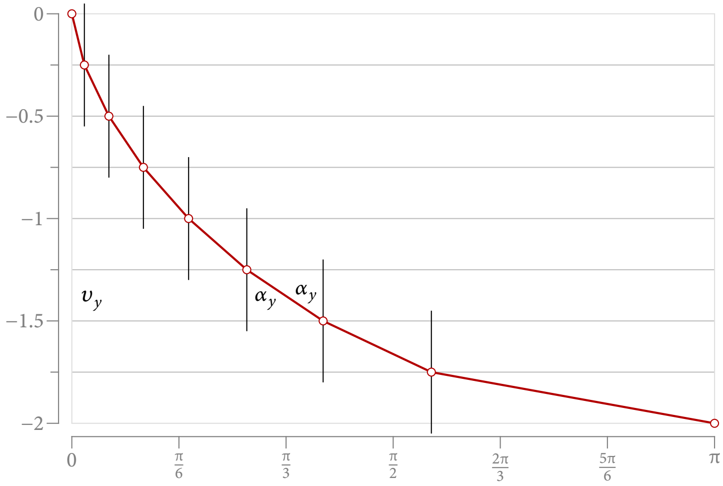

FIGURE 9.15 Johann Bernoulli's method for solving the brachistochrone problem using an optical reinterpretation: a beam of light travelling between the same two points, passing through layers of different densities so that its velocity parallels that of the descending point.

Johann's direct solution also yielded what he called a 'very beautiful'[1] synthetic geometric demonstration that the cycloid is the brachistochrone. But he did not publish it until 1718:[2] Leibniz had found the solution so beautiful and extraordinary that he counselled against publication,[3] with the aim of 'so frustrating those who are not very grateful and who are accustomed to profiting from the inventions of others and claiming the glory of them'.[4] This was a singular episode whose import must be emphasized: here, mathematical beauty contributed to the *suppression*, albeit temporary, of mathematical knowledge.

◆

In his eulogy for Johann, Jean le Rond d'Alembert (1717–83) defended mathematics as an aesthetic pursuit, independent of any potential applications:

> 'We will doubtless be asked the goal and usefulness of all this sublime research. [...] But doesn't geometry have a real beauty in itself, independent of any true or pretended utility?'[5]

He pointed out that Archimedes was better known for his work on parabolae and spirals than on moving bodies and levers, and that Newton and Descartes are immortal for their purely theoretical contributions, but more practical inventors are forgotten.[6] Thus 'respectable men', which presumably included d'Alembert and his audience, should defend even the most abstract mathematical researches from 'idle people, frivolous by nature, and incapable of appreciating them'.[7]

[1] Joh. Bernoulli, letter to Varignon, dated 27 Jul. 1697, repr. in Bernoulli, *Der Briefwechsel*, vol. 2, p. 117; trans. C.

[2] Bernoulli, 'Remarques sur [...] problems sur les isoperimetres', p. 266.

[3] *Loc. cit.*

[4] Joh. Bernoulli, letter to Varignon, dated 27 Jul. 1697, repr. in Bernoulli, *Der Briefwechsel*, vol. 2, p. 117; trans. C.

[5] d'Alembert, 'Éloge de Bernoulli', p. 353–4.

[6] *Ibid.*, p. 354.

[7] *Ibid.*, p. 354; trans. C.

LEONHARD EULER

> ‘ I marvel at the number of beautiful works that you bring out. ’
>
> — Alexis Clairault, letter to Euler, dated 12 Apr. 1741, repr. in Euler, *Opera Omnia*, vol. IV.A.5, p. 90 (trans. C).

Leonhard Euler (1707–83) was easily the most prolific mathematician in history. He is perhaps the only figure who might legitimately be placed alongside Archimedes, Newton, and Gauss.[1] He was not destined for mathematics: his father intended him to become a pastor, but he had been drawn to mathematics while attending the University of Basel. Johann I Bernoulli recognized his ability and helped persuade the elder Euler to allow his son to pursue mathematics.[2] His work was not limited to pure mathematics, but ranged over physics, astronomy, and mechanics. The publication of his *Opera Omnia* started in 1911 and continues still.

Euler frequently used aesthetic terms to describe the mathematical work of himself and others, both in scholarly papers and in personal correspondence. An entire book, or at least a lengthy chapter, would be required for a comprehensive survey, so a few highlights must suffice here.

Results

The kind of results that Euler appreciated aesthetically ranged from visualizable theorems of solid geometry to abstract propositions of analysis to practical rules of algebra. For instance, in the second volume of his *Introductio in Analysin Infinitorum* (1748) he found the following result an ‘elegant theorem’:[3]

THEOREM C. *If any cylinder* [*with an elliptical base*] *is cut by any plane, then the product of the principal axes of the section is to the product of the principal axes of the base of the cylinder as* 1 *is to the cosine of the angle made by the plane of the section with the plane of the base.* (See Figure 9.16.) $\boxed{\text{C}}$

Around the same time, he wrote that a ‘beautiful and important consequence’[4] of the Fundamental Theorem of Algebra was the following: *For real polynomials* $P(x)$ *and* $Q(x)$*, the existence of indefinite integrals of the form* $\int P(x)/Q(x)\,\mathrm{d}x$ *as a combination of polynomials, logarithms, and inverse tangents.* (The Fundamental Theorem ensures that $Q(x)$ can be factored into linear or quadratic real polynomials; partial fraction techniques can then be used to break up the integral.)

Slightly later, he complimented Gabriel Cramer (1704–52) on the beauty of what is now known as Cramer's rule for the solution of a system of n linear equations with n unknowns.[5]

[1] Kline, *Mathematical Thought*, vol. 2, p. 401.

[2] Calinger, *Leonhard Euler*, pp. 25–6.

[3] Euler, *Introductio in Analysin Infinitorum*, p. 357; trans. C.

[4] Euler, ‘Recherches sur les Racines Imaginaires des Equations’, p. 107; trans. C.

[5] Euler, letter to Cramer, dated 15 Oct. 1750, repr. in Euler, *Opera Omnia*, vol. IV.A.7, p. 246 [ltr 17].

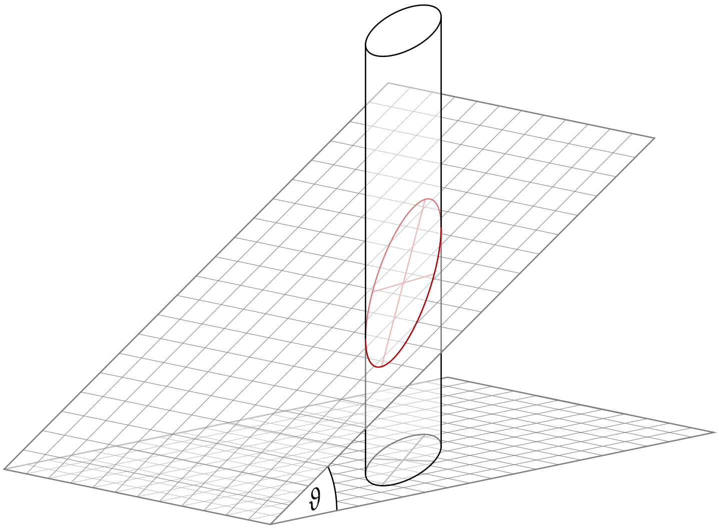

FIGURE 9.16
When an elliptical cylinder is cut by a plane at an angle ϑ to its base, the ratio of the product of the axes of the section is to the product of the axes at its base is 1 : cos ϑ.

Number theory

Aesthetic concerns certainly influenced the direction of Euler's mathematical career, and in particular seem to have been part of what eventually attracted him to number theory. After he completed his university studies, he was appointed to the Saint Petersburg Academy of Sciences, where he formed a friendship with Christian Goldbach (1690–1764). Goldbach was appointed tutor to Tsar Peter II, and when the Russian court moved to Moscow, Goldbach followed. He and Euler then began and maintained a correspondence, which contains numerous references to the beauty and elegance of results, formulae, proofs, and properties.

In a postscript to a December 1729 letter, Goldbach asked Euler if he knew of Fermat's conjecture that all numbers of the form $2^{2^n} + 1$ were prime.[1]* Goldbach had already tried unsuccessfully to interest Daniel Bernoulli (1700–82) in number theory, and had turned to trying to persuade Euler of its worth.[2] It seems that Euler had at that time not deeply engaged with Fermat's work,[3] and his initial response was unenthusiastic. He admitted that he could not prove Fermat's conjecture and seemed to doubt its significance.[4] But Goldbach persisted and Euler began making a serious study of Fermat's mathematics. His attention was caught by the result *Every natural number can be expressed as a sum of four squares.* With presumably deliberate understatement, he described it as a 'not inelegant theorem'.[5] Euler was thus lured — at least in part — by the aesthetic appeal of number theory.[6]

The 'not inelegant theorem' was actually then still an unproven conjecture. Euler was unable to find a proof, though he did prove results connected with the representation of numbers as sums of four squares, some of which he also found beautiful.[7] The 'theorem'

[1] Goldbach, letter to Euler, dated 1 Dec. 1729, repr. in Euler, *Opera Omnia*, vol. IV.A.4.I, p. 104 [ltr 2; trans. vol. IV.A.4.II, p. 590].

* ▸ p. 302

[2] Suzuki, 'Euler and Number Theory', p. 367.

[3] Calinger, *Leonhard Euler*, p. 80.

[4] Euler, letter to Goldbach, dated 19 Jan. 1730, repr. in Euler, *Opera Omnia*, vol. IV.A.4.I, p. 110 [ltr 3; trans. vol. IV.A.4.II, p. 599]; cf. Suzuki, 'Euler and Number Theory', p. 368.

[5] Euler, letter to Goldbach, dated 15 Jun. 1730, repr. in Euler, *Opera Omnia*, vol. IV.A.4.I, p. 113 [ltr 6, trans. vol. IV.A.4.II, p. 603]; trans. C.

[6] Calinger, *Leonhard Euler*, p. 78.

[7] See, for example, Euler, letter to Goldbach, dated 9 Jun. 1750, repr. in Euler, *Opera Omnia*, vol. IV.A.4.I, p. 466 [ltr 144, trans. vol. IV.A.4.II, p. 1014].

remained unproven until 1770, when Joseph-Louis Lagrange* (1736–1813) gave the first proof;[1] it subsequently became known as Lagrange's four-square theorem.

A similar example was Fermat's little theorem, which states that *If* p *is prime and* $x \in \mathbb{N}$ *is such that* $p \nmid x$*, then* $x^{p-1} \equiv 1 \pmod{p}$. In 1731, Euler judged the special case where $x = 2$ a 'not inelegant theorem'.[2] In a paper written in 1732/3, when he was sure that an equivalent form was true but still had not yet proved it, he called it an 'elegant theorem'.[3] Euler finally proved it in 1736 and subsequently gave three more proofs.[4]

Euler thus thought that *unproven* conjectures could have aesthetic value. In particular, he judged Fermat's last theorem, which would remain unproven for another two-and-a-half centuries, to be *very* beautiful:

> 'In Fermat there is another very beautiful [sehr schönes] theorem for which he claims to have found a proof. Indeed [...] the formula $a^n + b^n = c^n$ is impossible whenever $n > 2$. Now I actually have found proofs that $a^3 + b^3 \neq c^3$ and $a^4 + b^4 \neq c^4$ [...]; but the proofs for these two cases are so different from one another that I see no chance of deriving from them a general proof [...]. On the other hand, it can be seen — as it were through a veil but fairly plainly — that the greater n, the more impossible the formula must be. Meanwhile I have not even yet been able to prove that the sum of two fifth powers cannot be a fifth power.'[5]

Results and proofs

Even when a result was proven, it seems that, for Euler, it could be beautiful independently of its proof. In a letter to Goldbach, he wrote that the result *A number* $\sqrt{1 + 16a^2 + 16b^2}$ *cannot be of the form* $4n - 1$ was 'very beautiful', but its proof — which is actually quite short — was not so easy to spot.[6]

A proof could be both 'simple and beautiful', which was his judgement of a demonstration by Goldbach that for $m, n \in \mathbb{N}$ the quantity $(4n - 1)m - 1$ was never a square number.[7]

A theorem with a trivial proof could be beautiful, for that is how he described the theorem *A sum of four squares* $a^2 + b^2 + c^2 + d^2$ *for which* $a + b + c + d = 0$ *can be written as a sum of three squares*, although, as he pointed out, it is 'quite obvious':[8]

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 &= a^2 + b^2 + c^2 + (a + b + c)^2 \\ &= (a + b)^2 + (b + c)^2 + (c + a)^2. \end{aligned}$$

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* ▸ pp. 393–401

[1] Dickson, *Hist. Theory of Numbers*, vol. ii, p. 279.

[2] Euler, letter to Goldbach, dated 6 Dec. 1731, repr. in Euler, *Opera Omnia*, vol. iv.A.4.i, p. 134 [ltr 15, trans. vol. iv.A.4.ii, p. 634]; trans. C.

[3] Euler, 'Observationes de theoremate', p. 3.

[4] Lemmermeyer & Mattmüller, notes to Euler, *Opera Omnia*, vol. iv.A.4.ii, p. 694, n. 3.

[5] Euler, letter to Goldbach, dated 4 Aug. 1753, repr. in *ibid.*, vol. iv.A.4.i, p. 534 [ltr 169, trans. vol. iv.A.4.ii, p. 1091].

[6] Euler, letter to Goldbach, dated 28 Aug. 1742, repr. in *ibid.*, vol. iv.A.4.i, p. 203 [ltr 54, trans. vol. iv.A.4.ii, p. 720].

[7] Euler, letter to Goldbach, dated 15 Oct. 1743, repr. in *ibid.*, vol. iv.A.4.i, p. 285 [ltr 74, trans. vol. iv.A.4.ii, p. 816].

[8] Euler, letter to Goldbach, dated 4 Sep. 1751, repr. in *ibid.*, vol. iv.A.4.i, p. 489 [ltr 154, trans. vol. iv.A.4.ii, p. 1038].

Proofs, too, could be beautiful: Euler wrote to Goldbach that Nikolaus I Bernoulli (1687–1759) had sent him a beautiful proof that if (A) below holds, then so does (B):

$$s = 1 + \frac{a}{n+1} + \frac{a^2}{2n+1} + \frac{a^3}{3n+1} + \dots \tag{A}$$

$$\left.\begin{aligned} \frac{s^2}{2} &= \frac{1}{2} + \frac{a}{n+2}\Big(1 + \frac{1}{n+1}\Big) \\ &\quad + \frac{a^2}{2n+2}\Big(1 + \frac{1}{n+1} + \frac{1}{2n+1}\Big) + \dots . \end{aligned}\right\} \tag{B}$$

The idea of the proof is to substitute x^n for a, put $t = sx$, use (A) to obtain a power series for t, differentiate t with respect to x to obtain a new series of the form $\mathrm{d}t = \mathrm{d}x(\dots)$, multiply with the power series, then integrate, cancel x^2 from both sides, and obtain (B).[1] Finding beauty here suggests that for Euler beauty in proof included neat tricks that rapidly yielded the conclusion.

Elegance and efficiency

Johann I Bernoulli wrote to Euler[2] that Leibniz had once asked him for a shortcut for finding a partial sum of the harmonic series; that is, of calculating

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{x}. \tag{H}$$

Bernoulli said that he had offered a general solution: (H) equals

$$\begin{aligned} x - \frac{x(x-1)}{2\cdot 2} + \frac{x(x-1)(x-2)}{2\cdot 3\cdot 3} - \frac{x(x-1)(x-2)(x-3)}{2\cdot 3\cdot 4\cdot 4} \\ + \frac{x(x-1)(x-2)(x-3)(x-4)}{2\cdot 3\cdot 4\cdot 5\cdot 5} - \dots \pm \frac{1}{x}; \end{aligned}$$

or, more succinctly,

$$\sum_{k=1}^{x} (-1)^{k+1} \frac{1}{k}\binom{x}{k}.$$

In his reply, Euler said that the equality Bernoulli gave was 'very neat and elegant', but was not practical for calculating the sum of a specified number of terms. His own work yielded an easier method, since (H) also equals

$$\left.\begin{aligned} \gamma + \log x + \frac{1}{2x} - \frac{1}{2\cdot 3\cdot 2x^2} + \frac{1}{4\cdot 5\cdot 6x^4} - \frac{1}{6\cdot 7\cdot 6x^6} \\ + \frac{3}{8\cdot 9\cdot 10x^8} - \frac{5}{10\cdot 11\cdot 6x^{10}} + \dots . \end{aligned}\right\} \tag{G}$$

Here γ is Euler's constant:

$$\gamma = \lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{1}{i} - \log n\right) = 0.577\,215\,664\,901\dots,$$

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[1] Euler, letter to Goldbach, dated 7 Jan. 1745, repr. in Euler, *Opera Omnia*, vol. IV.A.4.I, pp. 227–8 [ltr 60, trans. vol. IV.A.4.II, p. 750–1].

[2] Bernoulli, letter to Euler, dated 16 Apr. 1740, repr. in *ibid.*, vol. IV.A.2, p. 348 [ltr 27].

though the notation and name are later. When x is large, the terms in (G) with powers of x in the denominator will be small, which facilitates calculation to high precision.[1]

That Euler found Bernoulli's equality elegant despite its impracticality suggests that the calculatory efficiency of a result was not a prerequisite for it to be elegant.

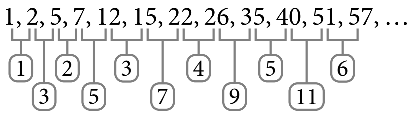
Beauty and surprise

The *sum of divisors* function σ maps a natural number n to the sum of its divisors (including n itself); thus $\sigma(n) = \sum_{d|n} d$. For example, $\sigma(28) = 1 + 2 + 4 + 7 + 14 + 28 = 56$. In 1747, Euler discovered a recurrence relation, which is certainly fascinating and which he found beautiful, involving the sum of divisors function:

$$\left.\begin{aligned} \sigma(n) = \sigma(n-1) &+ \sigma(n-2) - \sigma(n-5) - \sigma(n-7) \\ &+ \sigma(n-12) + \sigma(n-15) - \sigma(n-22) - \sigma(n-26) \\ &+ \sigma(n-25) + \sigma(n-40) - \sigma(n-51) - \sigma(n-57) \\ &+ \ldots; \end{aligned}\right\} \quad \text{(S)}$$

where the relation is determined by the following rules:

1) the signs + and – each appear twice in succession;
2) the sequence of subtrahends $1, 2, 5, 7, 12, 15, \ldots$ is such that their differences alternate between the sequence of natural numbers and the sequence of odd natural numbers:



3) only the terms are taken where the parameter of σ is non-negative;
4) if $\sigma(0)$ appears, its value is taken to be n.

He gave evidence of this result, and showed it to be a consequence of the result later known as 'Euler's pentagonal number theorem'.[2]* At the time, he had no proof of the second result, only discovering one three years later.[3] [Jean le Rond d'Alembert (1717–83) wrote to Euler that the then-unproven pentagonal number theorem seemed to him 'very beautiful'.[4]] The passage where Euler discussed the beauty of the relation (S) is worth quoting in full:

> 'one will be especially more surprised by this beautiful property, seeing no connection between the composition of my formula and the nature of the divisors with the sum of which the proposition is concerned. The sequence of the numbers 1, 2, 5, 7, 12, 15, etc. seems to have no relation to the subject

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[1] Euler, letter to Bernoulli, dated 16 Apr. 1740, repr. in Euler, *Opera Omnia*, vol. IV.A.2, p. 364 [ltr 28].

[2] Euler, letter to Goldbach, dated 1 Apr. 1747, repr. in Euler, *Opera Omnia*, vol. IV.A.4.I, pp. 384–6 [ltr 113, trans. vol. IV.A.4.II, p. 926–8]; see also Euler, 'Decouverte d'une Loi Tout Extraordinaire', pp. 248–9.

* ▸ p. 503

[3] Euler, letter to Goldbach, dated 9 Jun. 1750, repr. in Euler, *Opera Omnia*, vol. IV.A.4.I, pp. 469–70 [ltr 144, trans. vol. IV.A.4.II, p. 1016–18].

[4] d'Alembert, letter to Euler, dated 30 Mar. 1748, repr. in *ibid.*, vol. IV.A.5, p. 286 [ltr 14]; trans. C.

> with which it deals, and, as the law of these numbers is interrupted and they are a mix of two different regular sequences, *viz.* of
>
> $$1, 5, 12, 22, 35, 51, \text{etc. and } 2, 7, 15, 26, 40, 57, \text{etc.},$$
>
> it almost seems that such an irregularity could not be found in analysis. Moreover, the lack of a demonstration must increase the surprise, seeing it seems morally impossible to reach the discovery of such a property without having been guided there by a reliable method that could take the place of a perfect demonstration. I also confess that it was not by pure chance that I tumbled on this discovery, but another proposition of the same nature that must be judged true, though I cannot give a demonstration, opened the road to reach this beautiful property.'[1]

This passage is the only part of the article in which Euler discussed either beauty or surprise, so although he did not make an explicit connection between the two, it at least hints at a link. The beauty of the result was somehow associated to the unexpectedness of there being any such recurrence relation, and that such a relation could even be found. In particular, since $\sigma(n) = n + 1$ if and only if n is prime, the result also connects the primality of a number with the divisors of the lesser numbers that appear in the relation. This was also unexpected, for Euler thought that in the sequence of primes 'there reigns neither order nor rule'.[2]

Beauty, rigour, and correctness

One of the accomplishments that established Euler's reputation was his solution of the Basel problem, which asked for an exact value of the infinite series $\sum_{k=1}^{\infty} 1/k^2$. This question had defeated both Leibniz and Jakob I Bernoulli.[3] Manual computation is difficult because the series converges slowly, but in 1731 Euler had found an alternative series that converged much faster: $\sum_{k=1}^{\infty} 1/(k^2 2^{k-1}) + [\log 2]^2$, which allowed him to compute the value 1.644 934. His real triumph came in 1735:

> 'quite unexpectedly, I have found an elegant formula for the sum of the infinite series $1 + 1/4 + 1/9 + 1/16 + \ldots$, which depends upon the quadrature of the circle [...]. I have found that six times the sum of this series equals the square of the circumference of a circle whose diameter is 1'.[4]

That is, he had shown that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

[1] Euler, 'Decouverte d'une Loi Tout Extraordinaire', p. 248; trans. C.

[2] *Ibid.*, p. 241; trans. C.

[3] Dunham, *Euler*, p. 42.

[4] Euler, 'De summis serierum reciprocarum', pp. 73–4; trans. Calinger, *Leonhard Euler*, p. 119.

The 'elegant formula' was specifically the value of the sum: $\pi^2/6$. Its elegance was presumably connected with the surprise that there should be such a simple closed-form solution. [Evidence suggests that this result is still seen as 'one of the most beautiful results in mathematical analysis';[1] see Tables 22.2 and 22.3 on pages 868–73.] Euler has been criticized for a lack of rigour in his demonstration of this result, but at minimum he seems to have been aware of the potential problems when he first offered an argument for this result, and can plausibly be interpreted as having later supplied a proof whose main flaw was a paucity of detail.[2]

In general, Euler seems to have been sensitive to the aesthetic qualities of results relating to infinite series, for he also mentioned a 'beautiful relation' holding between the following series:

$$1^m - 2^m + 3^m - 4^m + 5^m - 6^m + 7^m - 8^m + \dots$$
$$\frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \frac{1}{7^n} - \frac{1}{8^n} + \dots .$$

The relation is that if the sum of the first series was known for some m, the sum of the second series can 'almost always' be determined for $n = m + 1$. He only proved special cases, but assured the reader that they would find his reasoning rigorous.[3]

Beauty for persuasion and guidance

Finding the correct definition of the logarithm of a negative real number was a problem that pre-dated Euler's work in mathematics. Leibniz thought that it was impossible to define. Johann I Bernoulli thought that $\log -x = \log x$ for $x \in \mathbb{R}^+$. Bernoulli's view was *prima facie* reasonable, since the real logarithm can be defined as the anti-derivative of the hyperbola $y = 1/x$, and there are symmetric branches of the hyperbola for $x > 0$ and $x < 0$. But Euler was not content with some of the paradoxical consequences of this definition.[4] In 1743–4, he completed his *Introductio in Analysin Infinitorum*, which contains only an incomplete description of the logarithm. In the years following, he discovered the modern conception of the complex logarithm as a multi-valued function:[5]

$$z \mapsto \log|z| + \mathrm{i}(\arg z + 2\pi n) \quad \text{for } n \in \mathbb{Z}. \tag{C}$$

In correspondence with Euler, d'Alembert defended the view that $\log -x = \log x$, and never quite abandoned it.[6] Even allowing that the logarithm might take multiple values, he thought that one of the values of $\log -1$ should be 0 (which is the value of $\log 1$ with the real logarithm).[7] In response, Euler made a partly aesthetic argument: he pointed out that if one considered the multiple values of $\log 1$, $\log \mathrm{i}$, $\log -1$, and $\log -\mathrm{i}$ given by (C), one found 'the most beautiful harmony', for adding any two of the values of $\log -1$ yielded one of the values of $\log 1$; similarly, adding any two of the values of $\log \mathrm{i}$

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[1] Maor, *To Infinity and Beyond*, p. 35.

[2] McKinzie & Tuckey, 'Hidden lemmas in Euler's summation'.

[3] Euler, 'Remarques sur un beau rapport'.

[4] Bradley, 'Euler, D'Alembert and the Logarithm Function', pp. 260–1.

[5] *Ibid.*, p. 263.

[6] *Ibid.*, p. 264.

[7] d'Alembert, letter to Euler, dated 26 Apr. 1747, repr. in Euler, *Opera Omnia*, vol. IV.A.5, p. 268 [ltr 8].

yielded one of the values of log 1; and so on, in a clear analogy of the established laws of logarithms of positive real numbers.[1] Euler later observed in a letter to Maupertuis that one could start from some remarks of Bernoulli and reach the 'beautiful' conclusion that $\pi = (\log -1)/\mathrm{i}$.[2]

Although Euler judged unproven conjectures beautiful or elegant, and seems to have been willing to use beauty as a guide, he thought that one should not follow it blindly. In a 1765 paper, he examined the expansion of $(1 + x + x^2)^n$ for $n \in \mathbb{N} \cup \{0\}$ and in particular the coefficient of x^n in the expansion, which is here denoted c_n.[3] He calculated the first terms of the sequence $(c_n)_{n\geqslant 0}$, and related sequences:

$$\begin{aligned} (c_n)_{n\in\mathbb{N}\cup\{0\}} &= 1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, \ldots;\\ (3c_{n-1})_{n\in\mathbb{N}} &= \quad 3\ \ 3\ \ 9\ \ 21\ \ 57\ \ 153\ \ 423\ \ 1179\ \ 3321\ \ \ldots;\\ (3c_{n-1} - c_n)_{n\in\mathbb{N}} &= \quad 2\ \ 0\ \ 2\ \ \ 2\ \ \ 6\ \ \ 12\ \ \ 30\ \ \ \ 72\ \ \ 182\ \ \ldots. \end{aligned}$$

These calculations suggested that $3c_{n-1} - c_n = F_n(F_n + 1)$ for $n \in \mathbb{N}$, where

$$(F_n)_{n\in\mathbb{N}} = \quad 1\ \ 0\ \ 1\ \ \ 1\ \ \ 2\ \ \ \ 3\ \ \ \ 5\ \ \ \ \ 8\ \ \ \ 13\ \ \ldots.$$

(The sequence $(F_n)_{n\in\mathbb{N}}$ is the Fibonacci sequence with initial terms $F_1 = 1$ and $F_2 = 0$, although Euler did not use this terminology.) But while $3c_{n-1} - c_n = F_n(F_n + 1)$ holds for $n \leqslant 10$, Euler saw that it fails for $n = 11$.[4] (Since $c_{11} = 8953$ and $F_{11} = 21$, but $3c_{10} - c_{11} = 464 \neq 21(21 + 1)$.) So the evidence for $n \leqslant 10$ leads to an untrue conjecture, which Euler thought was instructive enough to include it in the paper as 'a memorable example of false induction'.[5] (The term 'induction' here denoted reasoning from particular examples to general rules, not mathematical induction.) He recalled it in a 1783 paper:

> 'This example of illicit induction is all the more noteworthy, because not yet has such a case befallen me, in which such a *beautiful* [speciosa] induction has deceived me.'[6]

The example was 'noteworthy' because it was a rare phenomenon: in Euler's experience, this much inductive evidence, exhibiting such beauty, would normally point to a correct result. Euler had likely relied many times on beautiful inductions to show the way. Here, however, the aesthetic guide was untrustworthy.

[1] Euler, letter to d'Alembert, dated 19 Aug. 1747, repr. in Euler, *Opera Omnia*, vol. IV.A.5, p. 272 [ltr 9]; trans. C.

[2] Euler, letter to Maupertuis, dated 21 Jul. 1750, repr. in *ibid.*, vol. IV.A.6, p. 155 [ltr 69]; trans. C.

[3] Euler, 'Observationes Analyticae', pp. 52 sqq.

[4] *Ibid.*, p. 55.

[5] *Ibid.*, p. 54; trans. C.

[6] Euler, 'Varia Artificia in Serieruivi Indolelvi Inquirendii', p. 386, emphasis added.

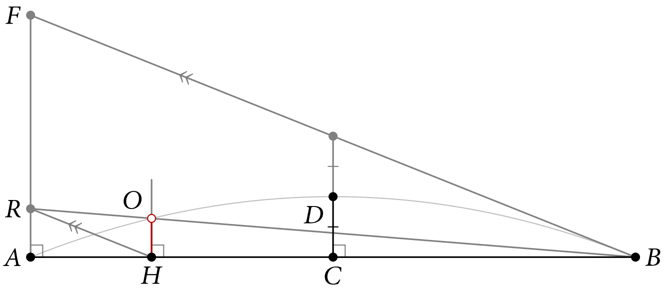

FIGURE 9.17
Dee's method for finding a segment of a diameter terminating on the parabola without having to draw the parabola itself.

ANCIENTS AND MODERNS

> ‘On s'étudiera à trouver dans nos ouvrages des beautés que nous n'avons point prétendu y mettre.’
>
> — Bernard de Fontenelle, ‘Digression sur les Anciens et les Modernes’, p. 251.

Although analytic geometry emerged with the work of Fermat and Descartes, it was not immediately embraced by other mathematicians. This was partly due to the limited circulation of Fermat's *Introduction to Plane and Solid Loci* and the perhaps deliberate obscurities in Descartes's *La Geometrie*. But the new science was also subject to attacks from mathematicians who cleaved to the methods of the Greeks.[1] The defenders of the old ways questioned the appropriateness and legitimacy of using algebraic methods and the manipulation of symbols to solve geometric problems.

A foreshadowing of this disagreement can be found in a 1558 manuscript on the theory of parabolae by the mathematician and occultist John Dee (1527–1608/9). One particular result describes how to determine a segment of a diameter terminating on the parabola, without having to draw the parabola itself; indeed, the construction allows one to find a collection of points on the parabola to draw its approximation. Dee's method starts with a segment AB, perpendicular to the axis of the parabola, and a sagitta CD, where C is on AB and the parabola passes through A, B, and D. Given a point H on AB, the aim is to find the point O on the parabola such that OH is a segment of a diameter. First, produce the segment CD to double its length. Draw the line though B and the terminus of the double, intersecting at F the perpendicular of AB through A. Then draw a line through H parallel to BF, intersecting AF at R. Then O is the intersection of BR and the perpendicular to AB through H (see Figure 9.17). Dee wrote of this method ‘I think this is a beautiful and easy mechanical way, for *we need here no operations with numbers*.’[2]

The debate — or confrontation — proper emerged in the seventeenth century, and peaked in the opposition of Thomas Hobbes and Isaac Barrow, who both favoured geometry over algebra, to Wallis, who enthusiastically adopted the new methods.[3] Hobbes in particular wrote that Wallis ‘mistook the study of *symbols* for the study of *geometry*’.[4]

[1] Boyer, *Hist. Analytic Geometry*, ch. VI.

[2] Dee, *Inventa*, p. 561, emphasis added.

[3] Pycior, *Symbols, Impossible Numbers, and Geometric Entanglements*, ch. VI; Mancosu, *Philosophy of Mathematics and Mathematical Practice*, pp. 86 sqq.

[4] Hobbes, ‘Six Lessons’, 187, emphases in orig.

Ancients and moderns

The debate was primarily foundational, though Leibniz also complained that algebra did not yield beautiful constructions.* In the late seventeenth century the debate expanded to encompass the validity of the calculus. There was little doubt about the power and success of the calculus, but its fluxional or infinitesimal grounding was criticized, most famously and most acutely by George Berkeley (1685–1753) in his 1734 tract 'The Analyst'.[1] In the eighteenth century some mathematicians who did not reject the validity of the calculus nevertheless, like Leibniz, evinced a strong aesthetic preference for the methods of antiquity.

Robert Simson (1687–1768), professor of mathematics at the University of Glasgow, dedicated his mathematical career to publishing the surviving works of Greek geometers and to reconstructing the lost ones.[2] His 1756 edition of Euclid's *Elements* indicates that he thought Euclid's original text to have been perfect, and that any blemishes were due to the errors of editors like Theon.[3] His hypothesized pure Euclidean text seems to have been for him an ideal of mathematical reasoning. He certainly had a strong preference for geometric reasoning over algebraic, and especially opposed mechanical algebraic proofs where the geometrical techniques of antiquity yielded simpler and more elegant solutions.[4] In correspondence with the amateur mathematician and encyclopaedist George Lewis Scott (1708–80), he wrote:

> 'You seem to think, that wherever magnitudes and their ratios are considered, that the algebraical method is better than that of the ancients in respect of facility, though not of accuracy. But what if the construction deduced from the first method, be nothing so good, either in respect of shortness or elegance, as that from the other? which is often the case.'[5]

This preference for geometric methods influenced him beyond the attention he gave to restoring and publishing ancient texts: his treatise *De Limitibus* was an attempt to secure the foundations of the calculus, formulated in a way that would have been accessible to a Greek geometer.[6]

Thomas Simpson (1710–61), who was rated highly by contemporaries as a mathematician and teacher,[7] wrote that when he carried out research into Newtonian physics and other areas, he proceeded by analytical methods, which were best suited for this kind of work. In writing, he gave synthetic geometrical proofs when possible and preferable, for thus 'a problem [...] acquires a degree of perſpicuity and elegance, not easy to be arrived at any other way'. But this aesthetic consideration did not override the fact that there were many situations for which the analytic method was the more suitable — or the only suitable — method. It was through modern analysis that mathematicians outside Britain had gone further than Newton and his successors, and the only complaint one could make was that some of their works lacked the neatness and rigour of ancient geometry.[8]

* ▸ p. 323

[1] Boyer, *Hist. Calculus*, ch. VI.

[2] Sneddon, 'Simson, Robert', p. 446.

[3] Simson, preface to Euclid, *Elements* (ed. Simson), n.p. Sneddon, 'Simson, Robert', p. 446.

[4] Trail, *Life and Writings of Robert Simson*, pp. 63–4, 67.

[5] Simson, letter to Scott, dated 27 Jul. 1764, quot. in *ibid.*, pp. 119–20.

[6] Sneddon, 'Simson, Robert', p. 446.

[7] Wallis, 'Simpson, Thomas', p. 444.

[8] Simpson, *Miscellaneous Tracts*, Preface, last page.

Ancients and moderns

Matthew Stewart (1717–85) entered the University of Glasgow in 1734 and studied under and became friends with Simson and with Francis Hutcheson[*] (1694–1746).[1] Stewart did not show an immediate taste for mathematics, but Simson recognized and directed his ability.[2] Simson passed to his student his interest in reconstructing Euclid's lost book on porisms, to which Stewart developed a new approach to the reconstruction. In 1747 he was elected to the chair of mathematics at Edinburgh on the strength of his book of geometrical propositions *General Theorems*.[3]

One of Stewart's aims was to explain difficult results in the most simple and elegant way, with the particular objective of applying geometry to problems considered soluble only by algebraic and analytical methods.[4] At least in the judgement of his obituarist John Playfair (1748–1819), Stewart's motivations for seeking such geometrical methods were twofold: he saw geometry as being superior in beauty,[5] and he and Simson 'were both apprehensive, that the beauty of their favourite science would be forgotten for the less elegant methods of algebraic computation'.[6]

Playfair seems to have agreed with Stewart's taste in geometry, and characterized the results in *General Theorems* as

> 'among the most beautiful, as well as most general propositions known in the whole compass of Geometry, and are perhaps only equalled by the remarkable *Locus* to the circle in the second book of APOLLONIUS, or by the celebrated theorem of Mr COTES.'[7]

The result in Apollonius' lost *Plane Loci* to which Playfair referred asserts that *The locus of points whose from two given points are in some given ratio forms either a circle or* (*when the ratio is* $1:1$) *a straight line*.[8] (See Figure 9.18.) The 'celebrated theorem' is presumably Roger Cotes's (1682–1716) rectification of the logarithmic curve.[9]

In his biography of Simson, the mathematician William Trail (1746–1831) considered the relation between geometry and algebra. He suggested that most contemporaneous mathematicians had believed that the elegance and clarity of the proofs of Euclid, Archimedes, and Apollonius must have rested upon finding solutions using some algebraic analysis that they had deliberately hidden in order to make their works seem more impressive. But Simson, wrote Trail, had refuted this idea.[10]

For Trail, the preference for geometry over algebra had an aesthetic component. He complained that favouring algebra over geometry 'supplanted a more elegant form of analysis and demonstration'.[11] For similar reasons, he considered Edmond Halley's (1656–1742) reconstructions of book VIII of Apollonius' *Conics* to be superior to other reconstructions of ancient Greek mathematical texts by François Viète[†] (1540–1603), Willebrord Snel (1580–1626), Marino Ghetaldi (1566/8–1626), Pierre de Fermat (1607–65), Franciscus van Schooten (1615–60), and John Wallis (1616–1703), which were 'defective in that

[*] ▸ pp. 357–62

[1] Playfair, 'Account of Matthew Stewart', p. 57; see also Mackay, 'Mathematical Correspondence'.

[2] Playfair, 'Account of Matthew Stewart', p. 57.

[3] Sneddon, 'Stewart, Matthew', p. 54.

[4] Sneddon, 'Stewart, Matthew', p. 54; Playfair, 'Account of Matthew Stewart', pp. 61–2.

[5] Playfair, 'Account of Matthew Stewart', p. 68.

[6] *Ibid.*, p. 75.

[7] *Ibid.*, pp. 59–60, emphases in orig.

[8] See Pappus, *Collection* 7, § 26.

[9] See Gowing, 'A study of spirals'.

[10] Trail, *Life and Writings of Robert Simson*, pp. 12–14.

[11] *Ibid.*, p. 27.

[†] ▸ pp. 272–4

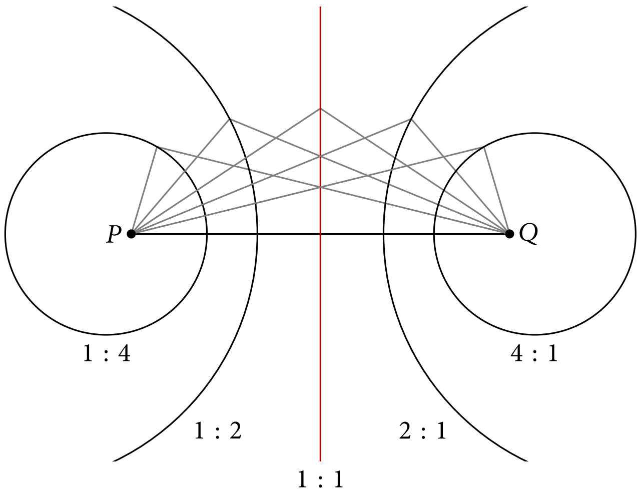

FIGURE 9.18
The theorem of Apollonius that the locus of points whose distances from two given points are in some given ratio forms either a circle or (when the ratio is 1 : 1) a straight line.

elegance and compleatness of solution which distinguish the analytical writings of the ancient Geometers';[1] Halley's was purely geometrical, owing to 'his superior taste and knowledge of the subject'.[2]

But not all writers had aesthetic reasons for favouring geometry over algebra. The mathematician and physicist John Leslie (1766–1832) thought that there were two reasons to study mathematics: exploring 'the beautiful relations of figure and quantity' and training oneself to think carefully and accurately. For the latter purpose, which was the more important for a liberal education, geometry in the mode of the Greeks was the best choice, not because of its beauty, but because it had a more lasting effect on the mind, unlike calculations that might lead to the same result.[3]

Retrospective views

At least one nineteenth-century historian thought that the new methods were superior to geometry: William Whewell (1794–1866). He admired Newton's geometrical methods in the *Principia*, but it was almost the admiration of a mortal for a demigod: he compared Newton's techniques to gigantic weapons from an ancient armoury, such as no modern could wield. But such arms were no longer necessary: the same end could be reached by algebraic manipulations, and he recognized 'the beauty and the power of those methods'.[4]

In his *History of the Inductive Sciences* (first edition 1837), he offered an explanation:

> 'As analysis was more cultivated, it gained a predominancy over geometry; being found to be a far more powerful instrument for obtaining results; and possessing a beauty and an evidence, which, though different from those of geometry, had great attractions for minds to which they became familiar.'[5]

[1] Trail, *Life and Writings of Robert Simson*, p. 17.

[2] *Ibid.*, p. 16.

[3] Leslie, *Elements of Geometry and Plane Trigonometry*, pp. iii–iv.

[4] Whewell, 'In replying to the toast', p. 105.

[5] Whewell, *History of the Inductive Sciences*, vol. 2, p. 72–3 [§ VI.VI.2].

Whewell seemed to suggest that it took time for analysis to be accepted over geometry, in part because the aesthetic criteria of the two fields are different. It was familiarity and success that led to acceptance. Whewell's passing reference here was thus a very early foreshadowing of James McAllister's (1962–) model of the adjustment of aesthetic criteria through aesthetic induction.*

W. W. Rouse Ball (1850–1925) did not indicate any such increased aesthetic acceptance of the new methods: he wrote in *A Short Account of the History of Mathematics* (first edition 1888) that while geometry in the Greek mode could supply elegant proofs of known results, analytic geometry had become important because of its power, simplicity, and success.[1]

JEAN-ÉTIENNE MONTUCLA AND *HISTOIRE DES MATHÉMATIQUES*

Jean-Étienne Montucla's (1725–99) great work *Histoire des Mathématiques* (first edition 1758; new edition 1799–1802) was a milestone in the historiography of mathematics.[2] While there had been earlier histories of mathematics, Montucla's breadth of reading and critical acumen surpassed all previous historians.[3] The range of his work was *encyclopédique*, for he shared the same aim of universality as Denis Diderot (1713–84) and Jean le Rond d'Alembert (1717–83), and held in high regard d'Alembert's 'Discours Préliminaire',[4] which served a manifesto that set forth the aspirations of the *philosophes* of the French Enlightenment. Montucla's history of mathematics maintained its primacy for over a century, eventually yielding to Moritz Cantor's (1829–1920) *Vorlesungen über Geschichte der Mathematik* (1880–1908).[5]

Montucla's first historical work was his *Histoire des Recherches sur la Quadrature du Cercle* (1754), which was well-received. The *Histoire des Mathématiques* itself first appeared in 1758. Its two volumes carried the history of mathematics to the end of the seventeenth century. In the four-volume new edition of 1799–1802, the first two volumes were revisions of the first edition, while the third and fourth volumes — which did not reach the same heights of quality as the original two[6] — covered the eighteenth century. The *Histoire* is not just a history of mathematics itself but rather a history of the mathematical sciences, and arguably a history of science from a mathematical perspective.[7]

Scope and loci of beauty and elegance

Throughout his histories, Montucla made aesthetic judgements of mathematics. His evaluations seem consistent over time: those in the *Histoire des Recherches sur la Quadrature du Cercle* are echoed in the first and second editions of *Histoire de Mathématiques*, though sometimes the expressions of excellence vary between aesthetic and non-aesthetic terms. Perhaps the clearest examples are his judgements on infinite series or products that (in effect, though he did not state them thus) yield π:

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* ▸ pp. 793–5

[1] Rouse Ball, *A Short Account of the History of Mathematics*, p. 218.

[2] Grattan-Guinness, 'Talepiece', p. 1666.

[3] Swerdlow, 'Montucla's Legacy', pp. 302–3.

[4] *Ibid.*, p. 301.

[5] Vogel, 'Montucla, Jean-Étienne', p. 501.

[6] Sarton, 'Montucla', p. 552.

[7] *Ibid.*, p. 536.

Viète's formula. François Viète was the first to give an infinite expression for π.* Montucla put it in the form

$$\frac{\pi}{2}=\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}}}}\cdots \qquad \text{(V)}$$

In both *Histoire des Recherches* and *Histoire des Mathématiques*, he praised Viète's discovery while pointing out that it would be tedious to use (V) to compute a decimal or rational approximation. This impracticality for calculation tempered his admiration: in *Histoire des Recherches*, he wrote that it had some beauty, but admitted that he did not insist on it.[1] In *Histoire des Mathématiques*, his praise was non-aesthetic: though (V) was impractical, it remained 'remarkable at least in theory'.[2]

Wallis's product. John Wallis found another infinite product yielding π, which Moncula gave as

$$\frac{\pi}{4}=\frac{2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdots}{3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdots}. \qquad \text{(W)}$$

In *Histoire des Recherches*, Montucla implicitly indicated that he saw beauty in the product (W) when he compared it to William Brouncker's (1620–84) continued fraction (*q.v.*), and in the *Histoire des Mathématiques* called it 'very elegant'.[3] He stated no caveats about usefulness.

Brouncker's continued fraction. Brouncker found a continued fraction formula, which Montucla presented in the following form:

$$\left.\frac{\pi}{4}=\cfrac{1}{1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\dots}}}}}\quad\right\}\ \text{(B)}$$

In *Histoire des Recherches*, Montucla wrote that (B) 'did not yield in beauty' to Wallis's formula (W).[4] In *Histoire des Mathématiques*, he praised (B) above (W) in possibly non-aesthetic terms, as 'more perfect'.[5]

The convergents to (B) are the partial sums of the Leibniz series (L) for π/4.† Montucla mentioned this series separately

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* ▸ p. 274

[1] Montucla, *Hist. Quadrature*, p. 44.

[2] Montucla, *Histoire*, vol. 1, p. 499; Montucla, *Histoire* New ed., vol. 1, pp. 611–12; trans. C.

[3] Montucla, *Histoire*, vol. 2, p. 302; Montucla, *Histoire* New ed., vol. 2, p. 352.

[4] Montucla, *Hist. Quadrature*, pp. 117–18; trans. C.

[5] Montucla, *Histoire*, vol. 2, p. 305; Montucla, *Histoire* New ed., vol. 2, p. 354.

† ▸ p. 316

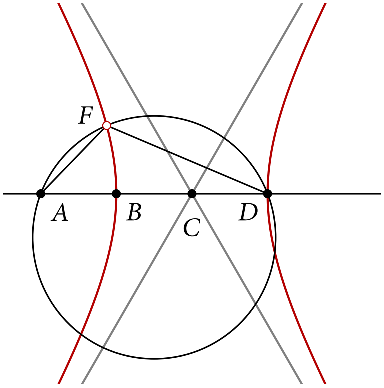

FIGURE 9.19
Trisection of an angle using a hyperbola.

> from Brouncker's continued fraction, and found it remarkably elegant in theory, but lacking in utility because of its slow convergence.[1] Perhaps strangely, he did not state the caveat about usefulness with regard to (B). One might speculate that he found the continued fraction formulation to have greater appeal, which overrode such objections.

There seems to have been a slight distinction between Montucla's uses of *beauty* and *elegance*. Throughout his works, Montucla applied *beautiful* to results, theories, problems, properties, and methods; and *elegant* to results, constructions, methods, solutions, texts,[2] and writing styles.[3] There was clearly overlap in their scope, and in particular he applied both terms to Wallis's product (W). But it seems that, for Montucla, elegance was more closely connected than beauty to explanation or at least to exposition.

But elegance could be grounded in beauty: Montucla recounted what he considered an elegant ancient method of trisecting an angle, depending on what he called a 'beautiful property' of the hyperbola, *viz.* that if the distance from each focus to the corresponding vertex is equal to the semi-axis (so that the hyperbola has eccentricity 2 and equation $x^2/a^2 - y^2/(a^2\sqrt{3}) = 1$ for some a), then the two asymptotes make an angle of $2\pi/3$ and for any point F on the hyperbola $\angle FAB = 2\angle FDB$, where A and B are respectively the focus and vertex of the branch of the hyperbola on which lies F, and D is the vertex of the other branch. It follows that any arc of a circle through A, F, and D will be such that the arc AF will be half of the arc FD. Thus, given an arc to trisect, one can draw the chord AD, divide it appropriately to obtain B and C, draw the asymptotes and so the hyperbola, which will cut the arc in the correct place (see Figure 9.19).[4] [This method of trisection was due to Pappus.[5]] This point must be emphasized: the *elegant* method stands atop the *beautiful* property.

Even if a result was so well-explained as to seem trivial, it could still be beautiful. For instance, Montucla thought that Archimedes' theorem that the volume and surface area of a sphere are two-thirds those of its circumscribing cylinder* had become pedestrian over

[1] Montucla, *Hist. Quadrature*, pp. 155–6.

[2] For example, Montucla, *Histoire*, vol. 2, pp. 376, 452.

[3] For example, *ibid.*, vol. 2, p. 433.

[4] Montucla, *Hist. Quadrature*, pp. 267–8; Montucla, *Histoire*, vol. 1, p. 195.

[5] Pappus, *Collection* 4, Prop. 34, pp. 152–5; see also Pappus, *Collection* 7, Prop. 237, pp. 365–9.

* ▸ pp. 106–8

time, but insisted that it remained beautiful: 'this beautiful geometrical discovery, I say beautiful, although today well-known and almost trivial'.[1]

Montucla seems never to have explicitly called a *proof* beautiful or elegant, only methods, solutions, and constructions. He did at least contrast the elegance of a result with its proof, showing that the two could be separated aesthetically. He gave the example of Albert Girard's (1595–1632) theorem: *The area of a triangle ABC on the surface of a unit sphere is* $A + B + C - \pi$. Montucla called the result 'very elegant' (and it surely is), but complained that Girard had proved it in a 'quite laborious and obscure' fashion.[2]

Synthetic elegance

The ancient methods of synthetic geometry seem to have been a canonical source of elegance for Montucla. He greatly admired Viète's solution to Apollonius' problem of finding a circle tangent to three given circles,[3] calling it 'very elegant'[4] and later describing Viète's (modestly-titled) book *Apollonius Gallus* [*The Gallic Apollonius*] as 'an exquisite model of geometrical elegance'.[5]

In general he thought that the very power of analytic geometry had led to a tendency to take it too far and reduce everything to mere calculation. He detected in the best mathematicians, like Fermat, Newton, and Colin Maclaurin (1698–1746), regrets about the harm thus done to geometrical elegance.[6] He held that anyone who compared the 'elegant solutions' of Apollonius' problem, such as that by Viète, which uses synthetic geometry, to that of Descartes, which uses analytic geometry, would be forced to concede that it would be wrong to scorn the older method.

This caution about analytic geometry did not mean that he had a overmastering preference for synthetic geometry. He wrote that certain late seventeenth century geometric texts had 'a charming elegance *for those who still have a taste for the style of ancient geometry*'.[7] This assessment suggests that, with the advent of analytic geometry in the early seventeenth century and its evident success, Montucla thought that tastes should have changed. He saw these texts, and presumably thus their authors and readers, as hold-outs against the new style.

Elegance and unity

One condition for elegance is discernible in Montucla's examples: unity. That is, a result or method was elegant if it unified hitherto disparate results.

For instance, Montucla recounted John Napier's (1550–1617) rule for a spherical triangle as shown in Figure 9.20 (left), which uses the circular mnemonic device in Figure 9.20 (right). The two parts of rule state that *The sine of any sector is equal to the product of the*

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[1] Montucla, *Histoire*, vol. 1, p. 233; Montucla, *Histoire* New ed., vol. 1, p. 223; trans. C.

[2] Montucla, *Histoire* New ed., vol. 2, p. 8; trans. C.

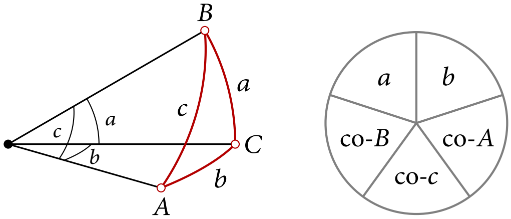
[3] See Heath, *Hist. Greek Math.*, vol. II, pp. 181–2.

[4] Montucla, *Histoire*, vol. 1, p. 263; Montucla, *Histoire* New ed., vol. 1, p. 251; trans. C.

[5] Montucla, *Histoire* New ed., vol. 1, p. 578; trans. C.

[6] Montucla, *Histoire*, vol. 1, p. 176; Montucla, *Histoire* New ed., vol. 1, p. 167.

[7] Montucla, *Histoire* New ed., vol. 2, p. 78, emphasis added.



A spherical triangle

A mnemonic device for Napier's rule

FIGURE 9.20
Illustration of Napier's rule for the fundamental identities of a spherical triangle.

tangents of the two sectors adjacent to it and *The sine of any sector is equal to the product of the cosines of the two sectors opposite to it*, where it is understood that the 'co-' prefix in the lower three sectors of the mnemonic device indicates a switch from sine/tangent to cosine/cotangent or vice versa. This rule subsumed and unified the ten fundamental identities for a spherical triangle:[1]

$$\left.\begin{array}{rr} \sin b = \tan a \cot A & \sin a = \sin A \sin c \\ \cos c = \cot A \cot B & \cos A = \sin B \cos a \\ \sin a = \cot B \tan b & \cos B = \cos b \sin A \\ \cos A = \tan b \cot c & \sin b = \sin c \sin B \\ \cos B = \cot c \tan a & \cos c = \cos a \cos b \end{array}\right\} \quad \text{(S)}$$

Montucla was clearly impressed by how Napier's rule removed the need to consult a table of identities (S), and wrote that it was the rule's *elegance* that made it 'imprint itself deeply in memory'.[2]

That unity was a contributor to beauty is also evident from his account of Pascal's treatise on conics,[3] wherein 'all that *Apollonius* had proved was elegantly deduced from a single general proposition'.[4]

Beauty and utility

In the *Histoire des Recherches sur la Quadrature du Cercle*, Montucla indicated that a result could be beautiful but not useful: he declined to explain results that, 'though beautiful and ingenious in theory, are not markedly useful', preferring to proceed to what he saw as more interesting topics.[5] In context, the term 'useful' ['utilité'] seems to indicate applicability *within* mathematics.

But in *Histoire des Mathématiques*, there are hints that Montucla saw utility or applicability as a factor contributing to beauty. At one point, he wrote that 'if the beauty of a discovery is measured by the sublimity of the objects to which it applies, there is little in mechanics as brilliant as what we are going to relate'.[6] The topic he proceeded to discuss in the section introduced by this quotation was the explanation of the orbits of the planets by Newton,[7] and so the 'sublimity' must presumably have applied to the heavenly bodies and their motions. The sentence is a conditional one, but Montucla clearly accepted the consequence, as shown by the tone of admiration in his subsequent account. What then of the condition? It is reasonable to conclude that Montucla at least partly endorsed it, for otherwise he could simply have stated the consequence ('there is little in mechanics as brilliant as what we are going to relate') and it would be rare to find disagreement. Therefore there seems to have been, for Montucla, at least some connection between beauty and applicability.

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[1] For a modern account, see Van Brummelen, *Heavenly Mathematics*, ch. 5.

[2] Montucla, *Histoire*, vol. 2, p. 15; Montucla, *Histoire* New ed., vol. 2, p. 25.

[3] See Pascal, 'Essay pour les Coniques'; trans. Pascal, 'Essay on Conics'.

[4] Montucla, *Histoire*, vol. 2, p. 53; Montucla, *Histoire* New ed., vol. 2, p. 62, emphasis in orig.

[5] Montucla, *Hist. Quadrature*, 272; trans. C.

[6] Montucla, *Histoire*, vol. 2, p. 410; Montucla, *Histoire* New ed., vol. 2, p. 438.

[7] Montucla, *Histoire*, vol. 2, pt IV, bk VII, § V; Montucla, *Histoire* New ed., vol. 2, pt IV, bk VII, § V.

◆

An Almost Infinite Variety

10

Mathematics and the emergence of modern aesthetics

> ‘In every Part of the World which we call Beautiful,
> there is a surprizing Uniformity
> amidst an almost infinite Variety.’
>
> — Francis Hutcheson,
> *An Inquiry Concerning Beauty, Order &c.*, § II.V.

> ‘Variety of Uniformities makes compleat Beauty’
>
> — Christopher Wren FRS, fragment printed in
> Christopher Wren MP, *Parentalia*, p. 352.

In 1735, Alexander Gottlieb Baumgarten (1714–62) coined the term *aesthetic* from the Greek *aisthētikos* [αἰσθητικός],[1] meaning *of* or *for sense-perception*, *sensitive*, *perceptive*, in his essay *Meditationes Philosophicae de Nonnvllis ad Poema Pertinentibvs*:

> ‘*things known* are to be known by the superior faculty as the object of logic; *things perceived* [are to be known by the inferior faculty, as the object] of the science of perception, or AESTHETIC.’[2]

Aesthetic as the science of perception: this definition is much wider than the later usage that makes aesthetics the philosophy of concepts such as beauty, elegance, sublimity, and taste, and sees it as connected to, overlapping with, or possibly identical to the philosophy of fine art. Certainly Baumgarten's definition encompasses the foundational texts of modern aesthetics that predate the name: some of the treatises in Shaftesbury's (1671–1713) *Characteristicks of Men, Manners, Opinions, Times*, Jean-Pierre de Crousaz's (1663–1750) *Traité du Beau*, Francis Hutcheson's (1694–1746) *An Inquiry Concerning Beauty, Order &c.* Shaftesbury broadly dealt with taste, Crousaz and Hutcheson with beauty. And they and many later authors accepted intellectual beauty and at least acknowledged mathematical beauty. Later, starting from Georg Hegel's (1770–1831) lectures on ‘Philosophy of Art or Aesthetics’, the term *aesthetics* grew to mean just the philosophy of fine art.[3]

This chapter will examine the attitude to mathematical beauty in this early period, up to the work of Immanuel Kant (1724–1804)

[1] Cf. LSJ, ‘αἰσθ-ητικός’.

[2] Baumgarten, *Meditationes Philosophicae*, § CXVI [p. 39 (facsimile)]; trans. Baumgarten, *Reflections on Poetry*, p. 78, insertion in trans.

[3] Guyer, *Hist. Modern Aesthetics*, vol. 1, p. 4.

and a few later thinkers. Some figures who are relatively obscure today in the history of general aesthetics, like Jean-Pierre de Crousaz and Yves-Marie André[1] (1675–1764) were nevertheless important in the development of philosophical accounts of mathematical beauty. Conversely, many prominent writers on aesthetics, like Jean-Jacques Rousseau (1712–78) and Edmund Burke (1729–97), simply did not mention beauty in mathematics and so are not studied here. Some others seem to have explicitly or implicitly rejected it. For instance, Johann Christoph Gottsched (1700–66), who was an important figure in eighteenth-century criticism, explicitly held that taste did not apply to subjects where reason alone applies and where it is possible to make rigorous proofs, such as arithmetic and geometry.[2] Kant, as discussed below, once accepted and later rejected beauty in mathematics.

SHAFTESBURY[3]

The work of Anthony Ashley Cooper, the Third Earl of Shaftesbury (1671–1713), is as good a place as any to mark the beginning of modern aesthetics. Shaftesbury introduced the idea that the experience of the beauty of a thing is *disinterested*,[4] in the sense that it is independent of any desire for possession or use of that thing. This disinterestedness has certainly been a recurring theme in modern aesthetics, and has sometimes been thought to be its defining idea,[5] although it can be found at least as early as the work of Thomas Aquinas (1224/5–74 CE).[6]

The treatises that make up Shaftesbury's *Characteristicks of Men, Manners, Opinions, Times* show him to have been a neoplatonist and in particular that he identified the true, the good, and the beautiful. The most explicit statement of this position[7] is the following passage:[8]

> 'Whoever has any Impression of what we call *Gentility* or *Politeness*, is already so acquainted with the DECORUM and GRACE of things, that he will readily confess a Pleasure and Enjoyment in the very *Survey* and *Contemplation* of this kind. Now if in the way of polite Pleasure, *the Study* and *Love* of BEAUTY be essential; the Study and Love of SYMMETRY and ORDER, on which *Beauty* depends, must also be essential, in the same respect.
>
> […] Even in the imitative or *designing* Arts, […] the *Truth* or *Beauty* of every Figure or Statue is measur'd from the Perfection of Nature, in her just adapting of every Limb and Proportion to the Activity, Strength, Dexterity, Life and Vigor of the particular Species or Animal design'd.'[9]

The true, the good, and the beautiful were ultimately united for Shaftesbury because of the design, harmony, and order that pervaded the world. The world formed a unified whole where all things were related to each other in a system.[1]

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[1] Tsien, *The Bad Taste of Others*, p. 47.

[2] Beiser, *Diotima's Children*, p. 84; Guyer, *Hist. Modern Aesthetics*, vol. 1, p. 313.

[3] In all quotations from Shaftesbury, italics and small capitals are per the edition.

[4] Stolnitz, 'On the Origins of "Aesthetic Disinterestedness"', § 1.

[5] Guyer, *Hist. Modern Aesthetics*, vol. 1, p. 30.

[6] Thomas Aquinas, *Summa theologiae*, pt 1, Qu. 91, Art. 3 ad 3.

[7] Guyer, *Hist. Modern Aesthetics*, vol. 1, p. 39–40.

[8] But see also, for example, Shaftesbury, 'The Moralists', § III.ii ⟦pp. 223, 225⟧.

[9] Shaftesbury, 'Miscellaneous Reflections', ch. III.ii ⟦p. 109–10⟧.

Beauty, however, was not inherent in matter itself. Rather, beauty was impressed upon matter by skill and design: it was ultimately an organizing mind, or what it produced, that was beautiful:

> '"That *the Beautiful*, *the Fair*, *the Comely*, were never in the *Matter*, but in the *Art* and *Design*; never in *Body* it-self, but in the *Form* or *forming Power*." [...] What is it you admire but MIND, or the Effect of *Mind*?'[2]

This distinction follows from the threefold hierarchy of beauty that Shaftesbury envisaged. In the first, lowest, order are '*the dead Forms*, [...] form'd, whether by Man, or Nature; but have no forming Power, no Action, or Intelligence'.[3] In these, the perception of beauty was effected only by the form itself. The second order consists of '*the Forms which form*; that is, which have Intelligence, Action, and Operation'[4] Here the beauty arose both from form and from mind. At the pinnacle stood God, '*Supreme* and *Sovereign Beauty*',[5] the source from which all beauty descended: '*Divinity it-self,* [...] *is surely beauteous, and of all Beautys the brightest;* [...] *from whence the Beauty of Bodys is deriv'd*'.[6] God's *role* was roughly analogous to the Platonic Form of the Beautiful, but, for Shaftesbury, God's organizing and harmonizing *mind* was essential.

Shaftesbury's theory of beauty connected with mathematics in two places. Firstly, the sensed beauty in the world was ultimately dependent on our notion of order and proportion, which meant that numbers possessed a power (whether pythagorean or promethean) to influence the world:

> 'Nothing surely is more strongly imprinted on our Minds, or more closely interwoven with our Souls, than the Idea or Sense of *Order* and *Proportion*. Hence all the Force of *Numbers*, and those powerful *Arts* founded on their Management and Use.'[7]

This idea is a recurring motif in the treatises in the *Characteristicks*. There are repeated references to 'Numbers' in the context of harmony and style in the advice he offered to authors in his 'Soliloquy'.[8] More explicitly, he wrote:

> 'Let Poets, or the Men of Harmony, deny, if they can, this Force of *Nature*, or withstand this *moral Magick*. [...] For in the first place, the very Passion which inspires 'em, is it-self *the Love of Numbers, Decency and Proportion* [...]. For this is the Effect, and this the Beauty of their Art; "in vocal Measures of Syllables, and Sounds, to express the Harmony and Numbers of an inward kind [...]."'[9]

Furthermore, Shaftesbury elevated intellectual pleasures over sensory ones:

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[1] Shaftesbury, 'The Moralists', § II.iv ⟦pp. 161–3⟧.

[2] *Ibid.*, § III.ii ⟦p. 226⟧.

[3] *Ibid.*, § III.ii ⟦p. 227⟧.

[4] *Ibid.*, § III.ii ⟦p. 227⟧.

[5] *Ibid.*, § III.ii ⟦p. 228⟧.

[6] *Ibid.*, § II.iv ⟦p. 166⟧.

[7] *Ibid.*, § II.iv ⟦p. 160⟧.

[8] Shaftesbury, 'Soliloquy', §§ I.iii, II.i, II.ii, III.ii, III.iii ⟦pp. 129, 135, 146–7, 195, 204–7, 218⟧.

[9] Shaftesbury, 'Sensus Communis', § IV.ii ⟦p. 85⟧.

> 'The *Pleasures of the Mind* being allow'd, therefore, superior, to those of *the Body*; it follows, "That whatever can create in any intelligent Being a constant flowing Series or Train of mental Enjoyments, or Pleasures of the Mind, is more considerable to his Happiness, than that which can create to him a like constant Course or Train of sensual Enjoyments, or Pleasures of the Body."
>
> Now the mental Enjoyments are either actually *the very natural Affections themselves in their immediate Operation*: Or they wholly in a manner *proceed from them*, and are no other than *their Effects*.
>
> If so; it follows, that the natural Affections duly establish'd in a rational Creature, being the only means which can procure him a constant Series or Succession of the mental Enjoyments, they are the only means which can procure him a certain and solid *Happiness*.'[1]

This was not to say that all sensory pleasures are assigned to his first order, nor all intellectual pleasures to his second order. Rather, intellectual pleasures could be collectively elevated over sensory ones for the same reason that the second order was elevated over the first: the role of mind was much greater in the former than the latter. This applied in particular to mathematics:

> 'There is no-one who, by the least progress in Science or Learning, has come to know barely the Principles of *Mathematicks*, but has found, that in the exercise of his Mind on the Discoverys he there makes, tho merely of speculative Truths, he receives a Pleasure and Delight superior to that of Sense.'[2]

Shaftesbury did not develop a theory of mathematical beauty. He did not explicitly identify loci of mathematical beauty, although the word 'Discoverys' perhaps points to beauty in theorems and proofs rather than in objects. Nor did he discuss criteria for mathematical beauty beyond the general notions of harmony, proportion, order, and symmetry upon which beauty is based. But he exhibited mathematical beauty as one of the clearest evidences of the disinterestedness of the pleasure obtained from the beautiful:

> 'When we have thorowly search'd into the nature of this contemplative Delight, we shall find it of a kind which relates not in the least to any private Interest of the Creature, nor has for its Object any Self-good or Advantage of the private System. The Admiration, Joy, or Love, turns wholly upon what is exterior, and foreign to our-selves. [...] the original Satisfaction can be no other than what results from the Love of Truth, Proportion, Order, and Symmetry, in the Things without. If this be the Case, the Passion ought in reality to be rank'd with *natural Affection*. [...] it must be judg'd to be, what it truly is,

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[1] Shaftesbury, 'An Inquiry', § 2.I.i ⟦p. 58⟧.

[2] *Ibid.*, § 2.II.i ⟦p. 60⟧.

> “A natural Joy in the Contemplation of those *Numbers*, that *Harmony*, *Proportion*, and *Concord*, which supports the universal Nature, and is essential in the Constitution and Form of every particular Species, or Order of Beings.”’[1]

Perhaps, in the end, it was only this disinterestedness that gave Shaftesbury a reason to discuss mathematical beauty, for his conception of beauty was ultimately bound up with order and harmony in the world, and since, at the time, the truths of mathematics were seen as truths about the world, beauty in mathematics was just one part of beauty in the world.

JEAN-PIERRE DE CROUSAZ

One of the first scholars to depart from the idea that aesthetic pleasure arises from some form of cognition of truth was Jean-Pierre de Crousaz (1663–1750). Crousaz taught mathematics and philosophy at the Académie de Lausanne from 1685 to 1724, during which period he published his *Treatise on Beauty* [*Traité du Beau*]. This study of beauty only touched upon mathematical beauty, but was a precursor to the later study by Hutcheson.

For Crousaz, to say ‘this is beautiful’ expressed ‘a certain rapport of an object with agreeable sentiments, or with the ideas of approbation’.[2] Crousaz distinguished ‘sentiments’ from ‘ideas’; the latter affect the mind, but

> ‘sentiments dominate us, they take hold of us, they decide our mood and render us happy or unhappy, according to whether they are sweet or vexacious, agreeable or disagreeable’.[3]

Ultimately, therefore, whether something was beautiful depended on whether contemplating it produced a certain pleasure in the mind: it was not a cognition of truth. Crousaz then set out to characterize beauty. The human mind liked variety (‘diversité’ or ‘varieté’) in its ideas, because variety ‘animates it and stops it from falling into boredom and languor’.[4] But to avoid fatiguing the mind, there also had to be uniformity (‘uniformité’) amidst the variety.[5] Therefore, in short, the primary characteristic of beautiful things was that they presented the mind with uniformity amidst variety. This characterization did not imply that conscious awareness of uniformity amidst variety *produced* the state of mind that led one to say ‘this is beautiful’; it simply described the kind of things that *caused* that state of mind.

Crousaz held that uniformity amidst variety gave rise to a perception of beauty through regularity, order, and proportion.[6] In regular polygons, for instance, the diversity intermingled with variety exhibited by its sides and angles formed regularity.[7]

[1] Shaftesbury, ‘An Inquiry’, § 2.II.i ⟦p. 60⟧.

[2] Crousaz, *Traité du Beau*, § II.iii ⟦p. 7⟧; trans. Guyer, *Hist. Modern Aesthetics*, vol. 1, p. 75.

[3] Crousaz, *Traité du Beau*, § II.iv ⟦p. 8⟧; trans. mod. Guyer, *Hist. Modern Aesthetics*, vol. 1, p. 75.

[4] Crousaz, *Traité du Beau*, § III.ii ⟦p. 12⟧; trans. C.

[5] *Ibid.*, § III.ii ⟦p. 12⟧.

[6] *Ibid.*, § III.iii ⟦p. 14⟧.

[7] *Ibid.*, § III.iii ⟦p. 14⟧.

Crousaz applied his analysis of beauty to a variety of subjects, analysing beauty in science, virtue, eloquence, and music.[1] Physics was the field in which one found the most beauty, for this subject exhibited the order of the entire universe, as created by God.[2] Crousaz treated beauty in mathematics as essentially inseparable from beauty in physics.[3] Mathematics therefore seems to have stood equal to physics in terms of its beauty: 'in the speculations of this Science more than any other the Mind delightedly discovers uniformities that ever stand together among infinite diversities'.[4] Crousaz explicitly declined to give examples of beautiful mathematics, and thus it is impossible to discuss his views on the distribution of beauty in mathematics.[5] It thus remains unknown whether, for example, he thought that certain areas or results held more beauty than others, or what, for him, were the loci of beauty in mathematics.

FRANCIS HUTCHESON

Francis Hutcheson (1694–1746) was the first to attempt a comprehensive theory of beauty, and mathematics played an important role in his work. Hutcheson was born in Ireland but in 1711 went up to the University of Glasgow. The University had fallen to a nadir around the start of the eighteenth century as result of loss of income and a period in which professors were appointed more for orthodoxy than accomplishment, and was just beginning to recover.[6] Perhaps fortuitously for Hutcheson, Robert Simson (1687–1768) was appointed professor of mathematics in the year of Hutcheson's matriculation. Simson was a competent mathematician who later produced critical editions and translations of ancient Greek mathematical works. After six years at Glasgow, Hutcheson returned to Ireland; he would come back to Glasgow to become professor of moral philosophy in 1729. In 1725, he published *An Inquiry into the Original of Our Ideas of Beauty and Virtue in Two Treatises*. This book, which went through four editions in his lifetime, comprises *An Inquiry Concerning Beauty, Order &c.* which will be the main focus here, and *An Inquiry Concerning the Original of our Ideas of Virtue or Moral Good*.

Hutcheson's work can be characterized, not inaccurately, as a systematized version of Shaftesbury's work.[7] (The first edition of *An Inquiry into the Original* said on its title page that in it 'the principles of the late Earl of Shaftesbury are Explain'd and Defended'.) Hutcheson inherited from Shaftesbury the notion of a *sense* of beauty. Specifically, for Hutcheson, 'beauty' was a pleasant idea that was raised in the mind, and a *sense of beauty* was the faculty that allowed one to receive this idea.[8] This sense was internal, passive,[9] and separate from the external powers of perception like sight and hearing. Hutcheson gave two distinct evidences for this description: first, two people could

[1] Crousaz, *Traité du Beau*, chs viii–xi ⟦pp. 83–302⟧.

[2] *Ibid.*, § viii.iii ⟦p. 93⟧.

[3] *Ibid.*, § viii.iv ⟦pp. 100–1⟧.

[4] *Ibid.*, § viii.iv ⟦pp. 100–1⟧; trans. C.

[5] *Ibid.*, § viii.iv ⟦p. 101⟧.

[6] Scott, *Francis Hutcheson*, pp. 11–12.

[7] Guyer, *Hist. Modern Aesthetics*, vol. 1, p. 99.

[8] Hutcheson, *Inquiry Concerning Beauty*, Art. i.ix ⟦p. 23⟧.

[9] *Ibid.*, Art. vi.x ⟦p. 67⟧.

receive the same external sensory inputs of sight or sound, but could perceive in it different degrees of beauty; second, beauty was perceived where the external senses are uninvolved, in universal truths, general causes, principles of action, and theorems.[1] The sense of beauty was autonomous and disinterested: the idea of beauty arose directly from the perception, not from reflection on those perceptions, and did not depend on any knowledge of what was perceived, though of course such knowledge might lead to additional pleasures: either pleasure from foreseen usefulness of what was perceived, or 'that peculiar kind of Pleasure, which attends the Increase of Knowledge'.[2] As a consequence of this autonomy of the sense of beauty, it was clear that Hutcheson did not see the response to beauty as a cognitive one: it did not arise from any insight into the true form of the object. Rather, it was simply the free emotional play of the mind.[3]

However, although Hutcheson did not think of the sense of beauty as stimulated by the recognition of certain properties of what was perceived, it was nevertheless possible to give a theory of beauty that described characteristics of objects that were perceived as beautiful. This is not to say that it was cognition of these properties that triggered the sense of beauty, but simply that objects with these properties activated the sense of beauty. The distinction was made clear by Hutcheson's suggestion that figures perceived as most beautiful are those in which there was 'Uniformity amidst Variety',[4] for he went on to write that 'the Pleasure is communicated to those who never reflected on this general Foundation'.[5] That is, the beauty was perceived even without a cognition of uniformity amidst variety. He argued that this criterion of uniformity amidst variety applied to the perception of beauty in nature, in animals and plants, and in the harmony of sounds.[6]

Beauty in mathematical objects

But the first illustration Hutcheson gave of the criterion of uniformity admist variety concerned polygons and polyhedra. First, beauty increased if uniformity was constant and variety was increased: an equilateral triangle was exceeded in beauty by the square, which was exceeded by the (from context, regular) pentagon, which was surpassed by the (regular) hexagon: here uniformity was constant and variety increased. The pattern was not simply an increase in beauty corresponding to the number of sides, for the lack of parallelism in the sides of regular polygons with an odd number of sides diminished their beauty: these shapes had lesser uniformity.[7]

Similarly, Hutcheson held that for regular solids, beauty was correlated with the number of faces:

> 'So in Solids, the Eicosiedron surpasses the Dodecaedron, and this the Octahedron, which is still more beautiful than the

[1] Hutcheson, *Inquiry Concerning Beauty*, Arts I.x–xi ⟦pp. 23–4⟧.

[2] *Ibid.*, Art. I.xiii ⟦p. 25⟧.

[3] See the discussion in Guyer, *Hist. Modern Aesthetics*, vol. 1, p. 100 sqq.

[4] Hutcheson, *Inquiry Concerning Beauty*, Art. II.iii ⟦p. 28⟧.

[5] *Ibid.*, Art. II.xiv ⟦p. 35⟧.

[6] *Ibid.*, Arts II.v–xiii ⟦pp. 30–5⟧.

[7] *Ibid.*, Art. II.iii ⟦p. 29⟧.

> Cube; and this again surpasses the regular Pyramid: The obvious Ground of this, is greater Variety with equal Uniformity.'[1]

The assertion that variety increased and uniformity remained constant in this example highlights a problem. If uniformity involved only the condition 'equal faces and angles between faces' or 'inscribed in a sphere', then it was indeed fixed for all the regular solids. Yet Hutcheson had already said that parallelism was part of uniformity. In the cube, parallelism was visible in each face and each edge of the solid was parallel to three other edges. In an octahedron, a dodecahedron, or an icosahedron, there was no parallelism in each face, while each edge of the solid was parallel to only one other edge. Perhaps Hutcheson was aware of this difficulty, for his one example of a regular polygon whose odd number of faces had diminished beauty was the regular heptagon, rather than the equilateral triangle or regular pentagon, the regular polygons with an odd number of edges that appear as faces of the regular solids. Perhaps he considered parallelism less important in polygons with a small number of sides.

Weakening uniformity lessened beauty: thus less beauty was seen in irregular polygons and polyhedra than in regular ones. However, sometimes weakening the uniformity could allow for an increase in variety: an archimedean solid* such as a truncated octahedron (which Hutcheson called by the older name 'Exoctaedron'), with its eight hexagonal and six square faces, or a truncated dodecahedron ('Eicosidodecaedron'), with its twelve decagonal and twenty triangular faces, could be almost as beautiful as a regular solid.[2]

* ▸ pp. 109–12

These solids were beautiful *as mathematical objects*. The most beautiful solids — the icosahedron, say — were not generally chosen to serve as pedestals of statues, for they did not seem so stable as a cube or cuboid. This contrast was an instance of how the beauty of an *embodied* mathematical object could be tempered by what Hutcheson termed *relative* beauty, which depended in this instance on the intention of stability.[3]

Absolute beauty, on the other hand, meant beautiful in itself, unlike the relative beauty of an object which imitated or represented another. In a landscape one might perceive absolute beauty; in a painting of it, relative beauty. [Note that in using the term 'absolute', Hutcheson did not exclude *comparisons* of beauty. Objects in which one perceived absolute beauty could still be judged more or less beautiful than other objects.] Thus, a mathematical object could have absolute beauty, while a physical representation of it possessed relative beauty, which depended on how successfully it represented the object and, potentially, how well it served some purpose.

Beauty in theorems

Up to the third edition of *An Inquiry into the Original of Our Ideas of Beauty and Virtue*, Hutcheson maintained that absolute

[1] Hutcheson, *Inquiry Concerning Beauty*, Art. II.iii ⟦p. 29⟧.

[2] *Ibid.*, Art. II.iii ⟦p. 29⟧.

[3] *Ibid.*, Art. IV.v ⟦pp. 44–5⟧.

beauty was possessed by theorems.[1] However, in the fourth edition, he restricted the division of beauty into absolute and relative to 'Beauty, in Corporeal forms', so that theorems no longer fit into this classification. He devoted an entire section to the beauty of theorems, for such beauty epitomized the criterion of uniformity amidst variety:

> 'The Beauty of Theorems, or universal Truths demonstrated, deserves a distinct Consideration, being of a Nature pretty different from the former kinds of Beauty; and yet there is none in which we shall see such an amazing Variety with Uniformity'.[2]

Theorems possessed *infinite* variety, for in each theorem there was an infinite multitude of particular truths. He gave the particular example of Proposition i.47 of Euclid's *Elements* (the pythagorean theorem): this result concerns infinitely many dissimilar right-angled triangles of infinitely many sizes, which was an example of an 'Infinity of Infinites'[3] of truths. This endless multiplicity of truths was not limited to geometry, but applied too in algebra and the calculus: theorems applied to infinitely many species of curves, of infinitely many sizes, made up of infinitely many points.[4]

That theorems possess uniformity — or *unity*, which term Hutcheson used in discussing theorems[5] — was indicated by the fact that trying out particular examples yielded no pleasure in beauty. It was only when the examples were united in a result that this pleasure was possible.[6]

So to be beautiful, a theorem must possess unity as well as variety, and the unity must be of a high enough degree. For example, consider the theorems:

1) *A cylinder has greater volume than an inscribed sphere, which in turn has greater volume than a cone of equal base and height.*
2) *The ratios of volumes of a cylinder, its inscribed sphere, and a cone of equal base and height are* $3 : 2 : 1$.

Both these theorems are true and apply to precisely the same objects; thus they were equal in terms of variety. In theorem 1 there was a paucity of unity: the theorem did not connect its objects precisely enough. In theorem 2, with the exact ratios: 'how beautiful is the Theorem, and how are we ravish'd with its first Discovery!'[7] [Theorem 2 is effectively due to Archimedes.*]

Hutcheson also noted 'another Beauty in Propositions', when a theorem has many easily deducible corollaries.[8] He gave as examples the pythagorean theorem and a proposition from the first book of Euclid's *Elements*: '*Parallelograms which are on the same base and in the same parallels are equal to one another.*'[9] This proposition immediately leads to the result that triangles on the same base and with the same altitude (which can be considered as half-parallelograms) have the same area, and is a key to Euclid's theory of equal areas.

[1] Hutcheson, *Inquiry Concerning Beauty*, Art. i.xvii ⟦p. 27⟧.

[2] *Ibid.*, Art. iii.i ⟦p. 36⟧.

[3] *Ibid.*, Art. iii.ii ⟦p. 36⟧.

[4] *Ibid.*, Art. iii.ii ⟦p. 36–7⟧.

[5] *Ibid.*, Arts iii.iii–iv ⟦p. 37⟧.

[6] *Ibid.*, Art. iii.iii ⟦p. 37⟧.

[7] *Ibid.*, Art. iii.iv ⟦p. 37⟧.

* ▸ pp. 106–8

[8] *Ibid.*, Art. iii.v ⟦p. 38⟧.

[9] Euclid, *Elements*, Prop. i.35.

Yet the idea that the beauty of a theorem could be dependent on its consequences seems difficult to reconcile with Hutcheson's overarching view that the perception of beauty was independent of 'any Knowledge of Principles, Proportions, Causes, or of the Usefulness of the Object'.[1] Hutcheson did not explore this potential problem, but a possible solution is that Hutcheson restricted consideration to *easily deducible* corollaries, and intended to consider here corollaries that are so close to the theorem that they are encompassed in the same act of perception.

Surprise could enhance the pleasure in the beauty of a theorem, but did not subsume it.[2] Theorems that were surprising gave a greater pleasure in the feeling of discovery, but results that were unsurprising gave less, and those that could be intuited, such as '*a Line bisecting the vertical angle of an Isosceles triangle, bisects the Base*'[3] gave little. But surprise was not the only pleasure in theorems, nor the only source of joy in truth. The pleasure in the surprise was greatest when the theorem was first learned — when the uniformity amidst variety was first encountered — whereas the pleasure in the knowledge was constant.[4] A second kind of surprise, 'which adds to the Beauty of some Propositions less universal, and may make them equally pleasant with more universal ones',[5] lay in the discovery of a truth that was previously thought false. However, in no case was surprise sufficient for beauty, for a novel *example* had not the beauty one perceived in a theorem.[6]

Hutcheson's analysis of the final cause of the sense of beauty made it a gift of a beneficient God, who so shaped the human mind that those things that were more advantageous to us were perceived as beautiful. In particular, the beauty perceived in uniformity amidst variety encouraged humans to seek general theorems rather particular truths, which influenced people to avoid the tedium and toil of gathering individual facts, and seek universal rules. This search was a rational act, for

> 'The manner of Knowledge by universal Theorems, and of Operation by universal Causes, as far as we can attain it, must be most convenient for Beings of limited Understanding and Power.'[7]

Hutcheson held that it was because God was good that we gained pleasure in pursuing what cold analysis, minus any hedonic considerations, would conclude was acting rationally in our own interest.[8]

Originality

As discussed above, Hutcheson drew on Shaftesbury's *Characteristicks*, but otherwise did not cite explicitly other contemporaneous work. He later acknowledged that the writer and poet Joseph Addison (1672–1719) had, in an essay published in 1712,[9] anticipated many of the ideas that he had independently developed on the sense of beauty.[1] However, at no point did Addison touch on the beauty of theorems or of geometric objects. The influence of Crousaz on Hutcheson is more problematic to unpick. There is a clear echo of Crousaz's 'l'uniformité au milieu de la diversité'[2] in Hutcheson's 'Uniformity amidst Variety'.[3] Indeed, Hutcheson was accused of plagiarizing Crousaz's *Traité du Beau*.[4] In his published defence, Hutcheson conceded that he had read Crousaz years before he started writing *An Inquiry into the Original of Our Ideas of Beauty and Virtue*, but thought that he had drawn all his important ideas from ancient authors and from Shaftesbury,[5] adding:

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[1] Hutcheson, *Inquiry Concerning Beauty*, Art. I.xiii ⟦p. 25⟧.

[2] *Ibid.*, Art. III.vii ⟦p. 40⟧.

[3] *Ibid.*, Art. III.iv ⟦p. 37⟧.

[4] *Ibid.*, Arts III.vi–vii ⟦p. 40⟧.

[5] *Ibid.*, Arts III.vii ⟦p. 40⟧.

[6] *Ibid.*, Arts III.iv, vii ⟦pp. 38, 40⟧.

[7] *Ibid.*, Art. VIII.ii ⟦p. 79⟧.

[8] *Ibid.*, Art. VIII.ii ⟦pp. 80–1⟧.

[9] Addison, 'On the Pleasures of the Imagination'.

> 'As for M. de Crousaz's book, I had only a distant recollection of his general idea of beauty as *unity amidst variety*, in which he is as little original as I am'.[6]

Whatever ideas he consciously or unconsciously drew from Crousaz, Hutcheson explored mathematical beauty much more thoroughly.

Hutcheson was therefore the first modern philosopher to attempt a theory of beauty into which mathematical beauty, both in objects and results, could be fitted. Whatever the shortcomings of his theory, his priority here remains a monument.

YVES-MARIE ANDRÉ

Yves-Marie André (1675–1764), who was known as 'le Père André', was a Jesuit whose sympathy with Gallicanism, Cartesianism, and Jansenism led to his being assigned to scientific studies.[7] He became professor of mathematics at the University of Caen. His *Essay on Beauty* [*Essai sur le beau*], based on discourses he read to the Académie de Caen, was initially published in 1741, with an extended edition published posthumously in 1770. The 1741 edition brought fame to André. In the second volume of the *Encyclopédie*, published in 1752, Denis Diderot praised André's system as being the most accepted, most widely-known, and best-argued of those he knew, and outlined it at some length; indeed, he praised it above the arguments of Plato, Augustine, Christian Wolff (1679–1754), Crousaz, Shaftesbury (whom he identified only as '[t]he author that gave us the *Essay On Merit and Virtue*'), and Hutcheson.[8]

André began the *Essay* confidently, by emphasizing that the question he examined was not 'what is beautiful?' but 'what is beauty?', explicitly citing Plato's *Greater Hippias*.[9][*] He then proceeded to outline a typology of beauty. The first division was threefold: *essential beauty*, independent of any institution; *natural beauty*, independent of human opinion; and *beauty of human institution* or *artificial beauty*, which was in some degree arbitrary.[10] A separate twofold

[1] Thorpe, 'Addison and Hutcheson on the Imagination', pp. 216 sqq.

[2] Crousaz, *Traité du Beau*, § III.ii ⟦p. 12⟧.

[3] Hutcheson, *Inquiry Concerning Beauty*, Art. II.iii ⟦p. 28⟧.

[4] [Anonymous], 'Nouvelles Litteraires (1725)', pp. 281 sq.

[5] Hutcheson, undated letter, repr. in [Anonymous], 'Nouvelles Litteraires (1726)', pp. 514–15.

[6] Hutcheson, undated letter, repr. in [Anonymous], 'Nouvelles Litteraires (1726)', p. 515; trans. Raynor, 'Hutcheson's Defence', p. 178, emphasis in orig.

[7] For biographical details, see Campbell, 'André'.

[8] Diderot, 'Beau', pp. 173–5.

[9] André, *Essay on Beauty*, p. 1.

[*] ▸ p. 60

[10] *Ibid.*, pp. 2, 31.

division comprised *sensible beauty*, which one perceived in bodies; and *intellectual beauty*, which one perceived in minds.[1]

At least in visual beauty, essential beauty could be recognized in the preference for 'regularity, order, proportion, symmetry',[2] which fits with the maxim André adopted 'in its full extent' from Augustine of Hippo that 'it is unity that constitutes […] the form and the essence of beauty in all types of beauty. *Omnis porro pulchritudinis forma unitas est.*'[3] This principle is deeply embedded in André's system: he reiterated it several times in the Essay.[4] Applying it to visual beauty led one to see that there was essential beauty: 'a geometric beauty' whose idea 'shaped *the art of the Creator*'.[5] That is, the principles of essential beauty, which were in some sense geometric, were independent of God. It was only natural beauty that depended on God's will.

André wrote that '*beauty*, in a work of the mind' was 'what is entitled to appeal to reason and reflection by its own excellence, by its light or by its exactness, and […] by its intrinsic pleasantness'.[6] Within his threefold division of essential, natural, and artificial beauty, essential beauty consisted in '[t]ruth, order, honesty, and decency'.[7]

For 'pure intelligences, or at least men more rational than sensible', truth by itself would appeal:

> 'it would have enough to charm them by its light, by the order of the principles that demonstrate it, or by that of the consequences that burst from it like the rays of the Sun. This is the only beauty that is asked for in a work of mathematics'.[8]

Thus beauty in mathematics was classified as essential beauty; by contrast, natural beauty in works of the mind was divided into beauty in images, feelings, and movements.[9] Mathematics as a science (that is, as a practice, rather than a body of knowledge or a platonic universe) also partook of artificial beauty in terms of clarity, which was the first beauty of expression.[10] The style of mathematical exposition could also be a source of artificial beauty: to adapt André's phrase, the mathematical Muses are allowed to laugh.[11] André thought that Archimedes put 'ease and lightness into the mathematical style';[12] he also praised the styles of, among others, Galileo Galileo[*] (1564–1642), René Descartes[†] (1596–1650), and Guillaume de l'Hospital (1661–1704).[13]

André held that the love of mathematical beauty drove mathematicians towards the most beautiful discoveries. Although he credited Thales and Pythagoras with contributions that are now seen as historically questionable,[14] his list of such discoveries illustrates the kind of beauty he saw in mathematics. His first examples were: the very method of rigorous geometry; the theory of proportions; the unified theory of the *Elements* of Euclid.[15] He praised highly Archimedes for his work on the quadrature of the circle (presumably referring to the bounds he calculated for the value of π[16]) and the parabola,[17] his theories of centres of gravity and of floating bodies,[18] and his measurement of the surface area of a sphere:[‡] 'the most beautiful discovery, or at least the most useful that had been made in geometry since its birth.'[1] André also thought that the desire for mathematical beauty had motivated the astronomical and geographical discoveries of Hipparchus[*] (first quarter of 2nd century–after 127 BCE), Eudoxus[†] (c. 400–c. 347 BCE), Eratosthenes[‡] (c. 276–c. 195 BCE), Nicolaus Copernicus[§] (1473–1543), Galileo, Johannes Kepler[¶] (1571–1630), and 'the two Cassinis' (Gian Domenico (1625–1712) and his son Jacques (1677–1756)); the mathematical discoveries of Diophantus (*fl.* c. 250 CE), Descartes, Bonaventura Cavalieri (1598?–1647), and John Wallis (1616–1703); and the mechanical discoveries of Christiaan Huygens[♠] (1629–95):[2] 'In a word, there is no academy in Europe where the love of mathematical beauty has not given in our days new conquests to the kingdom of truth.'[3]

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[1] André, *Essay on Beauty*, p. 2.

[2] *Ibid.*, p. 3.

[3] André, *Essay on Beauty*, p. 5; cf. Augustine, letter to Cœlestinus, dated 390 CE, repr. in Augustine, *Works*, vol. VI, p. 42.

[4] André, *Essay on Beauty*, pp. 23, 42, 63.

[5] *Ibid.*, p. 5, emphasis in orig.

[6] *Ibid.*, p. 31, emphasis in orig.

[7] *Ibid.*, p. 32.

[8] *Ibid.*, p. 33.

[9] *Loc. cit.*

[10] *Ibid.*, p. 37.

[11] Cf. *ibid.*, p. 115.

[12] *Loc. cit.*

[*] ▸ pp. 277–9

[†] ▸ pp. 303–5

[13] *Ibid.*, pp. 115–16.

[14] *Ibid.*, p. 128.

[15] *Loc. cit.*

[16] Dijksterhuis, *Archimedes*, pp. 223–4.

[17] *Ibid.*, ch. XI.

[18] André, *Essay on Beauty*, pp. 128–9.

[‡] ▸ pp. 106–8

DAVID HUME

The philosopher David Hume (1711–76) did not consider beauty in mathematics in his essay 'Of the Standard of Taste', which was his one work focused on aesthetics.[4] In another essay he made a mention of beauty in connection with mathematics:

> 'A man may know exactly all the circles and ellipses of the Copernican system, and all the irregular spirals of the Ptolomaic, without perceiving that the former is more beautiful than the latter. Euclid has fully explained every quality of the circle, but has not, in any proposition, said a word of its beauty. The reason is evident. Beauty is not a quality of the circle. It lies not in any part of the line, *whose* parts are all equally distant from a common centre. It is only the effect, which that figure produces upon a mind, whose particular fabric or structure renders it susceptible of such sentiments. In vain would you look for it in the circle, or seek it, either by your senses, or by mathematical reasonings, in all the properties of that figure.'[5]

This argument was not in itself a rejection of mathematical beauty; Hume's point was rather that the beauty of a circle or of the Copernican system is not a conclusion drawn mathematically, any more than the beauty of a proof is a consequence that appears in the deduction. But Hume seems to have been unaware of beauty in mathematics, and would doubtless have scorned it as a motivation, for he did not think mathematicians generally gained pleasure from their work:

> 'But tho' the exercise of genius be the principal source of that satisfaction we receive from the sciences, yet I doubt, if it be alone sufficient to give us any considerable enjoyment. The truth we discover must also be of some importance. 'Tis easy to multiply algebraical problems to infinity, nor is there any

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[1] André, *Essay on Beauty*, p. 129.

[*] ▸ p. 118

[†] ▸ pp. 84–6

[‡] ▸ p. 94

[§] ▸ pp. 274–5

[¶] ▸ pp. 279–94

[♠] ▸ pp. 307

[2] *Loc. cit.*

[3] *Loc. cit.*

[4] Guyer, *Hist. Modern Aesthetics*, vol. 1, p. 124.

[5] Hume, 'The Sceptic', p. 181.

> end in the discovery of the proportions of conic sections; tho' few mathematicians take any pleasure in these researches, but turn their thoughts to what is more useful and important.'[1]

DENIS DIDEROT

The philosopher and man of letters Denis Diderot (1713–84), who was well-versed in mathematics,[2] was the editor of the first seven volumes of the *Encyclopédie* alongside the mathematician and physicist Jean le Rond d'Alembert (1717–83), and the sole editor for the remainder.[3] He himself wrote hundreds of articles for the *Encyclopédie*, including the article on 'Beautiful' ['BEAU, adj. (*Métaphysique*)'][4] for the second volume (1752), in which he briefly discussed mathematical beauty. He was aware of the work of previous authors who had examined beauty, and in particular had translated Shaftesbury's 'An Inquiry Concerning Virtue, or Merit' into French.[5] As already mentioned, he thought André's theory of beauty excelled all others.

Diderot was critical of Hutcheson's application of 'uniformity amidst variety' to geometrical figures, since in a polygon with many sides the viewer lost track of how the sides interact,[6] and this seemed to raise problems with Hutcheson's view that, for regular polygons, a greater number of sides entailed greater beauty.

Furthermore, Diderot argued that 'uniformity amidst variety' could not apply at all to theorems and demonstrations: if it did, then Hutcheson's argument that a theorem had uniformity because it gave rise to many particular truths could also be applied to an *axiom*. In particular, one could consider an axiom of which the theorem is a corollary alongside infinitely many other theorems. Such an axiom should have greater beauty because it possesses uniformity amidst variety in greater degree. Yet it seemed that no-one spoke of 'beautiful axioms', only 'beautiful theorems'.[7]

Diderot's argument here is unconvincing. As an example of uniformity amidst variety, Hutcheson discussed how the pythagorean theorem was a single theorem that applied to an infinity of triangles.* That is, Hutcheson held that the infinity of particular truths were the *instantiations* of the theorem, not its logical consequences. Therefore in applying Hutcheson's 'uniformity amidst variety' criterion to an axiom, one should not consider the theorems that follow from it, but rather its instantiations. The instantiations of an axiom do not seem to have much variety. For example, consider Euclid's first postulate: 'To draw a straight line from any point to any point.'[8] There is very little variety in its instantiations: in some sense, once one has seen a single instance of a line drawn between two points, one has seen them all. Hence, whatever uniformity there may be in an axiom, there is

[1] Hume, *A Treatise of Human Nature*, § 2.3.10, para. 4 ⟦p. 287⟧.

[2] Krakeur & Krueger, 'The Mathematical Writings of Diderot'.

[3] Lough, *The* Encyclopédie, pp. 14–27.

[4] Diderot, 'Beau'; trans. Diderot, 'Beautiful'.

[5] Guyer, *Hist. Modern Aesthetics*, vol. 1, p. 279.

[6] Diderot, 'Beau', p. 172.

[7] *Loc. cit.*

* ▸ p. 360

[8] Euclid, *Elements*, Post. 1.

almost no variety; thus the lack of 'beautiful axioms' does not count against Hutcheson's criterion.

In Diderot's *Encyclopédie* entry, the only positive statement directly addressing his own views on mathematical beauty is a passing remark that

> 'by a *beautiful problem* one means a problem that is difficult to solve; by a *beautiful solution*, the simple and easy solution of a difficult and complicated problem.'[1]

This observation served Diderot as a counterexample to greatness being a part of beauty, and he did not expand upon it. But these characterizations of beautiful problems and solutions are important for their modern sound: they would not be out of place in twentieth-century discussions of mathematical beauty such as those of G. H. Hardy* (1877–1947).

Diderot's characterizations of beautiful problems and solutions were compatible with his general rule:

> 'I thus call *beautiful* outside of me all that contains in itself that which awakens in my understanding of the idea of relation; *&* *beautiful* in relation to me, all that awakens that idea.'[2]

Clearly a difficult problem could not be easily solved directly, and thus contained notions that gave rise to thoughts of how related concepts could be brought to bear, and thus awoke the idea of relation. A simple and easy solution could apply related (and perhaps unexpected) concepts, which thus similarly awoke the idea of relation. However, the lack of further information about Diderot's view on beauty in mathematics makes a deeper analysis impossible.

Finally, in one of his mathematical works, Diderot seemed to caution against giving too much attention to the aesthetic value of mathematics at the price of practicality:

> 'I fear [...] that we have attached an imaginary elegance to employing in the construction of equations a curve of a certain kind, in cases where a curve of a higher kind also satisfied, and was traced with more ease.'[3]

HENRY HOME

The jurist Henry Home (1696–1782), who took the title Lord Kames on his appointment to the Court of Session (the highest civil court in Scotland), was not only a great and eminent legal scholar and judge, but a first-rate historian and philosopher. One of the objectives of his *Elements of Criticism* was to argue against Hutcheson's views on morality. This work considered aesthetic value

[1] Diderot, 'Beau', p. 178, emphases in orig.; trans. C.

* ▸ pp. 498–523

[2] *Ibid.*, p. 176, emphases in orig.; trans. C.

[3] Diderot, 'Examen de la Développante du Cercle', p. 133; trans. C.

in mathematics in passing, and what he said on this topic was almost diametrically opposed to Hutcheson's views.

In the first place, Home considered that 'beauty' was a term that was properly applied only to the objects of sight.[1] It was only figuratively that one said 'a beautiful sound, a beautiful thought or expression, a beautiful theorem, a beautiful event, a beautiful discovery in art or science'.[2]

As regards regular polygons, Home took the view that a square was less beautiful than a circle, but more beautiful than a hexagon or octagon.[3] Hutcheson had taken the opposite view, for these latter shapes had the same degree of uniformity and greater variety than the square. But Home held that in viewing a polygon, the attention was divided among the sides and angles, so fewer sides led to greater simplicity, while more sides led to less simplicity, and the mind could not perceive distinctly a many-sided regular polygon.[4] Here there is an echo of Diderot's complaint about applying uniformity amidst variety to polygons.[*] The relevant volume of the *Encyclopédie* was published in 1752, which makes it possible that Home had read the entry before writing *Elements of Criticism*, whose first edition appeared in 1762.

For the circle, there was no division of the attention: 'the circumference of a circle, being a single object, makes one entire impression'.[5] In each case, simplicity contributed to beauty. One point on which Home agreed with Hutcheson was that parallelism was important for beauty: an equilateral triangle was less beautiful than a square.[6]

However, Home here considered polygons as visual objects, not mathematical ones. Aside from his general restriction of beauty to objects of sight, his position is made clear by his observation that, although a square has greater regularity than a parallelogram, it was the latter shape that was used for doors and windows. (Home seems to have meant specifically rectangular parallelograms.) '[H]ere we find the beauty of utility, prevailing over that of regularity and uniformity.'[7] Thus his discussion may not have applied to aesthetic value of polygons as mathematical objects.

Home's approach to was to consider 'such attributes, relations, and circumstances, as in the fine arts are chiefly employ'd to raise agreeable emotions'.[8] Beauty was only one such attribute; others included grandeur and sublimity, motion and force, novelty, resemblance, uniformity, and variety. Although theorems did not fall within the remit of the fine arts, some of these concepts remained applicable: theorems could be 'delightful by their simplicity, and by the easiness of their application to variety of cases'.[9]

In particular, Home considered the concept of uniformity amidst variety, and reached the conclusion that it was important for human understanding, on grounds that were not too distant from Hutcheson's, though traced to an impersonal Nature rather than to a Deity:

> 'Nature therefore [...] has made too great uniformity or too great variety in the course of perceptions, equally unpleasant:

[1] Home, *Elements of Criticism*, HE6 vol. 1, pp. 196–7, vol. 2, p. 522.

[2] *Ibid.*, HE6 vol. 1, p. 197.

[3] *Ibid.*, HE6 vol. 1, pp. 202–3.

[4] *Ibid.*, HE6 vol. 1, pp. 203.

[*] ▸ p. 365

[5] *Ibid.*, HE6 vol. 1, p. 203.

[6] *Ibid.*, HE6 vol. 1, p. 204.

[7] *Ibid.*, HE6 vol. 1, p. 203.

[8] *Ibid.*, HE6 vol. 1, p. 195.

[9] *Ibid.*, HE6 vol. 1, p. 205.

> and indeed, were we addicted to either extreme, our internal constitution would be ill suited to our external circumstances.'[1]

However, Home disputed the idea that (as he put it) 'beauty consists in uniformity amid variety'.[2] In particular, Home held that variety contributes nothing to the beauty of a mathematical theorem, while the sphere is the most uniform solid, and has little variety, but much beauty.[3] Indeed, the Hutchesonian criterion for beauty

> 'is but obscurely expressed, is only applicable to a number of objects in a group or in succession, among which indeed a due mixture of uniformity and variety is always agreeable; provided the particular objects, separately considered, be in any degree beautiful, for uniformity amid variety among ugly objects, affords no pleasure.'[4]

Uniformity amidst variety remained a quality that theorems could possess, for the various particular cases that a theorem covered could be the 'objects in a group or in succession'. However, having this quality was quite distinct from being beautiful.

JOSHUA REYNOLDS

The painter Joshua Reynolds (1723–92) gave a series of lectures at the Royal Academy of Arts, of which he was president from 1768 until his death, mainly concerned with painting and with its role as a way of communicating truth. However, it is worth noting one of the comparisons Reynolds made to illustrate the desire for truth in painting:

> 'The natural appetite or taste of the human mind is for TRUTH; whether that truth results from the real agreement or equality of original ideas among themselves; from the agreement of the representation of any object with the thing represented; or from the correspondence of the several parts of any arrangement with each other. It is the very same taste which relishes a demonstration in geometry, that is pleased with the resemblance of a picture to an original, and touched with the harmony of musick.
>
> All of these have unalterable and fixed foundations in nature, and are therefore equally investigated by reason, and known by study; some with more, and some with less clearness, but all in exactly the same way. A picture that is unlike, is false. Disproportionate ordonnance of parts is not right; because it cannot be true, until it ceases to be a contradiction to assert, that the parts have no relation to the whole. Colouring is true, when it is naturally adapted to the eye, from

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[1] Home, *Elements of Criticism*, HE6 vol. 1, p. 320.

[2] *Ibid.*, HE6 vol. 1, p. 324.

[3] *Ibid.*, HE6 vol. 1, pp. 324–5.

[4] *Ibid.*, HE6 vol. 1, p. 325.

> brightness, from softness, from harmony, from resemblance; because these agree with their object, NATURE, and therefore are true; as true as mathematical demonstration; but known to be true only to those who study these things.'[1]

Note that the phrase 'touched with the harmony of musick' emphasizes that the enjoyment of truth in geometrical proof is an emotional response. Reynolds was not speaking here to a audience of mathematicians or philosophers, but presumably felt that his audience would understand the comparison.

Reynolds closely linked beauty and truth, because his conception of truth was broad enough to encompass accuracy of representation in art, even allowing for a degree of 'improvement' through removal of blemishes from the original, and the internal balance and symmetry of a work of art.[2]

For Reynolds, the whole purpose of art was to elevate the mind to things higher than the material world:

> 'Because these arts, in their highest province, are not addressed to the gross senses; but to the desires of the mind, to that spark of divinity which we have within, impatient of being circumscribed and pent up by the world which is about us.'[3]

This attitude seems very reminiscent of Plato's view of mathematics, and may have owed something to the influence of Zachariah Mudge (1694–1769), a clergyman with platonist views to whom Reynolds was said to have ascribed his interest in abstract speculation.[4] That said, Reynolds's statement about the aim of art is not necessarily connected with his acceptance of beauty in mathematics.

THOMAS REID

Thomas Reid (1710–96) succeeded Adam Smith (1723–90) as professor of moral philosophy at Glasgow in 1764, having previously been a professor at King's College, Aberdeen. After his resignation from Glasgow in 1781, he compiled his lectures for publication. One of the resulting books, *Essays on the Intellectual Powers of Man*, contains his discussions of beauty. Reid maintained a division of aesthetic properties into novelty, grandeur, and beauty, a trichotomy that had been introduced by Addison in 1712.[5] However, he mentioned mathematics only in connection with beauty.

Cause and response

Reid made explicit an important distinction that had not previously been emphasized. In contemplating a beautiful object, it was necessary to distinguish the emotional response it produced

[1] Reynolds, 'Discourse VII', pp. 200–1, emphases in orig.

[2] Guyer, *Hist. Modern Aesthetics*, vol. 1, p. 209.

[3] Reynolds, 'Discourse XIII', pp. 142–3.

[4] Burke, letter to Malone, dated 5 Apr. 1797, repr. in Burke, *Correspondence*, vol. IX, p. 326.

[5] Guyer, *Hist. Modern Aesthetics*, vol. 1, pp. 32, 221.

from the properties that caused that response. 'When I hear an air in music that pleases me, I say, it is fine, it is excellent. This excellence is not in me; it is in the music. But the pleasure it gives is not in the music; it is in me.'[1] He emphasized this point in opposition to philosophers who attempted to reduce perceptions to 'mere feelings or sensations'[2] in the perceiver, with no correspondent in the object perceived. Reid held that this reduction was contrary both to common sense and to the use of language, for even those who held this position had to express themselves in a way that contradicted their position, as if beauty was a property of the object.[3]

Characteristics of beauty

Unlike Home, Reid held that the term *beauty* did properly apply to objects other than those of sight, basing this upon how the term was commonly used, and the observation that a blind person could make judgement about beauty.[4] Indeed, the term *beauty* is applied to a wide variety of objects: 'The beauty of a demonstration, the beauty of a poem, the beauty of a palace, the beauty of a piece of music, the beauty of a fine woman'.[5] Reid emphasized that there was no distinction between these uses of the word except the object to which it was applied. Further, beauty could be seen in contrasting loci: in morals or nature; in sensory or mental objects; in human and divine works; in non-living, living, or intelligent beings; in the bodies or minds of humans.

Taste might be corrupted, and so judgements might be wrong when measured against a true standard of beauty in nature.[6] And the perception of beauty could require knowledge: a skilled painter or sculptor could see more of the beauty in a work of one of their fellows.[7] Any 'real excellence' could be perceived as beautiful from the proper perspective.[8] This observation did not lessen the problem of characterizing beauty, for it was just as hard to characterize 'real excellence'. But it was difficult even to give *any* criterion for beauty:

> 'What can it be that is common to the thought of a mind and the form of a piece of matter, to an abstract theorem and a stroke of wit?
>
> I am indeed unable to conceive any quality in all the different things that are called beautiful, that is the same in them all. There seems to be no identity, nor even similarity, between the beauty of a theorem and the beauty of a piece of music, though both may be beautiful.'[9]

However, Reid thought that there had to be some commonality to all things called beautiful, depending once again on the common use of language. If there were as many kinds of beauty as kinds of object to which the concept is applied, why would the same word be used for all of them? There was no commonality between, say, a theorem and a piece of music, but both were beautiful. The commonality was

[1] Reid, 'Of Taste', p. 490.

[2] *Loc. cit.*

[3] *Ibid.*, pp. 490, 492.

[4] *Ibid.*, p. 502.

[5] *Ibid.*, p. 491.

[6] *Ibid.*, p. 492.

[7] *Ibid.*, p. 500.

[8] *Ibid.*, p. 491.

[9] *Ibid.*, p. 498.

therefore not in the objects themselves, but in two aspects of their relation to the contemplating subject:

> '*First*, When they are perceived, or even imagined, they produce a certain agreeable emotion or feeling in the mind; and, *secondly*, This agreeable emotion is accompanied with an opinion or belief of their having some perfection or excellence belonging to them.'[1]

Reid declined to judge whether there was some necessary connection between the emotional response and the perception of excellence, or whether God had willed matters so that the emotion accompanied the perception.

Reid allowed, in the course of his discussion, that both theorems and proofs could be beautiful. Indeed, he seems to have been the first to mention mathematical proofs in a philosophical treatment of beauty. However, he did not give examples of beautiful theorems or proofs. He did mention elsewhere, in a discussion of simplicity, the result that there are only five regular solids, but his remark that 'this notion of the Pythagoreans and Platonists has undoubtedly great beauty and simplicity'[2] seems to have referred instead to the idea that the five regular solids were the key to understanding nature.*

There were two ways in which one came to think an object beautiful: *instinctive*, and *rational*. The first way was when an object struck us at once as beautiful, without any prior thought and without any subsequent way to justify our judgement of it as beautiful. Reid's examples were the beauty seen in 'the plumage of birds and of butterflies, in the colours and form of flowers, of shells'.[3] The second way was when the judgement was based on 'some agreeable quality of the object which is distinctly conceived, and may be specified',[4] such as when an expert mechanic examined a well-constructed machine, admired its design and the quality of its manufacture, and declared it beautiful. While the emotional response was the same, the judgement was caused by, and could be defended with reference to, the excellence of certain qualities of the machine.[5] The two ways were not completely separate: an initial instinctive judgement might later become rational, as the object was examined further and certain hitherto unseen qualities were perceived.[6]

Reid did not consider whether both forms of judgement applied to beauty in mathematics, but there seems to be no obstacle to judging instinctively a theorem or proof to be beautiful, and only later subjecting it to analysis that reveals aspects (such as deep ideas or connections with other areas of mathematics) which then lend themselves to a rational judgement of beauty. This point is important: the initial judgement and a later reflective judgement of beauty in mathematics — regardless of whether the Reidian terms 'instinctive' and 'rational' are the best adjectives to describe them — may involve very different mental processes.

[1] Reid, 'Of Taste', 498, emphases in orig.

[2] Reid, 'Of Judgment', p. 471.

* ▸ pp. 67–70

[3] Reid, 'Of Taste', p. 500.

[4] *Ibid.*, p. 501.

[5] *Loc. cit.*

[6] *Loc. cit.*

For Reid, just as judgements of beauty divided into instinctive and rational, beauty itself divided into original and derived, depending on whether the object was beautiful in itself or reflected the beauty of another thing.[1] Things of intellectual beauty, which presumably included the beautiful parts of mathematics, could be perceived in those sensory objects that bear their image.[2] Although Reid did not draw this conclusion explicitly, his position points to a theorem, for example, having original beauty, and a diagram of an instance of that theorem having a corresponding derived beauty.

MOSES MENDELSSOHN

Moses Mendelssohn (1729–86) was a philosopher who contributed to many areas, including metaphysics, politics, theology, and aesthetics. He was, quite self-consciously, the defender of rationalist values in general and of the rationalist vein in aesthetics in particular.[3] He discussed aesthetics in several treatises, but never wrote a systematic full-length presentation of his aesthetic theory.[4]

He was clearly aware to some extent of beauty in mathematics, and wrote of the mathematician's experience of the beauty found in proof:

> 'If [...] he thinks all at once over the chain of inferences that he worked through, if he calculates how the truths might be assembled in the best order, segment by segment, and how one flows from all the rest and all the rest from one, what fullness of sensuous gratification must then stream from his brain over his entire body! His representation will cease to be distinct; it is not possible for him to look over the entire chain all at once with complete sobriety. But the astonishing multiplicity effecting the most beautiful order animates every fiber of his brain in a splendid concord. It energizes the play of all the nerves; the mathematician swims in rapture.'[5]

This passage seems to have been his only explicit discussion of beauty in mathematics, but it fits into the rest of his aesthetic theory. He was almost certainly aware of Hutcheson's views on the beauty of theorems and mathematical objects, for he rejected the explanation offered by 'that English philosopher' for the commonalities of beautiful things, namely, an appeal to a sense of beauty that had been emplaced by God.[6] Even if the experience of beauty ultimately derived from some divine intention, it was necessary to explain the efficient causes of the experience.[7]

Mendelssohn argued that the mind has a preference for perfection over imperfection in things perceived: 'Each concept of perfection, harmony, and flawlessness is preferred by our soul to the deficient, the imperfect, and the discordant.'[1] Such perfection was not in itself beauty, but knowledge of this perfection was called beauty if it was *sensuous*, meaning that it was felt by the senses so that many of its features were perceived together and inseparably.[2] A beautiful object had to have multiple parts: monotony lacked appeal. The parts had to fit together harmoniously as a whole, in the sense that there had to be apparent order and regularity in how they combine. And the whole could not be too small or too big: in the former case, 'the mind misses the multiplicity', in the latter, 'it misses the unity in the multiplicity'.[3] Here, in different words, the Crousazian 'uniformity amidst variety' reappeared, and Mendelssohn seems to have thought that the pleasure is caused by the mind's attention switching back and forth between the perceiving the assemblage of parts and grasping the whole:[4]

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[1] Reid, 'Of Taste', p. 501.

[2] *Loc. cit.*

[3] Beiser, *Diotima's Children*, p. 196.

[4] Guyer, *Hist. Modern Aesthetics*, vol. 1, p. 347.

[5] Mendelssohn, 'On Sentiments', p. 54.

[6] Mendelssohn, 'On the Main Principles of the Fine Arts and Sciences', p. 171, esp. n. 2.

[7] *Ibid.*, p. 171.

> 'I contemplate the object of the pleasure, I reflect upon all sides of it, and strive to grasp them distinctly. Then I direct my attention at the general connection among them; I swing from the parts to the whole. The particular distinct concepts recede as it were back into the shadows. They all work on me but they work in such a state of equilibrium and proportion to one another that the whole alone radiates from them, and my thinking about it has not broken up the manifold, but only made it easier to grasp.'[5]

In the letters that make up 'On Sentiments', where the criterion of unity in multiplicity appears, Mendelssohn's spokesman Theocles explains how this criterion arises from our finite mental powers: our intellectual limitations cause us to perceive a multiplicity where greater minds might see an unappealing monotony, and also place a limit on the degree of the multiplicity we can unpack.[6]

Two particular remarks of Mendelssohn foreshadowed later discussions of the aesthetics of mathematics, one negative, the other positive. First, Mendelssohn seems to have seen no scope for aesthetic value as a guide to mathematical discovery:[*]

> 'The profound mathematician who excavates the most hidden truths improves his soul. But the senses take no part in the joy as long as he proceeds laboriously from truth to truth. In this sequence of his reflections one distinct concept gives way to another. Sheer work! Sheer troublesome work!'[7]

Beauty in proof could be seen in retrospect, but could not be seen during the process of discovery, for each step in the proof was a distinct concept and so was not sensuous in Mendelssohn's sense; thus aesthetic considerations could not suggest the way ahead.

Mendelssohn also made comments that seem to suggest that mathematics could be a refuge from the world, distantly presaging a view that emerged fully in the twentieth century.* He initially referred to the philosopher's 'contemplation of the structure of the world', but the example he gave was the story of the death of Archimedes, absorbed in his diagrams,[1] and he specifically referred to nature's beauty:

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[1] Mendelssohn, 'On the Main Principles of the Fine Arts and Sciences', p. 172.

[2] *Loc. cit.*

[3] *Ibid.*, p. 175.

[4] Cf. Guyer, *Hist. Modern Aesthetics*, vol. 1, p. 346.

[5] Mendelssohn, 'On Sentiments', pp. 14–15.

[6] *Ibid.*, p. 23.

[*] ▸ pp. 651–713

[7] *Ibid.*, p. 54.

> 'The contemplation of the structure of the world thus remains an inexhaustible source of pleasure for the philosopher. It sweetens his lonely hours, it fills his soul with the sublimest sentiments, withdrawing his thoughts from the dust of the earth and bringing them nearer to the throne of divinity. Because of his contemplations he must perhaps dispense with honor, sensual ecstasy, and riches; for him they are but dust upon which he treads with his feet. Preoccupied like Archimedes, he says to the persecutors who stand behind him with their swords drawn: *Whatever you do, don't mess up my circle!* But how must he prepare himself for this fullness of pleasure? Well, my dear young man, here is the way to true pleasure! Apply my doctrine to the beauty of nature in general.'[2]

IMMANUEL KANT

The life of Immanuel Kant (1724–1804) is often divided into two epochs: the *pre-Critical* and *Critical* periods, the dividing line being the 1781 publication of his *Critique of Pure Reason*, which studied epistemology and metaphysics, arguing that any viable metaphysics must be both rationalist and empiricist. The Critical period saw him publish a number of important works, raising him to become the dominant figure in German philosophy. These works included a second edition of the *Critique of Pure Reason* (1787), the *Critique of Practical Reason* (1788), which concerned ethics, and the *Critique of the Power of Judgment* (1790), which examined aesthetics.

Kant studied at the University of Königsberg (1740–7) and subsequently worked as a tutor and *Privatdozent* (unpaid lecturer) before eventually being appointed by the University to a chair in metaphysics and logic in 1770. According to notes of his lectures on anthropology made by one of his students in the 1760s or 1770s, he said:

> 'We have beautiful ideas about objects that have no beauty at all, e.g. mathematical figures are not beautiful, geometrical demonstrations can through their shortness have a beauty, because of their completeness, natural light, and ease of comprehension. The pleasure in the easiness of proofs produces their beauty. There is here an agreement with the subjective laws of understanding, so that I can comprehend something very easily.'[3]

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* ▸ pp. 547–50

[1] See Plutarch, *Marcellus*, STE 308–9; § XIX.3–6.

[2] Mendelssohn, 'On Sentiments', pp. 15–16, emphasis in orig.

[3] Kant, *Anthropologie* (Ms. Collins), p. 183; trans. C.

Kant was certainly aware that others saw beauty in mathematical figures, since he was familiar with and an admirer of Hutcheson's *Inquiry*.[1] He did not explain here why he disagreed, but maintained this view later in life and gave an explanation for it when he wrote the *Critique of the Power of Judgment*. But his view of the beauty in proof was in the same rationalist mould as Mendelssohn's: the beauty was connected with how easily our minds could grasp it. Kant would later change his mind and deny that there was beauty in proof.

Mathematics was taught poorly at the University of Königsberg.[2] Although the subjects Kant lectured and tutored included mathematics,[3] he seems to have had little experience of mathematical discovery.[4] Nevertheless, mathematics came to play a crucial role in Kant's philosophy, for mathematical truths are synthetic *a priori*.[5] The *a priori* truths are those that are true independent of experience; the *a posteriori* truths are those whose truth depends on experience. If experience had been different, the former would certainly still be true and the latter would possibly be false.[6] *Analytic* truths are those whose truth is guaranteed by the meaning of the terms involved; at most an effort of analysis is requires to show that the predicate is contained in the subject. *Synthetic* truths are those where the predicate contains something more.[7] Kant's quintessential example of a synthetic *a priori* truth was the result that the interior angles of a triangle are equal to two right angles, which is part of Proposition I.32 of Euclid's *Elements*. Such a result was not found by analysing the terms involved:

> 'Give a philosopher the concept of a triangle, and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on this concept as long as he wants, yet he will never produce anything new. He can analyze and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts.'[8]

What the geometer did to establish the result was not analysis, but *construction*:[9] for a triangle $\triangle ABC$, they extend the side BC of the triangle to D to create two adjacent angles $\angle ACB$ and $\angle DCA$ that sum to two right angles, then draw CE parallel to AB, and finally apply the facts that $\angle ECA = \angle BAC$ and $\angle DCE = \angle CBA$ (see Figure 10.1).

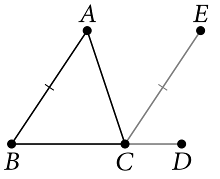

FIGURE 10.1
Construction for proof of Euclid, *Elements*, Prop. I.32.

Critique of the Power of Judgment

Neither of the earlier *Critiques* had suggested that a third lay ahead, and in the introduction to the *Critique of Pure Reason*, Kant had argued that beauty could not be the subject of a science and that there could be no criteria or rules of taste.[10] There may been some change in his view of the tasks of philosophy which led him to write the *Critique of the Power of Judgment* with its extended treatment of aesthetic judgement, or perhaps he simply had a desire for completeness.[1]

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[1] Winegar, 'Kant and Hutcheson on Aesthetics and Teleology', p. 72.

[2] Kuehn, *Kant*, ch. 2.

[3] *Loc. cit.*

[4] Wenzel, 'Mathematics and Aesthetics in Kantian Perspectives', p. 103.

[5] Kant, *Critique of Pure Reason*, KCP A10/B14–15.

[6] *Ibid.*, KCP A2/B2.

[7] *Ibid.*, KCP A6/B10.

[8] *Ibid.*, KCP A716/B744.

[9] *Ibid.*, KCP A716–17/B744–5.

[10] *Ibid.*, KCP A21/B35–6.

In the *Critique of the Power of Judgment*, Kant seems to have adopted Edmund Burke's (1729–97) approach in considering two aesthetic categories: the beautiful and the sublime. Interest in the sublime had developed after the rediscovery and publication in the seventeenth century of *On the Sublime* [*Peri Hypsous* Περὶ῞Υψους], a first century CE treatise conventionally attributed to Longinus, but Burke had placed the sublime — a quality of grandeur or greatness, perhaps overwhelming — on the same level as beauty.[2] Kant, when he wrote the *Critique of the Power of Judgment*, divided its first half into 'Analytic of the Beautiful' and 'Analytic of the Sublime'. (He did not treat in the same detail elegance, though this latter term occurs in his discussion of mathematics.)

In the former part, Kant began by analysing the judgement of taste in four 'moments'. In the first moment, he pointed out that this judgement was not a cognitive one, but an aesthetic one, for it was based solely on a feeling of pleasure or displeasure and therefore had to have a subjective ground.[3] This judgement was *disinterested*: a thing was judged beautiful not because it was agreeable (and so would gratify some desire) or useful for some end;[4] this distinction was one that Shaftesbury had made first and with which Hutcheson had agreed.

In the second moment, Kant noted that a judgement of taste claimed universal assent: when one made such a judgement about a thing, one expected others to concur; indeed, one spoke of beauty as if it were a property of the thing and the judgement a logical one. But the judgement was *not* logical and did not involve the concepts that the thing fell under; hence the beauty of a thing could not follow from any of the concepts that applied to it.[5] Hence: 'That is *beautiful* which pleases universally without a concept.'[6]

In the third moment, he considered what relation existed between ends and judgements of taste. He concluded: 'BEAUTY is the form of the PURPOSIVENESS of an object, insofar as it is perceived in it WITHOUT REPRESENTATION OF AN END.'[7] The judgement of taste rested only on the form of purposiveness. Pleasure in the beauty of a thing did not depend on any recognition of a practical or moral *end* that it served. Kant held that purposiveness could exist without an end; rather, a thing was purposive if it could only be explained by assuming 'a causality in accordance with ends'.[8] Thus a beautiful thing was one that stimulated the free play of the mind in a way that satisfied the fundamental cognitive purpose of finding unity in aesthetic experience but served no more specific purpose.[9]

Further, the pleasure in aesthetic judgement was *a priori* because, although the feeling was determined by something, it was not an *effect* of that thing.[10] And judgements of taste could not be based on emotion, which would happen if the purposiveness arose from the feeling of pleasure, and not the other way around.[1]

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[1] Guyer, *Hist. Modern Aesthetics*, vol. 1, pp. 426–8.

[2] *Ibid.*, vol. 1, pp. 148–9.

[3] Kant, *Critique of the Power of Judgment*, KGS vol. 5, pp. 203–4 [§ 1].

[4] *Ibid.*, KGS vol. 5, pp. 204–11 [§§ 2–5].

[5] *Ibid.*, KGS vol. 5, p. 211–19 [§§ 6–9].

[6] *Ibid.*, KGS vol. 5, p. 219 [§ 9].

[7] *Ibid.*, KGS vol. 5, p. 236 [§ 17]; bold emphases changed to small capitals.

[8] *Ibid.*, KGS vol. 5, p. 220 [§ 10].

[9] Guyer, *Hist. Modern Aesthetics*, vol. 1, p. 435.

[10] Kant, *Critique of the Power of Judgment*, KGS vol. 5, p. 222 [§ 12].

But Kant, in the third moment, also allowed a distinction between *free beauty* and *adherent beauty*.[2] The former was what he had considered thus far; the latter differed in that it did 'presuppose a concept of the end that determines what the thing should be, hence a concept of its perfection'.[3] Kant did not explain this presupposition exactly, and there are various possibilities.[4] But adherent beauty involved how well the thing fulfils its perceived end.

In the fourth moment, Kant returned to how judgements of taste claimed universal assent. He held that the expectation of agreement was, in fact, reasonable: since judgements of taste were based on the play of cognitive faculties of imagination and understanding, and since cognitions can be communicated, there was good reason to assume a common sense in judgements of taste.[5] That is, the feeling of beauty was 'a NECESSARY satisfaction'.[6]

Beauty and mathematical objects

In the *Critique of the Power of Judgment*, Kant held that there was no such thing as 'beautiful science', for the pleasure in the feeling of beauty arose from the free play of the cognitive faculties, not from the determinative judgements used in science.[7]

In particular, therefore, there could be no beauty in mathematics. It seems that there were several reasons for this. The simplest is that many passages in the *Critique of the Power of Judgment* point to judgements of beauty being about objects of the senses.[8] But Kant was not explicit on this point, so it is not clear if he intended to rule out beauty of non-sensory things, including mathematical objects, theorems, proofs, and so forth.

But the very nature of the objects of mathematics seems to make them incompatible with Kant's theory of beauty. For instance, figures such as circles, polygons, and polyhedra, which Hutcheson and others had thought beautiful, could not, for Kant, be beautiful:

> 'Now geometrically regular shapes — a circle, a square, a cube, etc. — are commonly adduced by critics of taste as the simplest and most indubitable examples of beauty; and yet they are called regular precisely because they cannot be represented except by being regarded as mere presentations of a determinate concept, which prescribes the rule for that shape (in accordance with which it is alone possible). Thus one of the two must be wrong: either the judgment of the critics that attributes beauty to such shapes, or ours, which finds purposiveness without a concept to be necessary for beauty.'[9]

Therefore it was because these figures could be represented only as presentations of concepts that they could not be beautiful. Whatever preference there might be for a circle over a 'scribbled outline' or a

[1] Kant, *Critique of the Power of Judgment*, KGS vol. 5, p. 223–6 [§§ 13–14].

[2] *Ibid.*, KGS vol. 5, p. 240 [§ 16].

[3] *Ibid.*, KGS vol. 5, p. 230 [§ 16].

[4] Guyer, *Hist. Modern Aesthetics*, vol. 1, p. 439.

[5] Kant, *Critique of the Power of Judgment*, KGS vol. 5, p. 238 [§ 21].

[6] *Ibid.*, KGS vol. 5, p. 240 [§ 22]; bold emphasis changed to small capitals.

[7] *Ibid.*, KGS vol. 5, p. 304 [§ 44].

[8] Wenzel, 'Beauty, Genius, and Mathematics', pp. 420–1.

[9] Kant, *Critique of the Power of Judgment*, KGS vol. 5, p. 241.

square over an irregular quadrilateral could be explained by 'only common understanding and no taste at all'.[1]

But mathematical objects in general could only be represented as presentations of concepts. Kant's explanation of his view of the construction of mathematical concepts has been accurately described as 'opaque'.[2] Fortunately, the details are less important here than the simple fact that, for Kant, the mental acts of doing mathematics were grounded in the *concepts*:

> 'to CONSTRUCT a concept means to exhibit *a priori* the intuition corresponding to it. For the construction of a concept, therefore, a NON-EMPIRICAL intuition is required, which consequently, as intuition, is an INDIVIDUAL object, but that must nevertheless, as the construction of a concept (of a general representation), express in the representation universal validity for all possible intuitions that belong under the same concept.'[3]

For instance, to construct the concept of a triangle meant to picture a particular triangle that corresponded to the concept; this particular triangle had to represent all possible triangles that were subsumed under the concept of the triangle. An algebraic proof was 'a characteristic construction' that used symbols to denote the concepts.[4] In short, there was no mathematical object 'without a concept'. Mathematical objects could only be represented as presentations of concepts. And if one ceased to think of, say, a triangle as corresponding to the concept, one ceased to think of it as a *mathematical* object. So the idea that there could be beauty in mathematical objects was antithetical to Kant's conclusion in his analysis of beauty that judgements of taste could not depend on concepts.

[The philosopher Christian Wenzel offered an interpretation of the incompatibility of beauty and mathematical objects: that because the objects of mathematics are precisely defined and can be grasped in an entirely rational way, they lack a kind of '*ineffable* and *infinite* possible variety' necessary for the play of the cognitive faculties.[5]]

There was a final possible incompatibility: Kant repeatedly wrote that a judgement of taste was made based on a *given* representation of a thing.[6] If this was a precondition for beauty, it excluded the objects of mathematics, which were *constructed* by the mathematician.

Kant recognized that properties of mathematical objects were often *called* beautiful, on account of what he called 'a certain *a priori* purposiveness [...] for all sorts of cognitive use'.[7] The purposiveness of mathematical objects is 'objective and intellectual',[8] which could be considered as 'merely formal (not real), i.e., as purposiveness that is not grounded in a purpose'.[9] That the purposiveness was objective is clear: it was a relation between mathematical objects. (In contrast, objects of aesthetic appreciation had subjective purposiveness, because of the relation to the appreciating subject.) The purposiveness was also formal because, although the objects could be and were used for

[1] Kant, *Critique of the Power of Judgment*, KGS vol. 5, p. 241.

[2] Young, 'Construction, Schematism, and Imagination', p. 159.

[3] Kant, *Critique of Pure Reason*, KCP A713/B741; bold emphases changed to small capitals.

[4] *Ibid.*, KCP A734/B762.

[5] Wenzel, 'Beauty, Genius, and Mathematics', 421, emphases in orig.

[6] Wenzel, 'Beauty, Genius, and Mathematics', p. 420; Kant, *Critique of the Power of Judgment*, KGS vol. 5, p. 203 [§ 1].

[7] Kant, *Critique of the Power of Judgment*, KGS vol. 5, p. 366 [§ 62].

[8] *Ibid.*, KGS vol. 5, p. 362 [§ 62].

[9] *Ibid.*, KGS vol. 5, p. 364 [§ 62].

the solution of myriad mathematical problems, they were not *created* for such an end.[1]

The judgement that led to application of the aesthetic term *beautiful* to a mathematical object was thus not a genuinely aesthetic judgement, for it involved a concept and so was not grounded in the *subjective* purposiveness in play of cognitive faculties, but rather in awareness of an *objective* purposiveness, namely its utility. It was not beauty but rather a 'RELATIVE PERFECTION'.[2] [This notion seems to have been close to Kant's notion of adherent beauty, but he did not explore the distinction, if there was one.]

Beauty and proof

Following his explanation of why properties of mathematical objects are incorrectly called beautiful, Kant wrote:

> 'The designation of an INTELLECTUAL BEAUTY can also not be allowed at all, for otherwise the word "beauty" would have to lose all determinate meaning, or intellectual satisfaction would have to lose all preeminence over sensible satisfaction. One could rather call a DEMONSTRATION of such properties beautiful, since by means of this the understanding, as the faculty of concepts, and the imagination, as the faculty for exhibiting them, feel themselves strengthened *a priori* (which, together with the precision which is introduced by reason, is called its elegance): for here at least the satisfaction, although its ground lies in concepts, is subjective, whereas perfection is accompanied with an objective satisfaction.'[3]

Demonstration (the word is identical in German) was Kant's preferred term for mathematical proof.[4] *Prima facie*, this quotation reads like Kant denied any kind of intellectual beauty, then retreated to allow beauty in proof. But it seems instead to have been a concession that it was *less wrong* to call a proof beautiful than to call a property beautiful, for the '*at least* satisfaction [...] subjective, not objective'.[5] But the satisfaction still depended on concepts, and so could at most resemble, and not truly be, an *aesthetic* pleasure.

Another interpretation

However, Kant's discussion about beauty and demonstrations left open the possibility that there might be a theory of beauty in proofs that is compatible with Kant's thought. In 2013, the philosopher Angela Breitenbach described such a Kantian theory of beauty in proofs. The theory depends on the contrast between Kant's views of how mathematical deduction proceeds and of how mathematical objects are cognized.

For Kant, knowledge of particular properties of mathematical objects did not comprise analytic truths, but synthetic *a priori* truths.

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[1] Kant, *Critique of the Power of Judgment*, KGS vol. 5, p. 364 [§ 62].

[2] *Ibid.*, KGS vol. 5, p. 366 [§ 62]; bold emphasis changed to small capitals.

[3] Kant, *Critique of the Power of Judgment*, KGS vol. 5, p. 366 [§ 62]; bold emphasis changed to small capitals; trans. mod. per Breitenbach, 'Beauty in Proofs', pp. 961–2.

[4] Kant, *Critique of Pure Reason*, KCP A734–5/B762–3.

[5] Kant, *Critique of the Power of Judgment*, KGS vol. 5, p. 366 [§ 62], emphasis added.

They could not be deduced analytically, but only synthetically, which required them to be represented in intuition.[1] The deduction of such properties — a mathematical demonstration — had to be 'an apodictic proof, insofar as it is intuitive' and had to 'proceed through the intuition of the object'.[2]

The process of deduction began with the direct construction in intuition of the object according to a concept; it ended with a cognitive judgement that the object had certain properties. The initial construction supplied an individual object that represented *all* objects that corresponded to the concept. In imagination, just as on paper, the particular constructed object 'serves to express the concept without damage to its universality'.[3] Thus the imagination generated non-conceptual objects that nevertheless accorded with the concepts, and subsequent reflection on the objects yielded insights about their properties.[4] Breitenbach argued that *between* the initial construction from a concept and the eventual cognizing of a property, the imagination made a distinctive contribution that was not grounded in determinate concepts.[5] The steps in a proof starting from the initial construction did not comprise reading from empirical data (for the knowledge obtained is *a priori*). Nor did they consist of analysis of the concepts (for the knowledge obtained is synthetic). Thus the imagination produced unities *independent of concepts*. This, Breitenbach maintained, could be viewed as the mathematical reasoner

> 'imaginatively *playing around* with the mathematical objects constructed in a priori intuition before, subsequently, recognising — conceptually — that they have particular properties or stand in certain relations to one another.'[6]

Here, then, was the locus of the free play of the imaginative faculties.[7] Both the mathematician's discovery of the proof, and the student's re-creation of the proof when working through it, involved this free activity of the imagination:[8]

> 'Just as in aesthetic judgment, in mathematical demonstrations the imagination acts freely, unconstrained by and yet in harmony with the understanding.'[9]

Admiration of mathematical demonstrations came about because, in one's reflection on the objects, the conceptual understanding was aided by the imagination. There was astonishment and admiration in the harmony that mathematical proofs made evident between the concepts given by the understanding — which gave rise to difficult problems they involved — and the constructions carried out using the representation in the intuition — which could give rise to solutions.[10] In the case of mathematical *objects*, the admiration of the harmony was not the admiration of free play, but of cognitive judgements, grounded upon concepts, and so was the admiration of 'relative perfection'.[11] But the admiration of a demonstration was different. There *was* free play: the experience of beauty was evoked by the free play of the intellect, not by the mathematical objects, not by their properties, and not by their suitability for solving problems.[1] Thus there was a genuine aesthetic experience.

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[1] Kant, *Critique of the Power of Judgment*, KGS vol. 5, p. 364 [§ 62]; see also Breitenbach, 'Beauty in Proofs', pp. 963–4.

[2] Kant, *Critique of Pure Reason*, KCP A734–5/B762–3.

[3] *Ibid.*, KCP A734–5/B762–3.

[4] Breitenbach, 'Beauty in Proofs', p. 966.

[5] *Ibid.*, p. 965.

[6] *Ibid.*, p. 966, emphasis added.

[7] *Ibid.*, pp. 966–7.

[8] *Ibid.*, p. 967.

[9] *Loc. cit.*

[10] Kant, *Critique of the Power of Judgment*, KGS vol. 5, p. 365 [§ 62]; Breitenbach, 'Beauty in Proofs', pp. 963–4, 968–9.

[11] Breitenbach, 'Beauty in Proofs', p. 965.

As Breitenbach pointed out, Kant's discussion of mathematical beauty was focused on geometry, not arithmetic, and that there is scholarly disagreement over whether Kant thought intuition was essential for *all* proofs and for the *whole* of proofs. If intuition were used just for the initial construction, not in the subsequent steps, then Breitenbach's interpretation may run into trouble.[2] If intuition were used only in some proofs, or some *kinds* of proofs, then other proofs would not involve the free play of the imagination and so could not generate an aesthetic response. For Breitenbach, such a restriction of mathematical beauty to only some proofs was troubling, and she thought it inconsistent with the actual phenomenology of responses to proofs.[3]

Elegance in mathematics

But although Kant seems to have rejected the idea that there was beauty in mathematics, he still accepted that there was elegance:

> 'One could rather address to the modern geometers a reproach of the following nature: not that they derive the properties of a curved line from its definition without first being assured of the possibility of its object (for they are fully conscious of this together with the pure, merely schematic construction, and they also bring in mechanical construction afterwards if it is necessary), but that they arbitrarily think for themselves such a line (e.g., the parabola through the formula $ax = y^2$), and do not, according to the example of the ancient geometers, first bring it forth as given in the conic section. This would be more in accordance with the elegance of geometry, an elegance in the name of which we are often advised not to completely forsake the synthetic method of the ancients for the analytic method which is so rich in inventions.'[4]

Kant wrote this passage in 1789, having suspended his writing of the *Critique of the Power of Judgment* in order to reply to criticisms of the *Critique of Pure Reason* made by Johann August Eberhard (1739–1809). Thus his rejection of beauty in mathematics had not entailed a rejection of *elegance*. Note that his 'reproach' seems to have been grounded purely in a concern for elegance: the synthetic method was no more rigorous than the analytic; it was merely desirable in order to agree with the 'elegance of geometry'.[5] [Kant here was addressing an ongoing debate among mathematicians over the relative merits of the methods of synthetic geometry used by the ancients and the more modern techniques of analytic geometry.*]

[1] Breitenbach, 'Beauty in Proofs', p. 968.

[2] *Ibid.*, pp. 969–70.

[3] *Ibid.*, p. 970.

[4] Kant, *On a Discovery*, KGS vol. 8, p. 192.

[5] Cf. Brittan, 'Algebra and Intuition', p. 330.

* ▸ pp. 342–6

Thus a judgement of elegance, for Kant, could not be an aesthetic judgement. And this fits with what he wrote in his discussion of the incorrect use of 'beautiful' for proofs, quoted above: the elegance of a proof depended on the precision it has, and this precision was an objective satisfaction. It is possible that Kant's change of mind on beauty in proof stemmed from his separation of the subjective and objective satisfactions of purposiveness.[1] What he had previously seen as beauty (subjective) in proof he came to see as elegance (objective).

ARCHIBALD ALISON

Archibald Alison (1757–1839) developed a theory of taste that was grounded in the emotional response to a percept. He held that there was no sense of beauty, for aesthetic responses were not simple, but complex. The perception of beauty started a train of thoughts. Each part of the train was able to cause emotion, and the whole was accompanied by the 'Emotion of beauty'.[2]

His account of beauty touched upon mathematics at one point only, when, during an extensive examination of beauty of form, he considered the beauty of geometrical figures such as polygons and curves. *Uniformity* signified the similarity of the parts of a thing considered separately; *regularity* signified similarity between parts constituting the whole of a thing. Regularity thus implied uniformity, but the converse did not hold: there might be uniformity between the parts of a thing without there being a general similarity.[3] The qualities of uniformity and regularity are both indicative of *design*, from which the beauty of form arose.[4] The degree of beauty depended on the 'difficulty' of the attainment of uniformity, which increased with the number of parts involved, for uniformity among a large number of parts carried a greater suggestion of design.

Alison thought that the beauty of a geometrical figure was determined by considering it as a figure, ignoring whatever role it may have in science or art.[5] Applying his conception of uniformity yielded the relative beauties of various figures:

> 'An Equilateral Triangle is more beautiful than a Scalene or an Isosceles, a Square than a Rhombus, an Hexagon than a Square, an Ellipse than a Parabola, a Circle than an Ellipse; because the number of their uniform parts are greater, and their Expression of Design more complete.'[6]

In general, regular figures were more beautiful than irregular ones, and those with a greater number of parts more beautiful than those with fewer.[7] But there was a limit: figures that have so many parts as to cause confusion are no longer beautiful.[8]

[1] Wenzel, 'Beauty, Genius, and Mathematics', p. 426.

[2] Alison, *Essays on the Nature and Principles of Taste*, vol. I, pp. xxii–xxiii, 74–5.

[3] *Ibid*., vol. II, pp. 63–4.

[4] *Ibid*., vol. II, pp. 59–60, 64.

[5] *Ibid*., vol. II, pp. 66–7.

[6] *Ibid*., vol. II, pp. 67.

[7] *Loc. cit.*

[8] *Loc. cit.*

Alison's treatment of the beauty of mathematical figures most obviously differs from Hutcheson's* in that he considered only uniformity, not uniformity amidst variety. But the difference was more fundamental in that Alison, unlike Hutcheson, did not consider the figures *qua* mathematical objects.

DUGALD STEWART

The work of Dugald Stewart (1753–1823) was a capstone to eighteenth-century British aesthetics, synthesizing aspects of previous thinkers.[1] Stewart studied mathematics and moral philosophy at the University of Edinburgh, then attended Thomas Reid's classes in philosophy at Glasgow. From 1772 he acted as substitute to his father Matthew Stewart† (1717–85), who was professor of mathematics at the University of Edinburgh. The younger Stewart was elected to the chair in his own right in 1775. However, his main interest remained moral philosophy (his *Collected Works* contain no mathematical works), and in 1785 he was elected professor of moral philosophy at Edinburgh and resigned the chair in mathematics.[2]

Although Stewart has sometimes been considered a follower of Reid, one of his fundamental insights places him in diametric opposition to Reid's perspective about the use of language. Stewart accepted that it was proper to apply the words 'beauty' and 'beautiful' in a wide variety of contexts:

> 'We speak of beautiful colours, beautiful forms, beautiful pieces of music: We speak also of the beauty of virtue; of the beauty of poetical composition; of the beauty of style in prose; of the beauty of a mathematical theorem; of the beauty of a philosophical discovery.'[3]

Prima facie, these uses had little in common, other than referring to something which gave rise to 'a certain refined species of pleasure'.[4] Philosophers, though, had taken the use of language to imply that there was some common quality or qualities that distinguished the class of things that could be called 'beautiful', and had tried to identify such properties. Stewart held that this approach was based on what he termed a 'prejudice' that modern philosophy had inherited from scholasticism:

> 'when a word admits of a variety of significations, these different significations must all be *species* of the same *genus*, and must consequently include some essential idea common to every individual to which the generic term can be applied.'[5]

In place of this 'prejudice', Stewart put forth an alternative argument of how a single term could come to be applied to widely different

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* ▸ p. 358

[1] Guyer, *Hist. Modern Aesthetics*, p. 235; Kivy, *The Seventh Sense*, p. 207.

† ▸ pp. 343–4

[2] Veitch, *A Memoir of Dugald Stewart*, chs 1–2.

[3] Stewart, 'On the Beautiful', p. 191; cf. Stewart, 'On the Beautiful', p. 253.

[4] Stewart, 'On the Beautiful', p. 191.

[5] *Ibid.*, pp. 193–4, emphases in orig.

objects. In brief, his idea was that the same term could come to be applied to *A* and *E*, despite their having nothing in common, through common qualities being found in *A* and *B*, in *B* and *C*, in *C* and *D*, and in *D* and *E*.[1]

Thus Stewart did not seek a theory that reduced beauty to certain qualities of objects or to the effects of those qualities, but rather attended to 'the natural history of the Human Mind'[2] in putting forth a theory of how the term 'beauty' came to be so widely applied. Regrettably, he did not extend this investigation to beauty in mathematics: at the end of his essay 'On the Beautiful', he remarked that he had originally intended to give further observations on beauty as it applied to other topics including 'Geometrical Propositions', but instead decided to reserve these topics for future discussion.[3]

However, something of his views on beauty in mathematics can be gleaned from other works. First, unlike any of his predecessors since Hutcheson and André, he left an example of a result he thought beautiful: the

> 'beautiful theorem of Huygens, which demonstrates that the time of a complete oscillation of a pendulum in the cycloid, is to the time in which a body would fall through the axis of the cycloid, as the circumference of a circle is to its diameter'.[4]

This result, enunciated differently,[*] was thought beautiful by the historian of mathematics Jean-Étienne Montucla[†] (1725–99), who was a contemporary of Stewart.

Stewart also gave a condition for beauty in an 'arrangement' in pure mathematics; this seems to have been his term for a theory, or perhaps a less formal collection of concepts, results, and proofs. In pure mathematics, 'all the various truths are necessarily connected with each other',[5] at least via the fundamental postulates, and

> 'an arrangement is beautiful in proportion as the principles are few; and what we admire perhaps chiefly in the science, is the astonishing variety of consequences which may be demonstrably deduced from so small a number of premises.'[6]

Such a response to mathematics did not apply to natural philosophy, where the admiration was not directed to the way the results fit together in a systematic way, but rather to the underlying unity of nature that gave rise to simple laws that limited human minds could understand.[7] However, Stewart did not deny that individual theories could have aesthetic value, or that aesthetic evaluation could play a role in theory choice: the Copernican system was not empirically superior to other hypotheses, but rather aesthetically superior due to its simplicity and beauty.[8]

Stewart held that the elegance of a proof, however, was entirely divorced from its epistemic value. Mathematicians might seek new proofs of a theorem, but their aim was not to add to the evidence of its truth:

[1] Stewart, 'On the Beautiful', pp. 195–6.

[2] *Ibid.*, p. 253.

[3] *Ibid.*, p. 275.

[4] Stewart, *Elements*, Part First, ch. VI, § 5 ⟦vol. II, p. 400⟧.

[*] ▸ p. 309

[†] ▸ pp. 346–51

[5] *Ibid.*, Part Third, ch. I, § 3 ⟦vol. IV, p. 214⟧.

[6] *Ibid.*, Part Third, ch. I, § 3 ⟦vol. IV, p. 214⟧.

[7] *Ibid.*, Part Third, ch. I, § 3 ⟦vol. IV, pp. 214–15⟧.

[8] *Ibid.*, Part Second, ch. IV, § 4 ⟦vol. III, p. 300⟧.

> ‘In point of simplicity, and of what geometers call elegance, these various demonstrations may differ widely from each other; but *in point of sound logic, they are all precisely on the same footing*. […] the first which occurs (provided they be all equally understood) commands the assent not less irresistibly than the last.’[1]

VICTOR COUSIN

Victor Cousin (1791–1867) was the proponent of an eclectic method of philosophy whose declared but hopeless aim was to extract the true parts from each school and from them to synthesize a higher system. He gave a course of lectures *On the True, the Good, and the Beautiful* [*Du Vrai, du Beau et du Bien*] in 1818 at the Sorbonne. The 1836 book of the same title was based on notes taken in the original class, while Cousin substantively rewrote the 1854 edition.[2]

Cousin knew well the work of Hutcheson,[*] Reid,[†] and Stewart.[‡] He considered Hutcheson to have been the first modern philosopher of aesthetics,[3] though Reid was the greater influence on him, both generally and in terms of his theory of beauty.[4] Like Reid, Cousin thought that beauty was an objective property of an object that the human mind perceives. When one judged something beautiful, the situation was the same as judging something true: the judgement was not of a varying perception, but an absolute judgement imposed by reason.[5] Ethical and epistemic judgements were equally objective. Indeed, truth, goodness, and beauty only appeared different to because they were experienced them differently: ‘Truth exists by itself; realized in human actions, it becomes goodness; incorporated in sensible forms, it becomes beauty.’[6] And this connection was, in the end, a consequence of God being the source of truth, goodness, and beauty.[7]

When one encountered a beautiful object, the faculties of the mind attached themselves to the object and stayed there with ‘unmixed satisfaction’: ‘In this contemplation, the soul feels once more a sweet and tranquil joy, a kind of blooming’.[8] But what characterized the quality that produced this response? Cousin thought that the most plausible theory was that which considered beauty to be composed of unity in variety. His first example was a beautiful flower, which had unity — order and proportion and symmetry — and variety — subtle shades and rich details.[9]

Cousin’s second example was mathematics: ‘Even in mathematics, what is beautiful is not an abstract principle, it is this principle pulling with it a long chain of consequences.’[10] Cousin did not elaborate precisely what he meant by ‘what is beautiful’ in mathematics, but the mention of consequences suggests *results* as having been the most plausible locus. But note the use of the word ‘[e]ven’: Cousin did not think beauty in mathematics to be a controversial idea, but was rather emphasizing that it fit with the characterization of beauty as unity (the 'principle') in variety (the consequences).

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[1] Stewart, *Elements*, Part Second, ch. III, § 1 ⟦vol. III, p. 186⟧, emphasis added.

[2] Janet, *Victor Cousin et son Œuvre*, pp. 58–63.

[*] ▸ pp. 357

[†] ▸ pp. 369–72

[‡] ▸ pp. 369–72

[3] Will, *Flumen Historicum*, p. 55.

[4] Guyer, *Hist. Modern Aesthetics*, vol. 2, p. 230.

[5] Cousin, *Du Vrai, du Beau et du Bien* (2nd ed.), p. 140.

[6] Cousin, *Du Vrai, du Beau et du Bien* (1st ed.), p. 298; trans. C.

[7] Cousin, *Du Vrai, du Beau et du Bien* (1st ed.), pp. 226, 261; cf. Cousin, *Du Vrai, du Beau et du Bien* (2nd ed.), p. 169.

[8] Cousin, *Du Vrai, du Beau et du Bien* (2nd ed.), p. 146; trans. C.

[9] *Ibid.*, p. 160.

[10] *Ibid.*, p. 160; trans. C.

And because the qualities of beauty and goodness and truth only *seem* different, not only was there ultimately no difference between moral truth and mathematical truth,[1] but '[w]hat the mathematician and the philosopher do, the artist does also'.[2]

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[1] Cousin, *Du Vrai, du Beau et du Bien* (1st ed.), p. 317.

[2] *Ibid.*, p. 299; trans. C.

11

This Sublime Science

Mathematical beauty, poetry, and the age of Romanticism

> ‘the enchanting charms of this sublime science
> are revealed in all their beauty
> only to those who have the courage to delve deeply into it.’
>
> — Carl Friedrich Gauss, letter to Sophie Germain, dated 3 Apr. 1807,
> repr. in Del Centina & Fiocca,
> ‘The correspondence between Germain and Gauss’,
> app. IV (trans. C).

> ‘FAUST. Does some more inward sense than sight perceive
> the overflowing fountainhead of beauty?’
>
> — Johann Wolfgang von Goethe,
> *Faust*, part two, Act II, ll. 6487–8
> (trans. S. Atkins).

Stendhal (Marie-Henry Beyle; 1783–1842) once wrote that Romanticism was the art of creating literary works that give the greatest pleasure to people of the current generation; Classicism, in contrast, aimed to give the greatest pleasure to their great-grandfathers.[1] This distinction does not serve to define Romanticism, if indeed there is even any common thread from which one could weave a definition: the historian of ideas Arthur O. Lovejoy (1873–1962) insisted upon a multiplicity of *Romanticisms*.[2] But it does suggest something of the feeling of the Romanticists: there was a self-conscious retreat from the equally self-conscious Classicism of the Enlightenment. If the eighteenth century reached towards reason alone, the Romanticists looked to the united mind and heart.[3]

The aesthetics of mathematics does not fit easily into the Classicism–Romanticism divide. There had been an ongoing conflict between those who looked to the geometry of the ancients as the ideal and those who took up analytic geometry and algebra.* But there was an emerging awareness that mathematics could be an art form, with particular parallels being drawn between mathematics and poetry. This awareness did not give rise to a defence or justification of mathematics on aesthetic grounds, which came later in the nineteenth century.

In philosophical aesthetics, there was a decline in the role of mathematical beauty. After Immanuel Kant† (1724–1804), philosophical attention seems to have dwindled: the position of mathematical beauty

[1] Stendhal, *Racine and Shakespeare*, p. 38.

[2] Lovejoy, ‘On the Discrimination of Romanticisms’, pp. 235–6.

[3] Barzun, *From Dawn to Decadence*, p. 466.

* ▸ pp. 342–6

† ▸ pp. 374–82

in the thought of Archibald Alison[*] (1757–1839), Dugald Stewart[†] (1753–1828), and Victor Cousin[‡] (1791–1867) was an epilogue to the earlier time.

This shift was doubtless part of the increasing domination in aesthetics of the philosophy of *art*, rather than of *beauty*. The thought of Georg Wilhelm Friedrich Hegel (1770–1831) was to become immensely influential, and the focus of his lectures on *Aesthetics* — a title which he confessed was rather unsatisfactory — was the philosophy of art.[1] Hegel considered the beauty of art, which was born of the spirit, to be superior to the beauty of nature, which only reflected the beauty of the spirit.[2] Had he considered beauty in mathematics, it seems likely that he would have relegated it to the same status as beauty in nature.

Of the most prominent philosophers of aesthetics in the generation after Hegel — Karl Wilhelm Ferdinand Solger (1780–1819), Friedrich Theodor von Vischer (1807–87), Hermann Lotze (1817–81), and Karl Rosenkranz (1805–79)[3] — only Vischer seems to have treated of beauty in mathematics, and then only in negative terms.

Vischer began his six-volume *magnum opus* *Aesthetik* (1846–57) by defining aesthetics as the science of the beautiful, not the philosophy of art,[4] and, although the general temper of his work was arguably Hegelian, he gave due attention to beauty in nature.[5] But he seemed almost perplexed by the idea that there could be beauty in mathematics (which might be related to his self-described 'unmathematical disposition'[6]). He hinted at this puzzlement when he reviewed Aristotle's discussion of the chief forms of beauty in the *Metaphysics*.[§] Vischer initially disregarded their application to mathematics, and proceeded to describe how, in his view, Aristotle had confused the beautiful and the good, and ended by emphasizing that Aristotle had *even* attributed these qualities to mathematics.[7] He made his position still clearer when he argued against Francis Hutcheson's[¶] (1694–1746) characterization of beauty as unity in variety, which he saw as overly dry and crystalline. Vischer thought that Hutcheson had erred because he did not have contingency varying the dictates of his rule.[8] And it was because of this mistake that Hutcheson had said openly that there was beauty in mathematical theorems and figures.[9]

In light of the diminished attention paid to mathematical beauty by the philosophers, this chapter is devoted primarily to the attitudes of mathematicians, and to a lesser extent those of poets, and to one man who was a great mathematician and tried to be a great poet.

The late eighteenth and early nineteenth centuries were a time of great advance in mathematics and its applications. Perhaps the three masters, at least in analysis, were Joseph-Louis Lagrange (1736–1813), Pierre-Simon Laplace (1749–1827), and Carl Friedrich Gauss (1777–1855).[10] Gauss and Lagrange were considered by later writers to have been paragons of mathematical elegance. The historian of mathematics W. W. Rouse Ball (1850–1925) thought that 'Lagrange is perfect both in form and matter [...]. Gauss is as exact and elegant as Lagrange'; Laplace, in contrast, was 'indifferent to style'.[1] George Sarton (1884–1956) also said that Lagrange 'was a consummate artist and his standard of mathematical elegance was exceptionally high'.[2] Augustus De Morgan (1806–71) agreed:

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[*] ▸ pp. 382–3

[†] ▸ pp. 383–5

[‡] ▸ pp. 385–6

[1] Hegel, *Aesthetics*, [vol. 1] p. 1.

[2] Hegel, *Aesthetics*, [vol. 1] p. 2; cf. Guyer, *Hist. Modern Aesthetics*, vol. 2, pp. 139–40.

[3] Guyer, *Hist. Modern Aesthetics*, vol. 2, pt 3, ch. 4.

[4] Vischer, *Aesthetik*, vol. 1, p. 1.

[5] Guyer, *Hist. Modern Aesthetics*, vol. 2, p. 160.

[6] Vischer, *Aesthetik*, vol. 6, p. xii; trans. C.

[§] ▸ pp. 89–90

[7] *Ibid.*, vol. 1, pp. 102–3 [§ 36].

[¶] ▸ pp. 357–62

[8] *Ibid.*, vol. 1, p. 106 [§ 36]; cf. vol. 2, pp. 87–8 [§ 266].

[9] *Ibid.*, vol. 1, p. 106 [§ 36].

[10] Rouse Ball, *A Short Account of the History of Mathematics*, p. 373.

> 'The genius of Laplace was a perfect sledge hammer in bursting purely mathematical obstacles; but [...] it gave neither finish nor beauty to the results. [...] Laplace never attempted investigation of a subject without leaving upon it the marks of difficulties conquered: sometimes clumsily, sometimes indirectly, always without minuteness of design or arrangement of detail'.[3]

Lagrange and Gauss are thus the natural foci for discussion of the attitudes to beauty of mathematicians of this time.

POETRY, PROSE, AND MATHEMATICS

Friedrich Schiller

Friedrich Schiller (1759–1805) exercised a great and lasting influence on poetry, drama, and philosophy, both in his own right and through his friendship with Johann Wolfgang von Goethe (1749–1832). Schiller's theoretical writings ventured into different areas of philosophy, but were largely focused on aesthetics.[4] His poem 'Archimedes and the Student' ['Archimedes und der Schüler'] set out a particular view of mathematics which was self-consciously adopted by Carl Friedrich Gauss (1777–1855) and drew adherents during the nineteenth century:

> 'To Archimedes came an inquisitive youth
> "Initiate me," he said to him, "into the divine science,
> That bore such splendid fruit for the nation
> And shielded the walls of the city from the sambuca!"
> "Divine you call the science? It is," replied the sage,
> "But it was so, my son, even before it served the state.
> If you want only fruit from her, even mortals can provide it;
> Who courts the goddess, seeks not in her the woman."'[5]

[The 'sambuca' [σαμβύκη *sambukē*] was a ship-mounted siege engine used to raise attacking soldiers to the top of the city walls; its use during the Roman siege of Syracuse failed in the face of Archimedes' machines.[6]]

As conceived by Schiller, Archimedes considered mathematics to be 'divine' ['göttlich'], and this value preceded and was independent of any worldly utility. Schiller did not explicate what for him made mathematics divine. Aesthetic value might have been a factor for Schiller; it certainly was for Gauss when he quoted 'Archimedes and the Student' in his 1808 inaugural lecture at Göttingen.* Schiller's valuing the beauty of mathematics apart from utility would fit with his celebrated observation that 'man is only in earnest when it comes to the pleasant, the good, the perfect; but when it comes to beauty he plays […], and *he is only a complete man when he plays*'.[1]

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[1] Rouse Ball, *A Short Account of the History of Mathematics*, p. 373.

[2] Sarton, 'Lagrange's Personality', p. 483.

[3] [De Morgan], Review of *Théorie Analytique des Probabilités*, p. 348; for attribution of this unsigned review, see De Morgan, *Memoir of Augustus De Morgan*, p. 406.

[4] Moland, 'Friedrich Schiller'.

[5] Schiller, 'Archimedes und der Schüler', trans. C.

[6] Plutarch, *Marcellus*, STE 306, § XV.1.

Johann Wolfgang von Goethe

Schiller's friend Johann Wolfgang von Goethe (1749–1832) has sometimes been considered to have been an opponent of mathematics. The mathematician, biologist, and classicist D'Arcy Wentworth Thompson (1860–1948) once wrote that 'Goethe, lover of nature as he was, ruled mathematics out of place in natural history.'[2] But Goethe's attitude was more complex. He valued pure mathematics because of its internal self-consistency. His criticisms of mathematics were strong but limited, and concerned its inappropriate application to nature, where there would always be some remnant of a natural phenomenon that could not be encapsulated in a formula. He also objected to overspecialization and abstraction, whereby an overmastering interest in formalism was an obstacle to deep understanding.[3] One of the 'Reflections in the Spirit of the Wanderers' on 'Art, Ethics, Nature'[4] from Goethe's fourth novel, *Wilhelm Meister's Journeyman Years*, touches on this distinction: 'Mathematics, like dialectics, is an instrument of the inner higher sense, while in practice it is an art like rhetoric.'[5] And both the theoretical role of purely intellectual mathematics and its practice seem to have been connected to an ability to see beauty in scientific truth:

> 'The mathematician is complete only insofar as he is a complete human being, as he is sensible in himself of the beauty inherent in truth. Only then will his work be thorough, transparent, perceptive, pure, clear, graceful, even elegant. All that is needed if one would be like La Grange.'[6]

William Wordsworth

William Wordsworth (1770–1850), who was one of the formative figures in the Romantic period of English literature, admired and enjoyed mathematics. He received an excellent schooling in mathematics, to the extent that when he went up to Cambridge he was far in advance of his fellows, a fact to which he later attributed a tendency towards idleness in his mathematical studies.[7] He had a particular attraction for geometry and none for arithmetic or algebra,[8] and references to geometry appear in his work. In his famous semiautobiographical poem 'The Prelude', he recalled a time spent in its contemplation:

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* ▸ p. 402

[1] Schiller, *On the Aesthetic Education of Man*, § 15, emphasis in orig.

[2] Thompson, *On Growth and Form*, p. 2.

[3] Miller, intro. to Goethe, *Scientific Studies*, pp. xv–xvi; McCarthy, 'Erscheinung, erscheinen (manifestation)'; Goethe, 'Theory of Color', § 752 ⟦p. 277⟧.

[4] Goethe, *Wilhelm Meister's Journeyman Years*, p. 294.

[5] *Ibid.*, p. 311.

[6] *Ibid.*, p. 312.

[7] Moorman, *William Wordsworth*, vol. 1, pp. 28, 90–3.

[8] *Ibid.*, vol. 1, p. 97.

> 'On poetry and geometric truth,
> And their high privilege of lasting life,
> From all internal injury exempt,
> I mused, upon these chiefly [...]'.[1]

Notice how he thought that geometry and poetry were alike immortal. But he also pointed to the *pleasure* of geometry, and lamented that he had not pursued it further:

> 'Yet may we not entirely overlook
> The pleasure gathered from the rudiments
> Of geometric science. Though advanced
> In these inquiries, with regret I speak,
> No farther than the threshold, there I found
> Both elevation and composed delight'.[2]

And some later lines, imagining a ship-wreck survivor distracting himself from his situation with geometry,[3] seem to point more directly to *aesthetic* pleasure and also to a platonic reality:

> '[...] Mighty is the charm
> Of those abstractions to a mind beset
> With images, and haunted by herself,
> And specially delightful unto me
> Was that clear synthesis built up aloft
> So gracefully; even then when it appeared
> Not more than a mere plaything, or a toy
> To sense embodied: not the thing it is
> In verity, an independent world,
> Created out of pure intelligence.'[4]

In the famous preface to the 1802 edition of the collection *Lyrical Ballads*, Wordsworth also admitted that all knowledge was 'built up by pleasure'.[5] But mathematics, especially geometry, seems to have been a special case, because in the same preface he opposed poetry to science:

> 'Poetry is the breath and finer spirit of all knowledge; it is the impassioned expression which is in the countenance of all Science.'[6]

Geometry, as he wrote in 'The Prelude', could sit alongside poetry. It was perhaps the case that he objected to the scientific division and classification of the universe.[7] Geometry, being something more fundamental that explored the unity of the world, was different.

Edgar Quinet

The historian, poet, and philosopher Edgar Quinet (1803–75) admitted to having been prejudiced against mathematics:

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[1] Wordsworth, 'The Prelude', bk 4, ll. 65–8.

[2] *Ibid.*, bk 6, ll. 115–20.

[3] *Ibid.*, bk 6, ll. 142 sqq.

[4] *Ibid.*, bk 6, ll. 158–67.

[5] Wordsworth, 'Preface', pp. 105–6.

[6] *Ibid.*, p. 106.

[7] Moorman, *William Wordsworth*, vol. 1, p. 583.

having heard so many times that mathematics withered the imagination, he had come to believe it.[1] Jean-Baptiste Chachuat (1770–1842), one of his teachers, showed him that the higher mathematics had 'ample and indeed vast' imagination, and not only taught him mathematics, but led him to love it.[2] This benefitted him under less understanding teachers: he never lost his way, attributing this success to his feeling for 'the sublimity, the inexpressible poetry of mathematics'.[3] In algebra in particular, he was

> 'struck by the art with which mathematicians remove, reject, and eliminate little by little everything that is unnecessary to arrive at expressing the absolute, with the smallest possible number of terms, all while maintaining in the arrangement of these terms a distinction, a parallelism, a symmetry that seems to be the elegance and visual beauty of an eternal idea.'[4]

In the application of algebra to geometry, the amazement induced by conic sections was even greater than that caused by the *Thousand and One Nights*. The idea of expressing a curve in algebraic terms seemed to him as beautiful as the *Iliad*.[5]

SOPHIE GERMAIN

Despite initial opposition from her family and the myriad obstacles that her times placed in the path of a woman seeking a scientific education, Sophie Germain (1776–1831) became a great mathematician and made lasting contributions to number theory and the theories of acoustics and elasticity. After having come to the attention of Joseph-Louis Lagrange[*] (1736–1813) and Adrien-Marie Legendre (1752–1833), both of whom admired her work, she started corresponding with Carl Friedrich Gauss[†] (1777–1855).[6] Her first letter, signed with a pseudonym, began by discussing her generalization of what she termed a 'beautiful theorem'[7] from Gauss's *Disquisitiones Arithmeticae*.[8]

There is thus no doubt that Germain saw beauty in number theory. But her place in the history of mathematical beauty depends rather on her *General Considerations on the State of the Sciences and Letters in Different Periods of their Culture* [*Considérations Générales sur l'État des Sciences et des Lettres aux Différentes Epoques de leur Culture*], one of her two philosophical works, both of which were published posthumously. The *General Considerations*, a study of the unity of thought, won the approbation of Auguste Comte (1798–1857),[9] who was a pioneer in the modern philosophy of science. In this work, Germain argued that there was no fundamental difference between the motivations or the methods of the sciences, letters, and fine arts, and that a feature of their commonality was their direction by an

[1] Quinet, *Histoire de mes Idées*, p. 210.

[2] *Ibid.*, p. 211; trans. C.

[3] *Ibid.*, p. 212; trans. C.

[4] *Ibid.*, pp. 213–4; trans. C.

[5] *Ibid.*, p. 214.

[*] ▸ pp. 393–401

[†] ▸ pp. 401–8

[6] Kramer, 'Germain, Sophie', p. 375.

[7] Germain, letter to Gauss, dated 21 Nov. 1804, repr. in Del Centina & Fiocca, 'The correspondence between Germain and Gauss', app. I; trans. C.

[8] See Gauss, *Disquisitiones Arithmeticae (Werke)*, § 357.

[9] Kramer, 'Germain, Sophie', p. 376.

innate feeling.[1] She saw parallels between poetry and mathematics and the other sciences: she reached a position not unlike Goethe, Wordsworth, or Quinet, but from the opposite direction.

Germain wrote that the innate feeling common to science and art was grounded in order and proportion,[2] and sometimes also in simplicity and in unity.[3] These qualities were necessary for both beauty and truth;[4] more precisely, in different contexts, they led to different kinds of judgements:

> 'In moral things, we derive the rule of good; in intellectual things, we draw knowledge of truth; in things of pure pleasure, we find the character of beauty.'[5]

Thus it was unsurprising that literary and scientific production were akin:

> 'the highest literature, like the discoveries that enrich science, were inspired by a feeling of order and proportion which is the regulator of all intellectual progress.'[6]

The mind desired that it be clear how any work, in the sciences or the humanities, fitted together. It should be simple, and simplicity, which was the cause of elegance, produced delight.[7] Any stroke of genius was pleasing because it revealed hitherto veiled relationships between things.[8]

A poet would strive for unity of idea and action as a means to beauty, for such unity would entail order and proportion.[9] The geometer would attempt to lay out clearly the demonstration of a theorem, and 'the unity of composition will nowhere else be so perceptible'.[10] Thus, in at least one respect, mathematics was the *ne plus ultra* of scientific-literary achievement. But there was actually a precise analogy between mathematics and literature: both had style, and not everyone had equal skill in producing them.[11] The ability to produce and judge mathematics was analogous to the same ability for literature: it was simply a matter of taste, though applied outside what has habitually been considered its scope.[12] In terms of composition, choosing words in a text corresponded to choosing symbols in mathematics.[13] Sentences corresponded to formulae; both could vary in elegance.[14] Mathematics, rather than aiming at any kind of harmony between words, should 'present between its various elements relationships of order and simplicity that are easy to grasp at first glance'.[15]

JOSEPH-LOUIS LAGRANGE

That Joseph-Louis Lagrange (1736–1813) became a mathematician, and one of the greatest, was due to a confluence of circumstances. He himself said that if he had been wealthy — which

[1] Germain, *Considérations Générales*, p. 90.
[2] *Ibid.*, p. 78.
[3] *Ibid.*, pp. 79, 104.
[4] *Ibid.*, p. 79.
[5] *Ibid.*, pp. 104–5; trans. C.
[6] *Ibid.*, p. 81.
[7] *Ibid.*, p. 79.
[8] *Ibid.*, pp. 81–2.
[9] *Ibid.*, pp. 83–4.
[10] *Ibid.*, p. 84; trans. C.
[11] *Ibid.*, p. 86.
[12] *Ibid.*, p. 87.
[13] *Ibid.*, p. 86.
[14] *Loc. cit.*
[15] *Ibid.*, p. 87; trans. C.

he would have been, were it not for his father's unwise financial speculations — he would not have devoted himself to mathematics. His father pushed him towards a career in law, but Lagrange recognized his own abilities in mathematics and turned first to geometry and then to analysis.[1] Throughout his life he found mathematical beauty in many of the fields in which he worked. It is clear that sometimes the beauty served to motivate him; at other times, he seems to have admired it in a more passive way.

Number theory

Lagrange lamented what he saw as the near-abandonment of number theory in the eighteenth century. Leonhard Euler[*] (1707–83) was the only mathematician he knew who had worked on it, but Euler's 'beautiful and numerous discoveries' were compensation for the indifference other mathematicians seemed to have had for the subject.[2]

Among Euler's contributions, thought Lagrange, were proofs of most of the beautiful number-theoretic results that Pierre de Fermat[†] (1607–65) had left without proof.[3] In particular, among Fermat's remarks on Diophantus' *Arithmetica*, Lagrange found many beautiful theorems, although he only mentioned one specific result for which Fermat gave a proof: *The difference of two biquadratic numbers is never a square number.*[4] More succinctly, this says that the equation $x^4 + y^2 = z^4$ has no integer solutions (which follows from Fermat's last theorem).

Lagrange's aesthetic appreciation of Euler's number-theoretic work was not limited to the latter's proving results stated by Fermat. For instance, Lagrange called 'very beautiful' a method due to Euler of finding infinitely many integer solutions of equations of the form $A + Cq + Bq^2 = p$, provided that one solution is known.[5] He also said that Euler had found by induction a rule that, if it were generally true, would be 'one of the most beautiful theorems of Arithmetic'.[6] This rule was that every equation of the form $A = p^2 - Bq^2$, for given integers A and B, is always soluble in the integers when A is a number of the form $4nB + a^2$ or $4nB + a^2 - B$, where a and n are some integers, or more generally the prime factors of A are of this form.[7] Euler had given no proof, and Lagrange unsuccessfully sought one, finally chancing upon a counterexample: when $A = 101$ and $B = 79$, so that $A = 4 \cdot (-4) \cdot B + (38)^2 - B$, the equation has no solution.[8] [Euler knew well that induction could mislead, even when beautiful.[‡]]

The aesthetic appeal of number theory seems to have been one of the reasons Lagrange was drawn to it. He wrote to Jean le Rond d'Alembert (1717–83) in 1768 that in the preceding days, seeking to diversify his studies, he had tackled some number-theoretical problems, encountered more difficulties than he expected, but ended up finding some 'very beautiful theorems'.[9] And consider Lagrange's

[1] Itard, 'Lagrange', p. 560.

[*] ▸ pp. 334–41

[2] Lagrange, 'Additions aux Éléments d'Euler', p. 6; trans. C.; cf. p. 157.

[†] ▸ pp. 297–303

[3] *Ibid.*, p. 6.

[4] Lagrange, 'Sur quelques Problèmes de l'Analyse de Diophante', p. 377; Fermat, 'Observations', pp. 340–1; Fermat, *Œuvres*, vol. III, pp. 271–2.

[5] Lagrange, 'Sur la solution des Problèmes indéterminés du second degré', pp. 457–8.

[6] Lagrange, 'Nouvelle méthode pour résoudre les Problèmes indéterminés', p. 657; trans. C.

[7] See Euler, 'De resolutione formularum quadricarum indeterminatarum per numeros integros'.

[8] Lagrange, 'Nouvelle méthode pour résoudre les Problèmes indéterminés', p. 657.

[‡] ▸ p. 341

[9] Lagrange, letter to d'Alembert, dated 15 Aug. 1768, repr. in Lagrange, *Œuvres*, vol. 13, pp. 118–19; trans. C.

reaction to Wilson's theorem, which states that *If p is prime, then* $(p-1)! \equiv -1 \pmod{p}$. For Lagrange, this result was 'a very beautiful theorem of arithmetic'.[1] He encountered it in the work of Edward Waring (c. 1736–1798), who gave no proof. Because of this lack, and because of the elegance and usefulness of the result, Lagrange was motivated to seek a proof himself.[2] Euler wrote to Lagrange that his 'beautiful demonstrations' of Wilson's theorem 'had given him a very great pleasure'.[3]

In 1804, he wrote to Gauss of his reaction to the *Disquisitiones Arithmeticae*. After telling Gauss that the book showed him to be a mathematician of the highest class, he said that he looked upon Section VII as 'containing the most beautiful analytical discovery that had been made in a long time'.[4] Lagrange referred to the last section of *Disquisitiones Arithmeticae*, which concerns what are now called *cyclotomic polynomials*. The n-th cyclotomic polynomial $\Phi_n(x)$ is the unique irreducible polynomial that divides $x^n - 1$ but does not divide $x^k - 1$ for any $k < n$. The roots of $\Phi_n(x)$ are the primitive n-th roots of unity, namely $e^{2i\pi k/n}$ for $k \leqslant n$ and $\gcd(k, n) = 1$. The name *cyclotomic*, or 'circle-cutting', comes from the roots all lying on the unit circle in the complex plane. The theory that Gauss built up culminated in a characterization of the regular polygons that can be constructed using straightedge-and-compass: *The regular n-gon is constructible if and only if n is a product of a power of* 2 *and distinct primes of the form* $2^m + 1$. [If $2^m + 1$ is prime, them m itself must be a power of 2 and so $2^m + 1$ is a so-called *Fermat prime*.* Only five Fermat primes are known, namely $2^{2^r} + 1$ for $r \in \{0, 1, 2, 3, 4\}$.] The sequence of numbers n for which the regular n-gon is constructible thus begins $1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, \ldots$ (with the 1-gon and 2-gon counted as degenerate polygons).[5]

[This theorem and its proof seem to have attracted lasting aesthetic appreciation: over two centuries later, the philosopher Ian Hacking (1936–2023) called it 'a beautiful example of the application of algebra to geometry'.[6] The heptadecagon (17-gon) was the constructible polygon with the fewest sides for which no construction had previously been known. Gauss did not give an explicit construction, but such constructions were published later by others. The simplest known, measured by Émile Lemoine's (1840–1912) coefficient of simplicity,† is shown in Figure 11.1.]

In the same letter, referring to results in an article by Gauss on number theory, Lagrange wrote that 'they seem to me as beautiful as they are difficult to demonstrate', and that he had long since abandoned this type of research, but that it still held great attraction for him.[7]

* ▸ p. 302

† ▸ pp. 466–7

[1] Lagrange, 'Démonstration d'un Théorème Nouveau', p. 425; trans. C.

[2] *Ibid.*, p. 426.

[3] Euler, letter to Lagrange, dated 5 Oct. 1773, repr. in Lagrange, *Œuvres*, vol. 14, p. 235; trans. C.; Euler, *Opera Omnia*, vol. IV.A.5, p. 496 [ltr 34]; trans. C.

[4] Lagrange, letter to Gauss, dated 31 May 1804, repr. in Lagrange, *Œuvres*, vol. 14, p. 299; trans. C.

[5] OEIS, A003401.

[6] Hacking, *Why Is There Philosophy of Mathematics at All?*, p. 33.

[7] Lagrange, letter to Gauss, dated 31 May 1804, repr. in Lagrange, *Œuvres*, vol. 14, p. 299; trans. C.

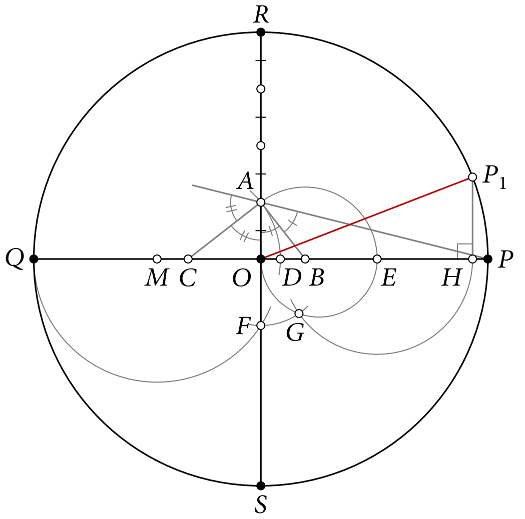

FIGURE 11.1
Construction of the heptadecagon (Callagy, 'The central angle of the regular 17-gon'): A is such that $AO = \frac{1}{4}OR$; AB, AC respectively internal and external bisectors of $\angle OAP$; D, E such that $CD = CA$ and $BE = BA$; M the midpoint of QD; F such that $MF = MQ$; G on semicircle with diameter OE such that $OG = OF$; H such that $EH = EG$; P_1 such that $P_1H \perp OP$. Then $\angle POP_1 = 2\pi/17$.

Algebra

Lagrange praised in aesthetic terms the results that had been built up by the efforts to solve the quintic equation, and, more generally, the whole theory of equations. In Lagrange's time, Niels Henrik Abel (1802–29) had not yet proven the insolubility of the general quintic equation, but there had been centuries of fruitless effort to find a solution. Lagrange did not see this work as wasted, partly because of the aesthetic value of what *had* been found:

> 'Yet these efforts are far from having been in vain. They have given rise to the many beautiful theorems which we possess on the formation of equations, on the character and signs of the roots, on the transformation of a given equation into others of which the roots may be formed at pleasure from the roots of the given equation'.[1]

More broadly, in the theory of equations, Laplace thought that Gauss's result that[2] $x^n - 1 = 0$ could be solved by solving as many equations as there are prime factors in $n-1$ was a 'beautiful discovery'.[3] And he thought beautiful the theorem that all imaginary roots of equations reduce to the form $A+B\sqrt{-1}$, where A and B are real.[4] (This result is a component of what would later be called the Fundamental Theorem of Algebra).

Analysis

Lagrange's book *Leçons sur le Calcul des Fonctions* indicates that he had great aesthetic appreciation for the theory of complex numbers, and for the connections between complex trigonometric and exponential functions, which had been one of the

[1] Lagrange, 'Leçons élémentaires sur les Mathématiques', p. 225; trans. Lagrange, *Lectures on Elementary Mathematics*, Lec. III.

[2] Gauss, *Disquisitiones Arithmeticae*, Arts 342 sqq.

[3] Lagrange, 'Sur la résolution des équations algébriques', 308; trans. C.

[4] Lagrange, 'Sur la forme des racines imaginaires', p. 479.

achievements of eighteenth-century analysis. He thought de Moivre's formula

$$(\cos(x) + \mathrm{i}\sin(x))^m = \cos(mx) + \mathrm{i}\sin(mx) \tag{M}$$

'beautiful' and wrote that it was 'as remarkable for its simplicity and its elegance as for its generality and fecundity'.[1] Abraham de Moivre (1667–1754) did not state (M) in complete generality; it was Euler who first gave it in his 1748 *Introductio in Analysin Infinitorum* [*Introduction to Analysis of the Infinite*], which also contains his famous formula $\mathrm{e}^{\mathrm{i}z} = \cos x + \mathrm{i}\sin z$, together with what are in some sense the converse formulae expressing the sine and cosine functions in terms of complex exponentials:[2]

$$\sin z = \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}}, \qquad \cos z = \frac{\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}}{2}.$$

For Lagrange, these two formulae constituted 'one of the most beautiful analytic discoveries that have been made in this time'.[3] And the expression of arcsine and arccosine in terms of complex logarithms, namely

$$\begin{aligned}\arcsin(z) &= \mathrm{i}\log\left(\sqrt{1-z^2} - \mathrm{i}z\right),\\ \arccos(z) &= -\mathrm{i}\log\left(\sqrt{1-z^2} + z\right),\end{aligned}$$

was 'one of the most beautiful discoveries'[4] of Johann I Bernoulli (1667–1748).

It was not only formulae that Lagrange found beautiful in analysis: he complimented d'Alembert on the beauty of the *methods* he used in the integral calculus.[5] His aesthetic praise could, however, be tempered by observations about his own method's advantages:

> 'Your method for integrating the equation
>
> $$Py + Q\frac{\mathrm{d}y}{\mathrm{d}x} + R\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \ldots = X$$
>
> is very beautiful; mine is totally different and it has the advantage of giving all at once the values of y, whereby we can also apply it to infinite equations.'[6]

As already mentioned, Lagrange sought a proof of Wilson's theorem because of its elegance and utility. A statement he made in a letter to d'Alembert shows that he could also see beauty in theorems in analysis without knowing their proofs, which was a motivation for him to seek a proof when none was at hand: 'Your theorems of integral Calculus seem to be very beautiful, and I am very curious to see the demonstrations; in waiting, I attempt to find them myself'.[7]

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[1] Lagrange, *Leçons sur le Calcul des Fonctions*, p. 110; trans. C.

[2] Euler, *Introductio in Analysin Infinitorum*, § 138.

[3] Lagrange, *Leçons sur le Calcul des Fonctions*, p. 109; trans. C.

[4] *Ibid.*, p. 110, trans. C.

[5] Lagrange, letter to d'Alembert, dated 30 Sep. 1771, repr. in Lagrange, *Œuvres*, vol. 13, p. 211; trans. C.

[6] Lagrange, letter to d'Alembert, dated 20 Mar. 1765, repr. in *ibid.*, vol. 13, p. 37; trans. C.

[7] Lagrange, letter to d'Alembert, dated 2 Feb. 1770, repr. in *ibid.*, vol. 13, p. 163; trans. C.

Calculus of variations

The roots of the calculus of variations arguably reach back to the isoperimetric problems of Greek antiquity, which asked which curve of a given length surrounded the largest area. It is easy to state this problem; it is easy to intuit the circle as the answer; it is not easy to solve rigorously. Johann I Bernoulli's solution to the brachistochrone problem was an early landmark, but it depended on the earlier insight of Pierre de Fermat.

Leonard Euler's 1744 book *Methodus Inveniendi Lineas Curvas* codified the as-yet unnamed discipline. This work transformed the calculus of variations from a collection of special methods devised by Jakob I (1655–1705) and Johann I Bernoulli into a treatment of very general classes of problems and essentially founded it as a distinct branch of mathematics.[1] But Euler stated a desire for a non-geometric method for tackling such problems.[2]

It was Lagrange who set out to fulfil Euler's wish.[3] Lagrange admired *Methodus Inveniendi*, writing that it was 'very precious for the number and the beauty of the examples it contained'.[4] He pointed to Euler's insight of an underlying uniformity in problems that had previously been treated separately as leading to his discovery of a 'beautiful theorem' on how to solve the problem of finding a curve satisfying a maximum or minimum condition among only those curves satisfying certain properties; it suffices to incorporate the properties into the maximum or minimum condition.[5]

These comments seem to indicate an important role for aesthetic value as a guiding and a motivating factor, at least for Lagrange, in the early history of the calculus of variations.

Geometry

Lagrange considered beautiful Girard's theorem relating the area and the angles of a spherical triangle, which Montucla had thought elegant.[6]* He also thought beautiful John Wallis's (1616–1703) proof of this result.[7]

Another example makes it clear that Lagrange saw beauty in aspects of solid geometry that were directly linked to — and indeed motivated by — applications. In any stereographic projection of a sphere onto a plane, any circle on the sphere that does not pass through the point of projection is projected to a circle on the plane (see Figure 11.2). Hence to find the image of such a circle under projection it suffices to find the images of three distinct points on the circle. This fact Lagrange thought a 'beautiful property of the stereographic projection',[8] though it is not quite clear whether he referred to the result that circles project to circles, or that only three points on the projection of a circle are necessary to determine it fully.

A passage in Lagrange's *Mécanique Analytique* also sheds light on his views on the relationship between the applicability of a piece

[1] Goldstine, *Hist. Calculus of Variations*, p. 68.

[2] Euler, *Methodus Inveniendi*, p. 52.

[3] Goldstine, *Hist. Calculus of Variations*, p. 111.

[4] Lagrange, *Leçons sur le Calcul des Fonctions*, p. 388; trans. C.

[5] *Ibid.*, pp. 388–9.

[6] Lagrange, 'Solutions de quelques Problèmes relatifs aux triangles sphériques', p. 337.

* ▸ p. 349

[7] *Loc. cit.*

[8] Lagrange, 'Sur la construction des Cartes géographiques', p. 639.

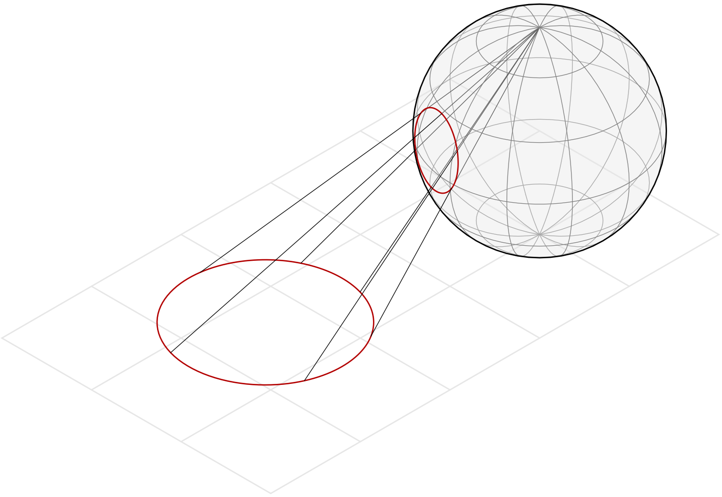

FIGURE 11.2
Stereographic projection from a sphere onto a plane: any circle on the sphere that does not pass through the projection point maps to a circle on the plane.

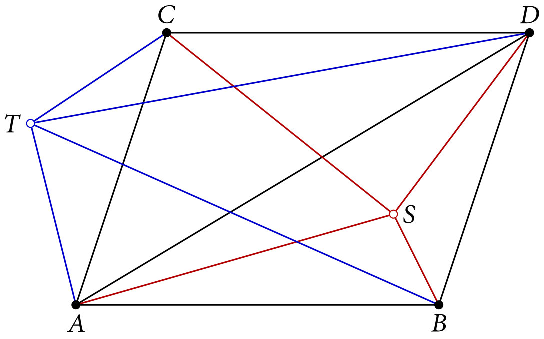

FIGURE 11.3
The point S lies within $\angle BAC$, so the area of ASD is the difference that of ASC and of ASB. The point T lies outside $\angle BAC$ and its opposite, so the area of ATD is the sum of the that of ATC and ATB. (Varignon, *Nouvelle Mechanique ou Statique*, vol. 1, § 1, Lem. XVI)

of mathematics and its beauty. While discussing the principles of levers, he made use of the following geometrical result stated by Pierre Varignon (1654–1722):[1]

THEOREM L. *Given a parallelogram ABDC, if a point S lies within the angle BAC or its opposite, then the area of the triangle ASD is the difference of the areas of ASC and of ASB; if a point T lies outside the angle BAC and its opposite, then the area of the triangle ATD is equal to the sum of that of ATC and of ATB.* (See Figure 11.3.) $\boxed{\text{L}}$

Lagrange said of Theorem L that it was 'in itself a beautiful theorem of Geometry, independently of its application to Mechanics'.[2] Thus, for Lagrange, a result could be beautiful in itself, before applications were contemplated. But his decision to reassure his readers thus suggests that he thought that applications could further enhance that beauty, or that he thought his readers would think so.

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[1] Varignon, *Nouvelle Mechanique ou Statique*, vol. 1, § 1, Lem. XVI.

[2] Lagrange, *Mécanique Analytique*, vol. 1 ⟦11⟧, p. 14; trans. C.

Classical and celestial mechanics

Lagrange has been celebrated for his reformulation of Newtonian mechanics using a smooth function L, known to posterity as the *Lagrangian*, of the space of configurations of that system, where L describes the difference of the kinetic and potential energies of the system. With this new framework, and with $\mathcal{F}$ being a function describing constraints such as friction, the behaviour of the system through time is given by Lagrange's equations (for each coördinate j of the configuration space):

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\vec{q}}_j} - \frac{\partial L}{\partial \vec{q}_j} - \frac{\partial \mathcal{F}}{\partial \dot{\vec{q}}_j} = 0. \tag{L}$$

Alfred North Whitehead (1861–1947) would later describe these equations as having 'beauty and almost divine simplicity'.[1] But Lagrange still appreciated aesthetically older work on mechanics. For instance, in his *Mécanique Analytique* he said that the first proposition of Newton's *Principia* was a 'beautiful theorem'.[2] The result of Newton to which Lagrange referred is a more general formulation of Kepler's second law of planetary motion:*

PROPOSITION A. *'The areas which bodies made to move in orbits describe by radii drawn to an unmoving center of forces lie in unmoving planes and are proportional to the times.'*[3] $\boxed{\text{A}}$

And he again praised the results of Newton elsewhere in *Mécanique Analytique*:

> 'One of the first and most beautiful results of the theory of Newton on the system of the world consists in that all the orbits of celestial bodies are of the same nature and differ from each other only by reason of the force of projection that these bodies can be supposed to have received at the creation of things.'[4]

In a similar vein, Lagrange seems to have had a preference for his own analytical approach over synthetic geometry. He was proud that there were no diagrams in his *Mécanique Analytique*.[5] However, he said that the results of Colin Maclaurin's (1698–1746) 'beautiful synthetic method' on the attraction of a spheroid agreed exactly with the what he himself had found analytically.[6] It is worth emphasizing this point: despite Lagrange's analytical approach, he still admired the beauty of Maclaurin's synthetic proof. He had a high opinion of Maclaurin's work, saying of his 1740 treatise 'Concerning the Physical Cause of the Flow and Ebb of the Sea' that 'one can compare it to all that is most beautiful and ingenious left to us by Archimedes'.[7]

He praised the analytical work of others, and had a particular admiration for Johann Heinrich Lambert's (1728–77) 'beautiful treatise'[8] on the motion of comets, *Insigniores Orbitae Cometarum Proprietates*. He wrote to d'Alembert that it contained what he described as

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[1] Whitehead, *Science and the Modern World*, p. 78.

[2] Lagrange, *Mécanique Analytique*, vol. 1 ⟦vol. 11⟧, p. 260; trans. C.

* ▸ p. 286

[3] Newton, *Principia*, bk 1, Prop. 1, Thm 1.

[4] Lagrange, *Mécanique Analytique*, vol. 2 ⟦vol. 12⟧, p. 62; trans. C.

[5] Lagrange, *Mécanique Analytique*, pp. xi–xii; Lagrange, *Analytical Mechanics*, p. 7.

[6] Lagrange, 'Sur l'attraction des Sphéroïdes elliptiques', p. 639; trans. C.

[7] *Ibid.*, p. 619; trans. C.

[8] Lagrange, 'Sur le Problème de la détermination des orbites', pp. 444, 448; trans. C.

‘the most elegant and the most useful theorems’ hitherto known on the subject[1] and as ‘some very beautiful theorems’.[2] There were two related results he particularly distinguished.

In a scholarly paper, he judged as a ‘beautiful theorem’[3] Lambert’s method for determining the time a comet would take to describe a certain segment of a parabola from the length of the chord subtending the segment and the sum of the lengths of the rays from the focus to the extremities of the segment.[4] This result, he wrote, ‘by its simplicity and its generality, must be regarded as one of the most ingenious discoveries that have been made in the Theory of the system of the world’.[5]

The corresponding result for elliptical motion,[6] where the time a comet takes to describe a segment of an ellipse is dependent on the lengths of the major axis, the chord subtending the segments, and the rays from the focus to the extremities of the segment, was also a ‘beautiful theorem’;[7] in correspondence with d’Alembert, he declared it beautiful above all other results in Lambert’s book.[8]

CARL FRIEDRICH GAUSS

The last of the traditional three greatest mathematicians in history is Carl Friedrich Gauss (1777–1855). On his death, the king ordered struck a commemorative medallion with the inscription ‘GEORGIUS V / REX HANOVERAE / MATHEMATICORUM / PRINCIPI [George v, king of Hanover, to the prince of mathematicians]’,[9] and, though Gauss was not the first mathematician to receive this accolade,[10] it remained attached to him.[11]

The ‘prince’ was born into a family of labourers in Brunswick. It became evident that he was an arithmetic prodigy when, at the age of three, he corrected an error in his father’s wage calculations.[12] His schoolteacher saw his abilities when he instantly summed an arithmetic series.[13]

[Many accounts claim that the series was $1 + 2 + \dots + 100$ and that the young Gauss saw that the i-th and $(101 - i)$-th terms always added to give 101, and thus that the sum of the series was $50 \cdot 101 = 5050$. However, none of the earliest accounts make this claim.[14] What seems to be a piece of folklore is sometimes given as an example of mathematical beauty.[15]]

Through the actions of his teacher and others, Gauss’s abilities came to the attention of people of high rank and eventually to Karl Wilhelm Ferdinand, the Duke of Brunswick, who awarded him a stipend that freed him to pursue study and research.[16] After attending the Collegium Carolinum in Brunswick and then the University of Göttingen, he published his *Disquisitiones Arithmeticae* in 1801.

[1] Lagrange, ‘Sur le Problème de la détermination des orbites’, p. 444; trans. C.

[2] Lagrange, letter to d’Alembert, dated 12 Sep. 1769, repr. in Lagrange, *Œuvres*, vol. 13, p. 149; trans. C.

[3] Lagrange, ‘Sur le Problème de la détermination des orbites’, p. 447; trans. C.

[4] Lambert, *Insigniores Orbitae Cometarum Proprietates*, Corol. I to Prob. xli [p. 126].

[5] Lagrange, ‘Sur le Problème de la détermination des orbites’, p. 447; trans. C.

[6] Lambert, *Insigniores Orbitae Cometarum Proprietates*, Prob. xli [p. 124].

[7] Lagrange, ‘Sur une manière particulière d’exprimer le temps’, p. 558.

[8] Lagrange, letter to d’Alembert, dated 12 Sep. 1769, repr. in Lagrange, *Œuvres*, vol. 13, p. 149.

[9] Eves, *An Introduction to the History of Mathematics*, p. 479.

[10] For example, Euler: see Joh. Bernoulli, letter to Euler, dated 23 Sep. 1745, repr. in Euler, *Opera Omnia*, vol. iv.a.2, p. 449 [ltr 37].

[11] See Bell, *Men of Mathematics*, ch. 14.

[12] May, ‘Gauss, Carl Friedrich’, p. 298.

[13] Sartorius von Waltershausen, *Gauss zum Gedächtniss*, p. 12.

[14] Hayes, ‘Gauss’s Day of Reckoning’.

[15] See Polster, *Q.E.D.*, p. 34.

[16] May, ‘Gauss, Carl Friedrich’, p. 298.

The *Disquisitiones* immediately established Gauss's reputation as a mathematician of the highest calibre. Number theory had previously been a disparate collection of isolated results. Gauss recognized his debt to earlier modern mathematicians,

> ‘of whom those few men of immortal glory P. DE FERMAT, L. EULER, L. LAGRANGE, A. M. LEGENDRE (and a few others) opened the entrance to the shrine of this divine science and revealed the abundant wealth within it.’[1]

But Gauss brought a system to the existing work, solved some difficult problems, and set the course for number theory for rest of the nineteenth century.[2] In the decade that followed, Gauss broadened his scope, moving fearlessly into astronomy, mechanics, and optics.

In 1808, Gauss became director of the observatory at Göttingen. In his inaugural lecture, he declared that mathematics in general and astronomy in particular had a value that was prior to and independent of any utility. He cited Schiller's ‘Archimedes and the Student’* and it is clear that he saw as his model the same Archimedes that the poet had portrayed:

* ▸ pp. 389–90

> ‘The happy great minds who created and expanded astronomy as well as the other beautiful parts of mathematics were certainly not inspired by the prospect of future use: they searched the truth for its own sake and found in the very success of their efforts their reward and their happiness. I cannot avoid at this point reminding you of ARCHIMEDES, who was admired by his contemporaries mainly just because of his artful machines, because of the apparently magical workings they had, but who placed so little value on all this, compared with his magnificent discoveries in the field of pure mathematics — which in themselves usually had no visible benefits in the common sense of the term, at least then — that he did not write down for us anything about the former, while he developed the latter with affection in his immortal works. You must all know the beautiful poem by SCHILLER [*Archimedes and the Student*].’[3]

Gauss clearly evaluated his own work in aesthetic terms. His mathematical diary, which is a bare record of his work during the period 1796–1814, contains many references to ‘elegant’ proofs, formulae, solutions, results, methods, and connections.[4] He used ‘beautiful’ [‘pulchrum’] to describe connections and discoveries,[5] and once described a theorem as ‘most charming’ [‘venustissimi’].[6] In correspondence with Friedrich Bessel (1784–1846), he called beautiful the result that a complex integral $\int f(x)\,\mathrm{d}x$ has the same value along two paths provided that f does not have a pole in the region of the complex plane between the two paths.[7]

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[1] Gauss, *Disquisitiones Arithmeticae (Werke)*, p. 6; trans. Gauss, *Disquisitiones Arithmeticae*, p. xviii.

[2] May, ‘Gauss, Carl Friedrich’, p. 299.

[3] Gauss, ‘Astronomische Antrittsvorlesung’, pp. 192–3, insertion in orig.; partial trans. Ferreirós, ‘Ὁ Θεὸς Ἀριθμητίζει’, p. 247.

[4] See, respectively, Gray, ‘Gauss's mathematical diary’, nos 16, 44, 107, 118, 132, 146.

[5] *Ibid.*, nos 106, 108.

[6] *Ibid.*, no. 123.

[7] Gauss, letter to Bessel, dated 18 Dec. 1811, repr. in Gauss & Bessel, *Briefwechsel*, p. 157.

Exposition

Before analysing Gauss's judgements of beauty, it is necessary to make some remarks on Gauss's writing style. Abel reported that August Crelle (1780–1855) thought everything that Gauss wrote was atrocious, being so obscure that it was near-impossible to understand.[1] But perhaps this was simply because Gauss gave no hint of how he discovered his results and proofs, which lent an *ex cathedra* overtone to his writing. The astronomer Christopher Hansteen (1784–1873) reported in a biography of Abel (hence the remark being sometimes mistakenly attributed to him) that an unnamed German student said of Gauss: 'he is like the fox, who with his tail erases his tracks in the sand'.[2] Rouse Ball had the same view, noting that Gauss

> 'removes every trace of the analysis by which he reached his results, and studies to give a proof which, while rigorous, shall be as concise and synthetical as possible.'[3]

This policy seems to have been adopted deliberately by Gauss: according to Sartorius, he used to say that '[a] good building should not show its scaffolding when completed'.[4]

When he received János Bolyai's (1802–60) work on non-euclidean geometry,* Gauss wrote to Christian Ludwig Gerling (1788–1864) with his evaluation:

> 'In it I find all *my own ideas and* RESULTS, developed with remarkable elegance, although in a form so concise as to offer considerable difficulty to anyone not familiar with the subject.'[5]

This remark indicates that, for Gauss, elegance was not the same as clarity of exposition. Elegance seems to have been more closely tied to the logical progression rather than to the mode of writing.

Foundations of beauty

The beauty of a piece of mathematics was, for Gauss, independent of any applicability. That this was true in terms of applicability outside mathematics is made clear by his archimedean archetype of the mathematician. It was also implied by his support for imaginary numbers having the same status as real numbers: his motive was aesthetic, not practical:

> 'The matter is not of practical usefulness, but analysis is for me an independent science, which by rejecting these fictitious magnitudes would lose enormously in beauty and roundedness, and at every instant would be forced to add quite cumbersome restrictions to truths that otherwise would be generally valid.'[6]

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[1] Abel, letter to Hansteen, dated 5 Dec. 1825, repr. in Hansteen, 'Niels Henrik Abel', p. 37.

[2] *Ibid.*, p. 37, col. 3, n. **.

[3] Rouse Ball, *A Short Account of the History of Mathematics*, p. 373.

[4] Sartorius von Waltershausen, *Carl Friedrich Gauss*, p. 67.

* ▸ pp. 429–30

[5] Gauss, letter to Gerling, dated 14 Feb. 1832, repr. in Gauss, *Werke*, vol. VIII, p. 220, emphases in orig.; trans. mod. Carslaw, *The Elements of Non-Euclidean Plane Geometry*, p. 19.

[6] Gauss, letter to Bessel, dated 18 Dec. 1811, repr. in Gauss & Bessel, *Briefwechsel*, p. 156; trans. Boniface, 'A process of generalization', p. 486.

But there were also occasions when he praised the beauty of mathematics that lacked applications internal to mathematics. For instance, he noted that direct techniques for determining the solubility of congruences of the form $x^2 \equiv a \pmod{m}$ are more prolix than indirect techniques, and thus are ‘memorable not for their usefulness in practice but for their beauty’.[1] It is natural to see here a tension between the beauty and the lengthiness, since brevity has been commonly connected with beauty. Gauss perhaps meant that a direct technique *per se* had beauty, possibly linked to the ideas it contained, and but any application of it would turn out to be lengthy.

Beauty was also independent of the efficiency offered by modern mathematical notation and techniques. Such advances have helped mathematics progress, at the price of the beauty of the older mathematics:

> ‘It is the character of the mathematics of modern times (in contrast to Antiquity) that with our symbolic language and terminology we possess a lever, by which the most complex arguments are reduced to a certain mechanism. Thereby the science has won infinitely in richness, but has lost just as much in beauty and solidity, as the trade is usually practiced. How frequently that lever is applied in a merely mechanical way’.[2]

Number theory

Perhaps Gauss’s work on number theory offers the most insight into his views on mathematical beauty. He wrote that that number theory and the theory of elliptic functions were the ‘most beautiful and most fruitful in the whole extent of mathematics’,[3] and that all who studied these areas felt their ‘peculiar beauties’, though none so obviously as Euler, who, judged Gauss, expressed his delight in almost every paper he wrote on number theory.[4]

Number theory was, for Gauss, completely pure mathematics, in the sense that the notion of number was entirely mental and thus its study was completely *a priori*. But he wrote in correspondence with Bessel that his investigations into the foundations of geometry had led him to a stronger view that it cannot be wholly *a priori*. He hesitated to make his research public, not wanting to face ‘the outcry of the Boeotians’.[5] (*Boeotian* was a pejorative term that suggested a lack of sensitivity and refinement.[6]) Gauss later emphasized that being entirely or partially *a priori* was a contrast between number theory and geometry.[7]

It is thus perhaps unsurprising that Gauss had no philosophical reluctance to extend the scope of number theory beyond the integers. Nor did he have aesthetic objections. On the contrary, such an expansion could enhance beauty: he thought that theorems about biquadratic residues would only ‘shine in utmost simplicity and genuine beauty' ['summa simplicitate ac genuina venustate resplendent'] when arithmetic was extended to the Gaussian integers.[1]

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[1] Gauss, *Disquisitiones Arithmeticae* (*Werke*), § 152; trans. Gauss, *Disquisitiones Arithmeticae*, § 152.

[2] Gauss, letter to Schumacher, dated 1 Sep. 1850, quot. in Gauss, *Werke*, vol. x.1, p. 434; trans. mod. Ferreirós, ‘Ὁ Θεὸς Ἀριθμητίζει’, p. 244.

[3] Gauss, foreword to Eisenstein, *Mathematische Abhandlungen*, p. iii.

[4] Gauss, foreword to *ibid.*, p. iv.

[5] Gauss, letter to Bessel, dated 27 Jan. 1829, repr. in Gauss & Bessel, *Briefwechsel*, p. 490.

[6] See Roberts, *The Ancient Boeotians*, § 1.1.

[7] Gauss, letter to Bessel, dated 9 Apr. 1830, repr. in Gauss & Bessel, *Briefwechsel*, § 166, p. 497; trans. C.

Beauty and depth

Any account of Gauss's attitude to beauty in mathematics must discuss the law of quadratic reciprocity, which necessitates defining some terminology and notation. Let a be a natural number and p a prime number. Then a is a *quadratic residue* mod p, where p is a prime number, if it is a square mod p; that is, if there is a solution to the equation $x^2 \equiv a \pmod{p}$. Thus 3 is a quadratic residue mod 13 (because $x = 4$ is a solution), but 5 is not a quadratic residue mod 13. The *Legendre symbol* $\left(\dfrac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \equiv x^2 \pmod{p} \text{ for some } x \not\equiv 0 \pmod{p}; \\ -1 & \text{if } a \not\equiv x^2 \pmod{p} \text{ for any } x; \\ 0 & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

LAW OF QUADRATIC RECIPROCITY. *For odd prime numbers p and q,*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}. \qquad \square$$

Gauss was particularly fascinated by this result. He assigned epithets to emphasize its importance and value: in his *Disquisitiones Arithmeticae*, he called it the 'fundamental theorem';[2] in his mathematical diary, the 'golden theorem'.[3] Aesthetically, he wrote that it was 'one of the most elegant of its type',[4] 'among the most beautiful truths of the higher arithmetic'[5] and a 'most elegant theorem'.[6] More generally, he thought that the fundamental theorems of quadratic residues were 'among the most beautiful treasures of the higher arithmetic'.[7]

The law of quadratic reciprocity was the culmination of investigations that had started with Fermat. It was first stated in the form given above by Legendre in 1798, but the proof he offered was incomplete.[8]

Gauss's journey towards the law and the first valid proof had started a few years earlier, as he recalled in his preface to *Disquisitiones Arithmeticae*:

> 'when I first turned to this type of inquiry in the beginning of 1795 I was unaware of the modern discoveries in the field and was without the means of discovering them. What happened was this. Engaged in other work I chanced on an extraordinary arithmetic truth (if I am not mistaken, it was the theorem of art. 108). Since I considered it so beautiful in itself [per se pulcherrimam] and since I suspected its connection with even

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[1] Gauss, 'Theoria residuorum biquadraticorum', § 30.

[2] Gauss, *Disquisitiones Arithmeticae*, § 131.

[3] Gray, 'Gauss's mathematical diary', nos 16, 23, 30.

[4] Gauss, *Disquisitiones Arithmeticae*, § 151.

[5] Gauss, 'Theorematis Fundamentalis (treatise)', p. 49; trans. C.

[6] Gauss, 'Theorematis arithmetici demonstratio nova', p. 4; trans. C.

[7] Gauss, 'Theoria Residuorum Biquadraticorum', 67; trans. C.

[8] Baumgart, *The Quadratic Reciprocity Law*, pp. 5–6.

> more profound results, I concentrated on it all my efforts in order to understand the principles on which it depended and to obtain a rigorous proof.'[1]

The 'theorem of art. 108' is as follows:

THEOREM S. '-1 *is a quadratic residue of all prime numbers of the form* $4n + 1$ *and a nonresidue of all prime numbers of the form* $4n + 3$.'[2] $\boxed{\text{S}}$

Gauss cited the elegance of Theorem S as a reason for giving a second proof of it.[3]

Note that there was an implied connection between beauty and depth in Gauss's account of why he worked to understand the principles behind Theorem S. He seems to have thought that combined simplicity and depth contributed to the beauty of a result. He described certain theorems as

> 'among the most beautiful [pulcherrima] in the theory of binary forms, especially because, despite their extreme simplicity, they are so profound that a rigorous demonstration requires the help of many other investigations.'[4]

Although Gauss did not explore this point, the requirement to use 'many other investigations' seems to indicate that depth for Gauss involved connections to myriad other ideas.

Beauty arose from simplicity or ease of discovery, together with depth or difficulty of proof: for Gauss, this was a mark of number theory:

> 'It is characteristic of higher arithmetic that many of its most beautiful theorems can be discovered by induction with the greatest of ease but have proofs that lie anywhere but near at hand and are often found only after many fruitless investigations with the aid of deep analysis and lucky combinations.'[5]

He repeated this point in later life: in his foreword to a collection of Gotthold Eisenstein's (1823–52) papers on number theory and elliptic functions, he wrote that the higher arithmetic was replete with interesting results that at first resist proof and then yield only to difficult and unnatural approaches, with simpler methods remaining hidden.[6] He pointed out that, having formulated a conjecture based on evidence, the mathematician's first aim is to prove it by any means possible. But in number theory one should not rest content with the first proof.[7] The first proof could be a blunt instrument, with no finesse, just as long as it established the result:

> 'sometimes one does not at first come upon the most beautiful and simplest proof, and then it is just the insight into the wonderful concatenation of truth in higher arithmetic that is the chief attraction for study and often leads to the discovery

[1] Gauss, *Disquisitiones Arithmeticae (Werke)*, p. 6; trans. Gauss, *Disquisitiones Arithmeticae*, p. xviii.

[2] Gauss, *Disquisitiones Arithmeticae (Werke)*, § 108; trans. Gauss, *Disquisitiones Arithmeticae*, § 108.

[3] Gauss, *Disquisitiones Arithmeticae (Werke)*, § 108.

[4] Gauss, *Disquisitiones Arithmeticae (Werke)*, § 287; trans. Gauss, *Disquisitiones Arithmeticae*, § 287.

[5] Gauss, 'Theorematis fundamentalis', p. 159; trans. May, 'Gauss, Carl Friedrich', 299, emphasis in orig.

[6] Gauss, foreword to Eisenstein, *Mathematische Abhandlungen*, p. III.

[7] Gauss, 'Theorematis fundamentalis', p. 159.

> of new truths. For these reasons the finding of new proofs for known truths is often at least as important as the discovery itself.'[1]

Thus Gauss's motives for seeking new proofs are greater beauty, simplicity, and insight, which seem to have been partly connected.

This discussion opened an 1817 paper that Gauss wrote on quadratic reciprocity. Gauss never seemed satisfied with the proofs he found of the law of quadratic reciprocity itself. He gave two proofs in the *Disquisitiones Arithmeticae* (*Werke*). The first is a long and unenlightening division into eight cases, and it is not surprising that he was not content with it and gave another.[2] He published four more proofs during his life, and found but never published another two.[3]

Prime number theorem

Let $\pi(n)$ denote the number of primes less than or equal to n.

PRIME NUMBER THEOREM.

$$\pi(n) \sim \frac{n}{\log n}, \quad \textit{or, equivalently,} \quad \lim_{n\to\infty} \frac{\pi(n)}{n/\log n} = 1. \qquad \square$$

The Prime Number Theorem was first proved in 1896 independently by Jacques Hadamard (1865–1963) and Charles-Jean de la Vallée Poussin (1866–1964), using methods from complex analysis.[4] Essentially, the Prime Number Theorem says that if n is large, then $n/\log n$ is a good approximation to $\pi(n)$. But a better numerical approximation is given by the (*offset*) *logarithmic integral*, which is defined by

$$\mathrm{Li}(n) = \int_2^n \frac{\mathrm{d}t}{\log t}.$$

The precise analogue of the Prime Number Theorem holds:

$$\pi(n) \sim \mathrm{Li}(n), \quad \text{or, equivalently,} \quad \lim_{n\to\infty} \frac{\pi(n)}{\mathrm{Li}(n)} = 1.$$

But as n increases, $\mathrm{Li}(n)$ quickly approximates $\pi(n)$ much better than $n/\log n$ (see Figure 11.4).

The first published conjecture that resembled the prime number theorem was due to Legendre, who thought that $\pi(n)$ approximates $x/(A\log x + B)$ for constants A and B; numerical work led him to conclude that $A = 1$ and $B = 1.08366$.[5] Gauss never published his own investigation into the distribution of the prime numbers, but according to an 1849 letter he sent to the astronomer Johann Franz Encke (1791–1865), who had studied under him at Göttingen, he started working on the topic in 1792 or 1793, and had been led to the conclusion that $\pi(n)$ is approximated by $\mathrm{Li}(n)$.[6]

[1] Gauss, 'Theorematis fundamentalis', pp. 159–60; trans. May, 'Gauss, Carl Friedrich', 299, emphasis in orig.

[2] Gray, 'Depth — a Gaussian Tradition in Mathematics', p. 179.

[3] Baumgart, *The Quadratic Reciprocity Law*, ch. 15.

[4] Dickson, *Hist. Theory of Numbers*, vol. 1, p. 439; see Hadamard, 'Sur la distribution des zéros'; de la Vallée Poussin, 'Recherches analytiques sur la théorie dos nombres premiers'.

[5] Goldstein, 'Hist. Prime Number Theorem', p. 601.

[6] Gauss, letter to Encke, dated 24 Dec. 1849, repr. in Gauss, *Werke*, vol. 2, p. 444.

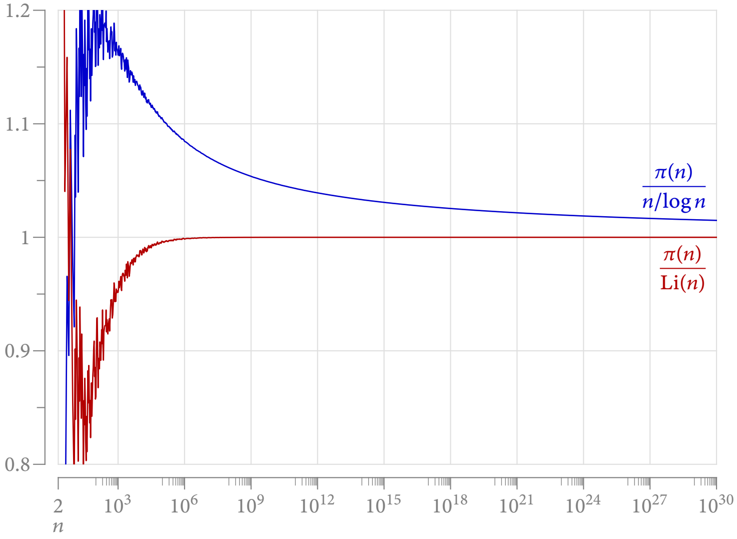

FIGURE 11.4
Comparison of $\pi(n)$, the number of primes not exceeding n, with the approximations $n/\log n$ and $\mathrm{Li}(n)$, where n is plotted logarithmically in the range $[2, 10^{30}]$.

The historian of mathematics Moritz Cantor (1829–1920) attended lectures given by Gauss in 1850–1,[1] and recalled that Gauss remarked, on an occasion when his desk was covered with logarithm tables: 'You have no idea how much poetry is contained in the calculation of a logarithm table.'[2] ['Sie ahnen nicht, wieviel Poesie in der Berechung einer Logarithmentafel enthalten ist.'] This comment suggests that Gauss had some aesthetic appreciation of the conjectures to which he had been led by the numerical calculations. [Cantor admitted that he had no notes of the lectures, describing the memories he shared as 'photographic snapshots of a time long past, without retouching'.[3] The 'time long past' signifies an interval of almost fifty years.]

DAVID RAMSAY HAY

David Ramsay Hay (1798–1866) was by profession neither a scientist nor a philosopher: he was a decorative artist, and was appointed decorator to Queen Victoria. He was the kind of figure who defies classification and is thus difficult to place into an intellectual history. But he cannot be overlooked, for he devised an aesthetic theory that tried to give a unified account of beauty in music, architecture, sculpture, and geometric forms, linking them together and reducing them to mathematical principles. Specifically, mathematics underlay *harmony*, a component of beauty:

> 'This mathematical proportion of parts is that particular element of beauty called harmony, which, in a general sense, means the adaptation of the parts of any thing one to another, or the

[1] Cantor, 'Carl Friedrich Gauss', p. 253.

[2] *Ibid.*, p. 254; trans. C.

[3] *Ibid.*, pp. 253, 254.

> just proportion of sounds, colours, forms, words, or sentiments.'[1]

While Hay's work has been called pythagorean,[2] it ranges far beyond the numerical foundation of proportion found in pythagoreanism. It is rather *sui generis*.

Hay's thought was presented in a series of books, copiously illustrated, that he published throughout his life. His theory grew in scope with time, with expansion of particular points motivated in part by discussions and reviews of his work and in part by his own reading. He was certainly familiar with philosophers of aesthetics like Francis Hutcheson[*] (1694–1746), Thomas Reid[†] (1710–96), and Archibald Alison[‡] (1757–1839).[3] The contemporary response to his work seems to have been largely positive, so there was little shift in his overall position and his later books are compatible with his earlier ones.

For Hay, beauty was not subjectively relative, and thus the science of beauty had to be based upon fixed principles.[4] The underlying principles derived from some mathematical notion that was an intrinsic part of human nature:

> 'there appears to be implanted in the human mind a governing principle of harmony of a mathematical nature, responsive to impressions made upon the organs of sense by certain combinations, motions, and affinities in the elements of matter. [...]
>
> These combinations, motions, and affinities, act by the harmony of numbers, exemplified in the agreement of certain arithmetical ratios.'[5]

While Hay's account of beauty had a broad range, he distinguished symmetrical and picturesque beauty. He thought that both arise from uniformity and variety, but in symmetrical beauty, such as is exhibited in a geometrical configuration, the former predominates; in picturesque beauty, such as that of an old oak tree, the latter.[6] His work was focused upon symmetrical beauty, which he seems to have considered more amenable to being understood by precise laws.

Hay's first book, *The Laws of Harmonious Colouring* (1828), points to the position he would maintain on a unified theory of beauty. He held that the effect of harmonious colours upon the viewer was analogous to the effect of harmonious sounds upon the hearer.[7] And just as three fundamental notes were the foundation of all harmony; three primary colours were the foundation of colour.[8] The primary colours of red, yellow, and blue had 'numerical proportional power' 5, 3, and 8, respectively. Thus orange, for instance, was produced by mixing red and yellow in a 5 : 3 ratio.[9] Hay presented a simplified version of a diagram by which the chemist George Field (1777–1854) had illustrated his own analogy between musical concord and colour harmony (see Figure 11.5).[10]

[1] Hay, *Proportion*, p. 10.

[2] Raz, 'Music of the Squares'.

[*] ▸ pp. 357–62

[†] ▸ pp. 369–72

[‡] ▸ pp. 382–3

[3] Hay, *The Science of Beauty*, p. 8; Hay, *Harmony of Form*, p. 2.

[4] Hay, *The Science of Beauty*, pp. 8–9.

[5] Hay, *The Principles of Beauty in Colouring Systematized*, p. 10.

[6] Hay, *First Principles of Symmetrical Beauty*, pp. 20–3.

[7] Hay, *The Laws of Harmonious Colouring*, p. 17.

[8] *Ibid.*, pp. 17–18.

[9] *Ibid.*, p. 19.

[10] Field, *Chromatics*, pp. 32–3.

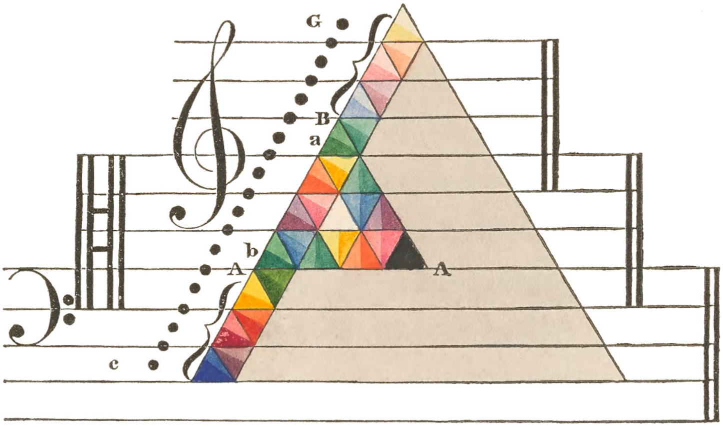

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FIGURE 11.5
Field's chromatic colour scale (Field, *Chromatics*, p. 33).

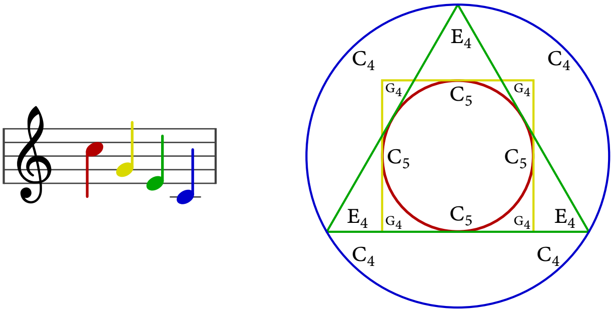

FIGURE 11.6
The correspondence Hay posited between geometrical shapes and musical notes (Hay, *Harmony of Form*, pl. III, Fig. 1).

But Hay went further than Field, for he extended his analysis to link geometrical objects to musical intervals and the numerical ratios that described them to geometrical configurations. Alongside the three primary colours and three basic notes he placed three fundamental geometrical objects: the circle, square, and equilateral triangle. And these fitted together in a way that, for Hay, presented a close analogue to the relationship between musical notes. If a circle represented C_4 (middle C), then C_5 (tenor C), an octave higher, would be represented by circle of half the radius.[1] Between these one could place an equilateral triangle (see Figure 11.6). For Hay, the relationship between the outer circle and triangle corresponded to the musical interval $5 : 4$, since their perimeters were in this approximate ratio; thus the triangle corresponded to E_4 (middle E). Similarly the relationship between the outer circle and the square circumscribed about the inner circle corresponded to $3 : 2$, since their perimeters were in this approximate ratio; thus the triangle corresponded to G_4 (middle G).[2] To be precise, if the inner circle had radius 1, then the outer circle had circumference $c_{\bigcirc} = 4\pi$, the triangle had perimeter $c_{\triangle} = 6\sqrt{3}$, and the square had perimeter $c_{\square} = 8$. And $c_{\bigcirc}/c_{\triangle} \approx 1.209$, and $c_{\bigcirc}/c_{\square} \approx 1.571$.

When Hay wrote his 1843 book *Proportion*, he thought that the Greeks had possessed 'a positive geometric principle of beauty systematically developed and applied in all their works',[3] but that the true

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[1] Hay, *Harmony of Form*, p. 18.

[2] *Ibid.*, pp. 19–20.

[3] Hay, *Proportion*, p. i.

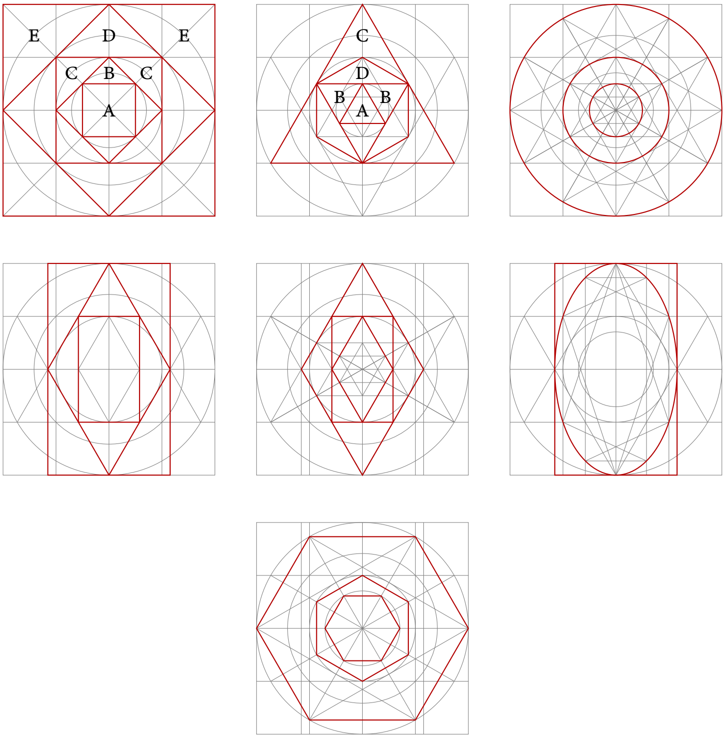

FIGURE 11.7
Hay's seven elements of geometrical beauty (Hay, *Proportion*, pl. XIII).

Greek system seemed to have been lost before Vitruvius* (early first century–c. 25 BCE) wrote.[1] In *Proportion*, Hay presented a detailed account showing how harmonies of geometrical figures, colours, and music corresponded to each other and to numerical ratios.[2] Starting with the fundamental circle, equilateral triangle, and square, he built up configurations involving rectangles, rhombi, and ellipses, leading to what he called '[t]he seven elements of Geometric Beauty' (see Figure 11.7).[3] The particular proportions of the rectangle, ellipse, and rhombus made them the 'leading figures' of their types, having a greater symmetry and beauty.[4]

An anonymous review in the *Athenæum* of Hay's book *Harmony of Form* generally praised his analysis, while pointing out difficulties arising from Hay's acceptance of approximation.[5] The reviewer went on to pay a high compliment: 'Mr. Hay, by an independent process of his own mind, versant about proportion and beauty, has arrived exactly at some of the identical proportions of the Platonic theory, as we understand and apply it'.[6] Hay was thus moved to consult Thomas Taylor's (1758–1835) translation of the *Timaeus*, which renders the phrase indicating that in the fairest scalene triangle† the square on the longer leg is three times the square on the shorter leg as: 'its longer side triply greater in power than the shorter'.[7] Hay interpreted this phrase as meaning that one leg has length three times that of the other, and pointed to the right-angled triangle with angle $\pi/10$ (a fifth of a right angle), the longer leg of which is $\tan(2\pi/5) = \sqrt{5+2\sqrt{5}} \approx 3.078$ the length of the shorter.[1] Since the division of a right angle into 5 was, for Hay, a harmonic division,[2] he concluded that this approximate correspondence with his interpretation of the *Timaeus* was a platonist validation of his theory.[3]

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* ▸ pp. 129–30

[1] Hay, *Proportion*, p. iv.

[2] *Ibid.*, pp. 18–19.

[3] *Ibid.*, pp. 66–7, pl. XIII.

[4] *Ibid.*, p. 67.

[5] [Anonymous], Review of *The Harmony of Form*, p. 543.

[6] *Ibid.*, p. 586.

† ▸ pp. 65–7

[7] Plato, *Timæus* (trans. Taylor), p. 527.

[Hay was aware of Adolph Zeising's (1810–76) *New Theory of the Proportions of the Human Body** (1854), which expounded a law of proportion based on the extreme and mean (or 'golden') ratio, though it is unclear how much of its content he knew directly. In the appendix to *The Science of Beauty* (1856), he quoted a largely negative review of Zeising's book by an author whom he described only as '[o]ne of the most learned and talented professors' of the University of Edinburgh.[4]]

WILLIAM ROWAN HAMILTON

It was his ability to learn languages that first pointed to William Rowan Hamilton's (1805–65) quite extraordinary intellect. He was learning Hebrew, Latin, and Greek at three; when he was ten, his father boasted of his son's mastery of over a dozen tongues.[5] The proud parent might have exaggerated, but there is no doubt that Hamilton was brilliant, not just in languages, but in mathematics. While a student at Trinity College, University of Dublin, he received the unique distinction of two awards, in Greek and in mathematical physics, of *optime*, indicating such complete mastery of the subject as to be beyond examination.[6]

And Hamilton knew well his own strength. At the age of seventeen, he indicated that he would prefer to be a creator like Archimedes or a great poet, instead of a soldier like Marcellus or a classicist.[7] Mathematics was a field where he could contribute something entirely new and where he could achieve greatness. Besides mathematics, the one area in which he thought he might excel was poetry, but William Wordsworth (1770–1850) dissuaded him from abandoning the former for the latter. As his biographer Thomas L. Hankins (1933–) noted, '[i]t was good advice, because Hamilton's poetry never matched the beauty of his mathematics.'[8]

Even at this early age, Hamilton recognized that there were aesthetic choices to be made in both the direction of research and the presentation of a mathematical work.[9] In an imagined dialogue that he wrote, he had Euclid declare to Pappus an aesthetic motivation for the scope of his work:

> '[*Euclid*.] [...] the ideas which I entertained of symmetry, and of the τὸ καλόν, induced me to attend only to regular figures, regarding none else as symmetric or beautiful.'[10]

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[1] Hay, *Proportion*, p. 77.

[2] Hay, *The Science of Beauty*, p. 40.

[3] Hay, *Proportion*, p. 77.

* ▸ pp. 555–6

[4] Hay, *The Science of Beauty*, pp. 105–8.

[5] Hankins, *Hamilton*, pp. 12–13.

[6] Graves, *Hamilton*, vol. 1, pp. 154, 209; Hankins, *Hamilton*, p. 30.

[7] Hamilton, letter to Hutton, dated 26 Aug. 1822, quot. in Graves, *Hamilton*, vol. 1, p. 110.

[8] Hankins, *Hamilton*, p. 25.

[9] Cf. Tomalin, 'William Rowan Hamilton'.

[10] Hamilton, 'Waking Dream', p. 665.

Pappus wonders where Euclid's results came from: did he develop them gradually, as other men might, or were they bestowed as a divine gift?[1] Euclid confesses that he wrote with the deliberate intention of disguising the process of discovery, to make geometry seem more engaging and wondrous.[2] And Hamilton's Euclid seems to be ambitious for his own personal fame: if he had left in place the 'scaffolding' used to assemble his work, he would not have been 'reverenced as […] the Mathematician of Alexandria'.[3] [According to his biographer Sartorius, Gauss thought similarly.[*] But since Hamilton was writing in 1823, long before Sartorius published, there is unlikely to have been any connection. And the attitude is an old one, reflected in the aphorism 'art est celare artem'; 'art is to conceal art'. This maxim is supposed to be ancient and is often attributed to Ovid; the closest match in his work is 'Si latet, ars prodest'; 'Art, if hidden, avails'.[4]]

While still an undergraduate, Hamilton was elected Andrews Professor of Astronomy and thus Astronomer Royal for Ireland and the director of Dunsink Observatory.[5] He made negligible contributions to observational astronomy;[6] his passion was mathematics, and it was there that he left his mark.

Aesthetics of mathematics

Hamilton's attitude to the role of aesthetic concerns in mathematics is evidenced in a 1829 exchange of letters with his friend Francis Beaufort Edgeworth (1809–46). In a tone of friendly antagonism,[7] Edgeworth held that, by considering astronomy not in terms of empirical adequacy, but in terms of its relation to the mind and 'the truth of beauty', he was led to the conclusion that apprehending the universe as a work of art was a 'higher exercise of mind, and more allied to divinity', than searching for physical laws. The laws of — for example — Newtonian physics might be simpler and more economical, but to creative minds, the Ptolemaic system was more beautiful, for it could be grasped as a whole, not as a disparate assembly governed by a law. And things that were called beautiful, such as artworks, were beautiful only as a whole.[8]

Scientific knowledge had no effect on those arts that aim at beauty. The knowledge of mathematics had advanced since the time of the Greeks, but the ancient arts were not scorned. Thus, advised Edgeworth, when it came to the mathematical sciences, 'do not claim for them any sublimer title as studies of beauty'.[9]

Edgeworth also raised the question: 'Why is the study of beauty *higher* than that of mathematical truth?'[10] He seemed to think that the presupposition was uncontroversial.

Hamilton disagreed. While he certainly did not identify truth and beauty, he did not separate them as Edgeworth would. He saw them as different facets or foci of our perception of the universe:

[1] Hamilton, 'Waking Dream', p. 662.

[2] *Loc. cit.*

[3] *Ibid.*, pp. 622–3.

[*] ▸ p. 403

[4] Ovid, *The Art of Love*, bk II, l. 313.

[5] Hankins, *Hamilton*, p. 45.

[6] *Ibid.*, pp. 47–8.

[7] Tomalin, 'William Rowan Hamilton', § III.

[8] Edgeworth, letter to Hamilton, dated 7 Aug. 1829, repr. in Graves, *Hamilton*, vol. I, p. 338.

[9] Edgeworth, letter to Hamilton, dated 7 Aug. 1829, repr. in *ibid.*, vol. I, p. 340.

[10] Edgeworth, letter to Hamilton, dated 7 Aug. 1829, repr. in *ibid.*, vol. I, p. 340, emphasis added.

> ‘Now it is this self-consistency, or consistency with its place in the universe, that comes properly under my head of Truth; its fitness or unfitness to excite sublime or tender emotions in the human mind, I refer to that of Beauty. The one may be said to be perceived by the mind, the other by the heart, of man. I believe that these two views of Nature have a mysterious and intimate connexion, [...] and I think that we may correctly say of the scientific and (of) the poetical man, that, while each contemplates both Truth and Beauty, yet the former habitually looks at things, or thoughts, rather as true than as beautiful; the latter as beautiful rather than as true.’[1]

Hamilton saw in mathematical theories ‘even more of “the intense unity of the energy of a living spirit” than the work of a poet or of an artist’. Newton’s *Principia*, for instance, would remain for mathematicians ‘a structure of beautiful thoughts’, even though experimental science might show that it ceased to agree so well with nature.[2]

Hamilton thought that his disagreement with Edgeworth was grounded in a fundamental difference in their psychologies. He conceded that his perception of beauty in mathematical theories was dependent on his ‘particular constitution of mind’. He wrote bluntly that he was inferior to Edgeworth in ‘poetical sensibility and power’ and Edgeworth was inferior to him in mathematical ability; thus Hamilton’s lesser feeling for artistic beauty and Edgeworth’s rejection of mathematical beauty. Hamilton confessed that he found it difficult to imagine what it would be like to do creative mathematics and *not* see any beauty, but he did not blame Edgeworth for seeking beauty where he might more easily find it, without having to make the effort of acquiring the necessary mathematical skill.[3]

In notes he made for his answer to Edgeworth, and in response to the contention that things that are graspable as a whole were more beautiful, Hamilton wrote that ‘the higher orders of Beauty seem at least to suggest infinity’. Even if there was no such suggestion, he admitted that there was intense pleasure in trying to grasp infinity or in contemplating things that suggested infinity.[4]

Although Hamilton thought that he lacked poetic taste, he continued to compose poetry throughout his life. When he was twenty-four, he explained to Wordsworth his reasoning:

> ‘I have always aimed to infuse into my scientific progress something of the spirit of poetry, and felt that such infusion is essential to intellectual perfection.’[5]

He specifically paired mathematics and poetry in a hierarchy of intellectual endeavour which he composed at age twenty-five ‘to represent the ascending scale of human thought’ and which he sent to his stu-

[1] Hamilton, letter to Edgeworth, dated 31 Oct. 1829, repr. in Graves, *Hamilton*, vol. 1, pp. 346–7.

[2] Hamilton, letter to Edgeworth, dated 20 Nov. 1829, repr. in *ibid.*, vol. 1, pp. 348–9.

[3] Hamilton, letter to Edgeworth, dated 20 Nov. 1829, repr. in *loc. cit.*

[4] Hamilton, memorandum, repr. in *ibid.*, vol. 1, p. 349.

[5] Hamilton, letter to Wordsworth, dated 1 Feb. 1830, quot. in *ibid.*, vol. 1, p. 354.

dent Edwin Wyndham-Quin, Viscount Adare (later Third Earl of Dunraven and Mount-Earl; 1812–71):[1]

Religion.
Metaphysics.
Mathematics and Poetry.
Physics and Literature.

By the term *literature* he doubtless referred to prose in distinction to poetry. Presumably he grouped together mathematics and poetry as similarly aethereal and located them above the more mundane physics and literature. This placement indicates a kind of extended platonic position in which both mathematics and poetry elevated the mind towards eternal metaphysical ideas.[2]

◆

Hamilton's perception of beauty in mathematics was not limited to pure mathematics. The mathematical expression of physical laws could have beauty, even when detached from physical interpretation:

> 'Among these laws of nature or laws of thought, those which relate to force and motion are eminent in utility and interest. Considered physically, the experience on which they are founded is grand and important: considered mathematically, the trains of thought to which they lead are beautiful and profound.'[3]

And physical laws were of concern to Hamilton. Although he was primarily a mathematician, he remained until the end of his life a professor of *astronomy*. In the first lecture in each of the annual series he delivered, he would discourse upon the history of and relationship between astronomy, mathematics, natural science in general, and philosophy. These introductory lectures, which varied each year, were adorned with poetic language and literary allusion, and were popular, not just with students of astronomy, but with other scholars and literary figures.[4] They give further insight into his views on the relation between truth and beauty in mathematics.

In 1830, he said that there was 'no antagonism and no hostility between truth and beauty', but that in their pursuit, it was necessary to aim at only one of them, and take pleasure should the other arise.[5] As he said in 1832, one should not be surprised when truth and beauty arise together:

> 'Be not surprised that there should exist an analogy, and that not faint nor distant, between the workings of the poetical and of the scientific imagination; and that those are kindred thrones whereon the spirits of Milton and Newton have been placed by the admiration and gratitude of man.'[6]

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[1] Hamilton, letter to Adare, dated 29 Dec. 1830, quot. in Graves, *Hamilton*, vol. i, p. 415.

[2] Tomalin, 'William Rowan Hamilton', § ii.

[3] Hamilton, memorandum dated September 1830, repr. in Graves, *Hamilton*, vol. i, p. 413, emphases in orig.

[4] *Ibid.*, vol. i, p. 497.

[5] Hamilton, 1830 Introductory lecture, quot. in *ibid.*, vol. i, p. 499.

[6] Hamilton, 'Introductory Lecture on Astronomy', p. 670.

As Hamilton said in his 1831 lecture, Newton's aesthetic contribution had been the mathematical unification of the cosmos: previously, even after Johannes Kepler (1571–1630), theoretical astronomy

> 'lay under the oppression of facts, material, unintellectual, disjointed; the old and beautiful array of circles and spheres of heaven had been overturned by observation and driven from the creed of men; simplicity of form was gone, and no other simplicity had yet been enthroned in its stead.'[1]

In Hamilton's view, Newton had been driven at least as much by aesthetic as by empirical considerations in formulating his physical theory. The product of his efforts was a work of art; he was like a sculptor or painter or poet in feeling out and selecting for simplicity and beauty:

> 'he seems to have been half determined by its mathematical simplicity, and consequent intellectual beauty, and only half by its agreement with the phenomena already observed.'[2]

But one should not take this too far: Hamilton pointed out in 1833 that in astronomy it was 'not [merely] the beauty of the mathematical reasoning, nor the practical accordance with phenomena' that gave the 'highest value and the deepest truth' to the theory of gravitation; rather it was a deeper engagement of the mind with the universe. The visible world of the phenomena was interpreted via an invisible world of mathematics, but the mind desired that this modelling of the world should be in harmony with our way of thinking. At the most basic level, a light in the night sky was imagined to be a body and have a position.[3] Thus, while Hamilton recognized both the mathematical beauty of physical law, and its empirical adequacy, he saw that neither was truly fundamental to science.

* * *

In his inaugural address as president of the Royal Irish Academy in 1838, Hamilton suggested that the highest part of the study of the beautiful would be

> 'the relation of Beauty to the mind, and the study of those essential Forms, or innate laws of taste, in and by which, alone, man is capable of beholding the beautiful.'[4]

He referred to an 'Idea of Beauty', which did not necessarily mean a Platonic Form, and which could inspire a person to be a creator of beauty, whether in poetry or sculpture or painting. And mathematicians so inspired have a special role because of their search for truth:

> 'The mathematician himself may be inspired by this in-dwelling beauty, while he seeks to behold not only truth but harmony;

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[1] Hamilton, 1831 Introductory lecture, quot. in Graves, *Hamilton*, vol. i, p. 502.

[2] Hamilton, 1831 Introductory lecture, quot. in *loc. cit.*

[3] Hamilton, 1833 Introductory lecture, quot. in *ibid.*, vol. ii, p. 68, insertion in ed.

[4] Hamilton, 'Inaugural Address', p. 735.

> and thus the profoundest work of a Lagrange may become a scientific poem.'[1]

Lagrange was, for Hamilton, the touchstone of mathematical achievement, and this achievement was at least partly dependent on the beauty of his work: in making a similar point some years earlier, Hamilton pointed to

> 'the beauty of the method so suiting the dignity of the results, as to make of his great work a kind of scientific poem.'[2]

For Hamilton, it seems that the way in which Lagrange had developed his theory and its methods had given it a beauty that was, if not identical, at least closely related to the beauty of a poem.

Hamilton's appreciation of Lagrange is evidenced in a letter he wrote to Augustus De Morgan, where he recalled the meeting of the British Association for the Advancement of Science fourteen years earlier:

> 'I heard Jacobi speak of me, at Manchester, in 1842, [...] as "le Lagrange de votre pays"'.[3]

[The abstract of Jacobi's lecture only calls Hamilton 'illustrious',[4] so this letter seems to have been the origin of the epithet 'the *Irish Lagrange*' for Hamilton. De Morgan's obituary of Hamilton defended this modified epithet on the basis of the similarity of Hamilton's and Lagrange's styles: 'power over symbols, combined with elegance of expression'.[5] A later obituary by the amateur mathematician and metaphysician C. M. Ingleby (1823–86) made a similar assertion,[6] but it was probably derived from the earlier memoir.]

And mathematicians and scientists should not work in isolation, thought Hamilton, but should seek out their literary counterparts: 'the scientific man learns elegance of method from the man of literature, and teaches him precision in return'.[7]

Beauty and elegance

Hamilton seemed to have seen mathematical beauty in theories, methods, and results, but not in proofs. For instance, he called 'very general and beautiful'[8] and 'very remarkable and beautiful'[9] Gauss's generalization of Girard's theorem* to geodesic triangles on other curved surfaces; this was a theorem that Gauss himself had thought 'most elegant [elegantissima]'.[10] Similarly, he considered Poinsot's ellipsoid, a geometrical model of the motion of a rotating body, to be 'one of the most beautiful of the results wherewith science has been enriched by that geometer'.[11] His admiration for the theories and methods devised by Lagrange, and for their 'beauty and utility'[12] have already been discussed.

But although Hamilton wrote on *beauty* in mathematics, he seems to have reached more often for *elegance* as a term of approbation. He characterized theorems,[1] proofs, and equations as elegant. For instance, Girard's theorem on the area of spherical triangles could be 'easily and elegantly proved by *lunes*',[2] and he thought that the deduction of the fundamental properties of conic sections using Dandelin spheres was 'a very elegant geometrical proof'.[3] And he thought the differential equations of motion in Lagrangian mechanics[*] were elegant.[4]

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[1] Hamilton, 'Inaugural Address', p. 735.

[2] Hamilton, 'On a General Method in Dynamics', p. 104.

[3] Hamilton, letter to De Morgan, dated 11 Jun. 1856, repr. in Graves, *Hamilton*, vol. III, p. 509.

[4] Jacobi, 'On a New General Principle of Analytical Mechanics', p. 4.

[5] [Anonymous], 'Sir W. R. Hamilton', p. 132; attrib. to De Morgan by Graves, *Hamilton*, vol. III, p. 217.

[6] Ingleby, 'The *Late* Sir William Rowan Hamilton', p. 163.

[7] Hamilton, 'Inaugural Address', p. 737.

[8] Hamilton, *Lectures on Quaternions*, p. 581.

[9] *Ibid.*, p. 609.

* ▸ p. 349

[10] Gauss, 'Disq. Gen. circa Superficies Curvas', p. 246.

[11] Hamilton, 'On Quaternions and the Rotation of a Solid Body', p. 385.

[12] Hamilton, 'On a General Method of Expressing the Paths of Light', p. 315.

Perhaps unusually, Hamilton was at least open to describing *diagrams* as elegant: in discussing an approximate but very accurate construction of the regular heptagon by Friedrich Gottlob Röber (1769–1833), Hamilton said that the diagram used in his construction was 'not very complex, and may even be considered to be elegant'.[5]

[Röber's construction first appeared in the book *Contributions to Research on the Basic Geometrical Forms in the Ancient Temples of Egypt and their Relationship to the Ancient Knowledge of Nature* [*Beitraege zur Erforschung der Geometrischen Grundformen in den Alten Tempeln Aegyptens und deren Beziehung zur alten Naturerkenntniss*] by his son Friedrich Röber, then later in a volume of his papers edited by the younger Röber. This latter Röber was important in the history of golden numberism,[†] and Hamilton dryly noted that the works of both the elder and the younger Röber were 'replete with the most curious speculations'.[6]]

At least with regard to equations, Hamilton's use of the term *elegance* seems to have been concerned with their efficiency and the conciseness of their form. Thus relatively minor syntactic alterations would alter his judgement of elegance. For instance, consider the following equation:

$$\left.\begin{aligned} \eta_{x,t} = {} & \frac{1}{2}\operatorname{vers}\left(\mu x - 2t\sin(\mu/2)\right) \\ & - \frac{\operatorname{vers}\mu}{4\pi}\int_0^\pi \frac{\sin(2x\vartheta - 2t\sin\vartheta)}{\sin\vartheta\,(\cos\vartheta - \cos(\mu/2))}\,\mathrm{d}\vartheta\,. \end{aligned}\right\} \quad \text{(H1)}$$

With the substitution $\nu = \mu/2$, equation (H1) becomes

$$\left.\begin{aligned} \eta_{x,t} = {} & \sin(x\nu - t\sin\nu)^2 \\ & - \frac{\sin\nu^2}{2\pi}\int_0^\pi \frac{\sin(2x\vartheta - 2t\sin\vartheta)}{\sin\vartheta\,(\cos\vartheta - \cos\nu)}\,\mathrm{d}\vartheta\,. \end{aligned}\right\} \quad \text{(H2)}$$

For Hamilton, this change made (H2) 'more elegant' than (H1).[7]

Quaternions

The *quaternions* comprise quantities $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$, subject to the rules $\mathrm{ij} = \mathrm{k}$, $\mathrm{jk} = \mathrm{i}$, $\mathrm{ki} = \mathrm{j}$, and $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$. From these rules it follows that $\mathrm{ji} = -\mathrm{k}$, $\mathrm{kj} = -\mathrm{i}$, and $\mathrm{ik} = -\mathrm{j}$. Thus multiplication of quaternions is not commutative; they are the unique finite-dimensional associative non-commutative division algebra over the reals. The physicist Roger Penrose (1931–) would later call them 'a very beautiful algebraic structure'.[8]

[1] Hamilton, letter to Herschel, dated 16 Jun. 1831, quot. in Graves, *Hamilton*, vol. 1, p. 436.

[2] Hamilton, *Lectures on Quaternions*, p. 337, emphasis in orig.

[3] Hamilton, letter to Adare, dated 6 Mar. 1832, repr. in Graves, *Hamilton*, vol. 1, p. 526.

[*] ▸ p. 400

[4] Hamilton, 'On a General Method in Dynamics', pp. 106, 117.

[5] Hamilton, 'On Röber's Construction of the Heptagon', p. 566.

[†] ▸ p. 556

[6] *Ibid.*, p. 566, n. †.

[7] Hamilton, 'Propagation of Motion in Elastic Medium', p. 554.

[8] Penrose, *The Road to Reality*, p. 200.

The set of quaternions is denoted $\mathbb{H}$ after Hamilton, whose discovery of them is perhaps one of most scrutinized moments of mathematical creativity, not least because of Hamilton's own near-contemporary and later accounts of it, and both the immediate and the lasting impact upon him.

The circumstances were as follows. At least six mathematicians seem to have realized independently that the complex numbers admit a natural interpretation as a plane, with the number $a+b\mathrm{i}$ (where $a, b \in \mathbb{R}$) identified with the coördinate (a, b).[1] The first of these six was Caspar Wessel (1745–1818), although his contribution went unnoticed until 1897.[2] Wessel tried to find an analogous 'three-dimensional' system comprising 'triplets' $a + b\mathrm{i} + c\mathrm{j}$, where j is a new imaginary unit distinct from i. He encountered difficulties that forced him into unsystematic limitations, but his achievement was impressive and allowed him to derive important results in spherical geometry.[3]

Hamilton described the reasoning that led him to the quaternions. Pursuing the geometrical intuition, one saw that multiplication by j, like multiplication by i, should correspond to a rotation by a quarter-circle, which implied that $\mathrm{j}^2 = -1$. But multiplication of quantities $a + b\mathrm{i} + c\mathrm{j}$ produced terms involving ij: for instance, it seems that

$$(a + b\mathrm{i} + c\mathrm{j})^2 = a^2 - b^2 - c^2 + 2ab\mathrm{i} + 2ac\mathrm{j} + 2bc\mathrm{ij}.$$

Hamilton wondered how to rid himself of this troublesome product:

> 'Behold me therefore tempted for a moment to fancy that $\mathrm{ij} = 0$. But this seemed *odd and uncomfortable*, and I perceived that the same suppression of the term which was *de trop* might be attained by assuming what seemed to me *less harsh*, namely that $\mathrm{ji} = -\mathrm{ij}$.'[4]

The one course was to Hamilton 'odd and uncomfortable'; the other 'less harsh'. These phrases do not use aesthetic terms, but they hint at some kind of possibly aesthetic hesitation. Hamilton therefore set $\mathrm{ij} = \mathrm{k}$ and $\mathrm{ji} = -\mathrm{k}$, and thought to see what value k might naturally have. Thinking through the consequences, he was led to the conclusion that k was the unit on the axis of the *fourth* dimension, and that the order of multiplication was '*not indifferent*'.[5]

The illumination struck him on 16 October 1843, while he walked to chair a meeting of the council of the Royal Irish Academy:

> 'I then and there felt the galvanic circuit of thought *close*; and the sparks which fell from it were the *fundamental equations between* i, j, k*; exactly such* as I have used them ever since. I pulled out on the spot a pocket-book, which still exists, and made an entry, on which, *at the very moment*, I felt that it might be worth my while to expend the labour of at least ten (or it might be fifteen) years to come.'[6]

Hamilton was certain enough of the correctness and importance of his idea to seek, at the very meeting to which he was walking, the

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[1] Crowe, *A History of Vector Analysis*, p. 5.

[2] *Loc. cit.*

[3] *Ibid.*, pp. 7–8.

[4] Hamilton, letter to J. T. Graves, dated 17 Oct. 1843, repr. in Hamilton, 'Letter to Graves on Quaternions', p. 107, emphases added.

[5] Hamilton, letter to J. T. Graves, dated 17 Oct. 1843, repr. in *ibid.*, p. 107–8, emphasis in orig.

[6] Hamilton, letter to Tait, dated 15 Oct. 1858, repr. in [Anonymous], 'Sir William Rowan Hamilton', 57, emphases in orig.; attrib. dest. Tait, Graves, *Hamilton*, vol. 2, pp. 433.

council's permission to read a paper on the subject at the next general meeting.[1]

In the days following his discovery, Hamilton imagined that quaternions would prove useful in both physics and mathematics. This enthusiasm endured: he devoted the rest of his life almost exclusively to developing the theory.[2] And part of his continuing motivation to complete the theory was aesthetic: for instance, he saw it as 'an inelegance and imperfection' that it seemed to be necessary to appeal to ordinary algebra to solve equations in the quaternions, and saw the developments of methods for solving such equations as an important avenue for research.[3]

Quaternions played an important role in the emergence of vector analysis. But they were not greeted with universal approbation. At the 1847 meeting of the British Association in Oxford, Hamilton spoke on the application of quaternions to calculating the position of the Moon. This provoked a lively debate among the mathematicians and astronomers present as to the merits of quaternions, with the polymath John Herschel (1792–1871) giving specially strong support to their study.[4] The following year, Herschel wrote to Hamilton on the subject of quaternions: 'I can admire and feel the beauty of your theorems, I almost fear now that I may never be able to acquire any command over the powerful instrument which has originated them'.[5]

Benjamin Peirce (1809–80), who was professor of mathematics at Harvard University and is generally considered to have been the first great American mathematician, was another enthusiast. He lectured on quaternions,[6] and characterized their algebra as having 'symmetry, elegance, and power'.[7]

Felix Klein (1849–1925) thought it was a beautiful result[8] that a general rotation-dilation in three-dimensional space could be represented by multiplication of quaternions $q \mapsto \alpha q \beta$, where the operand q and the parameters α and β are all quaternions, and indeed described as beautiful the expression of the transformation in this quaternion form.[9]

The position of the physicist William Thomson (Baron Kelvin; 1824–1907) is curious: he seems to have recognized the mathematical beauty of quaternions, but did not see their worth in physics. He had what he characterized as 'thirty-eight years' war' about quaternions with Peter Guthrie Tait (1831–1901), with whom he co-authored the highly-regarded physics textbook *Treatise on Natural Philosophy*:

> 'He had been captivated by the originality and extraordinary beauty of Hamilton's genius in this respect [...]. Times without number I offered to let quaternions into Thomson and Tait, if he could only show that in any case our work would be helped by their use. You will see that from beginning to end they were never introduced.'[10]

(Tait had at least called some of Hamilton's work on the quaternions '*wonderfully elegant*'.[11]) Thomson's objection seems to have been not

[1] Hamilton, letter to his son A. H. Hamilton, dated 5 Aug. 1865, repr. in Graves, *Hamilton*, vol. 2, p. 435.

[2] Crowe, *A History of Vector Analysis*, p. 30.

[3] Hamilton, *Lectures on Quaternions*, p. 522.

[4] [Anonymous], 'Seventeenth Meeting of the British Association', p. 711.

[5] Herschel, letter to Hamilton, dated 3 Apr. 1848, quot. in Graves, *Hamilton*, II, p. 599.

[6] Crowe, *A History of Vector Analysis*, pp. 125–6.

[7] Peirce, 'Linear Associative Algebra', p. 216.

[8] Klein, *Elementary Mathematics from a Higher Standpoint*, vol. 1, p. 71–2.

[9] Klein, 'Zur Nicht-Euklidischen Geometrie', p. 360.

[10] Thomson, quot. in Chrystal, 'Professor Tait', p. 306.

[11] Hamilton, letter to Tait, dated Dec. 1858, quot. in Graves, *Hamilton*, vol. III, p. 107, emphasis in orig.

to quaternions *per se*; rather, he doubted whether they had any practical benefit in physics. But elsewhere he seemed harsher, holding quaternions to have been harmful in spite of their beauty.

> 'Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Clerk Maxwell.'[1]

In his obituary of Tait, Thomson referred to the 'captivatingly ingenious and beautiful method of quaternions in Mathematics'.[2] Perhaps the key part of this description is 'in Mathematics', and Thomson's point was that the beauty of quaternions was limited to their use in mathematics itself. But perhaps he thought that the captivating beauty was a negative feature, as if the sirenic quaternions lured physicists away from their true course.

The bell tolled for the use of quaternions in physics when Oliver Heaviside (1850–1925) did not use them in his development of Maxwell's work, and argued in his *Electromagnetic Theory* that they were unsuitable for physical applications. He made a damning remark:

> 'If we put aside practical application to Physics, [...] Quaternions[3] furnishes a uniquely simple and natural way of treating *quaternions*. Observe the emphasis.'[4]

◆

An October 1846 poem entitled 'The Tetractys', which his obituarist C. M. Ingleby (1823–86) gave as one of three sonnets exemplifying Hamilton's 'poetic genius',[5] connected his conversations with Herschel about quaternions with the intellectual heritage of Pythagoras:

> 'Or high Mathésis, with its "charm severe
> Of line and number," was our theme; and we
> Sought to behold its unborn progeny,
> And thrones reserved in Truth's celestial sphere;
> While views before attained became more clear:
> And how the One of Time, of Space the Three,
> Might in the Chain of Symbol girdled be:
> And when my eager and reverted ear
> Caught some faint echoes of an ancient strain,
> Some shadowy outline of old thoughts sublime,
> Gently He smiled to mark revive again,
> In later age, and occidental clime,
> A dimly traced Pythagorean lore;
> A westward floating, mystic dream of FOUR.'[6]

[1] Thomson, letter to Hayward, dated Dec. 1892, quot. in Thompson, *The Life of William Thomson*, vol. II, p. 1138.

[2] Thomson, 'Obituary Notice of Professor Tait', p. 364.

[3] The capitalized word denoted the theory.

[4] Heaviside, *Electromagnetic Theory*, § 175 [vol. I, p. 301].

[5] Ingleby, 'The *Late* Sir William Rowan Hamilton', p. 168.

[6] Hamilton, 'The Tetractys', repr. in Graves, *Hamilton*, vol. II, p. 525.

THOMAS HILL

The mathematician and educator Thomas Hill (1818–91), in a very positive[1] review of Hamilton's *Lectures on Quaternions*, dwelt upon the parallels between mathematics and poetry. They are 'of the closest kindred':

> 'Poetry is a creation, a making, a fiction; and the Mathematics have been called, by an admirer of them, the sublimest and most stupendous of fictions. It is true; they are not only μάθησις [*mathēsis*], learning, but ποίησις [*poiēsis*], a creation.'[2]

[The term ποίησις can refer to creation or production in general, and also specifically to the art of poetry.[3]]

But the mathematician had a freedom that was denied to the poet. The poetic imagination was ultimately limited by the necessity of conveying the meaning through images of material things. But the mathematician was not bound by matter, could 'revel in the creation of forms of abstract beauty', and could undertake 'a bolder flight of imagination than that of any poet that ever lived'.[4] The most beautiful mathematical objects, such as curves, might never have been physically realized; they existed only in the imagination.[5] And a yet greater difference between the mathematician and the poet was the nature of the things they conceived: those of the poet were 'merely fictions', while 'those of the mathematician have a real and necessary existence'. Hill seems to have viewed mathematical creation as an act of insight into a platonic mathematical reality: 'Whatever the mathematician really imagines, is not imaginary, but real.'[6] What the mathematician conceived of, therefore, were objects that had always existed and always would.[7] Even when the objects concerned were not and could not be instantiated in the real world, their study could be 'one of the finest achievements of the human intellect'[8] and could still ultimately yield tools to understand the physical world. Quaternions, thought Hill, could be viewed as a branch of this 'Imaginary Calculus'.[9]

◆

[1] Crowe, *A History of Vector Analysis*, p. 38.

[2] Hill, Review of *Lectures on Quaternions*, p. 229; transliterations added.

[3] LSJ, 'ποίησις'.

[4] Hill, Review of *Lectures on Quaternions*, p. 232.

[5] *Loc. cit.*

[6] *Ibid.*, p. 233.

[7] *Ibid.*, pp. 233–4.

[8] *Ibid.*, p. 235.

[9] *Loc. cit.*

12

The Beauty of Things Not Seen

Aesthetics and the isolation of mathematics

> ‘the algebraist, in his researches,
> needs to be guided by the principle of faith,
> so well and philosophically defined as
> “the substance of things hoped for,
> the evidence of things not seen.”’
>
> — J. J. Sylvester, ‘A synoptical table’, p. 217.

> ‘πραγμάτων ἔλεγχος οὐ βλεπομένων’
>
> — Hebrews 11:1;
> chosen by J. J. Sylvester as the motto for the
> *American Journal of Mathematics*,
> used from vol. II (1879).

Pure mathematics emerged as a distinct field of study during the nineteenth century: this assertion is true, but adequate only as the broadest summary, not least because the word *pure* seems to have shifted in meaning, or at least in implication, during that time.

The inclusion of what are now known as the natural sciences within the scope of natural philosophy lasted until the second half of the nineteenth century. At the start of the century, mathematics was one of what Samuel Taylor Coleridge (1772–1834) called the pure sciences, which ‘represent pure acts of the mind’.[1] They were pure not because of a lack of utility, but because they were built up as mental constructs without reference to the world.[2] William Rowan Hamilton (1805–65) has emphasized this point in his 1832 introductory lecture on astronomy:[*]

> ‘These purely mathematical sciences of algebra and geometry, are sciences of the pure reason, deriving no weight and no assistance from experiment, and isolated, or at least isolable, from all outward and accidental phenomena.’[3]

Euclidean geometry, for instance, was thought to describe the world and had innumerable applications, but grew out of axioms that were considered true on an intuitive basis; this was why geometrical statements were, for Immanuel Kant (1724–1804), the synthetic *a priori*.[†]

▪ Parts of this chapter are based on one of the author’s essays in *An Annotated Mathematician’s Apology*, *viz.*: Cain, ‘Context of the *Apology*’.

[1] Coleridge, ‘Treatise on Method’, p. 675.

[2] *Ibid.*, p. 631.

[*] ▸ pp. 415–16

[3] Hamilton, ‘Introductory Lecture on Astronomy’, p. 665.

[†] ▸ p. 375

The detachment of mathematical truth from empirical truth is one of the main strands that marks out the beginnings of pure mathematics. The acceptance of non-euclidean geometries meant that theorems of euclidean geometry were no longer facts about the world, but facts about an abstract space.[1] Another strand was the acceptance that mathematical models of reality were simply approximations. For example, the evolution of the distribution of heat could not be precisely described by continuous functions; fundamentally, the particulate nature of matter meant that the universe was in a sense *discontinuous*.[2] Yet another thread was that mathematicians had been encountering concepts that had no real-world counterpart, such as complex numbers or higher-dimensional spaces, but which still seemed to merit investigation; this led to higher degrees of abstraction and the formulation of concepts such as abstract groups.[3]

These intellectual currents were reinforced by social changes. In Germany, organized into a confederation after the disruption of the Napoleonic Wars and the dissolution of the Holy Roman Empire, a new conception of humanist education emerged and combined with university reform.[4] Wilhelm von Humboldt (1767–1835) was a key figure in both the former and latter. Both he and Friedrich August Wolf (1759–1824) argued for a modified humanism that did not focus on utilitarian concerns, but rather on self-cultivation and moral formation; this is what later scholars called *neo*humanism.[5] Fundamental to neohumanism was *Wissenschaft*, in the broad sense of academic scholarship, comprising active research towards a total understanding of the entire world, and pursued for its own sake.[6]

Ernst Gottfried Fischer (1754–1831), who taught mathematics and physics at a *gymnasium* in Berlin, wrote a book *On the most Appropriate Establishment of Educational Institutions for the Educated Classes* [*Ueber die zweckmäßigste Einrichtung der Lehranstalten für die gebildetern Stände*] (1806), proposing a new type of school for those who required higher non-academic education. He saw education as having several purposes, including formation of the mental faculties. Pure mathematics was particularly suited to this, and should be central to the curriculum.[7] But he also believed that a humanist education should provide a basis for entering any of the higher skilled professions. At least part of the reason that France and the United Kingdom were more economically developed than the German states, he believed, was that in those countries mathematics and the natural sciences were taught as *theoretical* subjects. Thus Fischer thought that practical strengths were grounded in a humanistic, non-utilitarian education.[8]

Humboldt was a key figure in the foundation of the University of Berlin, which became a centre for mathematics in the early part of the nineteenth century. August Leopold Crelle (1780–1855), who advised

[1] Maddy, ‘How applied mathematics became pure’, pp. 19–20.

[2] *Ibid.*, pp. 26–33.

[3] *Ibid.*, pp. 18–19.

[4] Anderson, *European Universities*, p. 51.

[5] van Bommel, *Classical Humanism and the Challenge of Modernity*, pp. 3–5; Anderson, *European Universities*, p. 52.

[6] Nipperdey, *Germany from Napoleon to Bismarck*, pp. 43–5.

[7] van Bommel, *Classical Humanism and the Challenge of Modernity*, pp. 148–9.

[8] *Ibid.*, p. 150.

the Prussian Ministry of Education on mathematics, criticized the French emphasis on applied mathematics.[1] Crelle used his influence to advance the careers of pure mathematicians whose abilities or even genius he recognized,[2] and founded the *Journal für die reine und angewandte Mathematik* (still known among mathematicians simply as *Crelle's Journal*), in which most of the published authors were pure mathematicians.[3]

◆

The emerging autonomy of mathematics is evidenced by the creation, in the later nineteenth century, of mathematical societies, separate from general scientific societies. In 1865 was founded the London Mathematical Society; in 1872, the Société Mathématique de France; in 1883, the Edinburgh Mathematical Society; in 1888, the New York Mathematical Society, which became the American Mathematical Society six years later; in 1890, the Deutsche Mathematiker-Vereinigung. These societies were focused on research, and produced their own specialist journals.[4]

◆

The legitimation of the autonomy of pure mathematics was neither universal nor uniform. Carl Jacobi (1804–51) saw the value of mathematics as lying in the cultivation of the human intellect and the satisfaction of its desires, and contrasted Joseph Fourier's (1768–1830) emphasis on utility:

> 'It is true that M. Fourier had the opinion that the principal goal of mathematics was public utility and the explanation of natural phenomena, but a philosopher like him should have known that the one goal of science is the honour of the human spirit, and that in this capacity, a question in numbers is worth as much as a question of the system of the world.'[5]

There may have been a national variation in attitudes: Jacobi was German; Fourier French. And Jacobi encountered scepticism of his view at the 1842 meeting of the British Association for the Advancement of Science in Manchester:

> 'I had the courage there to assert that it is the honour of science to be of no use, which provoked a powerful shaking of heads.'[6]

(Jacobi was not unique in his position: Ernst Kummer (1810–93) agreed with him.[7]) Jacobi's reception in Manchester is something of a historical irony, since during the next century many of the most powerful justifications for doing mathematics independently of possible applications would be made by British mathematicians.

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[1] Scriba, 'Crelle, August Leopold', p. 466.

[2] *Loc. cit.*

[3] Gray, *Plato's Ghost*, p. 153.

[4] *Ibid.*, p. 36.

[5] Jacobi, letter to Legendre, dated 2 Jul. 1830, repr. in Jacobi, *Gesammelte Werke*, vol. 1, pp. 454–5; trans. C.

[6] Jacobi, letter to his brother M. H. Jacobi, dated 25 Sep. 1842, repr. in Jacobi & Jacobi, *Briefwechsel*, p. 90; trans. Turner, 'The Growth of Professorial Research', p. 152.

[7] Biermann, 'Kummer, Ernst Eduard', p. 522.

WILLIAM WHEWELL AND MATHEMATICS AT CAMBRIDGE

When Jacobi saw a 'powerful shaking of heads' at Manchester, pure mathematics was just beginning its rise in the United Kingdom. At the start of the nineteenth century, mathematics in Britain, which was centred on the University of Cambridge, was almost entirely isolated from mathematics elsewhere in Europe, and so the ascent of pure mathematics in Britain was delayed.[1] Mathematics in Cambridge was of the Newtonian school, was seen as a part of natural philosophy, and emphasized arguments where the meanings in the physical world of the terms employed were to be kept in mind, whereas continental techniques permitted the formal manipulation of symbols. An effort by a group of scholars including Robert Woodhouse (1773–1827), George Peacock (1791–1858), Charles Babbage (1791–1871), and John Herschel (1792–1871) introduced the notation and methods of continental analysis.[2]

[Frederick Pollock (1783–1870), who was senior wrangler (the person who gained that year's highest mark in Part I of the Cambridge Mathematical Tripos) in 1806 and who went on to success in law and politics, once offered to Augustus De Morgan (1806–71) an evaluation of those whom De Morgan had suggested were the key figures of this period of change in Cambridge mathematics. He praised Robert Woodhouse over Edward Waring (c. 1736–1798) partly on the ground that the former had given a beautiful proof of Wilson's theorem.* While Pollock qualified the attribution of the proof to Woodhouse with '(I think)' (and the present author has been unable to locate a proof of Wilson's theorem by Woodhouse), there was no such qualification of the aesthetic evaluation. He described Samuel Vince (1749–1821) as 'a bungler' who was 'utterly insensible of mathematical beauty'.[3]]

But even with the introduction of the new analysis, mathematics at Cambridge remained focused on applied mathematics. William Whewell (1794–1866), who had the greatest influence over mathematics teaching at Cambridge during the 1830s and 1840s, promoted applied mathematics and thought pure analysis was a threat to the liberal education offered by the Cambridge Tripos, in which the study of mathematics was a means of training the mind, and after which the path into any profession was open.[4]

Whewell recognized that mathematics could legitimately be pursued even with no application in view. He attributed the study of conic sections by Greek mathematicians to the beauty of the subject: they saw no connection between such subjects and nature, and so there was no gain beyond satisfying curiosity and pure intellectual enjoyment.[5] Even in Isaac Newton's *Principia*, which Whewell thought a 'store of refined and beautiful mathematical artifices',[6] he found parts

* ▸ p. 395

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[1] For a full study, see Heard, 'The Evolution of the Pure Mathematician'.

[2] Rouse Ball, *A History of the Study of Mathematics at Cambridge*, ch. VII.

[3] Pollock, letter to De Morgan, dated 7 Aug. 1869, repr. in De Morgan, *Memoir of Augustus De Morgan*, pp. 390–2.

[4] Becher, 'William Whewell and Cambridge Mathematics', pp. 3, 10.

[5] Whewell, *History of Scientific Ideas*, vol. 1, pp. 161 [§ II.XIV.3].

[6] Whewell, *History of the Inductive Sciences*, vol. 2, p. 62 [§ VI.V.1].

that were worthwhile solely for their aesthetic value, independent of their applications:

> ‘The propositions in which it is shown that the law of the inverse square for the particles gives the same law for spherical masses, have that kind of beauty which might well have justified their being published for their mathematical elegance alone, *even if they had not applied to any real case*.’[1]

The formation by stages of the University of London in the 1820s and 1830s led to a lessening of the dominance of Cambridge over mathematics in Britain, in particular because of Augustus De Morgan’s appointment as professor of mathematics there in 1828. The creation of specialist mathematics periodicals, beginning with the *Cambridge Mathematical Journal* around the middle of the nineteenth century, is an indicator of the emergence of mathematics out of natural philosophy and the self-recognition of its practitioners as researchers in a distinct field.[2]

NON-EUCLIDEAN GEOMETRY

> ‘ You should not tempt the parallels in this way, I know this way until its end — I also have measured this bottomless night, I have lost in it every light, every joy of my life ’
>
> — Farkas Bolyai, letter to János Bolyai, dated Spring 1820, quot. in Bolyai & Bolyai, *Geometrische Untersuchungen*, vol. 1, p. 76 (trans. D. Struik, ‘Bolyai, János’, p. 270).

John Wallis (1616–1703), the third Savilian Professor of Geometry at the University of Oxford, wrote a treatise on the parallel postulate and matters connected with it, in which he recalled with approval Henry Savile’s (1549–1622) statement that ‘[o]n the most beautiful body of geometry there are two blemishes’.[3][*] The third part of this treatise is a ‘proof’ of the parallel postulate.[4] Wallis proved that the parallel postulate is equivalent to the existence of triangles that are similar but not congruent, which, for Wallis, was sufficient to establish the parallel postulate, for he thought it was indisputable that there were non-trivial similarities. He also gave a metaphysical proof, arguing that similarity depends on quality, magnitude on quantity, and that these two categories are distinct.[5] Wallis believed he had accomplished a task left by the founder of his chair, having ‘vindicated’ Euclid: ‘Atque hæc, in Euclidis vindicias, fufficiant’.[6]

Girolamo Saccheri

One of the most celebrated attempts to prove the parallel postulate — an attempt that was subsequently recognized as ven-

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[1] Whewell, *History of the Inductive Sciences*, vol. 2, p. 134–5 [§ vi.ii.5], emphasis added.

[2] Heard, ‘The Evolution of the Pure Mathematician’, ch. 2.

[3] Savile, *Prælectiones*, p. 140; trans. Goulding, *Defending Hypatia*, p. 177.

[*] ▸ pp. 271–2

[4] Wallis, ‘De Postulato Quinto’, pp. 674–8.

[5] De Risi, intro. to Saccheri, *Euclid Vindicated*, p. 16.

[6] Wallis, ‘De Postulato Quinto’, p. 678.

Non-euclidean geometry

turing far into the territory of hyperbolic geometry — was that made by Giovanni Girolamo Saccheri (1667–1733).

Saccheri seems not to have been a very well-read scientist, and his references to classical or Renaissance authors seem to be second-hand.[1] But it is likely that he knew Savile's lectures, which were a popular introduction to geometry.[2] He titled his work Euclid Vindicated [*Euclides ab omni naevo vindicatus*], using the same word 'blemish [naevus]' that Savile had used, having likely read Savile's assessment in quotation in Wallis's treatise,[3] and perhaps also used 'vindicated' in reference to Wallis. Saccheri actually saw one more blemish than Savile in what he called the 'most beauteous nor ever sufficiently praised *Elements*':[4] the definition of equiproportionals.[5]

In his bid to prove the parallel postulate, Saccheri began by considering a quadrilateral with equal-length sides AC and BD perpendicular to its base AB, and showed the two upper angles are equal (see Figure 12.1).[6] (This starting-point had previously been used by Omar Khayyam (c. 1048–c. 1131) and Naṣir al-Dīn al-Ṭūsī (1201–74).[7]) There are then three cases to consider: the upper angles may be acute, right, or obtuse. The parallel postulate implies that they are right. Saccheri showed that the hypothesis that they were obtuse implies the parallel postulate; hence they were right. Thus the obtuse hypothesis could not hold by *consequentia mirabilis*, which to Saccheri was an especially strong *reductio ad absurdum*: 'The hypothesis of obtuse angle is absolutely false, because it destroys itself.'[8] Saccheri then turned to the acute-angle hypothesis. From this possibility he deduced that two straight lines either intersect, or have a common perpendicular on each side of which they diverge, or they diverge on one side and come asymptotically close on the other.[9] In the third case, the lines have a common asymptote at infinity on one side and a common point at infinity on the other. But Saccheri seems to have thought of this asymptote and point as a perpendicular and a point in the usual sense, and so (erroneously) concluded that: 'The hypothesis of acute angle is absolutely false; because repugnant to the nature of the straight line.'[10] Thus, for Saccheri, both the obtuse and acute hypotheses were thus refuted, and the parallel postulate followed.

[But even from Saccheri's incorrect perspective, the acute-angle hypothesis did not lead to a *consequentia mirabilis*, only to a common *reductio ad absurdum*. This conclusion was unsatisfactory, for Saccheri wanted to show the parallel postulate to be an *axiom*, which for him was an immediately and evidently *demonstrable* principle. A hundred-

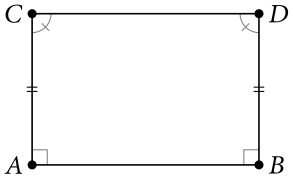

FIGURE 12.1
The quadrilateral set up by Saccheri: the segments AC and BD are of equal length and are perpendicular to AB, and thus $\angle ACD = \angle CDB$ (Saccheri, *Euclid Vindicated*, Prop. I.1).

[1] De Risi, intro. to Saccheri, *Euclid Vindicated*, pp. 24, 27.

[2] Goulding, *Defending Hypatia*, p. 177.

[3] De Risi, notes to Saccheri, *Euclid Vindicated*, 'Preface to the Reader', n. 2 [p. 250]; see Wallis, 'De Postulato Quinto', p. 655.

[4] Saccheri, *Euclid Vindicated*, p. 63.

[5] Saccheri, *Euclid Vindicated*, p. 65; cf. Euclid, *Elements*, Def. V.5 & Heath's notes.

[6] Saccheri, *Euclid Vindicated*, Prop. I.1.

[7] Rosenfeld, *A History of Non-Euclidean Geometry*, pp. 65 sqq., 75 sqq.

[8] Saccheri, *Euclid Vindicated*, Prop. I.14.

[9] Rosenfeld, *A History of Non-Euclidean Geometry*, p. 98.

[10] Saccheri, *Euclid Vindicated*, Prop. I.33.

page *reductio* proof did not qualify.[1] He therefore included a second *consequentia mirabilis* proof, possibly older than or at least written independently from the first.[2] But this proof was again flawed, this time because of an error in computing the length of a curve.[3]]

Saccheri's work had a clear aesthetic motivation in aiming to *vindicate* Euclid. He aimed to show that in the *Elements* were the resources to refute all the criticisms, and so show that the 'blemishes' were at most superficial and the 'most beauteous [...] *Elements*'[4] was untainted.

[History has been rather rough with Saccheri. The poet and philosopher Paul Valéry (1871–1945) wrote that Saccheri 'was suspicious, without admitting it to himself, of what was agreed upon in Euclid and half-opened a door to daring futures of geometry', and dismissed or pitied Saccheri and others as 'only Jesuits'.[5] The mathematician Raymond Wilder (1896–1982) thought Saccheri's work 'one of the best examples of the "law": "If the facts do not conform with the theory, they must be disposed of."'[6] And the historian of mathematics Howard Eves (1911–2004) wrote that 'Saccheri lamely forces into his development an unconvincing contradiction involving hazy notions about infinite elements'.[7] Certainly Saccheri was *wrong*, but is there really any ground for calling him self-deceiving?]

Emergence

In retrospect, Franz Taurinus (1794–1874) was a transitional figure in this development of non-euclidean geometry: while mathematicians like Saccheri had sought to prove the parallel postulate, Taurinus had developed a new geometry while maintaining the primacy, and presumably the physical reality, of euclidean geometry.[8] Two contemporaries of Taurinus had enough confidence in their discoveries to publish them and place their new geometries on an equal footing with euclidean geometry: Nikolai Ivanovich Lobachevsky (1792–1856) and János Bolyai (1802–60). Both started with the aim of proving the parallel postulate and came to develop geometrical theories in which there was more than one parallel to a given line through a given point; today, this would be called hyperbolic geometry. Bolyai, at least, seems to have had an aesthetic admiration for the emerging theory, as indicated by a letter he wrote to his father Farkas Bolyai (1775–1856) in 1823:

> 'I have made such wonderful discoveries that I have been almost overwhelmed by them, and it would be the cause of continual regret if they were lost. When you will see them, you too will recognize it. In the meantime I can say only this: *I have created a new universe from nothing*.'[9]

In 1831–2, Farkas Bolyai, doubting the validity of the system, sent his son's manuscript to Carl Friedrich Gauss* (1777–1855), who praised János's abilities but revealed that he had developed but never pub-

[1] De Risi, intro. to Saccheri, *Euclid Vindicated*, pp. 35–40.

[2] De Risi, intro. to *ibid.*, p. 30.

[3] Rosenfeld, *A History of Non-Euclidean Geometry*, p. 99.

[4] Saccheri, *Euclid Vindicated*, p. 63.

[5] Valéry, 'Variation sur une Pensée'.

[6] Wilder, *Evolution of Mathematical Concepts*, p. 10.

[7] Eves, *An Introduction to the History of Mathematics*, p. 497.

[8] Haas, 'Taurinus, Franz Adolph', p. 264.

[9] J. Bolyai, letter to F. Bolyai, dated 3 Nov. 1823, quot. in Bolyai & Bolyai, *Geometrische Untersuchungen*, vol. 1, p. 85, emphasis in orig.; trans. Bonola, *Non-Euclidean Geometry*, p. 98, emphasis in orig.

* ▸ pp. 401–8

lished much of the theory himself; this investigation was what had led him to think of geometry as not wholly *a priori*.[*] Janós was discouraged and depressed by this news, though he later allowed his manuscript to be published in the elder Bolyai's 1832 book *An Attempt to Introduce the Young Student to the Pure, Elementary and Sublime Elements of Mathematics, by an Intuitive Method, and with Suitable Proofs* [*Tentamen Juventutem Studiosam in Elementa Matheseos Purae, Elementaris ac Sublimioris, Methodo Intuitiva, Evidentiaque Huic Propria, Introducendi*], where it appeared under the title 'Appendix Explaining the Absolutely True Science of Space' ['Appendix Scientiam Spatii absolute veram exhibens'].[1] Bolyai thus either had the courage to face, or did not foresee, any 'outcry of the Boeotians'.[2]

Farkas Bolyai certainly aesthetically appreciated his son's discoveries: in the second edition of his textbook *Elements of Arithmetic* [*Az arithmetica Eleje*] (1843, first ed. 1830), he wrote:

> 'The Appendix is a work worthy of books. For the true geometer, it is a beautiful, necessary, original and colossal work'.[3]

But it would take until the posthumous publication of Gauss's correspondence for mathematicians generally to start taking seriously the idea of non-euclidean geometry. Abstract notions of geometry were subsequently developed and clarified by mathematicians including Felix Klein[†] (1849–1925) and Bernhard Riemann (1826–66). A key development came in 1868, with Eugenio Beltrami's (1835–1900) proof that the parallel postulate could not be proven from the other axioms.[4] Nevertheless, there was an aesthetic resistance to or reaction against the dethroning of Euclid.

Charles Lutwidge Dodgson

Charles Lutwidge Dodgson (1832–98) was a mathematician of some ability and perhaps overmastering thoroughness, but not a great discoverer.[5] He is better known as Lewis Carroll, the pen-name under which he published non-mathematical and literary works, most famously *Alice's Adventures in Wonderland* and its sequel *Through the Looking-Glass*.

Dodgson published little new research, and did not join the London Mathematical Society; his mathematical focus was teaching.[6] He was an arch-conservative in mathematics, and defended euclidean geometry both paedagogically and epistemologically. His 1879 book *Euclid and His Modern Rivals* aimed to show that the *Elements* was the only text necessary to teach elementary geometry, and that it should not be substituted by any modern text.[7] The book is a dialogue in which Euclid presents the case for his own work and a fictional German professor Niemand advocates for the modern rivals. They are interrogated and judged by Minos, named for one of the judges of Hades.

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[*] ▸ p. 404

[1] Struik, 'Bolyai, János', p. 270.

[2] Gauss, letter to Bessel, dated 27 Jan. 1829, repr. in Gauss & Bessel, *Briefwechsel*, p. 490.

[3] F. Bolyai, quot. in Bolyai & Bolyai, *Geometrische Untersuchungen*, vol. 1, p. 252; trans. C.

[†] ▸ pp. 459–62

[4] Greenberg, *Euclidean and Non-Euclidean Geometries*, pp. 185–6.

[5] Eperson, 'Lewis Carroll', pp. 99–100.

[6] Wilson & Moktefi, *The Mathematical World of Charles L. Dodgson*, ch. 3.

[7] Dodgson, *Euclid and His Modern Rivals*, p. ix.

Non-euclidean geometry

The arguments that are put forward for maintaining the body of euclidean geometry are often aesthetic. In response to modern rivals omitting theorems, Euclid sees the aesthetic damage that this kind of change would inflict on the theory:

> '*Euc.* As to I.7, I have several reasons to urge in favour of retaining it.
>
> […]
>
> And fourthly, there is a close analogy between I.7, 8 and III.23, 24. These analogies give to Geometry much of its beauty, and I think that they ought not to be lost sight of.'[1]

Dodgson seems to have suggested that aesthetic judgements vary with the experience of the student. Both Niemand and Minos agree that a certain non-euclidean method is excellent for the advanced student but not the novice:

> '*Nie.* For *beginners* we must admit that Euclid's method of treating this point is the best. But you will allow ours to be a legitimate and elegant method for the advanced student?
>
> *Min.* Most certainly. The whole of your beautiful treatise is admirably fitted for advanced students: it is only from the *beginner's* point of view that I venture to criticise it at all.'[2]

And again, with reference to another proof:

> '*Min.* Your proof of Prop. 2 is long, but beautiful. […] I fear that your proof of this theorem, though a model of elegance and perspicuity as a study for the advanced student, is wholly unsuited to the requirements of a beginner.'[3]

So, for Dodgson, elegance and clarity in proof did not coincide with accessibility.

Dodgson's conservatism with regard to mathematics was linked to his view of the permanence and the beauty of mathematics. He thought that the charm of pure mathematics was mainly due to 'the absolute *certainty* of its results',[4] and thought that the pythagorean theorem was 'as dazzlingly beautiful now as it was in the day when Pythagoras first discovered it'.[5]

His affirmation of the durability of mathematics explains his cleaving to Euclid. While conceding that non-euclidean geometries were just as consistent as euclidean, he was never reconciled to them. He sought, rather, to establish euclidean geometry as more natural by replacing Euclid's parallel postulate with a more natural alternative. His choice related to the area of a regular polygon inscribed in a circle. In the first edition of his book *A New Theory of Parallels*, he used a hexagon; from the third edition, he used a square.[6] Any regular polygon will do.[7] Dodgson's replacement axiom was:

[1] Dodgson, *Euclid and His Modern Rivals*, pp. 194–5.

[2] *Ibid.*, p. 56.

[3] *Ibid.*, p. 57.

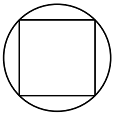
[4] Dodgson, *A New Theory of Parallels*, p. xv.

[5] *Ibid.*, p. xvi.

[6] *Ibid.*, pp. xii, 14.

[7] Wilson & Moktefi, *The Mathematical World of Charles L. Dodgson*, ch. 2.

> ‘In any Circle, the inscribed equilateral Tetragon is greater than any one of the Segments which lie outside it.’[1]



Dodgson thought that this axiom was superior to every other attempt to replace the parallel postulate, for it made use only of finite ideas.[2]

PROJECTIVE GEOMETRY

Projective geometry, which also emerged in the first half of the nineteenth century, has, at least in retrospect, been the subject of aesthetic acclaim. The mathematician and educator W. W. Sawyer (1911–2008) called it ‘one of the most beautiful parts of elementary mathematics’, which ‘abounds in beautiful impossibilities’ such as parallel lines meeting.[3] The historian of mathematics Morris Kline (1908–92) wrote that projective geometry was ‘one of the most beautiful branches of mathematics’.[4]

But contemporaneous mathematicians, including those involved in its development, also admired its beauty. Charles-Julien Brianchon (1783–1864) wrote a paper on proofs using perspective projections;[5] Jean-Victor Poncelet (1788–1867), who composed the first text on synthetic perspective geometry, thought that Brianchon’s work contained beautiful examples of how projections could transform geometrical questions into simpler ones.[6] Poncelet took the example of the duality of poles and polars of conic sections (see Figure 12.2), and realized that geometry could be generalized by introducing ideal points at infinity.[7] This change meant that two intersecting lines, which could be made parallel by a perspective projection, would still intersect. It also created the fundamental duality: any two points determine a line; any two lines determine a point.

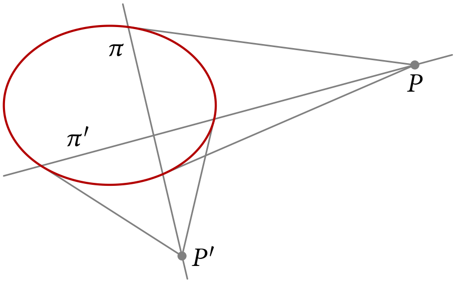


FIGURE 12.2
The duality of poles and polars: to any point P outside the ellipse, the line π through the points of contact of the tangents through P is the polar of P. Similarly, any line π that intersects the ellipse defines a pole P. Further, the pole π' of any point P' on π passes through P.

Michel Chasles (1793–1880) wrote in 1837 that Poncelet’s *Traité des propriétés projectives* (1822) contained ‘beautiful and numerous geometrical researches’.[8] He praised particularly the point–line duality, calling it ‘this beautiful law of spatial figures’.[9] Later writers seemed to agree: Eli Maor (1937–) named it 'one of the most beautiful achievements of projective geometry'.[1] Maor also said that this duality had introduced 'a measure of elegance into mathematics'.[2] Reuben Hersh (1927–2020) thought Euclid's axiom whereby two points determine a line and two *non-parallel* straight lines determine a point was, in comparison to the duality of projective geometry, 'unesthetic' and 'awkward and clumsy', because of the extra condition that the two lines are non-parallel.[3] He suggested that the move from euclidean to projective geometry was an example of altering definitions to obtain 'a more beautiful theory'.[4] [If Hersh meant to claim here that the introduction of projective geometry was aesthetically *motivated*, not just *admired*, then he reached beyond what contemporaneous evidence seems to support.]

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[1] Dodgson, *A New Theory of Parallels*, p. 14.

[2] *Ibid.*, p. xiv.

[3] Sawyer, *Prelude to Mathematics*, ch. 10.

[4] Kline, *Mathematics in Western Culture*, p. 158.

[5] See Brianchon, ‘Solution de plusiers Problèmes de géométrie’.

[6] Poncelet, *Applications d’Analyse et de Géométrie*, vol. 1, p. 116.

[7] Taton, ‘Poncelet, Jean Victor’, pp. 76–7.

[8] Chasles, *Aperçu Historique*, p. 191, n. 1; trans. C.

[9] *Ibid.*, p. 408; trans. C.

But the aesthetic evaluation of Poncelet's work was mixed. Thomas Stephens Davies (1795–1851) seemed to indicate a simultaneous aesthetic admiration of and hesitation on projective geometry when he discussed Brianchon's paper in 1848–9. On the one hand, he called it 'a theory which, for geometrical power and beauty in discussing the conic sections, is without a rival'.[5] On the other hand, he evinced a certain ambivalence about proofs using projections, which some might criticize as 'trickeries':

> 'It is sufficient to say just now, of such "trickeries," that they furnish unquestionable evidence of the truth of theorems respecting the conic sections when their correlatives for the circle have been proved; and though *our notions of elegance or geometrical purity may be violated by their use*, we are still *placed a step in advance*, as regards actual knowledge and perfect conviction, by the use of such processes as these. It is a strong incentive to perseverance in searching for a systematic demonstration of a theorem, to know before hand that it is absolutely true.'[6]

So a proof using perspective projection might deviate from accepted 'elegance or geometrical purity'. Since Davies saw the theory as unrivalled in *beauty*, for him *elegance* must have indicated a different aesthetic category. Given how his discussion progressed, elegance seems to have been bound up with the Euclidean axiomatic treatment of geometry. If a theorem about a conic section and concerning properties invariant under perspective projection had been proven for a circle, then a perspective projection — transforming the circle into the conic — provided 'unquestionable evidence' for the result and showed it to be 'absolutely true'. But this 'evidence' was merely motivation to seek a 'systematic demonstration'.

By later in the century, projective geometry seems to have become aesthetically accepted. A panel appointed to make recommendations on mathematics education in the United States argued, in its 1894 report, that projective geometry should be taught at school or undergraduate level, grounding its arguments largely on the aesthetic appeal of the subject:

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[1] Maor, *To Infinity and Beyond*, p. 112.

[2] *Ibid.*, p. 117.

[3] See, respectively, Hersh, 'Proving is convincing and explaining', p. 394; Hersh, *What is Mathematics, Really?*, p. 52.

[4] Hersh, 'Proving is convincing and explaining', p. 394; Hersh, *What is Mathematics, Really?*, p. 52.

[5] Davies, 'Historical Notices Respecting an Ancient Problem', p. 154.

[6] *Ibid.*, p. 225, emphases in orig.

> 'A place should also be found either in the school or college course for at least the elements of the modern synthetic or projective geometry. It is astonishing that this subject should be so generally ignored, for mathematics offers nothing more attractive. It possesses the concreteness of the ancient geometry without the tedious particularity, and the power of analytical geometry without the reckoning, and by the beauty of its ideas and methods illustrates the esthetic quality which is the charm of the higher mathematics, but which the elementary mathematics in general lacks.'[1]

This was the *only* one of the panel's recommendations to be based on aesthetic concerns.[2]

JAMES CLERK MAXWELL

In 1874, Francis Galton (1822–1911) attended a meeting of the Royal Society, of which he was already a fellow, and asked every scientist of high distinction there to answer a questionnaire that aimed to elucidate the degree to which scientific aptitude is inborn.[3] One of these scientists was James Clerk Maxwell (1831–79), who had developed the classical theory of electromagnetism. This theory was later encapsulated in a set of eponymous equations, which are oft held up as exemplars of beauty.[*] Asked for his 'mental peculiarities', Maxwell wrote that he had been 'delighted with the forms of regular figures and curves of all sort'.[4]

Maxwell's response is indicative of a visual aesthetic taste that recurred in his career, starting from his first publication, at the age of fourteen.[5] The scientific community had become interested in David Ramsay Hay's[†] (1798–1866) work and the particular problem of finding a straightforward mechanical method for drawing ovals. Maxwell devised a method for drawing certain Cartesian ovals, which are curves comprising points P such that

$$\mathrm{d}(P, A) + m\mathrm{d}(P, B) = \pm a,$$

where $\mathrm{d}(\blacksquare, \blacksquare)$ is the distance between two points, A and B are fixed points, and $m, a \in \mathbb{R}$ are constants. Maxwell's method applied when m was a positive rational number s/t (where, for practical reasons, $s, t \in \mathbb{N}$ should be small). He suggested fixing pins at A and B and winding a string around the pins and a pen an appropriate number of times, and drawing the curve in which the pen moves when the string is kept taut. This method generalizes a well-known technique for drawing an ellipse (see Figure 12.3). Maxwell's method was communicated to

[1] Newcomb et al., 'Mathematics', p. 116.

[2] *Ibid.*, *passim.*

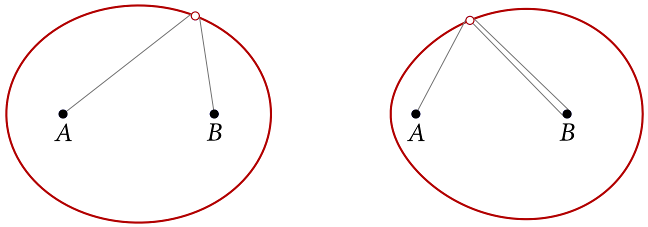
[3] Galton, *Memories of my Life*, pp. 291–3.

[*] ▸ pp. 884–5

[4] Maxwell, quot. in Hilts, 'A Guide to *English Men of Science*', p. 59.

[5] Everitt, 'Maxwell, James Clerk', p. 199.

[†] ▸ pp. 408–12



Drawing the ellipse with equation

$$d(P, A) + d(P, B) = 3.5.$$

with a string tied to A and B and looped around the pen.

Drawing the oval with equation

$$d(P, A) + 2d(P, B) = 5.$$

with a string tied to A and the pen and looped around B and the pen.

FIGURE 12.3
Drawing an ellipse and a Cartesian oval comprising points P with whose distances to foci A and B satisfy given linear expressions, using strings attached to or looped around the pen and the foci, with the pen being moved while the string is kept taut.

the Royal Society of Edinburgh by the physicist James David Forbes (1809–68), who thought it 'elegant'.[1]

After studying at Edinburgh, Maxwell went up to Cambridge, where in 1852 he was elected an Apostle.[2] The Cambridge Apostles are a secretive intellectual discussion group, founded in 1820 and still in existence, at each of whose meetings one member presents a paper on a chosen topic, which is then discussed by all.[3] Many great and eminent scholars became members: for instance, Roger Fry[*] (1866–1934; elected 1887), G. E. Moore[†] (1873–1958; elected 1894), Bertrand Russell[‡] (1872–1970; elected 1892).[4]

In the spring of 1854 Maxwell wrote an essay for the Apostles entitled 'Has everything beautiful in Art its original in Nature?' While this essay, like the others he offered, might be a speculative effort intended to provoke intellectual debate,[5] it does suggest some points about Maxwell's aesthetic attitudes. According to the essay, he believed that any beautiful human handiwork was produced by the 'laws of mind', and so were governed by what its Creator knew, although the imagination could reach beyond the basis of knowledge, and the moral faculty could emplace in art 'a still higher beauty, which expresses the glory of nature as the instrument by which our spirits are exercised, delighted, and taught'.[6] He also thought that the beauty of nature was absolutely compatible with its mathematical description, current and future:

> 'while I confess the vastness of nature and the narrowness of our symbolical sciences, yet I fear not *any* effect which either Science or Knowledge may have on the beauty of that which is beautiful once and for ever.'[7]

Although the same caveat about speculation could apply to this portion of the essay, it seems to have reflected a genuinely deeply held view, since Maxwell made a very similar point many years later in his October 1871 inaugural lecture at Cambridge:

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[1] Maxwell, 'On the Description of Oval Curves', p. 3; Forbes, letter to John Clerk Maxwell, dated 11 Mar. 1846, repr. in Campbell & Garnett, *The Life of James Clerk Maxwell*, p. 75.

[2] Deacon, *The Cambridge Apostles*, p. 203.

[3] *Ibid.*, p. 3.

[*] ▸ p. 729

[†] ▸ p. 478

[‡] ▸ pp. 479–87

[4] *Ibid.*, pp. 201–4.

[5] Campbell & Garnett, *The Life of James Clerk Maxwell*, p. 223.

[6] Maxwell, 'Has everything beautiful in Art its original in Nature?', p. 230.

[7] *Ibid.*, p. 230, emphasis added.

> ‘This habit of recognising principles amid the endless variety of their action can never degrade our sense of the sublimity of nature, or mar our enjoyment of its beauty.’[1]

Maxwell’s interest in curves also appeared in his Apostles essay:

> ‘A mathematician might express his admiration of the Ellipse. Ruskin agrees with him. […] All things are full of ellipses — bicentral sources of lasting joy, as the wondrous Oken might have said.’[2]

John Ruskin (1819–1900), writing in this particular instance on architecture, maintained that curves were more beautiful than straight lines, and that the beauty of a curve was dependent on how the curvature changed. Invariant curvature was unappealing; ‘the circle, is therefore the least beautiful of all curves’. Curves were beautiful when there was some variance in the curvature; ‘[t]he simplest of the beautiful curves are the conic, and the various spirals’.[3] Ruskin’s views on the relative beauty of the circle and ellipse stood in opposition to those Archibald Alison* (1757–1839). Maxwell’s aside to the biologist Lorenz Oken (1779–1851) seems not to have referred to any aesthetic judgement, but rather to a passage in Oken’s book *Elements of Physiophilosophy* which contrasted the monocentrality of God with the bicentrality of the world, as evidenced by the elliptical orbits of the solar system. The passage closes with the aphorism ‘Self-consciousness is a living ellipse.’[4] The explication of this statement is not relevant here; all that is important is to note that there is a faint echo of it in Maxwell’s essay.

H. J. S. SMITH

H. J. S. Smith (1826–83) made a relatively small but highly important contribution to mathematics.[5] He signifies an important milestone in the emergence of pure mathematics, since he was the first Savilian Professor of Geometry at Oxford whose most important work was in the inapplicable area of number theory.[6]

Smith certainly recognized mathematical beauty, and his *Collected Mathematical Papers* contain numerous aesthetic judgements. He also hinted at the ‘immortality’ that came from beautiful work in mathematics, a foreshadowing of a theme that G. H. Hardy (1877–1947) would discuss in a celebrated passage of *A Mathematician’s Apology*.† Smith thought that Ferdinand von Lindemann (1852–1939) had created an enduring fame for himself by proving that π is transcendental. Lindemann had proved that *If* $\alpha_1, \ldots, \alpha_n$ *are distinct algebraic numbers and* $A_1, \ldots, A_n$ *are non-zero algebraic numbers, then*

$$A_1 e^{\alpha_1} + \ldots + A_n e^{\alpha_n} \neq 0.$$

---

[1] Maxwell, quot. in Campbell & Garnett, *The Life of James Clerk Maxwell*, p. 355.

[2] *Ibid.*, p. 231.

[3] Ruskin, *Modern Painters*, vol. ii, pt iii.i, ch. vi, § 11.

* ▸ p. 382

[4] Oken, *Elements of Physiophilosophy*, § 232.

[5] North, ‘Smith’, p. 468.

[6] *Loc. cit.*

† ▸ p. 508

So if π were algebraic, then iπ would be too, and the equality $e^{i\pi} + e^0 = -1 + 1 = 0$ would contradict Lindemann's result. Smith recognized that Charles Hermite (1822–1901) had laid most of the foundation in a 'singularly profound and beautiful analysis',[1] by proving that e was transcendental and the special case of the result where the α_i are integers.[2] But it was Lindemann who made the final leap, which Smith found beautiful:

> 'It is difficult not to envy, as well as admire, people who do such beautiful things: Lindemann's name is sure of a place in every history of mathematics hereafter.'[3]

T. H. HUXLEY

The biologist T. H. Huxley (1825–95), one of the early champions of Charles Darwin's (1809–82) theory of evolution by natural selection, was a public figure who wrote brilliantly on a wide range of subjects. But his conception of the scope and activity of mathematics was limited; he thought it was concerned only with number and extension and consisted only in deduction and verification from self-evident propositions. The closest he came to praise was a concession that such deductions might be 'subtle'.[4] He certainly did not see the possibility for observation or experiment as guiding mathematical development.[5] He admitted that some unusual people experienced beauty or elegance in mathematics, creating pleasure just like that derived from artistic beauty:

> 'It may be that there are some peculiarly constituted persons who, before they have advanced far into the depths of geometry, find artistic beauty about it; but, taking the generality of mankind, I think it may be said that, when they begin to learn mathematics, their whole souls are absorbed in tracing the connection between the premisses and the conclusion, and that to them geometry is pure science. [...] But a great mathematician, and even many persons who are not great mathematicians, will tell you that they derive immense pleasure from geometrical reasonings. Everybody knows mathematicians speak of solutions and problems as "elegant," and they tell you that a certain mass of mystic symbols is "beautiful, quite lovely." Well, you do not see it. They do see it, because the intellectual process, the process of comprehending the reasons symbolised by these figures and these signs, confers upon them a sort of pleasure, such as an artist has in visual symmetry.'[6]

Huxley had a peculiar notion of how aesthetic pleasure in mathematics was generated: it arose in the *process* of understanding the meaning

[1] Smith, 'On the Present State and Prospects of Some Branches of Pure Mathematics', p. 182.

[2] Hermite, 'Sur la fonction exponentielle'.

[3] Smith, letter to Pearson, dated 20 Jul. 1882, quot. in Glaisher, intro. to Smith, *Collected Mathematical Papers*, vol. 1, p. xci.

[4] Huxley, 'On the Educational Value of the Natural History Sciences', pp. 57–8; Huxley, 'Scientific Education', p. 126.

[5] Huxley, 'The Scientific Aspects of Positivism', p. 667.

[6] Huxley, 'On Science and Art in Relation to Education', pp. 176–7.

that the mathematical text and diagrams communicated. For Huxley, the aesthetic pleasure did not arise from the meaning — the theorem or proof or theory — but in the act of comprehension. This idea suggests that Huxley imagined that the aesthetic *pleasure* was the same as in visual art, but that the aesthetic *experience* was different.

Huxley himself experienced an unsurpassable aesthetic pleasure in scientific theories that explained and unified many phenomena:

> 'I cannot give you any example of a thorough æsthetic pleasure more intensely real than a pleasure of this kind the pleasure which arises in one's mind when a whole mass of different structures run into one harmony as the expression of a central law.'[1]

But he seemed not to recognize a parallel between this aesthetic experience, which would also require a degree of advanced education, and that of mathematicians in their own subject.

JAMES JOSEPH SYLVESTER

James Joseph Sylvester (1814–97) once referred to his mathematical work as 'the worship of the True & Beautiful'.[2] His aptitude for mathematics was evident early: he was first in mathematics at the Liverpool Royal Institution School and won a prize of $ 500 for solving a problem in arrangements sent to him from the United States.[3] He placed second in the Cambridge Mathematical Tripos in 1837, but was unable to take his degree because he was Jewish and thus refused to declare his acceptance of the doctrine of the Church of England, as was required at the time. He thus took degrees at Trinity College, Dublin in 1841. He was elected to the Royal Society in 1839 on the basis of papers he had published on optical properties of crystals, the motion of fluids, and rigid bodies. But his greatest achievements were in algebra, most famously with his friend Arthur Cayley* (1821–96) in the theory of invariants. After being the first chair of mathematics at Johns Hopkins University 1876–83, he died as Savilian Professor of Geometry at Oxford.[4]

Sylvester is important in the history of mathematical beauty for his role in advancing an explicit argument that mathematics should be seen as an artistic pursuit and could be justified in the same way as other art. He seems to have seen beauty all over mathematics, in theorems, proofs, and theories. For him, the beauty of a result stood separate from its proof: he described theorems of Karl von Staudt (1798–1867) on polygons and polyhedra as'beautiful and important'[5] and 'simple and beautiful'[6] but held the proof method to be 'clumsy and circuitous'.[7] Given that he coined the term *matrix*, it is perhaps

[1] Huxley, 'On Science and Art in Relation to Education', p. 177.

[2] Sylvester, letter to Joseph Henry, dated 12 Apr. 1846, repr. in Parshall, *James Joseph Sylvester*, p. 16.

[3] [MacMahon], 'James Joseph Sylvester', p. xi.

* ▸ pp. 446–54

[4] North, 'Sylvester', pp. 216–18.

[5] Sylvester, 'On Staudt's theorems', p. 382.

[6] *Ibid.*, p. 383.

[7] *Loc. cit.*

fitting to note one instance of beauty he saw in that area. He described the following result as ‘a very beautiful and striking theorem’:[1]

THEOREM S. *For the* $n \times (n+1)$ *matrix*

$$\begin{bmatrix} T_1 & T_2 & T_3 & \cdots & T_{n+1} \\ T_2 & T_3 & T_4 & \cdots & T_{n+2} \\ T_3 & T_4 & T_5 & \cdots & T_{n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_n & T_{n+1} & T_{n+2} & \cdots & T_{2n} \end{bmatrix}$$

where $T_i = a_1^{r-i}b_1^{s+i} + a_2^{r-i}b_2^{s+i} + \ldots + a_{n-1}^{r-i}b_{n-1}^{s+i}$, *each of the* $n+1$ *determinants formed by deleting one column is identically zero.* $\boxed{S}$

[In a note attached to a problem he submitted to the *Educational Times* in 1882, Sylvester was reported as noting that the problem arises from ‘the very beautiful Ancient Egyptian method of expressing all fractions under the form of a sum of simple fractions’.[2]* Although this phrasing has been attributed to Sylvester,[3] it is ambiguous whether it was Sylvester or the editor who proclaimed the method ‘very beautiful’.]

Linkages

An area of mathematics that Sylvester seems to have found especially beautiful and which is now comparatively unknown was the theory of linkages.[4] This topic had a vogue in the 1870s, doubtless as a consequence of what the historian of mathematics Florian Cajori (1859–1930) called the ‘beautiful discovery’[5] of the Peaucellier–Lipkin linkage, a simple mechanism that transforms circular motion into linear motion (see Figure 12.4). It is named for the military engineer Charles-Nicolas Peaucellier (1832–1919), who posed the question of finding a ‘compound compass’ that produced straight-line motion in 1864[6] and applied his solution to measure distances in 1868[7], and the mathematician Lipman Lipkin (1846–76), who independently discovered it in 1871.[8]

The existence of such a mechanism has important benefits in engineering: it allows a part to be constrained to move in a straight line, but without using guide-rails and thus losing energy in friction. Sylvester said that he had seen a water-pump based on the linkage that performed much better than a conventional pump: ‘Its elegance, and the frictionless ease with which it can be worked (beauty as usual the stamp and seal of perfection) have made it the pet of the household.’[9] It was not only Sylvester who admired it: he related that when the physicist William Thomson (later Baron Kelvin; 1824–1907) was able to work a model of the linkage, he was reluctant to hand it back, saying: ‘No! I have not had nearly enough of it — it is the most beautiful thing I have seen in my life.’[10]

The beauty of the mechanism seems to have been due to in part to its almost *ex nihilo* appearance: it bore no resemblance to previ-

[1] Sylvester, ‘On canonical forms’, p. 209.

[2] Miller, *Mathematical Questions* 37, p. 42.

* ▸ pp. 30–1

[3] Gillings, *Mathematics in the Time of the Pharaohs*, p. 48.

[4] Sylvester, ‘On Recent Discoveries’; Sylvester, ‘On the plagiograph *aliter* the skew pantigraph’.

[5] Cajori, *A History of Mathematics*, p. 301.

[6] Peaucellier, ‘Lettre’.

[7] Peaucellier & Wagner, ‘Mémoire sur un appareil’.

[8] Lipkin, ‘Dispositif articulé’.

[9] Sylvester, ‘On Recent Discoveries’, p. 10, n. *.

[10] *Ibid.*, p. 11.

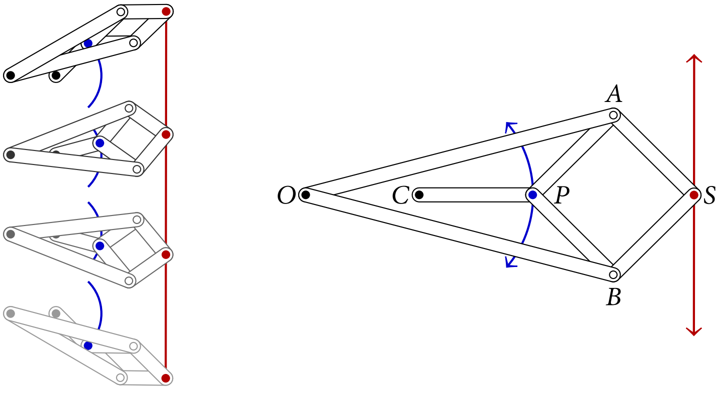


FIGURE 12.4
The Peaucellier–Lipkin linkage. The distances OC, CP, PA, PB, AS, BS are all equal. The pivots at O and C are fixed in place, but those at P, A, B, S, are free to move with the mechanism. Circular motion of P is transformed into linear motion at S, or vice versa.

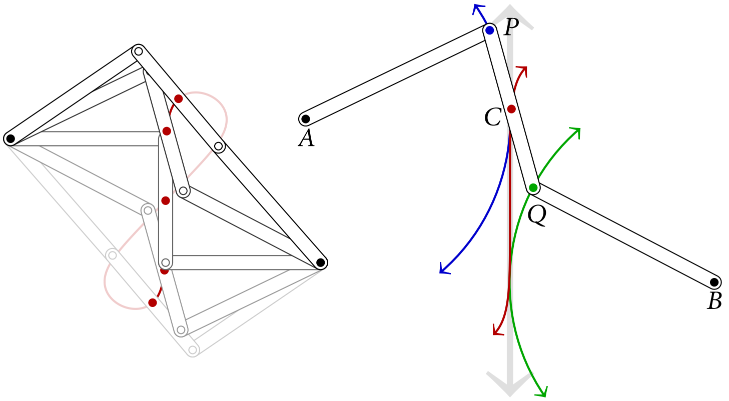


FIGURE 12.5
The linkage discovered by James Watt. The pivots at A and B are fixed, but those at P and Q are free to move with the mechanism. The distances AP and QB are equal, and the distances AB, PQ, and $AP + QB$ approximate the sides of a right-angled triangle. The midpoint C of PQ approximately traces linear motion through part of its range. Over the full range of the motion of the linkage, C traces a lemniscate.

ous linkages that approximated linear motion, such as James Watt's (1736–1819) (see Figure 12.5), and Sylvester suggested that, despite its simplicity and the two near-simultaneous independent discoveries, it might easily have remained undiscovered for another century.[1]

Sylvester also thought beautiful Alfred Bray Kempe's (1849–1922) result that two linkages of three pieces draw out the same curves up to similarity if the lengths of the pieces are the same (see Figure 12.6). This result, once noticed, can be deduced immediately from the principle of the ordinary pantograph. But, he asked rhetorically

> 'does the simplicity of the principle involved take away in any degree from the beauty of the result, or rather, is not the interest of the conclusion enhanced by the simplicity of the means by which it is arrived at? In fact, as Kant has observed, the groundwork of all mathematical proof consists in putting things together by a free act of the imagination; and the essence of Euclid is to be sought in the constructions which antecede the formal proofs. The real proof is the construction, and no one has the right to call Mr Kempe's discovery "a truism."'[2]

[1] Sylvester, 'On Recent Discoveries', p. 9, n. *.

[2] Sylvester, 'On the plagiograph *aliter* the skew pantigraph', p. 30, n. † cont. from p. 29.

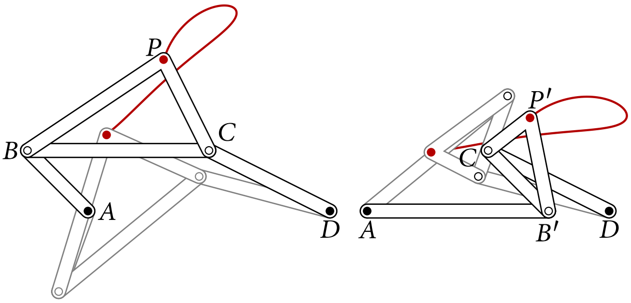


FIGURE 12.6
The curves generated by three-piece linkages are, up to similarity, dependent only on the lengths of the three pieces, not on their order. In both linkages, A and D are fixed pivots. The lengths AB and BC in the first linkage are respectively equal to $B'C$ and AB' in the second.

Kempe himself thought that the recent developments in the mathematical theory of linkages were of 'great beauty', and were valuable, but that it would be a mistake to consider them important. They *would have* been important a half-century earlier, when they would have helped the progress that engineering had been forced to make without them.[1]

Partitions

Another area in which Sylvester often saw beauty was the theory of partitions. A *partition* of a natural number (that is, positive integer) n is a finite nonincreasing sequence of natural numbers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ that sums to n; the λ_i are the *parts* of the partition. He called 'Euler's beautiful law'[2] the result that *The number of partitions of n with at most m parts equals the number of partitions of n in which no part exceeds m.* He was impressed by the proof suggested by Norman Ferrers (1829–1903), using what are now known as Ferrers diagrams. A *Ferrers diagram* is a visual representation of such a partition as an array of points, with the number of points in the rows corresponding to the parts, arranged in weakly decreasing (that is, non-increasing) order from top to bottom, and left-aligned. For example, the Ferrers diagram of (7, 5, 3, 2, 2) (which is a partition of 19) is

$$\begin{array}{l}\circ\,\circ\,\circ\,\circ\,\circ\,\circ\,\circ\\ \circ\,\circ\,\circ\,\circ\,\circ\\ \circ\,\circ\,\circ\\ \circ\,\circ\\ \circ\,\circ\end{array}\quad.$$

Using this representation, there is a natural bijection between the set of partitions of n with at most m parts and the set of partitions of n in which no part exceeds m. The bijection corresponds simply to reflecting the diagram along the diagonal:

$$\begin{array}{l}\circ\,\circ\,\circ\,\circ\,\circ\,\circ\,\circ\\ \circ\,\circ\,\circ\,\circ\,\circ\\ \circ\,\circ\,\circ\\ \circ\,\circ\\ \circ\,\circ\\ \\ \\ \end{array}\quad\mapsto\quad\begin{array}{l}\circ\,\circ\,\circ\,\circ\,\circ\\ \circ\,\circ\,\circ\,\circ\,\circ\\ \circ\,\circ\,\circ\\ \circ\,\circ\\ \circ\,\circ\\ \circ\\ \circ\end{array}\quad.$$

This proof Sylvester thought so 'simple and instructive', that he was sure that 'every logician will be delighted to meet with it here or elsewhere'.[3] The bijection is said to map each partition to its *conjugate*.

---

[1] Kempe, *How to Draw a Straight Line*, p. 6.

[2] Sylvester, 'On Mr Cayley's impromptu demonstration', p. 597.

[3] *Loc. cit.*

Sylvester also admired Fabian Franklin's (1853–1939) proof[*] of Euler's pentagonal number theorem, which makes use of partitions, calling it a 'beautiful demonstration'.[1]

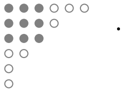
More often, though, Sylvester saw elegance in results on partitions. Let $p(\mathcal{O}, n)$ denote the number of partitions of n where all parts are odd, and let $p(\mathcal{D}, n)$ denote the number of partitions of n where all parts are distinct. He called an 'elegant theorem'[2] the result that $p(\mathcal{O}, n) = p(\mathcal{D}, n)$.[3] [His formulation of it, though, is decidedly lacking in elegance: 'a number may be partitioned in as many ways into only-once-occurring odd-or-even integers as into any-number-of-times-occurring only-odd integers'.[4]]

Sometimes, though, his aesthetic praise of a result seems excessive. Let $p(\mathcal{M}_q, n)$ denote the number of partitions of n into at most q parts. W. P. Durfee (1855–1941), a student of Sylvester, proved that the number of self-conjugate partitions of n is[5]

$$\sum_{q\in I(n)} p\left(\mathcal{M}_q, \tfrac{1}{2}(n-q^2)\right),$$

$$\text{where } I(n) = \begin{cases}\{1, 3, \ldots, \sqrt{n}\} & \text{if } n \text{ is odd;}\\ \{2, 4, \ldots, \sqrt{n}\} & \text{if } n \text{ is even.}\end{cases}$$

Sylvester called this a 'very elegant and interesting theorem'.[6] But unless the theorem is viewed in complete detachment from its proof, this praise seems almost hyperbolic, for the proof is nearly trivial when one sees that the Ferrers diagram of a self-conjugate partition consists of a $q \times q$ square (where q has the same parity as n) and two 'appendages', which are the Ferrers diagram of a partition of $\frac{1}{2}(n-q^2)$ into at most q parts and of its conjugate:



Sylvester's praise may have been influenced by the general power of Ferrers diagrams, and the notion of the *Durfee square* (the largest square 'inside' the partition), which he thought a very important tool.[7]

Characteristics of elegance

Concerning the method shown in Figure 12.7 for finding the barycentre of a quadrilateral, originally published in an unsigned note,[8] Sylvester wrote that it

> 'leaves nothing to be wished for in point of elegance and conciseness; [...] stands in advantageous contrast with all other methods of effecting the same end. It involves only four lines of construction and two bisections; in some elementary works on Mechanics, in use at our Universities, a method is given involving no less than 9 or 11 auxiliary lines.'[9]

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[*] ▸ pp. 504–5

[1] Sylvester, 'Sur les nombres de fractions', p. 86; trans. C.

[2] Sylvester, 'A constructive theory of partitions', p. 13.

[3] Andrews, *The Theory of Partitions*, Corol. 1.2.

[4] Sylvester, 'A constructive theory of partitions', p. 13.

[5] Durfee, 'On the Number of Self-Opposite Partitions'.

[6] Sylvester, 'Note on the paper of Mr Durfee's', p. 661.

[7] Sylvester, letter to Cayley, dated 16 Mar. 1883, repr. in Parshall, *James Joseph Sylvester*, p. 224.

[8] [Anonymous], 'Mechanical solutions of geometrical problems'.

[9] Sylvester, 'Observations on the method for finding the centre of gravity of a quadrilateral', p. 338.

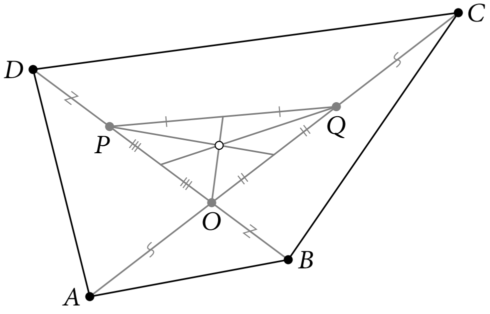

FIGURE 12.7
Construction to find the barycentre of a quadrilateral $ABCD$: the diagonals intersect at O; the points P and Q are such that $OB = PD$ and $OA = QC$. The barycentre of $ABCD$ coincides with the barycentre of the triangle OPQ.

Here Sylvester specified 'conciseness' separately from 'elegance', which suggests that he did not think that elegance *entailed* conciseness. This distinction is compatible with a judgement he made of another proof: 'very elegant but a little circuitous'.[1] But in discussing the construction to find the barycentre, Sylvester here seems to have justified his judgement of elegance with reference to the few auxiliary lines and operations required. (The lines to which he referred are the two diagonals AC and BD, two of the sides of OPQ, and bisections of two of these sides, which is all that is required to find the barycentre.) This justification suggests that he connected elegance in a method, not with brevity, but with *economy*.

This connection is also suggested by the elegance he saw in the result *When opposite sides of a quadrilateral inscribed in a circle a produced, the bisectors of the acute angles where they meet are orthogonal* (see Figure 12.8).[2] The geometric configuration exhibits a similar degree of economy: four lines segments extending the sides of the quadrilateral, and two bisections, this time of angles.

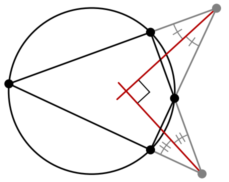

FIGURE 12.8
A theorem about a quadrilateral inscribed in a circle.

Beauty as a guide

By his own account, Sylvester was sometimes guided by aesthetic concerns in his mathematical discoveries. He wrote that '[i]t often happens that the pursuit of the beautiful and appropriate [...] is rewarded with a new insight into the true'.[3] As a particular example, he sought a representation of the quotients that appear in computing the Sturm sequence as functions of the factors of the original polynomial. In words that seem almost smug, he wrote:

> 'Guided by an instinctive sense of the beautiful and fitting, in a happy moment I have succeeded in grasping this much wished for representation'.[4]

The experience of beauty

Sylvester thought that when mathematics is studied correctly, so that one learns 'to think and feel as a geometer', an aesthetic response cannot be avoided: one finds 'the sentiment of the

[1] Sylvester, 'On Crocchi's theorem', p. 653.

[2] Sylvester, 'Note on the periodical changes of orbit', p. 558.

[3] Sylvester, 'On an improved form', p. 544.

[4] Sylvester, 'On the explicit value of Sturm's quotients', p. 638.

orderly and beautiful awakened and enhanced'.[1] And while advanced training is required to reach the rarefied heights, beauty in mathematics is accessible to those with only limited knowledge. For his inaugural lecture as Savilian Professor, he sought a topic suitable for a general audience

> 'so that those who are qualified by a moderate amount of mathematical culture to comprehend the drift of my discourse, may go away with the satisfactory feeling that their mental vision has been extended and their eyes opened, like my own, to the perception of a world of intellectual beauty, of whose existence they were previously unaware.'[2]

Sylvester equated the 'the pursuit of the beautiful and appropriate' with 'the endeavour after the perfect'.[3] Since the concept of perfection does not seem to admit ever-ascending degrees, the comparison suggests that, for Sylvester, neither did beauty.

Mathematics for mathematics's sake

Mathematics, thought Sylvester, should be counted among the liberal arts. All aesthetic pursuits had four centres: 'Epic, Music, Plastic, and Mathematic', which Sylvester placed at the vertices of a tetrahedron. In this geometrical metaphor, there was a plane — a commonality — through any three of these centres that excluded the fourth; there was a line — a narrower commonality — between any two that was orthogonal to the line through the other two. At the centre of the tetrahedron was a 'centre of gravity of the whole body of æsthetic', though Sylvester admitted that he had 'not had time to think out' what it was.[4] He published this model in 1878 and seems to have conceived it some time after 1864, when he drew a parallel between music and mathematics in a highly rhetorical style:

> 'May not Music be described as the Mathematic of sense, Mathematic as Music of the reason? the soul of each the same! Thus the musician *feels* Mathematic, the mathematician *thinks* Music, — Music the dream, Mathematic the working life'.[5]

But why should mathematics be considered a liberal art? Sylvester drew on Walter Pater's (1839–94) essay 'The School of Giorgione' (1877) (later incorporated into his book *The Renaissance*[6]), in which Pater expounded a theory of art and of aesthetic experience. Each art had a form of charm which was untranslatable to other arts, but which was still subject what Pater called *Anders-streben* [*lit.* 'other-striving'], an aspiration to transcend partially its own limits. [Pater ascribed the term to German critics,[7] although he probably coined it himself.[8]] Sculpture, for instance, could reach beyond the limit of form towards colour.[9] Art strove to be independent of intelligence alone. The ideal of art was that its subject-matter should not only be understood by the intelligence, but for the composition as a whole to be grasped by 'imaginative reason' where thoughts and their perceptual analogues go together. Music came closest to this ideal, for in music the matter and the form of art were identical.[1]

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[1] Sylvester, 'A probationary lecture on geometry', p. 9.

[2] Sylvester, 'On the Method of Reciprocants', p. 280.

[3] Sylvester, 'On an improved form', p. 544.

[4] Sylvester, 'Proof of the hitherto undemonstrated Fundamental Theorem of Invariants', p. 123.

[5] Sylvester, 'Algebraic Researches on Newton's Rule', p.419, n. ∗.

[6] Pater, *The Renaissance*, pp. 136–62.

[7] *Ibid.*, p. 140.

[8] Poueymirou, 'Walter Pater's *Anders-Streben*', para. 1.

[9] Pater, *The Renaissance*, pp. 140–1.

Sylvester saw the claim of mathematics to the status of an art to be an *Anders-streben*: mathematics could become distanced from its own logical limits and share in the features of the other liberal arts. In each branch of modern mathematics, there was a high degree of freedom in how to develop the subject: there was 'an almost unlimited scope left to the regulated play of the fancy'.[2] This characterization of the pure mathematician as a creative artist was made plausible by the attitude to art espoused by Pater, which separated moral truth in art from beauty:[3]

> '"To see the object as in itself it really is," has been justly said to be the aim of all true criticism whatever; and in æsthetic criticism the first step towards seeing one's object as it really is, is to know one's own impression as it really is, to discriminate it, to realise it distinctly. [...] What is this song or picture, this engaging personality presented in life or in a book, to *me*? What effect does it really produce on me? Does it give me pleasure? and if so, what sort or degree of pleasure? How is my nature modified by its presence, and under its influence?'[4]

And contemplation of beauty was one of the highest aims of life, for it could help us to fulfil ourselves in the time that we were given:

> 'Great passions may give one this quickened sense of life, ecstasy and sorrow of love, the various forms of enthusiastic activity, disinterested or otherwise, which come naturally to many of us. Only be sure it is passion — that it does yield you this fruit of a quickened, multiplied consciousness. Of such wisdom, the poetic passion, the desire of beauty, the love of art for art's sake, has most. For art comes to you professing frankly to give nothing but the highest quality to your moments as they pass, and simply for those moments' sake.'[5]

Beauty and the justification for mathematics

Sylvester used part of his 1869 address as president of the mathematics and physical science section of the British Association for the Advancement of Science to justify mathematics as a being based on observation, in opposition to T. H. Huxley's published view that mathematics consists only in new deductions from long-established axioms.* Although he did not discuss what would later be called mathematical platonism *per se*, he related the experience of mathematical observation to that of exploration or discovery in the physical sciences.[6]

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[1] Pater, *The Renaissance*, p. 145.

[2] Sylvester, 'Proof of the hitherto undemonstrated Fundamental Theorem of Invariants', p. 123.

[3] Cf. Heard, 'The Evolution of the Pure Mathematician', pp. 229–35.

[4] Pater, *The Renaissance*, p. x, emphasis in orig.

[5] *Ibid.*, pp. 252–3.

* ▸ pp. 437–8

[6] Sylvester, 'Presidential Address', pp. 655–7.

He offered a defence against an unnamed 'very clever writer' who doubted whether mathematics 'is, in itself, a more serious pursuit, or more worthy of interesting an intellectual human being, than the study of chess problems or Chinese puzzles':[1]

> 'The world of ideas which it discloses or illuminates, the contemplation of divine beauty and order which it induces, the harmonious connexion of its parts, the infinite hierarchy and absolute evidence of the truths with which mathematical science is concerned, these, and such like, are the surest grounds of its title to human regard, and would remain unimpaired were the plan of the universe unrolled like a map at our feet, and the mind of man qualified to take in the whole scheme of creation at a glance.'[2]

Furthermore, Sylvester held that pure mathematics was *about something*, a 'world of ideas', and was not just an intellectual game. In a 1869 paper, he pointed out that

> 'the results arrived at in the denumerational portion of this paper have been obtained almost exclusively by processes of observation, comparison, experiment, and induction, satisfying the sense of the perfect and beautiful.'[3]

This statement was an indication that aesthetic concerns guided or motivated at least the search for new results, but Sylvester continued, in an irony directed contra Huxley, that thus

> 'the theory must be regarded as essentially non-mathematical, it having been authoritatively laid down by a great naturalist and eminent public instructor in the *Fortnightly Review* for June 1869,[4] that Mathematics is "that study which knows nothing of observation, nothing of experiment, nothing of induction, nothing" [does any science know much?] "of causation."'[5]

Sylvester's remarks appear to be one of the earliest aesthetic *justifications* of the pursuit of mathematics, independently of practical applications.

ARTHUR CAYLEY

Arthur Cayley (1821–96) was senior wrangler at Cambridge in 1842 and became a fellow at Trinity College before departing for a career in law, where he met J. J. Sylvester,* who would become his friend. During his time working as a lawyer, he was a prolific mathematician, and in 1863 he was elected the first Sadleirian Professor of

[1] Sylvester, 'Presidential Address', p. 658.

[2] *Ibid.*, pp. 658–9.

[3] Sylvester, 'Outline trace', p. 663, n. †.

[4] See Huxley, 'The Scientific Aspects of Positivism', p. 667.

[5] Sylvester, 'Outline trace', p. 663, n. †, insertion in orig.

* ▸ pp. 438–46

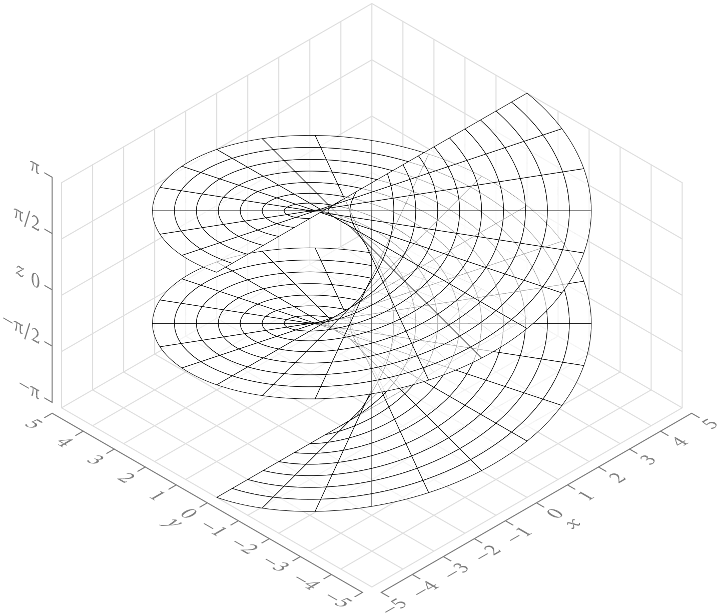

FIGURE 12.9
The portion of the helicoid described parametrically by

$$(\rho\cos(\vartheta), \rho\sin(\vartheta), \vartheta)$$

where $\vartheta \in [-\pi, \pi]$ and $\rho \in [0, 5]$.

Pure Mathematics at the University of Cambridge.[1] He holds a curious position in the history of mathematical beauty. He recognized it, used it to try to communicate something of the scope of mathematics, and accepted that mathematics should be pursued for its own sake. But elegance seems to have been the primary aesthetic category to which he appealed, and it is unclear what connection — if any — he saw it as having to beauty.

Beauty

Cayley's mathematical papers show that he saw beauty in results and in proofs. For instance, he considered beautiful Eugène Catalan's (1814–94) result that the helicoid (see Figure 12.9) and the plane are the only ruled minimal surfaces.[2] [A surface is *minimal* if every point on it has a neighbourhood, bounded by a simple closed curve, that has minimum area among all surfaces with that boundary; it is *ruled* if for every point on it, there is a straight line through that point lying in the surface.]

Further, Cayley seems to have been one of the earliest mathematicians to leave a record that he saw beauty in formulae: he called 'beautiful [belle]'[3] the following formula of Jacobi, which arises in the theory of elliptic functions:

$$\log\Theta(\alpha) - \log\Theta(0) = \tfrac{1}{2}\alpha^2\left(1 - \frac{E_{I}}{K}\right) - k^2\int_0^\alpha d\alpha\int_0^\alpha d\alpha\,\sin^2\operatorname{am}\alpha.$$

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[1] North, 'Cayley, Arthur', pp. 162–3.

[2] Cayley, 'Note sur les Surfaces Minima', p. 594.

[3] Cayley, 'Mémoire sur les Fonctions doublement périodiques', p. 156.

He also thought an equality due to Sylvester a 'very beautiful formula'.[1] The formula in question is:

$$\prod_{k=1}^{\infty}(1+ax^k) = 1 + \sum_{j=1}^{\infty} \frac{\prod_{i=1}^{j-1}(1+ax^i)}{\prod_{i=1}^{j}(1-x^i)} x^{(3j^2-j)/2} a^j. \tag{S}$$

Cayley wrote it in the long-winded form

$$\begin{aligned}
&(1+ax)(1+ax^2)(1+ax^3)\ldots\\
&\quad = 1 + \frac{1}{1-x}(1+ax^2)xa\\
&\qquad + \frac{1}{1-x\,.\,1-x^2}(1+ax)(1+x^4)x^5a^2\\
&\qquad + \frac{1}{1-x\,.\,1-x^2\,.\,1-x^3}(1+ax)(1+ax^2)(1+ax^6)x^{12}a^3\\
&\qquad + \ldots,
\end{aligned}$$

which does not give enough detail to show how the sequence of exponents $1, 5, 12, \ldots$ continues, while Sylvester gave it as[2]

$$\begin{aligned}
&1+ax\,.\,1+ax^2\,.\,1+ax^3\ldots\\
&\quad = 1 + \frac{1+ax^2}{1-x}xa + \frac{1+ax\,.\,1+ax^4}{1-ax\,.\,1-x^2}x^5a^2 + \ldots\\
&\qquad + \frac{1+ax\,.\,1+ax^2\ldots1+ax^{j-1}\,.\,1+ax^{2j}}{1-x\,.\,1-x^2\ldots1-x^{j-1}\,.\,1-x^j}x^{\frac{3j^2-j}{2}}a^j + \ldots.
\end{aligned}$$

Sylvester's form has the benefit of making clear the exponent $(3j^2 - j)/2$, which is the j-th pentagonal number; putting $a = -1$ into (S) yields Euler's pentagonal number theorem.* Cayley's choice of form seems much less pleasing than Sylvester's, although he specifically referred to the equation as Sylvester's. It seems likely that, for Cayley, the beauty of the formula did not attach to any particular form, but rather to the mathematical result it expressed.

Cayley thought that Augustin-Louis Cauchy's (1789–1857) second memoir on polygons and polyhedra[3] contained a 'very beautiful demonstration'[4] of the result now known as Cauchy's rigidity theorem, which states that *If two convex polyhedra are combinatorially equivalent such that corresponding faces are congruent, then the polyhedra are congruent.* (Two polyhedra are *combinatorially equivalent* if there are bijections between their sets of vertices, sets of edges, and sets of faces such that incidence is preserved. The assumption of convexity is necessary: it is possible for two polyhedra to be combinatorially equivalent and have congruent faces but not to be themselves congruent if at least one of them is non-convex; see Figure 12.10.) However, Cauchy proved the result by means of a lemma whose proof contains a gap;[5] presumably Cayley had not noticed this problem when he praised the proof. [Cauchy's proof, with a corrected demonstration of the lemma, appears in *Proofs from THE BOOK*.[6]†]

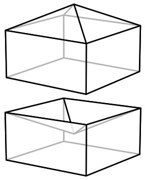

FIGURE 12.10
Combinatorially equivalent non-congruent polyhedra whose corresponding faces are congruent.

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[1] Cayley, 'Note on a partition-series', p. 217.

[2] Sylvester, 'A constructive theory of partitions', p. 33.

* ▸ p. 503

[3] Cauchy, '11.ᵉ Mémoire'.

[4] Cayley, 'Second Note on Poinsot's Four New Regular Polyhedra', p. 87.

[5] Alexandrov, *Convex Polyhedra*, p. 159.

[6] Aigner & Ziegler, *Proofs from THE BOOK* (6th edn), pp. 95–7.

† ▸ pp. 843–5

Cayley also located beauty in theories. His highest praise was reserved for the theory of proportion set forth in the *Elements*: 'there is hardly anything in mathematics more beautiful than [Euclid's] wondrous fifth book'.[1] [Note the stark contrast to what was probably Savile's view: that the theory of proportion was one of two blemishes, the other being the parallel postulate, in Euclid's geometry.[*]] He thought that the theory of cubic curves was just as beautiful as that of conic sections; he also remarked on the beauty of Chasles's theory of the characteristics of conics satisfying given conditions,[2] which was an important development towards answering the question of how many conics are tangent to five given conics.[3]

But the theory whose beauty Cayley mentioned most often was that of the quaternions.[4][†] Their beauty was not due to any physical application; rather, there was an intrinsic beauty in the concept itself:

> 'as I consider the full moon far more beautiful than any moonlit view, so I regard the notion of a quaternion as far more beautiful than any of its applications.'[5]

Quaternions were beautiful for the pure mathematician; for the applied mathematician, they were at best a shorthand or abbreviation, and not a particularly useful one, for the equivalent vectors.[6]

Elegance

Cayley clearly recognized beauty in mathematics, but much more often he referred to *elegance* both in his research and — especially — in his expository articles. Notably, in two reports he wrote for the British Association for the Advancement of Science on dynamics, the terms *elegance*, *elegant*, *elegantly* appear a total of thirty-five times;[7] *beauty* appears once.[8]

The only link between beauty and elegance discernible in Cayley's work was that he sometimes called a result *beautiful* and its proof *elegant*. For example, he called a result of Ludwig Schläfli (1814–95) 'a very beautiful theorem'[9] and said that the proof was 'absolutely simple and elegant'.[10] He applied the two aesthetic terms in the same way in his article on the mathematician John Landen (1719–90) for the ninth edition of the *Encyclopædia Britannica*: he wrote that Landen's theorem on rectifying a hyperbolic arc in terms of two elliptical arcs was a 'beautiful result',[11] and thought part of the reasoning that led to it 'ingenious and elegant'.[12] [Landen himself had been impressed by his own work, saying that it led to 'very elegant conclusions'.[13]] *Elegant* and its derived terms also appear in his article 'Equation' in the *Britannica*.[14] Thus, although none of the other articles he wrote for the *Britannica* mention beauty, he seemed to accept that aesthetic terms had a place in articles on mathematics for the general reader.

As another instance of the elegance he saw in proof, he thought that Cauchy had proved[15] 'very elegantly' that (in modern termin-

[1] Cayley, 'Presidential Address', p. 446.

[*] ▸ pp. 271–2

[2] *Ibid.*, pp. 449, 459.

[3] Kleiman, 'Chasles's Enumerative Theory of Conics', pp. 119 sqq.

[4] See Cayley, 'On certain results relating to Quaternions', p. 123; Cayley, 'Sur quelques Propriétés des Déterminants Gauches', p. 335.

[†] ▸ pp. 418–21

[5] Cayley, 'Coordinates versus Quaternions', pp. 541–2.

[6] Crowe, *A History of Vector Analysis*, p. 212.

[7] Cayley, 'On the Recent Progress of Theoretical Dynamics', *passim*; Cayley, 'Report on the Progress of the Solution of certain Special Problems of Dynamics', *passim*.

[8] Cayley, 'On the Recent Progress of Theoretical Dynamics', p. 170.

[9] Cayley, 'Sur um Théorème de M. Schläfli', 181; trans. C.

[10] *Ibid.*, 183; trans. C.

[11] Cayley, 'Landen', p. 584.

[12] *Loc. cit.*

[13] Landen, 'An Investigation of a general Theorem', p. 283.

[14] Cayley, 'Equation', pp. 495, 500, 507, 520.

[15] Cauchy, '1.$^{\text{er}}$ Mémoire', See.

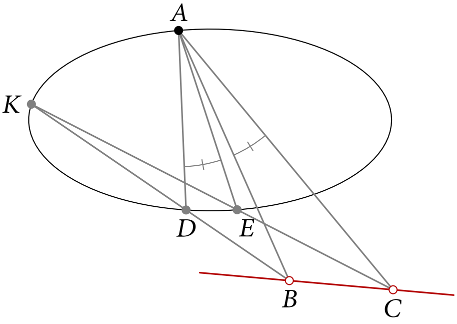


FIGURE 12.11
Illustration of Theorem C: for a point A on a conic section, there is a line such that for any other point K on the conic section and any straight lines through K intersecting the conic section at D, E and the line at B, C, the angles $\angle DAE$ and $\angle BAC$ are either equal or supplementary.

ology) all the regular star polyhedra* could be derived from the regular solids by stellation or greatening.[1]

But Cayley did not just see elegance in proofs, but also in theorems. As an example, he found 'very elegant' the following theorem:

THEOREM C. *'For every point A on a conic section there exists a straight line BC, not meeting the curve, such that, if through any other point K on the conic section there be drawn any two straight lines meeting BC in B, C, and the curve in D, E, the angles BAC, DAE are either equal or supplementary.'*[2] (See Figure 12.11.) $\boxed{\text{C}}$

Examining elegance

Discussing a result of George Boole (1815–64), Cayley wrote that the proof, 'though very elegant, does not appear to be the most direct which the theorem admits of'.[3] Two different implications can be drawn from this statement: that there was a more direct proof with still greater elegance, or that for Cayley there was no necessary connection between elegance in proof and directness.

The second interpretation is perhaps more likely, since other evidence points to Cayley not connecting elegance with simplicity or brevity. Take, for instance, Castillon's problem, which is to determine, given a circle and three points not on that circle, every triangle inscribed in the circle whose sides, extended if necessary, pass through those points (see Figure 12.12 (left)). Cayley discussed a variant of this problem where a circle has an inscribed and circumscribed triangle so that the sides of the inscribed triangle, when extended, pass through the vertices of the circumscribed triangle. The task then becomes the determination of one triangle from the other (see Figure 12.12 (right)). Cayley reproduced Thomas Clausen's (1801–85) solution, which consists of three-and-a-half pages of fairly dense calculation,[4] which surely can be considered neither short nor simple. For Cayley, though, it was 'very elegant'.[5]

This raises the question of what, if anything, 'elegant' meant for Cayley. It seems impossible to make any positive inference from the negative conclusions drawn above. There are some possibilities: Cayley might have taken *elegance* to signify a kind of conceptual economy,

* ▸ pp. 289–90

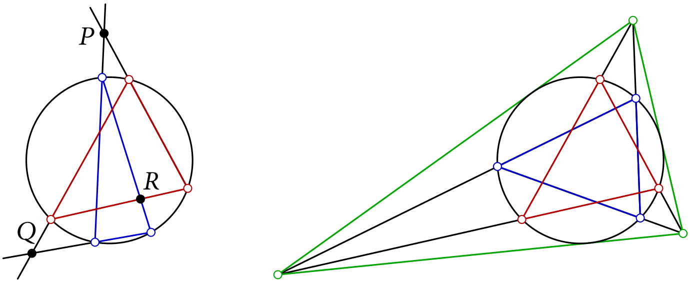
[1] Cayley, 'Second Note on Poinsot's Four New Regular Polyhedra', p. 86.

[2] Cayley, 'Problems and Solutions', p. 562.

[3] Cayley, 'On the Theory of Linear Transformations', p. 94.

[4] Cayley, 'On a particular case of Castillon's Problem', pp. 435–9.

[5] *Ibid.*, p. 435.



Castillon's problem for three points P, Q, R not on the circle

The variant Castillon problem relating inscribed and circumscribed triangles

FIGURE 12.12
Castillon's problem, in its original form, asks for the triangles inscribed in a circle whose sides, extended if necessary, pass through three given points not on the circle. A variation considers an inscribed triangle whose sides pass through the vertices of a circumscribed triangle, and aims to determine one triangle from the other.

which would leave open the possibility of long and complicated proofs being elegant.

It would also fit with a negative assessment of elegance he made, when he wrote about one of the proofs in Newton's *Principia*,[1] that the latus rectum of a conic section was found 'not very elegantly' using the latus rectum of another conic section.[2] Positing a second conic section could be seen as a conceptual inefficiency, which, under the hypothesis of Cayley seeing elegance as conceptual economy, would result in him judging the proof to be inelegant.

On the other hand, it is possible that Cayley simply judged elegance relative to alternative proofs, rather than absolutely, and so calling a long and complicated proof 'elegant' only indicated how much longer and more complicated other proofs were. This conclusion would fit with brevity being one of the goals of mathematics. [His obituarist George Bruce Halsted (1853–1922) reported that Cayley, with his experience both as a lawyer and as Sadleirian Professor, once said that 'the object of law was to say a thing in the greatest number of words, of mathematics to say it in the fewest'.[3] But the editor of the later volumes of Cayley's collected papers, A. R. Forsyth (1858–1942), wrote in a slightly later obituary that Cayley 'repudiated entirely' ever having held this view.[4]]

His views on elegance in *formulae* seem to indicate a similar independence from simplicity: he wrote, in discussing Leonhard Euler's (1707–83) work on the dynamics of a rotating body, that '[t]he formulæ, although complicated, are extremely elegant'.[5] But on some occasions, he discussed how transformations could increase the elegance of a formula. For instance, a 'somewhat more elegant form' could be given to a matrix equation by expressing it in an unified way (which was nevertheless an abuse of notation).[6] He also wrote that the equation

$$4(2k+1)^2 KK' \,.\, UU' - \Theta^2 = 0,$$

which is a locus arising from conic sections, 'may be written somewhat more elegantly under the form'[7]

$$4(2k+1)^2 KU \,.\, K'U' - \Theta^2 = 0.$$

[1] See Newton, *Principia*, bk I, § 3, Prop. 17.

[2] Cayley, 'Report on the Progress of the Solution of certain Special Problems of Dynamics', p. 516.

[3] Halsted, 'Arthur Cayley', p. 104.

[4] Forsyth, 'Arthur Cayley', p. x, n. †.

[5] Cayley, 'Report on the Progress of the Solution of certain Special Problems of Dynamics', p. 568.

[6] Cayley, 'On the Simultaneous Transformation of Two Homogeneous Functions of the Second Order', p. 430.

[7] Cayley, 'On a Locus derived from Two Conics', p. 29.

The increase in elegance from the interchange of two symbols seems to have been because the quantities KU and $K'U'$ have exactly parallel forms:

$$\begin{aligned} KU &= (BC - F^2, \ldots \;)\!\!(\alpha, \beta, \gamma)^2, \\ K'U' &= (B'C' - F'^2, \ldots)\!\!(\alpha, \beta, \gamma)^2 \end{aligned}$$

[The overlapping parentheses $)\!\!($ indicate that the two parenthesized expressions were an abbreviation for a homogeneous polynomial in α, β, γ; the precise definition is not relevant here.] In this case and in the case of the matrix equation, the increased elegance seems to have been grounded in an increased syntactic simplicity.

Justification for mathematics

Cayley maintained that mathematics was a useful mental exercise, and that pure mathematical research could be justified by producing advances that could later be of aid to the sciences. In a 1867 exchange of letters, Cayley debated this point with George Biddell Airy (1801–92), the Astronomer Royal from 1835 to 1881. Airy had a great respect for pure mathematics, but only insofar as it could be applied to practical ends, and was averse to mathematical research with no immediate practical application. One of Airy's complaints was that pure mathematicians retreated into isolation and did not contribute to the sciences:

> 'Now as to the Modern Geometry. With your praises of this science [...] I entirely agree. And if men, after leaving Cambridge, were designed to shut themselves up in a cavern, they could have nothing better for their subjective amusement. [...] But the persons who devote themselves to these subjects do thereby separate themselves from the world. They make no step towards natural science or utilitarian science, the two subjects which the world specially desires. The world could go on as well without these separatists.'[1]

But Cayley's most important contribution to the history of mathematical beauty was in his 1883 presidential address to the British Association. He turned to beauty in trying to communicate some idea of the scope of mathematics:

> 'It is difficult to give an idea of the vast extent of modern mathematics. [...] I mean extent crowded with beautiful detail — not an extent of mere uniformity such as an objectless plain, but of a tract of beautiful country seen at first in the distance, but which will bear to be rambled through and studied in every detail of hillside and valley, stream, rock, wood, and flower. But, as for anything else, so for a mathematical theory — beauty can be perceived, but not explained.'[2]

[1] Airy, letter to Cayley, dated 9 Dec. 1867, repr. in Airy, *Autobiography*, p. 277.

[2] Cayley, 'Presidential Address', p. 449.

The last few words are important: for Cayley, there were no reasons — or at least no *communicable* reasons — for judgements of beauty. He seems to have considered beauty to be one of several fundamental ideas that originated in the mind, and not from experience. Among these was the idea of equality, which to him seemed prior to any experience of equal things. The 'idea of the beautiful' was another, but he only mentioned it as an illustrative example, and did not expand upon it.[1]

Earlier in the same address, Cayley said that if he were to justify pure mathematics

> 'I should desire to do it [...] *not* by speaking to you of the utility of mathematics in any of the questions of common life or of physical science. Still less would I speak of this utility before, I trust, a friendly audience, interested or willing to appreciate an interest in mathematics in itself and for its own sake. I would, on the contrary, rather consider the obligations of mathematics to these different subjects as the sources of mathematical theories'.[2]

Cayley supported the idea (and here assumed his audience would also) that mathematics could be pursued independent of any applications, although he retained an interest in applications.[3] The physical sciences were a source of inspiration for the development of mathematics, which would in time give back to those theories. However, Cayley held to a platonic view of mathematics:

> 'I would myself say that the purely imaginary objects are the only realities, the ὄντως ὄντα, in regard to which the corresponding physical objects are as the shadows in the cave [...] at any rate the objects of geometrical truth are the so-called imaginary objects [...], and the truths of geometry are only true, and *a fortiori* are only necessarily true, in regard to these so-called imaginary objects'.[4]

Thus whatever pure mathematics might draw from the physical sciences, it was then not only pursued for completely independent reasons, but it was *about* something completely different from the physical sciences.

Cayley's platonism naturally fitted into his 'Whig history' view of pure mathematics, where nothing was ever lost and there was steady progress. This view is evident from his application to mathematics of the words of Alfred Tennyson (1809–92):[5]

> 'Yet I doubt not thro' the ages one increasing purpose runs,
> And the thoughts of men are widen'd with the process of the
>   suns.'[6]

[1] Cayley, 'Presidential Address', p. 431.

[2] *Ibid.*, pp. 430–1.

[3] Craik, *Mr Hopkins' Men*, p. 333.

[4] Cayley, 'Presidential Address', p. 433.

[5] *Ibid.*, p. 459.

[6] Tennyson, 'Locksley Hall', ll. 137–8; Cayley expanded the apostrophized words.

Legacy

The mathematician and theologian George Salmon (1819–1904), writing in the same year as Cayley delivered his presidential address, characterized Cayley's life as almost a kind of asceticism: Cayley, he said,

> 'has had courage to despise the allurements of avarice or ambition, and has found more happiness from a life devoted to the contemplation of beauty and truth.'[1]

Certainly Cayley's acceptance of the Sadleirian chair of mathematics entailed a reduction in his income: before taking up this post, he had been a successful lawyer.[2] Salmon placed mathematics between the arts and the applied sciences[3] but explicitly called Cayley 'a great artist',[4] and said that his work, like the work of other mathematicians, should not be judged based on the numbers that can appreciate it, just are artists are not so judged. The historian of science John Heard suggested that Salmon's portrayal of Cayley marked a turning-point when pure mathematicians started to become more confident in justifying the pursuit of pure mathematics independent of any applicability.[5]

J. W. L. GLAISHER

J. W. L. Glaisher (1848–1928) seemed to agree with the asceticism that Salmon ascribed to Cayley. At one point he dismissed the idea that money could influence pure mathematicians towards pursuing particular research: he complained about a prize offered on the wrappers of a volume of the *American Journal of Mathematics*, edited by Sylvester, for the proof or disproof of a particular conjecture. He dismissed this as an 'anachronism', and said that:

> 'It seems unlikely that any competent person would be tempted to investigate the subject by hope of the reward. Pure mathematics offers no mercenary inducements to its followers, who are attracted to it by the importance and beauty of the truths it contains; and the complete absence of any material advantage to be gained by means of it, adds perhaps even another charm to its study.'[6]

Note that the normative claim is about 'any competent person': those who would pursue financial gain through mathematics were thus not of the highest intellectual calibre. First-rate minds sought in mathematics values that included beauty but excluded material gain.

Glaisher, however, has been called a transitional figure, in that he completely accepted that pure mathematics could and should be pursued for its own sake, but that he sometimes wavered towards applied mathematics being a worthier calling than pure.[1] In 1890, as president of the mathematical and physical sciences section of the British Association for the Advancement of Science, he said that every mathematician cherished the hope that their field, however recondite, would find a practical application; yet, in the same address, he said that pure mathematics could not be justified on the basis of its applications.[2] He regretted that mathematical training at Cambridge was so focused on applications,[3] and said that:

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[1] Salmon, 'Arthur Cayley', p. 483.

[2] Forsyth, 'Arthur Cayley', p. xiv.

[3] Salmon, 'Arthur Cayley', p. 484.

[4] *Ibid.*, p. 483.

[5] Heard, 'The Evolution of the Pure Mathematician', p. 249.

[6] Glaisher, '*American Journal of Mathematics, Pure and Applied*', p. 195.

> 'it always appears to me that there is a certain perfection, and also a certain luxuriance and exuberance, in the pure sciences [...] which is conspicuously absent from most of the investigations which have had their origin in the attempt to forge the weapons required for research in the less abstract sciences'.[4]

Yet he also rejected the idea that researchers should be trained only in pure mathematics, so that only specialists could pursue the topic.[5] The only reason for a researcher to pursue pure mathematics was an aesthetic one: 'no one should devote himself to the abstract sciences unless he feels strongly drawn to them by his tastes'.[6] But he cautioned that the beauty of an area of mathematics could be hidden if that area was difficult of access.[7] Number theory was perhaps an example of this: Glaisher said that it was 'difficult and intricate [...] even in its elementary parts',[8] and, though it did not deal with physical reality, it yielded 'truths of the most fundamental kind, which must exist *in rerum naturâ*'.[9] It deserved to be appreciated and studied more, for it was, for Glaisher, a 'most beautiful'[10] theory.

These various aesthetic justifications for pure mathematics have a commonality in that they were all very individualistic. When the *mathematician* pursued beauty for its own sake, the reward was to the mathematician themself. Of course, others might read and appreciate the beautiful mathematics thus produced, but this benefit was not stated as a motivation. The closest approach made by Glaisher to a social justification for pure mathematics was the following observation:

> 'The search after abstract truth for its own sake, without the smallest thought of practical applications or return in any form, and the yearning desire to explore the unknown, are signs of the vitality of a people, which are among the first to disappear when decay begins.'[11]

Note, though, that this statement was not in itself a justification: a healthy society valued the pursuit by some of its members of pure mathematics, but it does not follow that a society could be made healthy by encouraging such research. (Compare the much later assertion by the historian G. M. Trevelyan (1876–1962): 'Disinterested intellectual curiosity is the life-blood of real civilization.'[12])

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[1] Heard, 'The Evolution of the Pure Mathematician', p. 128.

[2] Glaisher, 'Presidential address', pp. 722–3.

[3] *Ibid.*, p. 721.

[4] *Ibid.*, p. 720.

[5] *Ibid.*, pp. 724–5.

[6] *Ibid.*, p. 725.

[7] *Ibid.*, p. 724.

[8] *Ibid.*, p. 727.

[9] *Ibid.*, p. 726.

[10] *Ibid.*, p. 727.

[11] *Ibid.*, p. 725.

[12] Trevelyan, *English Social History*, p. viii.

GEORG CANTOR

Georg Cantor (1845–1918), who was the key figure in the creation of set theory, did not enunciate a clear position as to the foundations of mathematics. He was a formalist insofar as he thought that the existence of a mathematical object was guaranteed by an explicit construction; he suggested platonism when he chose as epigraphs for his first paper on 'Contributions to the Founding of the Theory of Transfinite Numbers' quotations from Francis Bacon and Isaac Newton about observing and recording, not inventing, the laws of nature;[1] he granted a degree of acceptance to idealism when he accepted a role for our mental acts.[2]

But he was pragmatic when it came to how the ontology of mathematics should affect its development. He held that by the *reality* of mathematical objects, including sets, one could refer to two notions. First, there was an object's *intrasubjective* or *immanent* reality: its reality as it was conceived mentally on the basis of definitions, and the way in which this conception shaped our cognition. Second, there was an object's *transsubjective* or *transient* reality: its reality as it reflected something external to the mind.[3] Mathematics was unique among the sciences because it needed only attend to the *immanent* reality of its concepts. As soon as a concept was determined by definitions, including in terms of how it related to already accepted concepts, the mathematician could view it as real.[4]

Cantor here enunciated the conceptual separation of pure mathematics from the world. But his argument was also a declaration and a justification that mathematics was *unconstrained* by any bond to the external world.

> 'Because of this remarkable feature [...] it especially deserves the name of *free mathematics*, a designation which, if I had the choice, would be given precedence over the now usual "pure" mathematics.'[5]

Cantor insisted upon this point: 'the *essence* of *mathematics* lies precisely in its *freedom*'.[6]

Cantor would have had no doubts about the existence of transfinite numbers, whatever philosophical objections other mathematicians had. Whether his exercise of freedom in choosing to explore set theory had any aesthetic component is unclear; he seems not to have discussed any such motivation in his written work. He certainly recognized mathematical beauty in results, mentioning in his obituary of Karl Ludwig Scheeffer (1859–85) an exemplar of his 'beautiful results',[7] and describing a development of Ivar Otto Bendixson (1861–1935) of one of his own results, which concerns what would now be called the iterated Cantor–Bendixson derivative, as being 'of great beauty'.[8] He also saw beauty in theories, observing that an attempt to point out a fundamental flaw in a theory can cause the emergence of

[1] Cantor, 'Contributions to the Founding of the Theory of Transfinite Numbers', p. 85.

[2] Grattan-Guinness, *The Search for Mathematical Roots 1870–1940*, pp. 119–20.

[3] Cantor, 'Foundations', pp. 895–6.

[4] *Ibid.*, p. 896.

[5] *Ibid.*, p. 896, emphases in orig.

[6] *Ibid.*, p. 896, emphases in orig.

[7] Cantor, 'Ludwig Scheeffer', p. 369; trans. C.

[8] Cantor, letter to Mittag-Leffler, dated 9 Sep. 1883, repr. in Cantor, *Briefe*, p. 128.

an even more beautiful successor,[1] but it is not clear *which* theories he found beautiful.

[Henri Poincaré (1854–1912) probably did not make the comment that is often attributed to him, that '[l]ater generations will regard [Cantor's] *Mengenlehre* as a disease from which one has recovered';[2] rather, it is likely that at the 1908 International Congress of Mathematician in Rome, he commented on the paradoxes affecting set theory and potential remedies. Each re-telling and re-translation coloured the next, until authors such as Morris Kline (1908–92) could affirm Poincaré's rejection of *Mengenlehre*.[3]]

LEOPOLD KRONECKER

Leopold Kronecker's (1823–91) place in the history of mathematical beauty is perhaps summed up in what Felix Klein (1849–1925) wrote about him in a letter to Kurt Hensel (1861–1941):

> 'This is the peculiar double nature of Kronecker: on the one hand the profound researcher who finds, or at least divines, the most beautiful results, on the other hand the dogmatist who believes he can erect barriers for the method of science in its thrust towards progress. Cf. his declaration against the irrational numbers.'[4]

The declaration to which Klein referred was: 'God made the integers, all else is the work of man' ['Die ganzen Zahlen hat der liebe Gott gemacht, alles anderes ist Menschenwerk']. According to his obituarist Heinrich Weber (1842–1913), Kronecker said this at the 1886 Berlin Congress of Scientists [Berliner Naturforscher-Versammlung].[5] [The printed proceedings of this conference only contains brief summaries of presentations and discussions. There is no mention of the statement in the summary of Kronecker's contribution.[6]]

Kronecker's statement was likely a philosophical objection to considering the set of real numbers in its entirety; he thought it *unnecessary*. He thought that mathematics should be grounded on as few philosophical commitments as possible.[7] He was uneasy with mathematics that considered infinities of points,[8] and opposed attempts to create a general theory of the irrationals:

> 'The above considerations seem to me opposed to the introduction of those concepts of Dedekind like "module", "ideal", etc.; similarly [they are opposed to] the introduction of various concepts by the help of which it has frequently been attempted in recent times (but first by *Heine*) to conceive and establish the "irrationals" in general. Even the *general* concept of an infinite series, for example one which increases according to

[1] Cantor, 'Historische Notizen', p. 359.

[2] See, for example, Kline, *Mathematical Thought*, vol. 3, p. 1003, ins. '[Cantor's]' in orig.

[3] Gray, 'Did Poincaré Say "Set Theory Is a Disease"?', esp. p. 21.

[4] Klein, letter to Hensel, dated 19 Dec. 1921, repr. in Petri & Schappacher, 'From Abel to Kronecker', p. 262; trans. Petri & Schappacher, 'From Abel to Kronecker', p. 251.

[5] Weber, 'Leopold Kronecker', p. 19; trans. Gray, *Plato's Ghost*, p. 153.

[6] Guttstadt, *Tageblatt der 59. Versammlung Deutscher Naturforscher und Aerzte*, p. 123.

[7] Edwards, 'Kronecker's Algorithmic Mathematics', p. 13.

[8] Dauben, *Georg Cantor*, p. 34.

> definite powers of variables, is in my opinion only permissible with the reservation that in every special case, on the basis of the arithmetic laws of constructing terms (or coefficients), just as above, certain assumptions must be shown to hold which are applicable to the series like finite expressions, and which thus make the extension beyond the concept of a *finite* series really unnecessary.'[1]

In general, he objected to considerations of arbitrary mathematical objects without an idea of how they were to be built. Thus in Cantor's work there was much to which he could object: how could one *reach* the first infinite ordinal ω starting from the finite ordinals $1, 2, 3, \ldots$?[2]

But Kronecker was not ready to abandon any established part of mathematics on such grounds. He believed that arithmetic, and the arithmetical parts of mathematics, which included algebra and analysis, could be grounded upon the integers.[3] That said, it is possible that he did not take entirely seriously this philosophically radical position.[4]

In his very first lecture on number theory, he said that

> 'If one defines e.g. the transcendental number π that originates from geometry, perhaps through the Leibniz series
>
> $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1},$$
>
> so emerges one of the most beautiful arithmetic properties of odd numbers, namely that of determining that geometrical irrational number; in this sense the well-known saying: "numero impari deus gaudet"[5] is to be understood.'[6]

[The 'saying' is a minor misquotation of what Leibniz wrote when he first published this series in 1682.*] So not only did Kronecker admit transcendental numbers to mathematics, but he found beautiful the connection of π and the odd numbers. Such acceptance and approval suggest that he was engaging in a deliberate hyperbole when he commented on Ferdinand von Lindemann's (1852–1939) proof that π is transcendental. According to Lindemann's account,

> 'Kronecker went even further and denied the existence of irrational numbers; thus, he once said to me in his lively and paradoxical way, "Of what use to us are your beautiful researches about the number π? Why consider such problems when in fact there are no irrational numbers?"'[7]

Lindemann was writing more than a decade after Kronecker's death and may not have recalled the exact form of words.[8] But given Kronecker's acceptance of the beauty of the property emerging from Leibniz's series, it is reasonable to believe that, even if he had philosophical objections to considering the set of all real numbers, he genuinely found Lindemann's proof beautiful.

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[1] Kronecker, 'Über einige Anwendungen der Modulsysteme auf elementare algebraische Fragen', p. 156, n. ∗; trans. Dauben, *Georg Cantor*, p. 68, insertion in trans.

[2] Grattan-Guinness, *The Search for Mathematical Roots 1870–1940*, p. 123.

[3] See Kronecker, 'Sur le Concept de Nombre'.

[4] Biermann, 'Kronecker, Leopold', p. 507.

[5] God rejoices in odd numbers.

[6] Kronecker, *Vorlesungen über Zahlentheorie*, vol. 1, p. 4; trans. C.

* ▸ p. 316

[7] Lindemann, notes to Poincaré, *Wissenschaft und Hypothese*, p. 20, n. 3 [p. 246]; trans. Edwards, 'Kronecker', p. 203.

[8] *loc. cit.* emphasizes this point.

FELIX KLEIN

> ‘ That the present has gone far beyond Klein,
> and has ascended higher than he saw,
> does not diminish the sublimity of his conception. ’
>
> — E. T. Bell, *Mathematics: Queen and Servant*, p. 144.

That the University of Göttingen became the centre for the exact sciences in Germany was largely due to the activity and skill, both mathematical and organizational, of Felix Klein (1849–1925). As a mathematician, Klein contributed to many areas of mathematics and mathematical physics and developed and exploited links between different fields.[1]

Klein had an appreciation of beautiful theorems in many areas. In geometry, for instance, he found beautiful August Möbius's (1790–1868) results on simultaneously inscribing two tetrahedra in each other, in the sense that the vertices of each tetrahedron lies in the plane of one of the faces of the other[2] (in what is now called a *Möbius configuration*).

He also put forward, but regrettably did not develop, the idea that aesthetic experience can be brought about by the *process* of mathematics, not just by its artefacts. He wrote that because of the ‘beautiful theorems’ that arise concerning algebraic curves of small order, *working* with these objects can ‘entail an aesthetic satisfaction’.[3] This idea seems to indicate that it is possible for an aesthetic experience to occur in the process of studying such objects, not only in the contemplation of theorems or proofs.

Klein's suggestion seems to have been the first statement of this facet of the aesthetic experience of mathematics, and the historian must lament that Klein did not explore this topic further, particularly because it would be desirable to have a clearer view of his opinion on the relationship between the ‘beautiful theorem’ and the ‘aesthetic satisfaction’. The latter seems to have been in some way dependent on the former, but it is not clear whether Klein was suggesting a psychological dependence — that the beauty of these theorems somehow helped induce the experience — or a mathematical one — that the theorems had logical consequences that induced the experience.

The Erlangen programme and the ordering of geometry

In 1872, for his inauguration to a chair of mathematics at Erlangen, Klein wrote a paper modestly entitled ‘A Comparative Review of Recent Researches in Geometry’, which may have been ahead of its time,[4] but which in the opinion of his obituarist Richard Courant (1888–1972), became ‘perhaps the most influential and widely-read mathematical paper’ of the six decades up to Klein's death.[5] In it, Klein put forward a new conception of geometry that had emerged during his discussions with Sophus Lie (1842–99):[6] to consider vari-

[1] Burau & Schoeneberg, ‘Klein, Christian Felix’, p. 396.

[2] Klein, *Development of Mathematics in the 19th Century*, p. 107; see Möbius, ‘Kann von zwei dreiseitigen Pyramiden […]’

[3] Klein, *Elementary Mathematics from a Higher Standpoint*, vol. III, p. 145.

[4] Birkhoff & Bennett, ‘Felix Klein and His “Erlanger Programm”’, p. 152.

[5] Courant, ‘Felix Klein’, p. 200.

[6] Klein, ‘A Comparative Review of Recent Researches in Geometry’, p. 216; Birkhoff & Bennett, ‘Felix Klein and His “Erlanger Programm”’, p. 150.

ous groups of transformations of a space and the properties that remain invariant under those transformations.[1] If the transformations comprise all translations and rotations, properties such as distances and angles remain invariant; under the larger set of affine (loosely speaking, skew) transformations, distances and angles are no longer invariant, but parallelism is. Under the still larger set of projective transformations, even parallelism is not preserved, but collinearity is. In order to prove a geometrical theorem, one can transform the configuration into an easier one using any transformations that preserve the properties involved in the statement of the result.

Klein's aim was systematizing and ordering the field of geometry. Since the emergence of non-euclidean geometries in the early nineteenth century, the most important unifying step, in Klein's view, had been made by Cayley, who had shown that metric and affine geometry were special cases of projective geometry.[2] Projective geometry could be seen as very powerful and allowed one to advance with 'ease and elegance' to important results, but Klein cautioned against imagining that it was a 'royal road' unknown to Euclid.[3] [Klein referred to the famous story, reported by Proclus, that Ptolemy I Soter supposedly asked Euclid if there was a shorter way to learn geometry than via the *Elements*, only to receive the reply that 'there was no royal road to geometry'.[4]] Indeed, Klein seems to have seen some aesthetic loss in moving away from euclidean geometry: if one defined lines as parallel if they meet at an infinitely distant point, as is the case in projective geometry, then one lost 'almost all of the elegant properties of Euclidean parallels'.[5]

In any case, Klein did not seek to promote any geometry over another. Rather, by ascending to a higher level of abstraction, Klein could see the systematic structure of all geometry. The greater abstraction involved a complete detachment of mathematical geometry from the space perceived in the physical world: the mental image constructed by space perception was 'not essential for purely mathematical investigations',[6] though it remained useful for illustration and is thus valuable for teaching.[7]

He thought that every mathematician and teacher of mathematics should be aware of the structure thus revealed. Why? He noted one particular reason: 'It certainly satisfies the aesthetic sense to have an orderly arrangement of the sort, which I have described.'[8]

Klein's aesthetic appreciation of geometry went beyond the theoretical. In his *Elementary Mathematics from a Higher Standpoint* (1908), he called the Peaucellier–Lipkin linkage[*] 'an elegant solution of the problem of generating a straight line'.[9] And he wrote that his then-student Robert Remak (1888–1942) had put forward 'a beautiful solution' to the problem of devising a mechanism that performs affine transformations.[10] Remak's solution was based upon the mechanism sometimes known (for reasons that defy investigation) as *Nuremberg scissors* [Nürnberger scheere]. The linkage is designed so that

[1] Klein, 'A Comparative Review of Recent Researches in Geometry', esp. pp. 217–19.

[2] Klein, *Elementary Mathematics from a Higher Standpoint*, vol. II, pp. 157.

[3] Klein, *Development of Mathematics in the 19th Century*, p. 124.

[4] Proclus, *Commentary on Euclid*, FRI 68.

[5] Klein, 'Zur Nicht-Euklidischen Geometrie', p. 361.

[6] Klein, 'A Comparative Review of Recent Researches in Geometry', p. 216.

[7] *Ibid.*, p. 244.

[8] Klein, *Elementary Mathematics from a Higher Standpoint*, vol. II, p. 182.

[*] ▸ pp. 439–40

[9] *Ibid.*, vol. II, pageno 117.

[10] *Ibid.*, vol. II, p. 88.

the cross-pivots lie on a straight line (see Figure 12.13). When the mechanism is actuated, the set of cross-pivots moves along the line according to a similarity transformation. If three such mechanisms are assembled into a triangle, the set of cross-pivots move in the plane according to an affine transformation. The same holds if any new such mechanism is inserted into the triangle.[1]

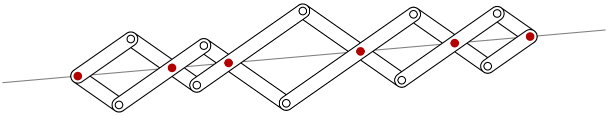

FIGURE 12.13

A set of Nuremberg scissors. When actuated, the cross-pivot points move along a line under a similarity transformation.

The enthusiastic and the indifferent

Klein recognized the isolation of mathematics, which he attributed to a special feature of mathematical thought: the impossibility of absorbing a given concept without recapitulating the concepts that lead to it. A layperson, however, only aimed 'to grasp approximately the essence of a subject which is strange to him, and to divine something of its particularity and beauty'.[2] He thought that, for mathematics, it was possible to communicate these qualities to an educated reader, but only via what he called a 'pious fraud', of presenting a shallow view of its historical development.[3]

Even mathematicians, divided into their specialities, could take this attitude. Klein thought that mathematicians could be divided into those enthusiastic about number theory, and those indifferent to it. For the enthusiasts,

> 'there is no other science so beautiful and so important, none which contains such clear and precise proofs, theorems of such impeccable rigor, as the theory of numbers.'[4]

For the indifferent class, which included the majority of students, number theory was too isolated and abstract, and was thus to be avoided.[5]

Thus in using the aesthetic experience to motivate students for mathematics, it was important to avoid the kind of formalism and completeness that would appeal only to the expert. In a discussion of the textbooks on analytic geometry written by George Salmon, Klein made it clear what ideal student texts should be like:

> 'calm and fluent accounts, in a comfortable and conversational style, [...] delightful and instructive walks through forests, fields, and gardens, in which the guide points out now this beauty, now that strange phenomenon, without forcing everything together into a rigid system of gapless completion [...] and also without digging up the most profitable plants to replant them in prepared soil according to the principles of a rational agronomy.'[6]

[1] Klein, *Elementary Mathematics from a Higher Standpoint*, vol. II, p. 88–9.

[2] Klein, *Development of Mathematics in the 19th Century*, p. 1.

[3] *Loc. cit.*

[4] Klein, *Elementary Mathematics from a Higher Standpoint*, vol. I, p. 40.

[5] *Loc. cit.*

[6] Klein, *Development of Mathematics in the 19th Century*, pp. 157–8.

Complete rigour was not necessary; interest, enjoyment, beauty *were*. The student who learned mathematics from such a text would be engaged and would absorb a strong foundation: 'We all grew up in these flower gardens; here we gathered the basic knowledge on which we were later to build'.[1]

DAVID HILBERT

On 8 August 1900, at the International Congress of Mathematicians (ICM) in Paris, David Hilbert (1862–1943) delivered a lecture that set the course of mathematics for at least the next century. When he was invited to address the ICM, he was already a mathematician of renown whose work had had an impact on varied fields. He conceived the idea of presenting a collection of problems that he thoughts mathematicians should focus on during the coming century and wrote to his friends Hermann Minkowski (1864–1909) and Adolf Hurwitz (1859–1919) to seek their opinions.[2] Minkowski prophesied — correctly — that the lecture would make Hilbert the 'Director-General' of twentieth-century mathematics.[3]

Hilbert's lasting influence on the development of mathematics, both through his own research, which influenced almost every branch of mathematics,[4] and through the challenge of the twenty-three problems he set out in Paris, makes it important to evaluate his attitude to mathematical beauty. There is no doubt that he was sensitive to it: in a moving obituary of Minkowski, Hilbert recalled the role of mathematical beauty in cementing their friendship:

> 'Since my first student days, MINKOWSKI was my best and most reliable friend, who was devoted to me with all of his own depth and loyalty. Our science, which was dear to us, had brought us together; it appeared to us like a blooming garden; in this garden there are paved paths which one can enjoy effortlessly by looking around, especially at the side of someone who feels the same. But we also liked to look for hidden paths and discover new, apparently beautiful prospects, and when one showed them to the other and we admired them together, our joy was complete.'[5]

Hilbert particularly recognized mathematical beauty in number theory and related fields. In lectures, he said that

> 'number theory is a finely structured, sky-high edifice that surpasses all other human intellectual products in beauty and perfection';[6]

and made a similar point, with greater specificity:

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[1] Klein, *Development of Mathematics in the 19th Century*, p. 158.

[2] Reid, *Hilbert*, pp. 69–70.

[3] Minkowski, letter to Hilbert, dated 28 Jul. 1900, repr. in Minkowski, *Briefe an David Hilbert*, p. 130.

[4] Weyl, 'David Hilbert. 1862–1943', p. 548.

[5] Hilbert, 'Hermann Minkowski', p. 39; trans. C., emphasis in orig.

[6] Hilbert, 'Ueber die Grundlagen des Denkens', p. 765; trans. C.

> ‘number field theory […] is a finely structured, sky-high edifice, connected with the most developed theories of analysis, which far surpasses in beauty and perfection all other intellectual products of mankind.’[1]

In his famous *Zahlbericht* — translated as *The Theory of Algebraic Number Fields* — he said that number field theory was like a ‘structure of wonderful beauty and harmony’; the ‘most richly endowed part’ is the theory of abelian and relatively abelian fields.[2] Olga Taussky (1906–95) reported that Hilbert declared at the 1932 ICM in Zürich that the theory of complex multiplication of elliptic modular functions, which connects number theory and analysis, was the most beautiful part of mathematics and indeed of the whole of science.[3] But he did not see beauty only in number theory and related fields; he called module theory beautiful[4] and wrote of the ‘beautiful theory’ of orthogonal invariants of Hurwitz.[5]

Hilbert also expressed pride in the beauty or simplicity of his own results.[6] He proclaimed that some of his own results regarding relatively abelian number fields of any relative degree and with relative discriminant 1 were theorems of ‘wonderful simplicity and crystalline beauty’.[7] For Hilbert a ‘particularly beautiful and profound example’[8] of the use of ideal elements in mathematics was the application of ideal prime elements to maintain (or recover) unique decomposition into primes in algebraic number fields.[9] The resulting factorization laws parallel the usual factorization of integers into primes, and had ‘a wonderful beauty and simplicity’.[10]

Minkowski also admired Hilbert’s work for its beauty. When he read the manuscript of the *Zahlbericht*, he wrote to Hilbert that the proof of the result that *Every number field which is an abelian extension of the rational number field is a cyclotomic field*[11] was ‘really extremely simple and beautiful’,[12] but recommended some improvements to the exposition so that the reader could enjoy it fully. At another time, he mildly chastised Hilbert for not disseminating his work sooner:

> ‘In the interest of the progress of science, I would like to ask you possibly not to keep back the most beautiful things for yourself for a long period, as it almost seems to be the case according to your pronouncements, but disclose everything you have achieved to others’.[13]

Set theory

Hilbert was a champion of Cantor’s set theory and his theory of transfinite numbers. His appreciation of Cantor’s work was at least partly aesthetic. He wrote that the theory of transfinite numbers was ‘the most admirable flower of the mathematical intellect and in general one of the highest achievements of purely rational human activity'.[1] He also said that one of Cantor's 'most beautiful and most important results'[2] was that *The set of all countable ordinals is the smallest uncountable ordinal.*[3]

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[1] Hilbert, ‘Die Grundlegung der elementaren Zahlenlehre’, p. 976; trans. C.

[2] Hilbert, *The Theory of Algebraic Number Fields*, p. x.

[3] Taussky, ‘Prof. David Hilbert, For.Mem.R.S.’, p. 182.

[4] Hilbert, ‘Logische Grundlagen der Mathematik’, p. 529.

[5] Hilbert, ‘Beweis für die Darstellbarkeit’, p. 511.

[6] Weyl, ‘David Hilbert’, p. 615.

[7] Hilbert, ‘Über die Theorie der relativ-Abelschen Zahlkörper’, p. 484; trans. C.

[8] Hilbert, ‘Über das Unendliche’, p. 685; trans. C.

[9] Hilbert, *Lectures on the Foundations of Arithmetic and Logic*, pp. 685–6.

[10] Hilbert, *The Theory of Algebraic Number Fields*, p. 9.

[11] *Ibid.*, Thm 131.

[12] Minkowski, letter to Hilbert, dated 21 Jul. 1896, repr. in Minkowski, *Briefe an David Hilbert*, p. 83; trans. Zassenhaus, ‘On the Minkowski-Hilbert Dialogue’, p. 448.

[13] Minkowski, letter to Hilbert, dated 13 Apr. 1898, repr. in Minkowski, *Briefe an David Hilbert*, p. 107; trans. Zassenhaus, ‘On the Minkowski-Hilbert Dialogue’, p. 448.

It is likely that his aesthetic admiration of Cantor's work was at least part of the reason why the first problem he set in his 1900 lecture was the proof of Cantor's Continuum Hypothesis. Hilbert stated it in the form: *Every set of infinitely many real numbers has either the same cardinality as the set of natural numbers or the same cardinality as the set of real numbers.*[4] The *problem* of whether the Continuum Hypothesis holds true was, for Hilbert, 'distinguished by its originality and inner beauty'.[5] Contemporaries of Hilbert also thought the Continuum Hypothesis beautiful: Gösta Mittag-Leffler (1846–1927) thought that if it were a true theorem, it would be 'one of the most important in all of Analysis and one of the most beautiful that has ever been found'.[6] [The Continuum Hypothesis turned out to be independent from the usual axioms of set theory.*]

Foundations

Hilbert usually focused completely on one area of mathematics before switching to another. One of his most important switches was from number fields to axiomatics. In his 1899 book *Grundlagen der Geometrie*, published in English in 1902 as *The Foundations of Geometry*, he set up new and comprehensive axioms for geometry, and studied their relationship. In particular, he showed that each was independent of the others by constructing models in which all of the axioms held but one.[7] He also showed that the axioms were consistent, in the sense that he used the algebraic numbers to construct a model in which all of the axioms hold.[8] But this proof was not an *absolute* demonstration of consistency, for it depends on the consistency of the system of algebraic numbers.

In December 1899, Hilbert wrote to Gottlob Frege (1848–1925) that 'the most beautiful and important propositions of geometry' concerned the provability or otherwise of axioms such as Euclid's parallel postulate, the axiom of Archimedes, and the Killing–Stolz axiom.[9] [The axiom of Archimedes asserts that *Given points A_0 and B and a point A_1 on the line A_0B, take points A_i for $i \geqslant 2$ so that A_i lies between A_{i-1} and A_{i+1} and A_iA_{i+1} is equal in length to A_0A_1. Then there is some A_n such that B lies between A_0 and A_n.*[10] The Killing–Stolz axiom is *If a rectangle is decomposed into triangles by means of straight lines, and one of the triangles is omitted, it is impossible to reassemble the original rectangle from the remaining triangles.*[11] The Killing–Stolz axiom can be proved using the axiom of Archimedes.[12]]

This interest paralleled a desire of Hilbert for an absolute proof of the consistency of a collection of axioms that sufficed to ground the whole of mathematical analysis and indeed the whole of Cantor's set theory, which was the second of his Paris problems.[13] Hilbert's motive seems to have been a desire to secure the logical underpinnings of the parts of mathematics he considered most valuable.

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[1] Hilbert, 'On the infinite', p. 373.

[2] Hilbert, 'Über das Unendliche', p. 738; trans. C.

[3] Cantor, 'Foundations', § 12.

[4] Hilbert, 'Mathematical problems', p. 446.

[5] Hilbert, 'On the infinite', p. 384.

[6] Mittag-Leffler, letter to Cantor, dated 9 Nov. 1881, quot. in Cantor, *Briefe*, p. 92.

* ▸ p. 705

[7] Hilbert, *The Foundations of Geometry*, §§ 10–12.

[8] *Ibid.*, § 9.

[9] Hilbert, letter to Frege, dated 29 Dec. 1899, repr. in Frege, *Philosophical and Mathematical Correspondence*, pp. 41–3.

[10] Hilbert, *The Foundations of Geometry*, p. 15.

[11] *Ibid.*, p. 45.

[12] *Loc. cit.*

[13] Hilbert, 'Mathematical problems', pp. 447–9.

This desire was a large part of his reason, or at least his justification, for rejecting intuitionism as a suitable philosophical foundation for mathematics. Intuitionism, which was formulated by L. E. J. Brouwer (1881–1966), considers that mathematics comprises making mental constructions that are ultimately based on the intuition of the passage of time, and of two moments separated by an interval. Valid mathematics can only be constructed within the intellectual limits of this foundation.[1] Infinite quantities were restricted to those that could be constructed. A valid proof must be constructive: proof by contradiction is not permitted in general, for intuitionism rejects the law of the excluded middle $p \vee \neg p$ applied to infinite sets.[2]

Hilbert recoiled from such a restrictive mathematics, the grounds that it would entail discarding much that was beautiful:

> 'What WEYL and BROUWER do, amounts in principle to striding along the former path of KRONECKER: they try to provide a foundation of mathematics by jettisoning everything that seems inconvenient to them and by setting up a dictatorship of prohibitions à la KRONECKER. This means, however, to chop up and mutilate our science, and we risk to lose a large part of our most valuable treasures if we follow such reformers.'[3]

Concepts such as irrational numbers, even arithmetical functions, and much of Cantor's set theory would be set aside by Brouwer and his acolytes. This price was too great for Hilbert, who declared instead that '[n]o one shall be able to drive us from the paradise that Cantor created for us.'[4] Nor was it just set theory whose beauty was threatened: with a restriction to the finite, one could build up a large part of number theory, but one lost the 'most beautiful, most important, and most fruitful' ideas and results.[5]

⁂

Brouwer himself admitted the aesthetic cost of adopting an intuitionist position:

> 'In general intuitionism brings about a complete recasting of mathematics, with the result, to our regret, that in many places its supple and elegant character is lost, and it has to assume much harsher, more tortuous and more complicated forms.'[6]

But he thought that intuitionism was the correct foundation of mathematics, and so the lamentable price had to be paid: 'Alas, the spheres of truth are less transparent than those of illusion.'[7]

Although in the early 1920s Hermann Weyl (1885–1955) had been a fervent supporter of Brouwer's foundational approach, his enthusiasm had cooled by the middle of the decade, and for reasons similar to Hilbert's.[8]

[1] Brouwer, *Cambridge Lectures on Intuitionism*, pp. 4–5.

[2] *Ibid.*, p. 5.

[3] Hilbert, 'Neubegründung der Mathematik', p. 159; trans. van Dalen, *L. E. J. Brouwer*, pp. 439–40.

[4] Hilbert, 'On the infinite', p. 376.

[5] Hilbert, 'Über das Unendliche', p. 749.

[6] Brouwer, 'Volition, Knowledge, Language', p. 444.

[7] *Loc. cit.*

[8] Pesic, 'Introduction'.

The 'twenty-fourth problem'

Hilbert recorded in a mathematical notebook, in an entry that probably dates to 1901,[1] that he had considered adding a twenty-fourth problem to the paper he gave at the 1900 ICM:

> 'The 24th problem in my Paris lecture was to be: Criteria of simplicity, or proof of the greatest simplicity of certain proofs.'[2]

This problem would have involved developing a theory that would allow one to explore the relationship between two proofs of the same result, of the 'area' lying between the proofs. By studying how one proof could be transformed into another, one could understand precisely how two proofs differed. And by considering the elementary operations that constituted proofs, one could compare them in terms of simplicity.[3]

There is no explicitly aesthetic statement in these notes, but, for Hilbert, simplicity was bound up with beauty. Instances where he praised a piece of mathematics as having both values have already been discussed;* another example is found in his praise of number theory:

> 'In the theory of numbers, we value the simplicity of its foundations, the exactness of its conceptions and the purity of its truths [...].
>
> [...][I]n the theory of numbers the oldest problem is often to-day modern, like a genuine work of art from the past. Nevertheless it is true now as formerly, a fact which Gauss and Dirichlet lamented, that only a small number of mathematicians busy themselves with the theory of numbers and attain to a full enjoyment of its beauty.'[4]

Thus there was at least an aesthetic undertone in the twenty-fourth problem.

Émile Lemoine and 'geometrography'

Hilbert seems not to have known that, in the last decade of the nineteenth century, Émile Lemoine (1840–1912) had formulated a measure of the simplicity of straightedge-and-compass geometric constructions. Although his motivation was not primarily aesthetic, there was a shade of aesthetic concern, for Lemoine characterized his theory, which he called *geometrography* [*géométrographie*] as 'a genuine *art of geometric constructions*',[5] and linked simplicity and elegance.[6]

Lemoine's system was based upon counting the following steps in a geometric construction:

R_1 Placing the edge of the ruler at a point.

R_2 Drawing a straight line.

[1] Thiele, 'Hilbert's Twenty-Fourth Problem', p. 5.

[2] Hilbert, quot. in *ibid.*, p. 2.

[3] *Loc. cit.*

* ▸ p. 463

[4] Reid, *The Elements of the Theory of Algebraic Numbers*, Hilbert, intro. to[xviii.

[5] Lemoine, *La Géométrographie*, p. 34, emphasis in orig.; trans. C.

[6] *Ibid.*, p. 34.

C_1 Placing a point of the compass at a given point.

C_2 Placing a point of the compass at an undetermined point on a line.

C_3 Drawing a circle (or part thereof).[1]

The numbers of steps involved in a construction could be expressed as

$$\ell_1 R_1 + \ell_2 R_2 + m_1 C_1 + m_2 C_2 + m_3 C_3.$$

Lemoine defined the *coefficient of simplicity* or just the *simplicity* of the construction as $\ell_1 + \ell_2 + m_1 + m_2 + m_3$, and the *coefficient of exactitude* or just the *exactitude* of the construction as $\ell_1 + m_1 + m_2$. For instance, the numbers of steps involved in drawing a straight line through two points is $2R_1 + R_2$ and so this (trivial) construction has simplicity 3 and exactitude 1. As Lemoine pointed out, both scales are inverted: a lower coefficient indicates a higher degree of simplicity or exactitude.[2]

For Lemoine, the advantage of the coefficient of simplicity was that it provided a *criterion* to judge and to compare simplicity.[3] He thought this necessary because there were famous constructions that were held to be simple and elegant but which were not actually the simplest: they were merely thought to be simple because they formed an easy picture and were remembered without difficulty.[4] As an example, he gave Chasles's construction of the axes of an ellipse from two conjugate diameters,[5] which has operation sum $14R_1 + 7R_2 + 21C_1 + 14C_3$; simplicity 56; exactitude 35. An alternative method[6] due to Amédée Mannheim (1831–1906) has operation sum $12R_1 + 6R_2 + 16C_1 + 9C_3$; simplicity 43; exactitude 28.

Lemoine's assertion of this or that being 'actually simpler' is clearly based on privileging his own measure of simplicity based on the number of construction steps over, say, simplicity as memorability or ease of understanding. The definition of the coefficient of simplicity was also based on assigning an equal value to each of the five atomic steps. In a contemporaneous explication of Lemoine's work, J. S. Mackay (1843–1914) suggested that it might be better to have a *partial* ordering of simplicity based upon comparing the numbers of corresponding atomic steps.[7]

Clarity

While it is not necessarily an aesthetic category, clarity had an important role in nineteenth-century mathematics. At minimum, there seems to have emerged an openly-declared desire to reformulate mathematical theories to increase clarity. Joseph Diez Gergonne (1771–1859) wrote to Adolphe Quetelet (1796–1874) in 1825 that 'one still does not have the last word of science on a theory, as long as one has not taken it to the point of being able to recount it to a passer-by in the street'.[1] Quetelet quoted him in an 1827 paper with evident approval;[2] Chasles inserted it into his 1837 *Aperçu Historique* with precise citation and less than precise quotation.[3] H. J. S. Smith expressed the same sentiment in his address to the mathematical and physical section of the British Association for the Advancement of Science in 1873, although he called it 'a brilliant exaggeration' and attributed it to an 'old French geometer'.[4] Hilbert seems to have come across it in Smith's *Collected Mathematical Papers* and repeated it in his lecture at the 1900 ICM, where he added that 'clearness and ease of comprehension' should be demanded even more for a mathematical problem.[5] [6]

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[1] Lemoine, *La Géométrographie*, p. 3.

[2] *Ibid.*, p. 3, n. ∗.

[3] *Ibid.*, p. 34.

[4] *Loc. cit.*

[5] See Chasles, *Aperçu Historique*, pp. 359–60.

[6] See Mannheim, 'Construire les axes d'une ellipse'.

[7] Mackay, 'The Geometrography of Euclid's Problems', p. 3.

The motivation for mathematics

In an address on receiving honorary citizenship of his home city of Königsberg in 1930,[7] the conclusion of which he repeated in a radio broadcast,[8] Hilbert agreed with the view that the value of mathematics was not grounded in applications. While he saw the value of the applications of mathematics in science, he said that '*all mathematicians* have refused to accept applications as a *measure* of the value of mathematics'.[9] But he did not follow the tendency to defend its pursuit on aesthetic grounds. Rather he seems to have accepted individual intellectual delight in the practice of mathematics as sufficient motivation, but also pointed out that it was this individual delight that led to the practical benefits of science.[10] He quoted Jacobi with approval: the sole purpose of science was the 'glory of the human spirit', and from this perspective, a problem in pure mathematics was just as valuable as a problem of applications.[11] And, for Hilbert, part of the satisfaction of tackling a problem was that there was *always* a solution. Just as he had said in his Paris lecture in 1900 that 'in mathematics there is no *ignorabimus*',[12] so he reiterated the point three decades later:

> 'In place of this foolish Ignorabimus, on the contrary let our watchword be:
> We must know,
> We shall know.'[13]

CONCLUSION

In parallel with the emergence of the defence of mathematics as an aesthetic pursuit, there was a semantic shift of the term *pure*. In the first half of the nineteenth century, when Samuel Taylor Coleridge and William Rowan Hamilton wrote about pure sciences, the word *pure* signified primarily a particular kind of epistemological foundation. In the later nineteenth century, a lesser connotation of *pure* came to predominate: a 'pure science' was done without regard to application or utility.[1] Pure science in this sense had to be driven by something other than a desire for obvious practical benefits, and aesthetic value could be put forth as such a motivation.

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[1] Gergonne, letter to Quetelet, dated 25 Feb. 1825, quot. in Barrow-Green & Siegmund-Schultze, '"The first man on the street"', p. 421; trans. C.

[2] Quetelet, 'Résumé d'une Nouvelle Théorie des Caustiques', p. 88, n. 1.

[3] Chasles, *Aperçu Historique*, p. 115.

[4] Smith, 'Address to the Mathematical and Physical Section of the British Association', p. 689.

[5] Hilbert, 'Mathematical problems', p. 438.

[6] For the descent of the expression from Gergonne to Hilbert, see Barrow-Green & Siegmund-Schultze, '"The first man on the street"'.

[7] Hilbert, 'Naturerkennen und Logik'.

[8] Vinnikov, 'We Shall Know'.

[9] Hilbert, 'Naturerkennen und Logik', p. 385; trans. C., emphases added.

[10] *Ibid.*, p. 386.

[11] *Ibid.*, p. 387.

[12] Hilbert, 'Mathematical problems', p. 445.

[13] Hilbert, 'Naturerkennen und Logik', p. 387; trans. C.

This development led into what the historian Jeremy Gray (1947–) characterized as a 'modernist transformation' of mathematics. Mathematics became autonomous. Pure mathematics, at least, had little reference to the world. Its practice emphasized formal aspects.[2] The writer Robert Musil (1880–1942), who was trained in engineering, composed in 1913 an essay on 'The Mathematical Man', which compared the utility and importance of mathematics with what he saw as the attitude and motivation of mathematicians. With few exceptions, every artefact of an industrial economy depended on mathematics in its design or production or distribution.[3] But to the practitioners of mathematics, all these consequences of mathematics were 'a matter of indifference'.[4] Mathematics proper was not concerned with utility: 'It is not goal-oriented, but uneconomical and passionate.'[5] The mathematician, thought Musil, was not interested in any eventual and distant application of their research, but only its truth and the importance that it had in a particular area of mathematics and its neighbours.[6] The passion that Musil noted in his essay could have manifested in values besides the aesthetic. But aesthetic value and aesthetic motivation were compatible with the transformation of mathematics and its autonomous nature.

Some, of course, would maintain that the autonomy of mathematics was in a sense superficial, for it never pulled entirely free from the external world. For Minkowski, both the foundations and the aesthetically valuable problems of mathematics were derived from experience of the physical world, and consequently the pure mathematician, although working in isolation from physics, would nevertheless be making progress on the physical questions. Minkowski made this point at the start of a course of lectures in 1904:

> 'Mathematics has the task of developing the tools necessary to grasp the logical coherence of external appearances. Its basic concepts, the axioms of physical quantities and of geometry, have arisen from experience. Mathematics constantly derives the most beautiful problems in applications from the natural sciences. And through a peculiar, preestablished harmony, it has been shown that, by trying logically to elaborate the existing edifice of mathematics, one is directed on exactly the same path as though by questions arising from the facts of physics and astronomy.'[7]

◆

[1] Cf. Medawar, 'Pluto's Republic', p. 15.

[2] Gray, *Plato's Ghost*, p. 1.

[3] Musil, 'The Mathematical Man', p. 41.

[4] *Loc. cit.*

[5] *Loc. cit.*

[6] *Loc. cit.*

[7] Minkowski, MS 'Differentialrechnung, S. S. 1904', quot. in Pyenson, 'Physics in the Shadow of Mathematics', p. 66.

$\prod_{n=1}^{\infty}\left(1+\frac{(-1)^{n+1}}{2n-1}\right)$
$=\left(1+\frac{1}{1}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)(1$
$1-\frac{1}{4(2n-1)^2}$
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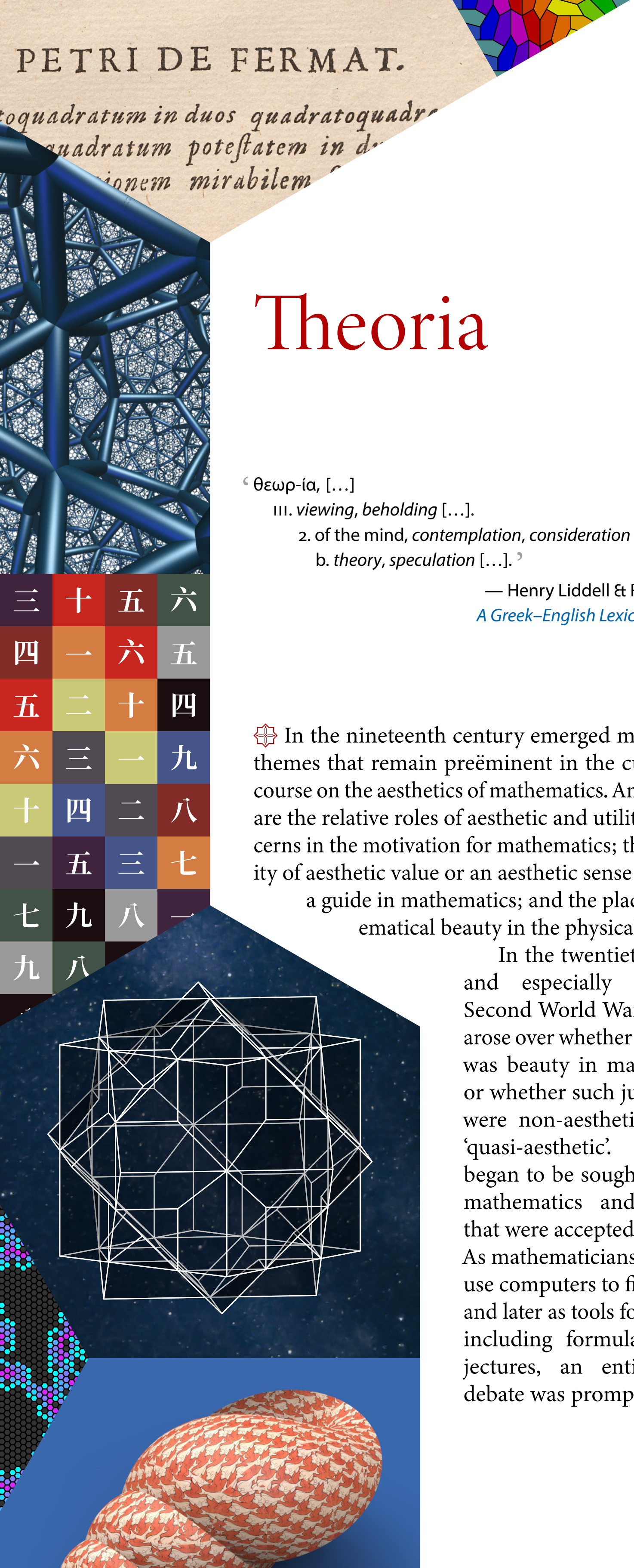

PETRI DE FERMAT.

Theoria

II

> ‘ θεωρ-ία, [. . .]
> III. *viewing*, *beholding* [. . .].
> 2. of the mind, *contemplation*, *consideration* [. . .].
> b. *theory*, *speculation* [. . .]. ’
>
> — Henry Liddell & Robert Scott,
> *A Greek–English Lexicon*, ‘θεωρ-ία’.

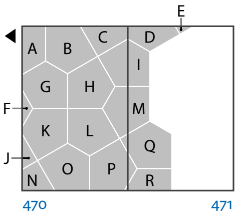
In the nineteenth century emerged many of the themes that remain preëminent in the current discourse on the aesthetics of mathematics. Among these are the relative roles of aesthetic and utilitarian concerns in the motivation for mathematics; the possibility of aesthetic value or an aesthetic sense serving as a guide in mathematics; and the place of mathematical beauty in the physical sciences.

In the twentieth century, and especially after the Second World War, a debate arose over whether there truly was beauty in mathematics, or whether such judgements were non-aesthetic or only ‘quasi-aesthetic’. Parallels began to be sought between mathematics and pursuits that were accepted as artistic. As mathematicians started to use computers to find proofs, and later as tools for research, including formulating conjectures, an entirely new debate was prompted: could proofs found by computer, or the results they proved, be beautiful? What was the relationship between the beauty of a proof and its explanatoriness?

The idea that beauty could be found in mathematics began to penetrate the popular consciousness, through works of visual art like those of M. C. Escher (1898–1972) and through computer-generated images of fractals, and also through works of popular and recreational mathematics.

While the chapters in the first part, HISTORIA, were arranged in a primarily chronological way, this second part concerns those aspects of mathematical beauty that are still relevant and whose history is still being written. The chapter division herein is thus primarily thematical.

> ‘the mere animal consciousness of the pleasantness I call Æsthesis;
> but the exulting, reverent, and grateful perception of it I call Theoria.
> For this, and this only,
> is the full comprehension and contemplation of the Beautiful’
>
> — John Ruskin, *Modern Painters*, vol. II, pt III.I, ch. 2, § 5 ⟦p. 47⟧.



A Final position in Anderssen vs Kieseritzky (London, 21 Jun. 1851). **B** Chaotic behaviour of a double pendulum. **C** Two infinite products described as beautiful by Finch, *Mathematical Constants*, p. 2. **D** Observation by Fermat to Question II.8 of Diophantus' *Arithmetica*. **E** Type 15 pentagonal tiling. **F** Detail from Gauss's mathematical diary (Gauß, 'Mathematisches Tagebuch', fol. 1v). **G** Tiling by Ammann's aperiodic set $A5$. **H** Drawing of a fern exhibiting self-similarity (Lowe, *Ferns*, vol. I, pl. XXIII). **I** Rectified icosahedral honeycomb in hyperbolic space. **J** Iterative approximation of Hilbert's space-filling curve. **K** Plot of the Riemann zeta function near 0. **L** 'The Book'. **M** Two mutually orthogonal Latin squares of order 10, illustrated by background colour and Chinese numerals. **N** $\{5, 4, 5\}$ hyperbolic honeycomb. **O** Mikael Hvidtfeldt Christensen, *Spherical Worlds*. **P** Langton's ant on a hexagonal grid with rule $L_1 L_2 NUL_2 L_1 R_2$. **Q** Union of three cubes. **R** Adaptation of 'fdecomite', *eschershell* (2016). A helical shell with a surface tessellation after M. C. Escher.

I © 2006 Claudio Rocchini. [CC-BY-4.0] O © 2012 Mikael Hvidtfeldt Christensen. [CC-BY-2.0] Q Background © 2014 'Grempz'. [CC-BY-2.0] R © 2016 'fdecomite'. [CC-BY-2.0] D, F, H, L, N Public domain.

The Cloister and the World

13

Beauty, utility, and the justification for mathematics

> ‘the perfection of mathematical beauty is such […],
> that whatsoever is most beautiful and regular
> is also found to be most useful and excellent.’
>
> — D’Arcy Wentworth Thompson, *On Growth and Form*, p. 1097.

> ‘Remember that the most beautiful things in the world
> are the most useless’
>
> — John Ruskin, *The Stones of Venice*, vol. i, ch. ii, § 17.

In 1918, in an address given at a celebration of Max Planck’s (1858–1947) sixtieth birthday, Albert Einstein (1879–1955) said:

> ‘I believe with Schopenhauer that one of the strongest motives that leads men to art and science is escape from everyday life with its painful crudity and hopeless dreariness, from the fetters of one’s own ever shifting desires. A finely tempered nature longs to escape from personal life into the world of objective perception and thought’.[1]

Digress from Einstein for a moment. The philosopher Arthur Schopenhauer (1788–1860) held that life is suffering,[2] and the fundamental reason was that a person was always prone to willing, and because there were but brief fulfilments of the lack from which willing arises, the subject of willing was ‘constantly lying on the revolving wheel of Ixion, is always drawing water in the sieve of the Danaids, and is the eternally thirsting Tantalus’.[3] Schopenhauer did, however, identify ways in which it was possible to create more peaceful states of mind. Aesthetic perception was a means of reaching such a state,[4] and in particular contemplation of the beautiful led to an easy transition to this state.[5]

Thus the reference to Schopenhauer suggests that Einstein believed that it was at least partly the aesthetic aspects of science that made it a place of escape. And thus, for Einstein, science as a refuge and science as an aesthetic pursuit were linked. With respect to math-

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- Parts of this chapter are based on the author’s three essays in *An Annotated Mathematician’s Apology*, *viz.*: Cain, ‘Context of the *Apology*’, ‘Reviews of the *Apology*’, ‘Legacy of the *Apology*’.

[1] Einstein, ‘Principles of Research’, p. 225.

[2] Schopenhauer, *The World as Will and Representation*, vol. 1, bk 4, § 56.

[3] *Ibid.*, vol. 1, bk 3, § 38.

[4] *Ibid.*, vol. 1, bk 3, § 34.

[5] *Ibid.*, vol. 1, bk 3, § 39.

ematics, the philosopher and logician Bertrand Russell (1872–1970) and the number theorist G. H. Hardy (1877–1947) also expounded at length and more explicitly this view, and Alfred North Whitehead (1861–1947) said in passing, almost as if it would be uncontroversial, that 'the pursuit of mathematics is a divine madness of the human spirit, a refuge from the goading urgency of contingent happenings'.[1] It is perhaps tempting to think it natural that it was in the first half of twentieth century, with the barbarities of two world wars, that this facet of mathematical aesthetics found its strongest advocates.

Yet the whole truth is more complicated. Three broad strands can be discerned among justifications for the pursuit of mathematics: one can defend mathematics as a *refuge*, as an *aesthetic pursuit*, or as a *useful science*. The previous chapter examined the growing tendency in the nineteenth century to justify pure mathematics on artistic grounds, and how the 'useful science' gave way to the 'aesthetic pursuit'. The events of the first half of the twentieth century led to parallel strengthenings of the attempts to justify mathematics as a refuge and as a useful science, and a weakening of attempts to justify it as an aesthetic pursuit. This is not to say that there was a lessening of the perceived beauty of mathematics, but simply that the idea of justifying it on these grounds lost traction: Hardy's celebrated essay *A Mathematician's Apology* was the swan song of this defence of mathematics.

From another perspective, the 'refuge' and the 'useful science' are opposed: the former is an individualistic justification, the latter a social one. The 'aesthetic pursuit' lies somewhere between the two: the enjoyment of beauty is an individual psychological state, but one can create beauty for others to contemplate. It is quite possible to accept both individualistic and social justifications for mathematics: for instance, Einstein did so for all forms of work, not just scientific work.[2] He warned against ignoring the value of work for society: 'If [...][people] use their intelligence from an individualist, i.e., a selfish standpoint, building up their life on the illusion of a happy unattached existence, things will be hardly better' than 'a state of insecurity, of fear, and of promiscuous misery.'[3] At the same time, he held that giving free rein to the individual was the best way to achieve both individual and social good:

> 'every individual should have the opportunity to develop the gifts which may be latent in him. Alone in that way can the individual obtain the satisfaction to which he is justly entitled; and alone in that way can the community achieve its richest flowering.'[4]

This chapter will focus on the 'aesthetic pursuit' defence of mathematics, only treating the other justifications insofar as they interact with this one. Particular foci will be Russell, whose early and later views on an aesthetic justification for mathematics were in almost complete opposition, and Hardy, whose discussions of beauty

[1] Whitehead, *Science and the Modern World*, p. 27.

[2] See, for example, Einstein, 'On Education', p. 62; Einstein, 'Principles of Research', p. 224.

[3] Einstein, 'Morals and Emotions'.

[4] *Ibid.*

in mathematics in *A Mathematician's Apology* shaped essentially all subsequent debate.

BEAUTY AND INUTILITY

According to a report written eight centuries after his time, Euclid (*fl.* c. 300 BCE), upon being questioned by a student about the benefit of studying mathematics, called a servant and said: 'Give him threepence, since he must make gain out of what he learns'.[1] This anecdote attests to a long divide between utilitarian and other motives for pursuing mathematics.

There are definite indications that by the end of the nineteenth century, the justifications for mathematics as an aesthetic pursuit and as a useful science may have been seen as being in tension. There seems to have emerged a kind of perverse pride in lack of applications of pure mathematics and of pure science more generally.

But even pure *natural* science could be defended as seeking to understand the physical world we inhabit. Mathematicians could not resort to such arguments when pursuing research into ever higher abstractions. Thus there had emerged a pattern of justifying pure mathematics on aesthetic grounds. Perhaps the connotation of the term *pure* as *undefiled*, the inapplicability of much of the new mathematics, the justification of mathematics as an artistic pursuit, and the beauty of the results all combined to create a canonical view that mathematics *should* be pure, pristine, beautiful, unsoiled, useless, …, and thus *good*. If the view expressed in the pernicious equation 'Useless = Good'[2] is accepted, the very lack of utility of a piece of mathematical or scientific research becomes its own justification.

This association of beauty, goodness, and purity, all unsullied by application, seems to have led to the opinion that, in mathematics, utility detracted from beauty. Evidence for such a view can only be anecdotal, but it seems to have been held by eminent figures: the mathematician E. W. Hobson (1856–1933) recalled 'a very great Pure Mathematician', whom he did not identify, but who would have been at Cambridge in the 1870s, saying that 'Bessel's functions are very beautiful functions, in spite of their having practical applications'.[3] (One speculates that hearing such views during his undergraduate years shaped Hobson's own view that utility detracts from the beauty of mathematics: in 1912, he said that number theory had 'never been soiled by any practical application' but wondered whether it would 'always remain undefiled'.[4])

For example, in the biography that prefaced the *Collected Mathematical Papers* of H. J. S. Smith (1826–83), C. H. Pearson (1830–94) reported that Smith once concluded a lecture by saying: '"It is the peculiar beauty of this method, gentlemen," […] "and one which endears

[1] Stobaeus, *Anthologium*, § II.xxxi.114 ⟦vol. 2, p. 228⟧; trans. Heath, *Hist. Greek Math.*, vol. 1, p. 357.

[2] Cf. Medawar, 'Pluto's Republic', p. 15.

[3] Hobson, *Mathematics*, pp. 4–5.

[4] *Ibid.*, p. 13.

it to the really scientific mind, that under no circumstances can it be of the smallest possible utility."'[1] But perhaps a degree of cynicism was at work: the physicist and mathematician Alexander Macfarlane (1851–1913) suggested that this remark, and a toast Smith once proposed ('[p]ure mathematics; may it never be of any use to any one'), were deliberate exaggerations in the face of utilitarian views.[2]

And some dissented from the view that beauty and utility were opposed. The October 1906 issue of *La Revue du mois*, a scientific and literary journal founded by Émile Borel (1871–1956), contains an unsigned article discussing the generalization of Lagrange's four-square theorem* and the value of such work.[3] The article relates how Edmund Landau (1877–1938) had proved the 'beautiful theorem'[4] that any positive definite polynomial with rational coefficients can be expressed as a sum of eight squares of such polynomials.[5] The writer of the article accepted that such results are 'curious' and 'give an impression of pure beauty to all who understand their meaning'.[6] But they wondered whether the aesthetic pleasure of a small number of people could justify mathematicians working on such problems, when there were questions that were just as beautiful and which were more directly applicable. They conceded that the immediate utility of mathematical work does not correspond to its eventual importance, but they argued that since physical quantities could only be known by approximation, results concerning only integers or rational numbers were likely to be the furthest from applicability. The writer saw in the likelihood of applicability a political significance in the choice of research fields, and approved of French mathematicians, unlike their German counterparts, turning away from such topics to work on questions relating to the differential and integral calculus.[7] [In making this point, the writer may have been influenced by the recent memory of the Tangier Crisis (31 March 1905–7 April 1906), when Germany challenged France's influence in Morocco, and the world took another step on the road to the Great War.]

HENRI POINCARÉ

For the mathematician, physicist, and philosopher Henri Poincaré (1854–1912), beauty was a motivation in the sciences generally. Writing in 1905, he explicitly made the point that, as a motivation, beauty was more important than utility in science:

> 'The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living.'[8]

[1] Pearson, 'Biographical Sketch', pp. xxxiii–xxxiv.

[2] Macfarlane, *Ten British Mathematicians*, p. 100.

* ▸ p. 577

[3] [Anonymous], 'Chronique', pp. 508–10.

[4] *Ibid.*, p. 509; trans. C.

[5] Landau, 'Über die Darstellung definiter Funktionen durch Quadrate'.

[6] [Anonymous], 'Chronique', p. 510; trans. C.

[7] *Ibid.*, p. 510.

[8] Poincaré, *The Value of Science*, Preface.

Furthermore, he held that focusing research purely on utility would be counterproductive in terms of the development of scientific results and the coherence of the theory.[1] The beauty of which Poincaré wrote was not the sensory beauty that nature possesses, although he valued that too, but the intellectual beauty that came 'from the harmonious order of the parts'.[2] Indeed, because it was intellectual beauty that motivated the scientist, it perhaps had greater value than natural beauty, through the ultimate good to society that was the consequence of scientific work.

As regards mathematics specifically, Poincaré thought that there were three interconnected aims in doing mathematics:

> 'They must furnish an instrument for the study of nature. But that is not all: they have a philosophic aim and, I dare maintain, an esthetic aim. They must aid the philosopher to fathom the notions of number, of space, of time. And above all, their adepts find therein delights analogous to those given by painting and music.'[3]

It is for these reasons that Poincaré held that mathematics should be studied for its own sake, including areas inapplicable to physics.[4] He thought that the aim of aesthetic value and the aim of applicability in physics were united, and that they are best pursued together or at least each should be pursued with an awareness of the other.[5]

But Poincaré took a position opposite to that later expressed by Hardy in *A Mathematician's Apology* about the position of applied versus pure mathematicians: while Hardy found the imaginary universes of pure mathematics more beautiful than physical reality,[6] Poincaré averred that the superior aesthetic value of physical reality meant that the scientist would not be distracted from their quest: 'One may dream a harmonious world, but how far the real world will leave it behind!'[7]

Poincaré discussed both beauty and elegance in mathematics;[8] it is unclear whether this was a deliberate distinction between two kinds of aesthetic value. In a proof or solution, the perception of elegance was produced by

> 'the harmony of the different parts, their symmetry, and their happy adjustment; it is, in a word, all that introduces order, all that gives them unity, that enables us to obtain a clear comprehension of the whole as well as of the parts. [...] Elegance may result from the feeling of surprise caused by the unlooked-for occurrence together of objects not habitually associated.'[9]

For Poincaré, the feeling of elegance was the emotional response to some parallel between the solution before us and 'the necessities of our mind', and it was this parallel that helped one to *use* the solution: 'aesthetic satisfaction is consequently connected with the economy of thought'.[10] Furthermore, if mathematics was to be of service to physics,

[1] Poincaré, 'Analysis and Physics', § 1.

[2] Poincaré, *The Value of Science*, Preface.

[3] Poincaré, 'Analysis and Physics', § 1.

[4] *Loc. cit.*

[5] *Loc. cit.*

[6] Hardy, *Apology*, § 26.

[7] Poincaré, *The Value of Science*, Preface.

[8] Poincaré, 'The Future of Mathematics'; Poincaré, 'Mathematical Discovery'.

[9] Poincaré, 'The Future of Mathematics'.

[10] *Ibid.*

> ‘it *must* be cultivated in the broadest fashion without immediate expectation of utility — the mathematician must have worked as artist.’[1]

Thus Poincaré linked the aesthetic appeal of mathematics to its usefulness, at least via the indirect development of other mathematics that can then provide ‘an instrument for the study of nature’.

G. E. MOORE

The philosopher G. E. Moore (1873–1958) argued in his *Principia Ethica* (1903) that ‘goodness’ is indefinable,[2] and that to attempt to define it in terms of concepts such as ‘desire’ or ‘pleasure’ is to commit the *naturalistic fallacy*.[3] Moore held that the naturalistic fallacy could be committed in aesthetic as well as in ethical reasoning:[4] beauty, like goodness, is indefinable. Nevertheless, judgements of both beauty and goodness are objectively true or false. Furthermore, experiencing beauty is a fundamental good:

> ‘By far the most valuable things, which we know or can imagine, are certain states of consciousness, which may be roughly described as the pleasures of human intercourse and the enjoyment of beautiful objects.’[5]

But Moore also thought that when — as in what he saw as humanity’s current state — only a small part of the good is attainable, the pursuit of some equally-attainable greater good had to take precedence of the pursuit of beauty. If no greater good is attainable, then the pursuit of beauty became a duty.[6]

The *Principia Ethica* does not deal with mathematical beauty, but it formed part of the intellectual background against which Russell and Hardy wrote on this topic, and it would have lent credence to the idea that a good life can be devoted to the pursuit of beauty. In particular, the objective Moorean view of judgements of beauty would harmonize with Hardy’s and the early Russell’s platonist views of mathematics.

Hardy and Russell both knew Moore well: all three were at Trinity College, and all three were former Apostles[7]* — ‘Angels’, to use the argot of the society. Russell held Moore in very high esteem,[8] and read and reviewed the *Principia Ethica*, positively but not uncritically.[9]

It is uncertain whether Hardy actually read the *Principia Ethica*, but it is likely, given his association with Moore and with the Bloomsbury Group,[10] the literary circle for whom Moore and the *Principia Ethica* held great value.[11]

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[1] Poincaré, ‘Analysis and Physics’, § 11, emphasis added.

[2] Moore, *Principia Ethica*, §§ 6 sqq.

[3] *Ibid.*, §§ 10–14.

[4] *Ibid.*, § 121.

[5] *Ibid.*, § 113.

[6] *Ibid.*, § 50.

[7] Deacon, *The Cambridge Apostles*, pp. 201, 203–4.

* ▸ p. 435

[8] Russell, *Autobiography*, pp. 53–54, 60.

[9] Russell, Review of *Principia Ethica*; Russell, ‘The Meaning of Good’.

[10] Snow, ‘Foreword’, p. 25.

[11] Rosenbaum, *Edwardian Bloomsbury*, *passim*, esp. p. 3.

THE TWO RUSSELLS

A particular passage written by the philosopher Bertrand Russell (1872–1970) often appears in discussions of mathematical beauty:

> ‘Mathematics, rightly viewed, possesses not only truth, but supreme beauty — a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.’[1]

These words are from Russell’s 1907 essay ‘The Study of Mathematics’, written during the period in which he was focused on the philosophy of mathematics. His views on the aesthetics of mathematics changed later in life, due partly to Ludwig Wittgenstein’s (1889–1951) influence, and partly to the First World War shifting his intellectual focus towards social and ethical concerns.

Early Russell and ‘The Study of Mathematics’

As a child, Russell developed a passion for mathematics, beginning when his older brother introduced him to geometry;[2] this passion remained with him and, he claimed, at one point saved him from suicide.[3] His early interest in the applications of mathematics gradually shifted to an interest in the principles that form the foundations of mathematics, and led to a journey of increasing abstraction towards mathematical logic:

> ‘I came to think of mathematics [...] as an abstract edifice subsisting in a Platonic heaven and only reaching the world of sense in an impure and degraded form. [...] I disliked the real world and sought refuge in a timeless world’.[4]

The very purity, perfection, and permanence of mathematics was, for Russell, a source of beauty. At the end of 1901, he wrote to his friend Helen Thomas (1871–1956):

> ‘The world of mathematics [...] is really a beautiful world; it has nothing to do with life and death and human sordidness, but is eternal, cold and passionless. To me, pure mathematics is one of the highest forms of art; it has a sublimity quite special to itself, and an immense dignity derived from the fact that its world is exempt from change and time. [...] And mathematics is the only thing we know of that is capable of perfection; in thinking about it we become Gods.’[5]

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[1] Russell, ‘The Study of Mathematics’, p. 60.

[2] Russell, *Autobiography*, p. 25; Russell, ‘My Mental Development’, p. 12.

[3] Russell, *Autobiography*, p. 32.

[4] Russell, *My Philosophical Development*, pp. 209–10.

[5] Russell, letter to Thomas, dated 30 Dec. 1901, repr. in Russell, *The Selected Letters*, vol. 1, p. 218 [ltr 98], emphasis in orig.

There are clear resonances in this letter with what Russell would later write in the passage previously quoted from 'The Study of Mathematics'. In particular, 'the sense of being more than man' is an attentuated expression of the same idea that lies behind 'we become Gods'.

In a 'Platonic eternal world, [...] mathematics shone with a beauty like that of the last Cantos of the *Paradiso*.'[1] His distaste for the real world and its 'human sordidness' formed part of an ascetic outlook that he held in the early years of the twentieth century, perhaps most clearly expressed in his 1903 essay 'A Free Man's Worship':

> 'To abandon the struggle for private happiness, to expel all eagerness of temporary desire, to burn with passion for eternal things — this is emancipation, and this is the free man's worship.'[2]

The work in which Russell most clearly discussed mathematical beauty was his essay 'The Study of Mathematics'. It was published in 1907 in *The New Quarterly*, a journal edited by Desmond MacCarthy (1877–1952), a member of the Bloomsbury Group. The essay is not specifically about the aesthetics of mathematics, but rather the motives for studying mathematics. Indeed, although Russell's papers and correspondence contain many references to aesthetics, he never published a book or paper specifically on the subject; it was '[j]ust about the only traditional branch of philosophy he did not write on'.[3] Even among works never published during his lifetime, only an essay read to the Apostles focused on an aesthetic theme. In this paper, Russell argued that beauty was not an intrinsic good, but was still an objective quality.[4]

Russell had doubts about aesthetics as a field of scholarly study: in private correspondence he wrote 'I feel sure learned aesthetics is rubbish [...] it ought to be a matter of literature and taste rather than science'.[5] Essays he wrote as a student sometimes considered aesthetics, but never engaged with the canonical works in the field.[6]

Russell felt himself to be deficient in the perception of beauty in painting and sculpture, although he enjoyed music, literature, and architecture.[7] His lack of aesthetic appreciation for the visual arts may have been connected to his thinking entirely in terms of words and language, not in pictures: he annotated a discussion of the 'visual mind' and 'auditory mind' in William James's *Principles of Psychology* with: 'Mine is the auditory. I never think except in words which I imagine spoken.'[8] Alan Wood (1914–57), Russell's biographer, agreed with this self-assessment.[9] Russell also wrote to his friend and lover Ottoline Morrell (1873–1938) that he had no 'power of *creating* beauty except to some extent in words'.[10]

Therefore, it is odd that Russell compared the beauty of mathematics, which he enjoyed, to that of sculpture, which he specifically said that he lacked the ability to appreciate, and contrasted it with music, which he was able to appreciate. [The earlier quotation in which he compared the beauty of mathematics to that of the *Paradiso* is from

[1] Russell, *Autobiography*, p. 701.

[2] Russell, 'A Free Man's Worship', p. 44.

[3] Slater, intro. to Russell, *Basic Writings*, p. ix.

[4] Russell, 'Was the World Good Before the Sixth Day?'; Spadoni, 'Bertrand Russell on Aesthetics', pp. 67–9.

[5] Russell, letter to Donnelly, dated 19 Oct. 1913, quot. in Slater, intro. to Russell, *Basic Writings*, p. ix.

[6] Spadoni, 'Bertrand Russell on Aesthetics', p. 56.

[7] Spadoni, 'Bertrand Russell on Aesthetics', esp. pp. 50, 70 and n. 6; Moran, 'Bertrand Russell meets his muse', p. 189.

[8] Spadoni, 'Bertrand Russell on Aesthetics', p. 57.

[9] *Ibid.*, p. 58.

[10] Russell, letter to Morrell, dated 27 Apr. 1911, quot. in Moran, 'Bertrand Russell meets his muse', p. 189.

his autobiography, written six decades after 'The Study of Mathematics'.] Thus Russell's comparison of beauty in mathematics and in sculpture cannot directly reflect his own aesthetic response to sculpture, but rather what he imagined the aesthetic response of others would be. In his autobiography, he wrote how his future brother-in-law Logan Pearsall Smith (1865–1946) 'indoctrinated him' about art and he 'learned the right thing to say'[1] about artists, which suggests that he lacked real understanding or appreciation of the art to which Smith introduced him.[2] The comparison with sculptural beauty may therefore reflect the attitudes he absorbed from Smith.

But none of this detracts from his description of the beauty of mathematics as 'cold and austere' and 'without appeal to any part of our weaker nature'; these qualities clearly fit with Russell's ascetic perspective at this time. If mathematics showed the way to a higher Platonic realm that was better than the real world, then its beauty had to be entirely intellectual and rational, not emotional. But the beauty being intellectual did not imply that mathematics research was intentionally directed toward beauty. Almost no mathematicians consciously sought beauty: when it was found, it was usually discovered by instinct, which, alas, was not always reliable.[3]

Russell analysed the aesthetic value both of proofs and of theories. The most beautiful mathematical proofs were such that the

> 'chain of argument is presented in which every link is important on its own account, in which there is an air of ease and lucidity throughout, and the premisses achieve more than would have been thought possible, by means which appear natural and inevitable.'[4]

Thus each part of the argument had to be fitting; it could not serve only a mechanical purpose of proving the result: 'for æsthetic perfection no part of the whole should be merely a means'.[5] The same was true for the development of a theory in a textbook. Every step had to be motivated by what preceded it; there should be no unnatural diversions from the line of reasoning. Just as in literature the plot had to develop unforcedly from what has gone before:

> 'unity and inevitability are felt as in the unfolding of a drama; in the premisses a subject is proposed for consideration, and in every subsequent step some definite advance is made towards mastery of its nature.'[6]

Further, a proof should be appreciated as an end in itself, for '[a]n argument which serves only to prove a conclusion is like a story subordinated to some moral which it is meant to teach'.[7]

For Russell, elegance referred to a particular aspect of proofs: 'elegance results from employing only the essential principles in virtue of which the thesis is true';[8] there was a 'defect' in an argument that employed superfluous premisses. Of course, this defect might not

[1] Russell, *Autobiography*, p. 69.

[2] Spadoni, 'Bertrand Russell on Aesthetics', p. 54.

[3] Russell, 'The Study of Mathematics', p. 61.

[4] *Loc. cit.*

[5] *Ibid.*, p. 70.

[6] *Ibid.*, p. 66.

[7] *Ibid.*, p. 70.

[8] *Loc. cit.*

be obvious from the proof itself: a proof could make use of all its premisses, while a different strategy might yield the conclusion from a smaller set of premisses. This part of the text suggests, but does not make explicit, that Russell viewed such a ‘defect’ as an aesthetic blemish, for the relevant paragraph is concerned with instructing students, and begins by recommending that students be persuaded of a result’s correctness ‘in the way which itself has, of all possible ways, the most beauty’.[1] Alfred North Whitehead (1861–1947), who was Russell’s co-author for the *Principia Mathematica*, explicitly wrote that unnecessary premisses ‘diminish our aesthetic pleasure in the mathematical reasoning’.[2] [Whitehead presumably referred to a distaste for *redundancy* among axiom sets, rather than a preference for a *minimum number* of axioms. For instance, groups can be characterized as sets with a binary operation / subject to

$$\left(x\Big/\left(((x/x)/y)/z\right)\right)\Big/\left(((x/x)/x)/z\right) = y;$$

the operation corresponds to right division.[3] But there seems little aesthetic value in this single axiom, and it certainly seems to have less than the standard group axioms.]

This concept of elegance applied to the whole of mathematics:

> ‘The discovery that all mathematics follows inevitably from a small collection of fundamental laws is one which immeasurably enhances the intellectual beauty of the whole; to those who have been oppressed by the fragmentary and incomplete nature of most existing chains of deduction this discovery comes with all the overwhelming force of a revelation; like a palace emerging from the autumn mist as the traveller ascends an Italian hill-side, the stately storeys of the mathematical edifice appear in their due order and proportion, with a new perfection in every part.’[4]

The evident enthusiasm in the preceding quotation must be tempered with Russell’s later account of the writing of the *Principia Mathematica*, to which, since it was a project to build mathematics upon a small number of logical axioms, the quotation must be relevant. Russell recalled how he often spent entire days staring at a blank sheet of paper, trying to think of solutions to the perplexities of the task.[5] The tone of ‘The Study of Mathematics’, and in particular of the quoted passage, may be partly ascribed to its being written in 1907, after the breakthrough of Russell’s discovery of the theory of types in 1906, but early in the subsequent toil of writing out the book, on which he worked for ‘ten to twelve hours a day for about eight months in the year, from 1907 to 1910’.[6] This conclusion is supported by the contrast between his attitude to the work when it was in its early stages in 1907 and what he thought in 1909, with the manuscript ready to go to the printers. In 1907, he wrote: ‘I work 9 or 10 hours most days [...]. But I

[1] Russell, ‘The Study of Mathematics’, p. 70.

[2] Whitehead, *Science and the Modern World*, p. 30.

[3] Higman & Neumann, ‘Groups as groupoids with one law’.

[4] Russell, ‘The Study of Mathematics’, pp. 67–8.

[5] Russell, *Autobiography*, p. 142.

[6] *Ibid.*, pp. 142–3.

get a great deal of pleasure out of my work';[1] in 1909, 'now I feel more or less as people feel at the death of an ill-tempered invalid whom they have nursed and hated for years'.[2] Toward the end of his life, recalling this period, he wrote that

> 'the pleasure to be derived from the writing of *Principia Mathematica* was all crammed into the latter months of 1900. After that time the difficulty and the labour were too great for any pleasure to be possible. [...] the only really vivid delight connected with the whole matter was that which I felt in handing over the manuscript to the Cambridge University Press.'[3]

The difficulty was such that, for the rest of his life, Russell felt that his 'intellect never quite recovered from the strain'.[4] Furthermore, Russell explicitly wrote that in order to resolve the contradictions that had arisen, he was forced to choose theories 'which might be true but were not beautiful' and that the 'splendid certainty' which he sought 'was lost in a bewildering maze'.[5] Nevertheless, after the *Principia Mathematica* was complete, Russell still had — or had perhaps rekindled — an aesthetic appreciation for the work, because in March 1912 he wrote about receiving a visit from Ludwig Wittgenstein (1889–1951), then a student, but one whose abilities Russell recognized:

> 'I think he has *genius*. [...] He is the ideal pupil — he gives passionate admiration with vehement and very intelligent dissent. He spoke with intense feeling about the *beauty* of the big book, said he found it like music. That is how I feel it, but few others seem to.'[6]

For Russell, mathematics was motivated by a search for beauty. He divided motivations into two groups, *possessive* and *creative*. The possessive impulse aimed to acquiring sole possession of something; the creative impulse tried to give something valuable to the world. He considered 'the best life that which is most built on creative impulses, and the worst that which is most inspired by love of possession.'[7] Since new knowledge was certainly valuable, the drive toward mathematical discovery was clearly a creative impulse. Beauty was an important motivation in pursuing mathematics. In particular, in pure mathematics one was not limited to what was applicable to the world: one can give free rein to

> 'reason's privilege of dealing with whatever objects its love of beauty may cause to seem worthy of consideration.'[8]

For the mathematician, the idea that the mathematics they discover might some day be useful could be a comfort in times of doubt. It could not, however, be a guide in their research: for example, the study of conic sections in antiquity was pursued without any glimmering that eighteen centuries later they would be used by Johannes Kepler* (1571–1630) in formulating his laws of planetary motion.[9] However,

[1] Russell, letter to Pretious, dated 25 Dec. 1907, repr. in Russell, *The Selected Letters*, vol. 1, p. 316 [ltr 146].

[2] Russell, letter to Donnelly, dated 18 Oct. 1909, repr. in *ibid.*, vol. 1, p. 326 [ltr 152].

[3] Russell, *Autobiography*, p. 146.

[4] *Ibid.*, p. 143.

[5] Russell, *My Philosophical Development*, p. 212.

[6] Russell, letter to Morrell, dated 18 Mar. 1912, repr. in Russell, *The Selected Letters*, vol. 1, p. 418 [ltr 186], emphases in orig.

[7] Russell, *Principles of Social Reconstruction*, p. 5.

[8] Russell, 'The Study of Mathematics', p. 70.

* ▸ pp. 279–94

[9] *Ibid.*, p. 72.

the study of mathematics did have a worthwhile effect in helping to inculcate a certain way of thinking:

> 'Every great study is not only an end in itself, but also a means of creating and sustaining a lofty habit of mind; and this purpose should be kept always in view throughout the teaching and learning of mathematics.'[1]

Here, perhaps, is a shade of Plato's prescription of mathematical education for the rulers in the *Republic*.[2][*]

Aside from applicability, beauty was not the only motivation for mathematics: on an individual level, mathematics could serve as a refuge from the troubles of the world, whither 'our nobler impulses can escape'.[3] Using mathematics as a refuge does not necessarily mean a complete devotion to the subject: one could visit the sanctuary without taking the tonsure. From personal experience, Russell knew how valuable the refuge was: 'Mathematics is a haven of peace without which I don't know how I should get on.'[4]

However, Russell did wonder whether it was ethical to devote oneself to mathematics, guided by 'love of beauty':

> 'In a world so full of evil and suffering, retirement into the cloister of contemplation, to the enjoyment of delights which, however noble, must always be for the few only, cannot but appear as a somewhat selfish refusal to share the burden imposed upon others by accidents.'[5]

His answer was twofold: 'some must keep alive the sacred fire',[6] and the fact, already discussed, that there was no way of telling in advance which parts of mathematics would prove useful.

But what does 'the few' imply here? Russell could have meant that few people are capable of doing or appreciating mathematics, or that, while many people are capable of it, few will be granted the opportunity of retreating to such contemplation. This passage yields no firm conclusion on this point, but the first possibility is more likely, since Russell wrote to Thomas about the world of mathematics in the 1901 letter quoted above[†] that '[t]he only difficulty is that none but mathematicians can enter this enchanted region, and they hardly ever have a sense of beauty.'[7] This statement suggests that 'the few' was restricted not just to mathematicians, but to a *minority* of mathematicians.

Russell's defence of mathematics stood on the cusp between individual and social. On the one hand, pursuing the mental states that beauty creates — '[t]he true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence'[8] — and taking refuge from the world in mathematics are individual motivations. On the other hand, the unknowable potential utility of each piece of mathematics is ultimately an appeal to a social end. Beauty can be both an individual and a social end, the latter because the creative impulse, for Russell, meant giving something to

[1] Russell, 'The Study of Mathematics', p. 73.

[2] See Burnyeat, 'Plato on Why Mathematics is Good for the Soul'.

[*] ▸ pp. 75–6

[3] Russell, 'The Study of Mathematics', p. 70.

[4] Russell, letter to Donnelly, dated 18 Feb. 1906, repr. in Russell, *Autobiography*, p. 175.

[5] Russell, 'The Study of Mathematics', p. 72.

[6] *Loc. cit.*

[†] ▸ pp. 479–80

[7] Russell, letter to Thomas, dated 30 Dec. 1901, repr. in Russell, *The Selected Letters*, vol. 1, p. 218 [ltr 98].

[8] Russell, 'The Study of Mathematics', p. 60.

the world. Even maintaining the 'sacred fire' is ultimately a social end, albeit a rather mystical one.

Later Russell and 'The Retreat from Pythagoras'

Always unashamed to change his mind,[1] Russell came to abandon most of these views. Although he still perceived beauty in mathematics, he rejected the idea that one could ethically devote one's life to pure mathematics, motivated by beauty and not utility. In *My Philosophical Development*, written in 1959, he described his 'gradual retreat from Pythagoras':[2] he withdrew from a mystical reverence for mathematics, eloquently expressed in 'The Study of Mathematics', to a view that mathematics consists only of tautologies and that it has no mystical aspect. He quoted extensively his early views in *My Philosophical Development*,[3] and dismissed them as 'largely nonsense', though he admitted that '[t]he aesthetic pleasure to be derived from an elegant piece of mathematical reasoning remains'.[4]

The main impetus for this shift in Russell's attitude was the outbreak of the First World War, 'the seminal event in Russell's life'.[5] The war — the millions of young men sent to their deaths — made it impossible for Russell to 'to go on living in a world of abstraction'.[6] He ceased to believe that a life could be devoted to mathematical research guided only by aesthetic criteria: 'The non-human world remained as an occasional refuge, but not as a country in which to build one's permanent habitation.'[7] He thought contemplation a good thing, but it was not enough for a good *life*. Contemplation had to lead to practice: 'While it remains secluded in the cloister it is only a means of escape.'[8]

But Russell wrote *My Philosophical Development* more than four decades after the end of the First World War, and may have recalled his change of mind as having been faster than it actually was. Although he declared in an April 1911 letter that 'mathematics is a cold and unresponsive love',[9] this seeming lament must be contrasted with what he wrote in another letter almost a year later:

> 'I like mathematics largely because it is *not* human and has nothing particular to do with this planet or with the whole accidental universe — because, like Spinoza's God, it won't love us in return.'[10]

And Virginia Woolf (1882–1941) recorded in her diary that Russell said in December 1921 that

> 'God does mathematics. That's my feeling. It is the most exalted form of art.
>
> […]
>
> Well theres [*sic*] style in mathematics as there is in writing. I get the keenest aesthetic pleasure from reading well written mathematics.'[11]

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[1] See Russell, *Dictionary of Mind, Matter and Morals*, Preface.

[2] Russell, *My Philosophical Development*, p. 208.

[3] *Ibid.*, p. 211.

[4] *Ibid.*, p. 212.

[5] Potter, *Bertrand Russell's Ethics*, p. 10.

[6] Russell, *My Philosophical Development*, p. 212.

[7] *Loc. cit.*

[8] Russell, 'If we are to survive this dark time—', p. 665.

[9] Russell, letter to Morrell, dated 3 Apr. 1911, repr. in Russell, *The Selected Letters*, vol. 1, p. 357 [ltr 163].

[10] Russell, letter to Morrell, dated 9 Mar. 1912, repr. in *ibid.*, vol. 1, p. 417 [ltr 185], emphasis in orig.

[11] Woolf, *Diary*, vol. 2, p. 147 (3 Dec. 1921).

But the context may indicate that he was referring to the style of mathematical exposition, not to the beauty of mathematics itself, especially since Woolf reported that he continued by noting that 'Lord Kelvin's style was abominable'.[1]

[Therefore Russell's remark that 'mathematics has the advantage of teaching you the habit of thinking without passion. […] You learn to use your mind primarily upon material where passion doesn't come in'[2] should be interpreted as a statement about the content of mathematics, not about the emotional response to that mathematics.]

In 1926, in justifying the role of universities as a venue for disinterested research, Russell wrote:

> 'If science and organization had succeeded in satisfying the needs of the body and in abolishing cruelty and war, the pursuit of knowledge and beauty would remain to exercise our love of strenuous creation. I should not wish the poet, the painter, the composer, or the mathematician to be preoccupied with some remote effect of his activities in the world of practice. He should be occupied, rather, in the pursuit of a vision, in capturing and giving permanence to something which he has first seen dimly for a moment which he has loved with such ardour that the joys of this world have grown pale by comparison. All great art and all great science springs from the passionate desire to embody what was at first an unsubstantial phantom, a beckoning beauty luring men away from safety and ease to a glorious torment. The men in whom this passion exists must not be fettered by the shackles of a utilitarian philosophy, for to their ardour we owe all that makes man great.'[3]

But this passage, which in isolation might resemble his earlier attitude in 'The Study of Mathematics', in context actually presented a utilitarian motivation for mathematics and science. The pursuit of knowledge for utilitarian ends was not 'self-sustaining': theoretical knowledge comes first; practical applications later. Thus, in order to achieve those applications, it was necessary to allow research for its own sake.[4] Similarly, although he made a passing remark in his 1935 essay '"Useless" knowledge' about Thomas Hobbes's (1588–1679) reaction to Euclid's *Elements* being a 'voluptuous moment, *unsullied* by the thought of the utility of geometry',[5] the essay as a whole points out practical and psychological benefits of education in 'useless' subjects, notably in promoting the habit of contemplation. Such an education not only increased the enjoyments and mitigated the vexations of life, but tended towards the utilitarian end of better choices of action.[6]

And Russell's turn away from mathematics and its philosophy had begun long before the lamps went out all over Europe. He and Ottoline Morrell, whom he had known slightly since childhood, got to know each other better during 1910 and became lovers in March 1911. He credited Morrell with broadening his interests and loosening his asceticism. In particular, she made him realize that he was capable of non-mathematical aesthetic appreciation, and that such appreciation was important.[1] In his autobiography, he recalled that he experienced the period from 1910 to 1914 as 'a process of rejuvenation, inaugurated by Ottoline Morrell and continued by the War', in the sense that he gained a new outlook and purpose.[2]

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[1] Woolf, *Diary*, vol. 2, p. 147 (3 Dec. 1921).

[2] Russell, Buchanan & Van Doren, 'Spinoza', pp. 113–4.

[3] Russell, *On Education*, pp. 243–4.

[4] *Ibid.*, p. 243.

[5] Russell, '"Useless" knowledge', p. 17, emphasis added.

[6] *Ibid.*, esp. pp. 24–7.

However, neither the war-wrought change in his attitude towards devotion to mathematics nor his expanded aesthetic appreciation actually undermined Russell's earlier views on mathematical beauty or his enjoyment of that beauty. As already noted, he could still obtain aesthetic pleasure from mathematics. Rather, his philosophical views on both mathematics and aesthetics were fundamentally altered by the influence of Wittgenstein and George Santayana (1863–1952).

Wittgenstein's work led Russell to regard all mathematics as tautological;[3] that 'to a mind of sufficient intellectual power, the whole of mathematics would appear trivial, as trivial as the statement that a four-footed animal is an animal'.[4] [Wittgenstein himself did not believe all mathematics was tautological.[5]] Thus any enjoyment of the beauty of a theorem, a proof, or a theory would be tempered by the awareness that the knowledge or reasoning was trivial, or at least would be trivial to a higher intellect. Thus, while the earlier Russell had gained pleasure in the unity demonstrated by all of mathematics following from a few assumptions, the later Russell dismissed the preference for small sets of premisses as '*merely* aesthetic'.[6] [The mathematician and historian of mathematics Morris Kline made a cutting dismissal of this attitude: 'It is literally correct to describe the logical structure of mathematics as tautology, but this statement is about as adequate as saying that Venus de Milo is just a big girl.'[7] In Kline's view, the 'all is tautology' attitude omitted everything of value.]

Santayana's *Winds of Doctrine*, which Russell held in high regard[8] but which scorned his moral philosophy, caused him to abandon his Moorean views on the objectivity of moral and aesthetic judgements.[9]

Thus not only had Wittgenstein led Russell to believe that there was no non-trivial mathematical knowledge, but now Santayana had caused him to suspect that the beauty he saw in mathematics was just as illusory as the knowledge he once thought it contained. The perceived beauty was not objective, just as the perceived knowledge was not absolute, but simply a consequence of our limited minds.

Where once the younger Russell desired 'to burn with passion for eternal things',[10]* the older Russell preferred 'to care for what is noble, for what is beautiful, for what is gentle'.[11] The older Russell's love of beauty was simpler, more moderate and (perhaps) more humane.

[1] Spadoni, 'Bertrand Russell on Aesthetics', § 7.

[2] Russell, *Autobiography*, p. 225.

[3] Russell, 'My Mental Development', p. 21.

[4] Russell, *My Philosophical Development*, pp. 211–2.

[5] Potter, 'The Logic of the *Tractatus*', § 6.6.

[6] Russell, *An Inquiry into Meaning and Truth*, p. 132, emphasis added.

[7] Kline, *Mathematics in Western Culture*, p. 459.

[8] Russell, letter to Dickinson, dated 13 Feb. 1913, repr. in Russell, *Autobiography*, pp. 215–6.

[9] Russell, *Portraits From Memory*, p. 96; Ruja, 'Russell on the Meaning of "Good"', p. 137; Spadoni, 'Bertrand Russell on Aesthetics', p. 74.

[10] Russell, 'A Free Man's Worship', p. 44.

* ▸ p. 480

[11] Russell, *Autobiography*, p. 702.

WAR AND AFTERMATH

J. W. N. Sullivan

Precisely because the First World War was the first industrial war and witnessed the use of new types of weapons, such as aircraft and chemical weapons, science began to be assigned a higher value by society in the Interbellum. The popular science writer J. W. N. Sullivan (1886–1937) noted that mathematics also benefitted from this tendency.[1] But he queried the grounds on which mathematics was valued: anything which society values was 'either useful or pleasant, or both'.[2] To use his examples, agriculture was valued because it was essential, music because it was pleasant. Society seemed to value mathematics because of its applications: that is, because it is useful.

Yet, as Sullivan pointed out, society's view was opposed to the main justification mathematicians offered for their own field: *viz.*, mathematics was an aesthetic pursuit. He saw that 'mathematics is an entirely free activity' and thus 'it is more just to call it an art than a science',[3] and that 'the mathematicians are impelled by the same incentives and experience the same satisfactions as other artists';[4] he further accepted the assertion of mathematicians that their field was a 'delightful one'[5] *for them*. However, he held that this delight was insufficient justification: there was no reason that society should support and admire mathematicians when the aesthetic pleasure they gain is accessible to so few.[6][*] For example, should not chess be supported by society in preference to mathematics? Chess professorships could be established, since 'there are probably more people who appreciate the "beauties" of chess than appreciate the beauties of mathematics.'[7]

In the end, Sullivan did elevate mathematics above chess and thus justified it, because mathematics was an art (and, by implication, chess was not):

> 'Art which is worthy of the name reveals to us some aspect of reality. [...]
>
> [...] the real function of art is to increase our self-consciousness; to make us more aware of what we are, and therefore of what the universe in which we live really is. And since mathematics, in its own way, also performs this function, it is not only æsthetically charming but profoundly significant. It is an art, and a great art. It is on this, besides its usefulness in practical life, that its claim to esteem must be based.'[8]

Thus while mathematicians may have a mainly aesthetic motivation for their work, the resulting mathematics, just like art, was nevertheless a social good. Thus Sullivan's arguments were another step away from a purely individualistic justification of mathematics.

Return for a moment to the key point Sullivan made in opposition to the idea that society should support mathematics as an aesthetic pursuit: if society wished to support activities that produced beauty, then other activities should have priority over mathematics, since few are able to appreciate its beauties. Arguments have been made that confront both Sullivan's inference and his premiss. Poincaré admitted that only a few can fully enjoy the aesthetic delights of mathematics, but said this does not distinguish it from other high arts.[1] Poincaré's position was echoed in 1958, but from the other direction, by Milton Babbitt (1916–2011), a composer who was trained in mathematics, in his celebrated essay 'Who Cares if You Listen?' Babbitt argued that the development of music, like that of mathematics, science, and philosophy, had passed beyond what can be appreciated by the non-specialist.[2] Modern composers would benefit themselves and their music if they did not continue to aim at a public audience,[3] and instead pursued 'a private life of professional achievement'; it was for the university to 'provide a home for the "complex," "difficult," and "problematical" in music'. Poincaré's and Babbitt's views therefore imply that the creation of something of aesthetic value, even if it was only accessible to a minority, was a valid justification for mathematics.

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[1] Sullivan, 'Mathematics as an Art', p. 2015.

[2] *Loc. cit.*

[3] *Ibid.*, p. 2020.

[4] *Loc. cit.*

[5] *Ibid.*, p. 2015.

[6] *Loc. cit.*

[*] ▸ pp. 574–600

[7] *Loc. cit.*

[8] *Ibid.*, p. 2021.

Other scholars opposed Sullivan's premiss. Alfred North Whitehead, for instance, condemned the idea that only 'a few eccentrics in each generation' loved mathematics as 'an erroneous literary tradition'.[4] G. H. Hardy agreed, though he misquoted Whitehead as referring to a 'literary superstition'.[5]

The astrophysicist E. A. Milne (1896–1950), in a lecture to the Cambridge University Natural Science Club on 6 February 1922, said that dead-ends in mathematical physics were still worth pursuing *if* they had beauty, for then they had intrinsic value:

> 'If now a problem in say physics leads to a piece of beautiful work in mathematical physics which yet clearly leads nowhere, is it for that reason not worth the doing? [...] I assert that it is worth the doing. It is a work of art. It is something of value in itself.'[6]

D. E. Smith

David Eugene Smith (1860–1944), a mathematician who wrote several books on the history and teaching of mathematics, also considered mathematics to be a creative and an aesthetic pursuit:

> 'When we say that "poetry" is "unfallen speech" we may properly add the dual proposition, "Pure geometry is undebased thought"; and when we say that "architecture is frozen music" we may add that it is also crystallized geometry. There are no cornerstones that mark off the domain of mathematics from the domains of music and poetry and architecture and the fine arts in general'.[7]

[1] Poincaré, 'Analysis and Physics', § 1.

[2] Babbitt, 'Who Cares if You Listen?', p. 40.

[3] *Ibid.*, p. 126.

[4] Whitehead, *Science and the Modern World*, p. 26.

[5] Hardy, *Apology*, § 10.

[6] Milne, 'The relations of mathematics to science', p. 57.

[7] Smith, 'The Poetry of Mathematics', p. 296.

Smith also took up a theme that recurs later in Hardy's consideration of mathematics and war. Modern warfare depended to some degree on mathematics, for matching shells to guns and in the working of rangefinders.[1] To this extent, harm could be attributed to mathematics. Conversely, all manner of commercial, industrial, and engineering practices were also dependent on mathematics, and so the idea of society functioning without mathematics was ridiculous.[2] However, only a basic level of mathematics was required by the educated citizen:

> 'He will need to know certain parts of economics and certain business customs which find place in arithmetic [...]; how to evaluate such simple formulas as are found in books on the radio, on nursing, and on the elementary trades [...]; he will need to know how to interpret a graph, and even how to make simple graphs relating to his business'.[3]

However, in general the citizen would not have to deduce formulae or make detailed statistical analyses.[4] Yet Smith also thought that mathematics was a necessary part of education, because the lessons learned in mathematics, including the appreciation of beauty, could be applied to other areas:

> 'If, for example, the knowledge of how to arrange a logical proof in geometry can be made of no value to us in other fields in which deductive logic can be applied; if the perfection of geometry does not give us an ideal of perfection that helps us elsewhere in our intellectual life; if the succinctness of statement of a geometric proof does not set a norm for statements in non-mathematical lines; if the contact with absolute truth does not have its influence upon the souls of us; if the very style of reasoning does not transfer so as to help the jurist, the physician, the salesman, the publicist, and the educator; if the habit of rigorous thinking, which usually begins first in demonstrative geometry, is not a valuable habit elsewhere; if a love for beauty cannot be cultivated in geometry so as to carry over to stimulate a love for beauty in architecture, — then let us drop demonstrative geometry from our required courses.'[5]

Smith took this argument to the extent of an almost platonic defence of mathematical education in shaping character: 'We teach it because it is one of the eternal verities, — one of those disciplines that arouse youth to a contemplation of the truths that endure.'[6] The mission and the duty of the mathematician was therefore to develop mathematics further, just as others had to develop the arts, sciences, letters so that each generation left the world a better place:[7]

> 'The call of mathematics is, then, to our physical well-being, and this is always recognized; but it is also to our spiritual well-being, and this we must not fail to recognize if our labors are not to be in vain.'[8]

[1] Smith, 'The Call of Mathematics', p. 283.

[2] *Ibid.*, pp. 283–284.

[3] *Ibid.*, pp. 284–5.

[4] *Loc. cit.*

[5] *Ibid.*, pp. 286–7.

[6] *Ibid.*, p. 287.

[7] *Ibid.*, p. 288.

[8] *Loc. cit.*

Thus, like Sullivan before him, Smith argued that the social value of mathematics goes beyond its immense practical applications. The social value extends to the effects of mathematics, including aesthetic and ethical lessons, in general education, and though this to the social benefits of a well-educated populace.

Norbert Wiener

In 1929, when he was a professor at the Massachusetts Institute of Technology, the mathematician and philosopher Norbert Wiener (1894–1964), remembered as the father of cybernetics, published an essay in the MIT magazine *The Technology Review* entitled 'Mathematics and Art', in which he held that mathematics was '*in the strictest sense of the word*, a fine art'.[1] His arguments, which were to an extent recapitulated in his later autobiography,[2] were grounded upon aspects of motivation, creativity, and contemplation.[3]

For Wiener, the choice of what mathematics to pursue was chiefly an aesthetic one,[4] which is illustrated by a decision early in his mathematical career. In 1920, he worked with Maurice Fréchet (1878–1973) on what would later be called Banach spaces. He abandoned the subject because it was too abstract to give him 'the highest esthetic satisfaction', and had no regrets, for he thought that a mathematician should devote their limited time to a subject that would grant 'the greatest inner satisfaction'.[5] Only when he came to write his second volume of autobiography, published in 1956, did he consider that a field that he had once found unsatisfying 'in an aesthetic rather than in any strictly logical sense' had developed sufficiently and acquired enough interesting theorems to start to be appealing.[6]

In his 1929 essay, Wiener started by arguing that mathematics could produce a genuine aesthetic response. He said that 'the thrill of delight' that he experienced in contemplating an outstanding painting, building, or dramatic tragedy was the same, in terms of both quality and intensity, as that produced by contemplating a mathematical theory that draws together apparently disparate ideas.[7] There was, at most, a difference of degree. He imagined arranging the arts in terms of their 'warmth' or 'coldness'. Architecture, for instance, was colder than sculpture, which was colder than painting. Mathematics was slightly colder than arabesques, whose 'fragile daintiness' he compared to much of the theories of algebraic geometry, numbers, and finite groups.[8] 'Arabesques' is a broad term for scrolling and interlacing decorative patterns, and Wiener may have meant specifically those that can be found in Islamic geometrical art,[*] for he mentioned them bringing 'glory to Moslem architecture'.[9] (Let it be emphasized that, as the term is commonly used, *a particular kind* of arabesque constitutes *a particular kind* of Islamic geometrical art. Neither subsumes the other.) In placing mathematics with the colder arts — those characterized by grandeur rather than lyric beauty[10] — Wiener was here doubtless influenced by Russell, under whom he had studied[1] and whose famous 'cold and austere' characterization* was quoted in Havelock Ellis's (1859–1939) book *The Dance of Life*, which argued that all human life had been and continued to be an artistic progress, and with which Wiener declared himself 'in complete accord'.[2]

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[1] Wiener, 'Mathematics and Art', p. 129, emphasis added.

[2] Wiener, *I Am a Mathematician*, pp. 60–3.

[3] Wiener, 'Mathematics and Art', pp. 129, 162.

[4] Wiener, *I Am a Mathematician*, p. 60.

[5] Wiener, *Ex-Prodigy*, p. 281.

[6] Wiener, *I Am a Mathematician*, p. 63.

[7] Wiener, 'Mathematics and Art', p. 130.

[8] *Loc. cit.*

[*] ▸ pp. 217–32

[9] *Loc. cit.*

[10] *Loc. cit.*

That mathematics is constrained by rigour was no disqualification from being a fine art, for great sculpture and architecture and drama were created within the strictures of canons.[3] A poet was not 'less free because his language has a grammar or his sonnets a form'.[4] Natural beauty could not be created by humans, and Wiener thought that any human art could only compare by overcoming obstacles. The demands of rigour were simply the obstacles placed in the mathematician's path.[5]

Just as a work of representational art could be beautiful both in itself and as a depiction, so mathematics 'besides the beauty of inner structure, has a further beauty as a representation of reality'.[6] Although the pure mathematician was an artist — a free creator — and the applied mathematician an artisan — a worker on an externally supplied problem[7] — this dual beauty arose not only in mathematical physics, but also in pure mathematics, for, even in pure mathematics, mathematical physics still often served as an unconscious guide.[8]

One might rationalize the motivating thought of an artist as imagining a thing as beautiful and deciding to show it to the world.[9] And one might rationalize the thought of the mathematician in the same way. But both rationalizations are false: 'it is not a matter of the artist laboring to express the emotion to the public but rather of the emotion endeavoring to express the artist'.[10] Wiener thought that an artist felt that the work was independent and was produced *through* them.[11] A mathematician felt similarly directed: 'the research demands to be done'.[12] And the mathematician was filled with the same aesthetic impulse:

> 'All through his moments and highest creative effort there is the same exaltation of contact with beauty that inspires his colleagues in the fine arts in the narrower sense.'[13]

For Wiener, the aesthetic motivation defined alike the scope of the mathematician's and the artist's work. Mathematicians of the highest class, like artists of the highest class, did not deign to restrict their interests to any one subject. For them, 'the world is rich in beauties of order' that clamoured to be expressed.[14]

Wolfgang Krull

In 1930, Wolfgang Krull (1899–1971) dedicated his inaugural lecture at the University of Erlangen to explaining his own personal view that the imagination and creativity of the mathematician and the artist are closely related,[15] although his aim and argument were rather different from those of the authors discussed

[1] Wiener, *Ex-Prodigy*, ch. XIV.

* ▸ p. 479

[2] Ellis, *The Dance of Life*, p. 129.

[3] Wiener, 'Mathematics and Art', p. 131.

[4] Wiener, *I Am a Mathematician*, p. 62.

[5] Wiener, 'Mathematics and Art', p. 131.

[6] *Ibid.*, p. 162.

[7] *Ibid.*, p. 131.

[8] *Ibid.*, p. 162.

[9] *Ibid.*, p. 130.

[10] *Loc. cit.*

[11] *Loc. cit.*

[12] *Ibid.*, p. 131.

[13] *Loc. cit.*

[14] *Loc. cit.*

[15] Krull, 'The aesthetic viewpoint in mathematics', pp. 48, 49.

previously. The most important difference was that he did not offer an explicit defence of or justification for mathematics, although some of his language — ‘a personal confession of faith’[1] — seems to find a faint echo, as considered later, in works such as Hardy’s *A Mathematician’s Apology*. He did imagine his listeners thinking

> ‘“Until now we always thought that the ultimate goal of mathematics was its application to practical problems. Now we see that [...] the major role is played by so-called aesthetic considerations. [...] Is it worth anything at all?”’[2]

But his answer, if it can be called that, was to lament the isolation of mathematicians and their work, and to hope that, just as the history of mathematics showed that it had become easier to understand, so it would continue to become more accessible in future.

The other major difference was that Krull argued for a parallel between mathematical and artistic imagination by emphasizing that a kind of mathematical beauty can be found in visual art. He paraphrased the group theorist Andreas Speiser (1885–1970) in claiming a close connection between the visual aspect of a tiling pattern and its underlying symmetry group,* and supported this claim using the example of a design approximating a logarithmic spiral (see Figure 9.12) carved on a Viking ship. [Krull may have been thinking of the ship excavated from a burial mound at Oseberg in 1904, whose stem and stern are carved spirals, approximately logarithmic (see Figure 13.1).[3]] He held that the aesthetic value of the carved spiral was due to the group of scaling-plus-rotation transformations that preserve the spiral: ‘I believe that it is precisely the mathematical group behind the spiral that is responsible for its aesthetic value’.[4]

Strictly speaking, invariance under scaling-plus-rotation or under translation is impossible for a necessarily bounded physical object. However, the representations Krull considered would suggest to a viewer — even a mathematically inexperienced one — the possibility of indefinite extension.

Krull’s use of the word ‘precisely’ suggests that he thought that the group and *only* the group is responsible. Any symmetry group can be viewed either as a particular collection of transformations which can be composed, or as an abstract group (a set equipped with a binary operation satisfying the group axioms). However, it cannot be true that Krull was thinking of the abstract group, for the same abstract group corresponds to the symmetries of an infinite straight line, which possesses little beauty. (The group is isomorphic to the set of real numbers under the operation of addition.) Thus the only sensible conclusion is that Krull thought that the beauty of the spiral was dependent on the particular collection of transformations that preserved its shape.

Krull thought that ‘[a] beautiful ornament should indeed set before the viewer an especially striking presentation of the totality of properties of the underlying group’.[5] This visual beauty of the physical

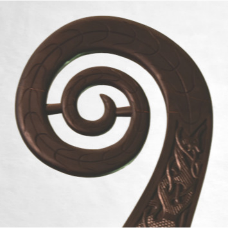

FIGURE 13.1
Spiral stern carving of the Oseberg ship.

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[1] Krull, ‘The aesthetic viewpoint in mathematics’, p. 48.

[2] *Ibid.*, p. 52.

* ▸ pp. 23–4

[3] Hagen, *The Viking Ship Finds*, pp. 6, 11, 13.

[4] Krull, ‘The aesthetic viewpoint in mathematics’, p. 49.

[5] *Loc. cit.*

artwork did not only reflect the underlying mathematical structure, but could motivate the creation of elegant mathematics through the investigation of that structure.[1] Further, Krull hinted at a deep connection between the aesthetic value of an ornament, and the aesthetic value in results and proofs: on the one hand, he said that

> '[m]athematicians [...] want to arrange and assemble the theorems so that they appear not only correct but evident and compelling. Such a goal, I feel, is aesthetic rather than epistemological.'[2]

On the other hand, 'a mathematician may find that an appropriate ornament is the most attractive way to present the mathematics'.[3]

In this single lecture, Krull set out a complex and nuanced position, in some ways an outlier, as to the value of mathematics. He admitted and regretted its remoteness. He celebrated beauty and admitted it as motive without proclaiming it a defence. He posited a connection between visual beauty of certain artworks and mathematical beauty. He admitted that practical applications did not motivate mathematicians. Finally, he did not even attempt, like other authors of the 1920s and 1930s, to defend mathematics on the grounds that there is no way to tell from which topics applications may arise in future.

E. T. Bell

E. T. Bell (1883–1960) did not object to mathematics being viewed as an aesthetic or creative pursuit. In his view, mathematics was a creative art and creative imagination was necessary for mathematical research,[4] and mathematics could be beautiful. But he distanced himself from platonism, reporting only that mathematicians *understand* the feeling that formulae have an independent existence; he himself seems to have inclined to the view that mathematics was invented rather than discovered.[5] In any case, Bell maintained that applications are important, although he held as a truth of experience that, for whatever reason, mathematics pursued for aesthetic ends turned out to be useful in the natural sciences.[6] He argued in favour of allowing mathematicians to follow their own interests, for experience tended to show that whatever mathematicians studied would turn out to be vital to science and industry in the future;[7] he even went so far as to counsel against focusing on mathematical research that was immediately applicable:

> 'Guided only by their feeling for symmetry, simplicity, and generality, and an indefinable sense of the fitness of things, creative mathematicians now as in the past are inspired by the art of mathematics rather than by any prospect of ultimate usefulness. However it may be in engineering and the sciences, in mathematics the deliberate attempt to create something of

[1] Krull, 'The aesthetic viewpoint in mathematics', p. 49.

[2] *Loc. cit.*

[3] *Loc. cit.*

[4] Bell, 'The Poetry of Mathematics', p. 561.

[5] Bell, *Men of Mathematics*, ch. 1.

[6] Bell, *The Queen of the Sciences*, pp. 1–3.

[7] *Ibid.*, pp. 81–4.

> immediate utility leads as a rule to shoddy work of only passing value. The important practical and scientific applications of mathematics are unsought byproducts of the main purposes of professional mathematicians.'[1]

To a later reader, the first two sentences in this passage, published in 1931, seem positively Hardian,[*] but Bell only defended the aesthetic motivation for mathematics as a means to the end of utility. Thus there is no contradiction between his views here and his later sardonically critical review of Hardy's *Apology*.[†]

In his later book *Mathematics: Queen and Servant of Science* (1951), he maintained that pure mathematics could be pursued for its own sake, but should not be walled off from applications:

> 'But art for art's sake does not tell the whole story. An old example is the classical theory of heat conduction [...], which led to the vast and ever-expanding theory of Fourier analysis. This is some of the purest of pure mathematics [...]. In applied mathematics Fourier analysis today has a far wider scope than Fourier ever imagined. It is indispensable, for instance, in all physics where wave motion underlies the pattern of events.'[2]

Indeed, Bell emphasized that the aesthetic appreciation of mathematics extended to its applications: for example, he described various applications as 'beautiful',[3] 'strikingly beautiful',[4] and 'beautiful and astonishing'.[5]

Lancelot Hogben

Lancelot Hogben (1895–1975), biologist and author of the popular mathematics book *Mathematics for the Million* (1937), maintained, like Sullivan, that few people can appreciate mathematics on an aesthetic level. While he admitted that '[t]he aesthetic appeal of mathematics may be very real for a chosen few', he said that it had little appeal for most people: if it had such appeal, teaching it would present no especial problem.[6] Hogben emphasized the latter point:

> 'Adolescents and children who are readily accessible to the austere aestheticism implied in the statement that a proof is elegant are rare. [...] Intense aesthetic satisfaction in mathematical pursuits is for most people an unattainable experience, or at the best an acquired taste. It is not, and cannot be, of itself a sufficient drive to proficiency. The teacher has to supply a powerful extrinsic motive of a more directly personal kind.'[7]

[Perhaps, though, social pressures prevented school pupils from admitting that they perceived beauty: H. E. Huntley[‡] (1892–1975) noted that when he described the pythagorean theorem as 'a thing of beauty' to an audience of sixth-formers, he was greeted with 'derisive laughter'; he ascribed this to such emotional statements being something that

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[1] Bell, *The Queen of the Sciences*, pp. 2–3.

[*] ▸ pp. 507–9

[†] ▸ p. 519

[2] Bell, *Mathematics: Queen and Servant*, p. 2.

[3] *Ibid.*, p. 181.

[4] *Ibid.*, p. 116.

[5] Bell, *Men of Mathematics*, ch. 4.

[6] Hogben, 'Clarity is not enough', p. 108.

[7] *Ibid.*, p. 107.

[‡] ▸ pp. 563–71

'isn't done' by sixth-formers, rather than to actual disagreement with the content of his statement.[1]]

Indeed, Hogben thought that the value of mathematics was bound up with applicability. Hogben's view of the history of mathematics was that mathematics only ever progressed in line with its applications:

> 'mathematics has advanced when has been real work for the mathematician to do, and that it has stagnated whenever it has become the plaything of a class which is isolated from the common life of mankind'[2]

Hogben seemed to assign value to mathematics only (or, at least, almost entirely) in terms of its applications. Although he did not say so explicitly, one could infer that, for him, an advance in mathematics that did not have an application would not be an advance in what he would call 'real mathematics'.

But Hogben undermined his own argument by setting up and demolishing a straw man: he denied that 'mathematics was invented by leisurely and idealistic Athenians out of sheer fascination with its utter uselessness'.[3] While it may have been claimed that mathematics was pursued in ancient Greece by a leisure class that viewed it as only an aesthetic pursuit or an intellectual game, this is very far from its 'uselessness' being a point of attraction.

[Regarding mathematical exposition, Hogben claimed that

> 'the most atrocious examples of turgid, prolix and circumlocutory English style can be collected by opening at random the pages of any text-book on mathematics.'[4]

The reader may get a certain malicious delight from George Orwell's (1903–50) use in 1946 of a sample of Hogben's own writing to illustrate the poor state of the English language.[5]]

Inutility and utility

Hogben's attitude may have been a reaction to what he saw as an Olympian disdain for utility and application in mathematics and pure science generally between the wars. Some anecdotal evidence points to broad acceptance of this view: C. P. Snow (1905–80), in his famous 1959 Rede lecture *The Two Cultures*, said that:

> 'Pure scientists have by and large been dim-witted about engineers and applied science. [...] Their instinct [...] was to take it for granted that applied science was an occupation for second-rate minds. I say this more sharply because thirty years ago I took precisely that line myself. [...] We prided ourselves that the science we were doing could not, in any conceivable circumstances, have any practical use. The more firmly one could make that claim, the more superior one felt.'[6]

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[1] Huntley, *The Divine Proportion*, p. 74.

[2] Hogben, *Mathematics for the Million*, p. 36.

[3] *Loc. cit.*

[4] Hogben, 'Clarity is not enough', p. 109.

[5] Orwell, 'Politics and the English Language'.

[6] Snow, *The Two Cultures*, § 3.

If this anecdote is an accurate reflection of attitudes of pure mathematicians at the time, then it points to a tension between the public and private justifications for mathematics. After the First World War, there seems to have been an awareness of the need to defend mathematics to the public in terms of social goods it produced, whether by comparing it to the social value of art, noting the role it could play in education, or emphasizing its practical applications.

But there were also statements about science and intellectual endeavour that did not disparage utility, but merely declared that their value did not lie solely in their practical ends; they should be pursued for their own sake, independently of utility. The classical scholar F. M. Cornford (1874–1943) said in a popular lecture that

> 'science as commonly defined [is] the pursuit of knowledge for its own sake, not for any practical use it can be made to serve.'[1]

One of the points the philosopher Julien Benda (1867–1956) made in his celebrated essay *La Trahison des Clercs*, which was published in English translation in 1928 as *The Treason of the Intellectuals*, was that intellectual activity had historically been valued to the degree in which, like art, it was worth pursuing for its own sake.[2] A component of the 'treason' Benda had discerned was that this view had become displaced by the teaching that '*intellectual activity is worthy of esteem to the extent that it is practical and to that extent alone*'.[3] There had been a shift from a position where utility was neutral in evaluating a science's worth as an intellectual endeavour, to one that made utility the sole measure of its value. Benda also thought that the modern state was to blame for not having 'maintained [...] a class of men exempt from civic duties, men whose sole function is to maintain non-practical values',[4] and that this failure had led to a degradation of society.[5]

The argument that it is impossible to forecast which areas of mathematics will be useful, previously offered by J. W. L. Glaisher* (1848–1928), by Russell, and in a slightly different form by Bell, was offered again as a defence of pure science and curiosity more generally by Abraham Flexner (1866–1959) in 1939 in his provocatively-titled essay 'The usefulness of useless knowledge': 'the pursuit of these useless satisfactions proves unexpectedly the source from which undreamed-of utility is derived'[6] A slightly different justification, encompassing the notion that it is impossible to forecast which areas of pure science will lead to harmful and which to beneficial applications, was given by Robert Strutt, the fourth Baron Rayleigh (1875–1947), in his presidential address to the British Association for the Advancement of Science in 1938.[7] A cynic might think that this argument was an excuse to allow mathematicians and scientists to pursue their work on aesthetic grounds, while justifying it to society with the prospect of possible applications in some indefinite future. In any case, there was a clear development of the perception that mathematics had to be justified in terms of producing some kind of social good.

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[1] Cornford, *Before and After Socrates*, p. 5.

[2] Benda, *The Treason of the Intellectuals*, § 3.

[3] *Ibid.*, § 3, emphasis in orig.

[4] *Ibid.*, § 3.

[5] *Loc. cit.*

* ▸ pp. 454–5

[6] Flexner, 'The usefulness of useless knowledge', p. 544.

[7] Rayleigh, 'Presidential Address', esp. p. 30.

G. H. HARDY AND *A MATHEMATICIAN'S APOLOGY*

G. H. Hardy (1877–1947) was one of the preëminent mathematicians of the first half of the twentieth century. He studied mathematics at Trinity College, Cambridge and was elected to a prize fellowship in 1900 and as a fellow of the Royal Society in 1910. He started his long collaboration with J. E. Littlewood (1885–1977) in 1912 and his tragically short one with Ramanujan* (1887–1920) in 1914. He became Savilian Professor of Geometry at Oxford in 1919 and returned to Cambridge as Sadleirian Professor of Pure Mathematics in 1931. Both independently and in his collaborations he made enormous contributions to number theory, analysis, and combinatorics.[1]

Hardy's essay *A Mathematician's Apology* is doubtless the best known account of mathematical beauty, of mathematics as a creative activity, and of the motivation for its study. Despite its fame, the *Apology* was not an *ex nihilo* discussion of beauty, for aesthetic concerns appear regularly in Hardy's writing throughout his career. For example, Hardy frequently used aesthetic terms to evaluate books he reviewed and the mathematics of the subjects of obituaries he wrote.[2] To take a single instance, although he deeply admired Camille Jordan's (1838–1922) *Cours d'Analyse*, which he acknowledged had sparked his passion for mathematics,[3] in his obituary of Jordan he nonetheless acknowledged that Jordan tended 'to neglect simplicity and symmetry of presentation'[4] and he quoted with approval Henri Lebesgue's (1875–1941) puckish claim: 'When M. Jordan encounters in an argument four quantities playing exactly the same role, he denotes them by u, A'', λ, e_3.'[5]

Furthermore, the importance Hardy assigned to aesthetic concerns appears in his more general discussions of the philosophy of mathematics. Hardy maintained an interest in the philosophy and history of mathematics;[6] indeed, he deplored what he saw as most mathematicians' lack of interest in philosophy.[7] Hardy was a firm mathematical platonist, and held that any philosophy acceptable to a mathematician must admit, in some way, that mathematics is part of objective reality.[8] He thought that research brings the mathematician, much more than the physicist, into close contact with reality, 'a reality far more intense and far more rigid than the dubious and elusive reality of physics'.[9] For Hardy, the objects of mathematics were not simply real, but were *more* real than the physical world was, or at least than those aspects of the physical world outside everyday experience. He rejected the idealism that some physicists accepted:

* ▸ pp. 667–72

[1] Titchmarsh, 'Godfrey Harold Hardy'.

[2] See Hardy, *Collected Papers*, vol. 7, pts 4–5.

[3] Hardy, *Apology*, § 29, p. 147.

[4] Hardy, 'Camille Jordan', p. xxiv.

[5] Lebesgue, Review of *Cours d'Analyse de l'École Polytechnique*, p. 16; trans. C.

[6] Grattan-Guinness, 'The interest of G. H. Hardy'.

[7] Hardy, 'The New Symbolic Logic'.

[8] Hardy, 'Mathematical proof', p. 4; see also Hardy, *Apology*, § 22.

[9] Hardy, 'The Theory of Numbers', p. 17.

> '"317" has nothing to do with sensation [...]: 317 is a prime, not because we think it so, or because our minds are shaped in one way rather than another, but *because it is so*.'[1]

The methods of the mathematician and the physicist are similar, in that the role of the mathematician is to *observe* the mathematical reality. Hardy twice[2] used the analogy of a observer gazing at a distant range of mountains, seeing one peak *A* clearly, glimpsing another peak *B* only occasionally, but gradually able to distinguish a ridge leading from *A* to *B* that can be traced, allowing the observer to fix *B* in his vision and to describe it to others. Taking this mathematician-as-observer view to its extreme led to the conclusion that

> 'there is, strictly, no such thing as mathematical proof; that we can, in the last analysis, do nothing but *point*; that proofs are what Littlewood and I call *gas*, rhetorical flourishes designed to affect psychology.'[3]

However, Hardy argued that in mathematics there were many proofs whose purpose was not to convince, but which were interesting for 'their formal and æsthetic properties'.[4] One might be interested in constructing a formal proof (that is, a sequence of logical statements starting from certain axioms and following certain rules of deduction) for a result one already knows is true. For instance, in Whitehead & Russell's *Principia Mathematica*, the proof[5] that $1 + 1 = 2$ would not convince any mathematician of the result: they would already be convinced of it; their interest lay rather in the *way* the result is proved. In general,

> 'Our object is *both* to exhibit the pattern and to obtain assent. We cannot exhibit the pattern completely, since it is far too elaborate; and we cannot be content with mere assent from a hearer blind to its beauty.'[6]

Forgive Hardy's mixed metaphor and witness the connection between his platonism, his view of proof as not aiming only to evoke assent, and his discontentment with assent that lacks aesthetic appreciation.

Seriousness and beauty

Hardy thought, unlike Hogben[*] or Sullivan,[†] that aesthetic appeal in mathematics was not something that was limited to a special few. A chess problem, he said, was simply an exercise in pure mathematics.[7] [Perhaps Hardy would have viewed a statement like 'a king and two knights cannot force checkmate against a lone king' as a chess theorem.] And since chess problems could be recognized as beautiful by many people, the ability to recognize mathematical beauty also had to be widespread.

But chess problems, while 'genuine mathematics',[8] were not 'serious' mathematics:

[1] Hardy, *Apology*, § 24, emphasis in orig.

[2] Hardy, 'The Theory of Numbers', p. 18; Hardy, 'Mathematical proof', p. 18.

[3] Hardy, 'Mathematical proof', p. 18.

[4] *Loc. cit.*

[5] Whitehead & Russell, *Principia Mathematica*, § *54.43.

[6] Hardy, 'Mathematical proof', p. 19.

[*] ▸ pp. 495–6

[†] ▸ pp. 488–9

[7] Hardy, *Apology*, § 10.

[8] *Ibid.*, § 11.

> '[t]he best mathematics is serious as well as beautiful — "important" if you like, but the word is very ambiguous, and "serious" expresses what I mean much better.'[1]

Seriousness did not lie in the practical consequences of a piece of mathematics, but in the '*significance* of the ideas it connects'.[2] For Hardy, mathematical ideas were significant if they were connected to many other mathematical ideas. The seriousness of a result was not *in* its consequences, although the consequences were evidence for its seriousness: a great writer might have enormous influence on the development of language, but this effect was not *why* they were great; it was merely evidence of greatness.[3]

Seriousness (or 'importance') in mathematics and beauty in mathematics were distinct and independent: a 1938 obituary that Hardy co-authored with Hans Heilbronn (1908–75) contrasted Edmund Landau's (1877–1938) *Handbuch* and *Ergebnisse*, respectively his 'most important' and 'most *beautiful*' books.[4]

It was primarily the quality of seriousness that elevated theorems above chess problems. Hardy admitted that, though it was 'obvious, to a trained intelligence, that [the theorem] has a great advantage in beauty also', it was more difficult to describe this advantage, precisely because the judgement of their relative seriousness interfered with the 'any more purely aesthetic judgement'.[5]

Hardy did not make entirely clear his views on the relationship between beauty and seriousness. On the one hand, he wrote:

> 'The beauty of a mathematical theorem *depends* a great deal on its seriousness, as even in poetry the beauty of a line may depend to some extent on the significance of the ideas which it contains.'[6]

This statement cannot imply that Hardy thought that seriousness was a necessary condition for beauty in mathematics generally: chess problems are trivial mathematics; there exist beautiful chess problems; hence trivial mathematics can be beautiful. Furthermore, Hardy drew a comparison between the beauty of poetry and of theorems. Since a poem could be beautiful but contain trivial ideas[7] or beautiful but contain significant ideas,[8] presumably Hardy held that the same can be true of theorems. Thus Hardy's position in the *Apology* seems simply to have been that, *ceteris paribus*, greater seriousness entails greater beauty.

According to his friend C. P. Snow (1905–80), Hardy that thought there was a connection in the other direction as well: that greater beauty entails greater seriousness:

> 'Hardy would have said […] [t]hat, in fact, if a mathematical theorem is beautiful then it is important. I once taxed him and said that if one would present a solution which finally proved Goldbach's theorem, and if it was ugly, would he accept it?

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[1] Hardy, *Apology*, § 11.

[2] *Ibid.*, § 11, emphasis in orig.

[3] *Ibid.*, § 11.

[4] Hardy & Heilbronn, 'Edmund Landau', pp. 307, 309; emphasis in original.

[5] *Ibid.*, § 18.

[6] *Ibid.*, § 11, emphasis in orig.

[7] *Ibid.*, § 10.

[8] *Ibid.*, § 11.

Hardy replied, “This is impossible. It couldn’t possibly be a proof of Goldbach’s theorem if it were ugly.”’[1]

Beauty in theorems and proofs

Among the results that Hardy considered beautiful were Fermat’s two-square theorem;[*] the law of quadratic reciprocity;[†] the infinitude of the prime numbers; the irrationality of $\sqrt{2}$; and the uncountability of the set of real numbers.[2] He gave another example of a beautiful theorem in his obituary of William Young (1863–1942), ‘perhaps the most *beautiful* that he ever proved’.[3] The trigonometrical series

$$\sum_{n=2}^{\infty} \frac{\sin n\vartheta}{\log n} = \frac{\sin 2\vartheta}{\log 2} + \frac{\sin 3\vartheta}{\log 3} + \frac{\sin 4\vartheta}{\log 4} + \dots \tag{S}$$

converges for every ϑ but is not a Fourier series. The corresponding cosine series

$$\sum_{n=2}^{\infty} \frac{\cos n\vartheta}{\log n} = \frac{\cos 2\vartheta}{\log 2} + \frac{\cos 3\vartheta}{\log 3} + \frac{\cos 4\vartheta}{\log 4} + \dots \tag{C}$$

does not even converge for $\vartheta = 0$. Nevertheless, Young proved that the series (C) *is* a Fourier series. A ‘singularly beautiful theorem’ was Landau’s result that all sufficiently large natural numbers can be expressed as a sum of eight positive cubic numbers.[4] [This result was a development in the question of finding the smallest number $g(k)$ such that every natural number can be expressed as a sum of at most $g(k)$ positive k-th powers. The question is called *Waring’s problem*, since it grew out of Edward Waring’s (c. 1736–1798) assertion that $g(3) = 9$ and $g(4) = 19$.[5] Hilbert proved the existence of $g(k)$ for any k.[6]]

When it came to formulae in particular, Hardy repeatedly pointed to the beauty of the Rogers–Ramanujan identities.[7][‡] He also mentioned the ‘exceedingly ingenious and beautiful character’ of other unspecified formal identities from the theory of partitions,[8] and noted that the behaviour of the function

$$F(x, s) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}$$

near $x = 1$ was described by the following ‘beautiful formula’[9] discovered by Ernst Lindelöf (1870–1946):[10]

$$F(x, s) = \Gamma(1 - s)\Big(\log \frac{1}{x}\Big)^{s-1} + \sum_{\nu=0}^{\infty} \zeta(s - \nu)\frac{(\log x)^{\nu}}{\nu!}.$$

The proofs that Hardy considered beautiful included Euclid’s proof of the infinitude of the primes;[11][§] the pythagorean proof of the irrationality of $\sqrt{2}$;[12][¶] Cantor’s diagonal argument for the uncountability of the real numbers;[♠] and Fabian Franklin’s (1853–1939) proof of Euler’s pentagonal number theorem, which will be examined in more detail below.

[1] Snow, ‘The Classical Mind’, p. 812.

[*] ▸ p. 7

[†] ▸ p. 405

[2] See Hardy, *Apology*, §§ 12–13. The last paragraph of § 13 makes it clear that Hardy saw beauty in the *results* on the infinitude of the prime numbers and the irrationality of $\sqrt{2}$, not just the *proofs*.

[3] Hardy, ‘William Henry Young’, pp. 229–30, emphasis in orig.

[4] Hardy, *Some Famous Problems*, p. 20; see also Dickson, *Hist. Theory of Numbers*, vol. II, p. 722.

[5] Dickson, *Hist. Theory of Numbers*, vol. II, p. 717; OEIS, A002804.

[6] Dickson, *Hist. Theory of Numbers*, vol. II, p. 722.

[7] Hardy, preface to Rogers & Ramanujan, ‘Proof of certain identities’, pp. 211–12.

[‡] ▸ p. 7

[8] Hardy & Ramanujan, ‘Asymptotic formulæ in combinatory analysis’, p. 76.

[9] Hardy, Review of *Le calcul des residus*, p. 233.

[10] Lindelöf, *Le Calcul des Résidus*, p. 139.

[11] Hardy, *Apology*, § 12.

[§] ▸ p. 103

[12] *Ibid.*, § 13.

[¶] ▸ p. 846

[♠] ▸ pp. 848

'Purely aesthetic qualities': the Hardian triad

Hardy identified three 'purely aesthetic qualities' in beautiful theorems and beautiful proofs:[1] *unexpectedness*, *inevitability*, and *economy*. His next sentences expanded on this, but in an unsystematic way:

> 'The arguments take so odd and surprising a form; the weapons used seem so childishly simple when compared with the far-reaching results; but there is no escape from the conclusions. There are no complications of detail — one line of attack is enough in each case'.[2]

To disentangle his meaning:

the arguments take so odd and surprising a form: It is clear that this quality contributed to unexpectedness. The proof might proceed in a direction that was not obvious from the theorem statement, or use ideas or concepts that seemed unrelated.

the weapons used seem so childishly simply when compared to the far-reaching results:
This contrast contributed both to unexpectedness and to economy. The tools used were not just simple, but seemed much too simple for the results they proved.

there is no escape from the conclusions: Such clear necessity was certainly an aspect of inevitability. It was not simply that the goal was reached, but that the route seemed so natural that there was no danger of wandering from it. The result seemed in no way coincidental.

there are no complications of detail — one line of attack is enough:
Avoiding division into different cases — 'one of the duller forms of mathematical argument'[3] — perhaps contributed to inevitability as well as to economy, for the act of finishing one case and backtracking to the start of another doubtless reduced the feeling of a relentless march towards the goal.

For the purposes of his argument in the *Apology*, Hardy did not distinguish between aesthetic properties of theorems and their proofs. However, his other writings show that he thought that unexpectedness could exist in a theorem independently of its proof: as discussed above, Young's theorem that the cosine series (C) is a Fourier series was unexpected in light of the corresponding sine series (S) not being a Fourier series. Hardy emphasized the beauty of this result, and presumably its unexpectedness contributed to this judgement. However, Hardy noted that the proof chosen by Young was 'not quite the most concise',[4] which suggests that he saw Young's proof as lacking economy. More dramatically, Hardy's account of the arrival of the first letter from Ramanujan in early 1913 supplies examples of results that were completely unexpected and which Hardy could not prove:

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[1] Hardy, *Apology*, § 18.

[2] *Loc. cit.*

[3] *Loc. cit.*

[4] Hardy, 'William Henry Young', p. 230.

> '(1.10)–(1.12) defeated me completely; I had never seen anything in the least like them before. [...] They must be true because, if they were not true, no one would have the imagination to invent them.'[1]

For illustration, '(1.11)' was

$$\frac{1}{1+}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+\dots} = \left\{\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right\} e^{2\pi/5}.$$

Hardy gave 'Euclid's proof of the existence of an infinity of prime numbers'[2] in the form of a *reductio ad absurdum*, beginning by saying (essentially): 'Suppose there are only finitely many prime numbers'. Euclid's proposition is *Prime numbers are more than any assigned multitude of prime numbers*[3] and his proof proceeds to show that there is a prime number outside of any given multitude (or set) $p_1, p_2, \dots, p_n$ of prime numbers.* The *reductio* in Euclid's proof is confined to proving that a prime number that divides $q+1$, where $q = \operatorname{lcm}\{p_1, p_2, \dots, p_n\}$, cannot be any of the given prime numbers p_i: the *reductio* is 'local' to this step, not 'global' in the proof. Indeed, although Euclid set out this local *reductio* explicitly, for a reader familiar with the proof, it almost vanishes into the detail: 'no p_i divides $q+1$, for dividing $q+1$ by p_i leaves remainder 1'. However, the 'global' *reductio* was important for Hardy's theme, for he wished to observe that

> '*reductio ad absurdum* [...] is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers *the game*.'[4]

This is not to suggest that Hardy wilfully distorted Euclid in order to support his case: many other modern mathematicians, possibly starting with Gustav Lejeune Dirichlet (1805–59), have presented Euclid's result as a *reductio*.[5]

Hardy's criteria can be illuminated by considering how they apply to a more sophisticated proof, which is not a *reductio*. The proof that will be examined is of the following result:

EULER'S PENTAGONAL NUMBER THEOREM.

$$\prod_{m=1}^{\infty}(1-x^m) = 1 + \sum_{n=0}^{\infty} c_n x^n,$$

$$\textit{where } c_n = \begin{cases} 0 & \textit{if } n \neq k(3k\pm 1)/2, \\ (-1)^k & \textit{if } n = k(3k\pm 1)/2. \end{cases}$$

The result is named thus because $k(3k-1)/2$ is a pentagonal number (see Figure 13.2) and $k(3k\pm 1)/2$ is a so-called generalized pentagonal number.

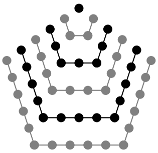

FIGURE 13.2
The first six pentagonal numbers 1, 5, 12, 22, 35, 51; that is, $k(3k-1)/2$ for $k = 1, 2, 3, 4, 5, 6$.

The following proof is due to Franklin[6] but follows an alternative exposition used by Hardy.[7] Hardy described it as 'beautiful'[8] and, writing with E. M. Wright (1906–2005), as 'very beautiful'.[9]

* ▸ p. 103

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[1] Hardy, *Ramanujan*, p. 9.

[2] Hardy, *Apology*, § 12.

[3] Euclid, *Elements*, Prop. IX.20.

[4] Hardy, *Apology*, § 12.

[5] Hardy & Woodgold, 'Prime simplicity'; Hardy, 'Three Thoughts on "Prime Simplicity"'; Siegmund-Schultze, 'Euclid's Proof'.

[6] Franklin, 'Sur le développement'.

[7] Hardy, *Ramanujan*, § 6.2.

[8] *Loc. cit.*

[9] Hardy & Wright, *Theory of Numbers*, § 19.11.

Proof. Multiply out the product $\prod_{m=1}^{\infty}(1-x^m) = (1-x)(1-x^2)(1-x^3)\cdots$. Let $\gamma(n)$ be the coefficient of x^n in the result. Every contribution to $\gamma(n)$ comes from multiplying some choice of $-x^i$, all different, whose exponents sum to n. That is, the contributions to $\gamma(n)$ correspond to partitions of n into distinct (that is, unequal) parts; if the number of parts is μ, the contribution to $\gamma(n)$ is $(-1)^{\mu}$. [For instance, 5 has three partitions into distinct parts: $5, 4+1, 3+2$; thus the contributions to $\gamma(5)$ come from $-x^5$, from $(-x^4)(-x) = x^5$, and from $(-x^3)(-x^2) = x^5$, which sum to x^5.]

Let $p(\mathcal{E}_{\mathrm{d}}, n)$ and $p(\mathcal{O}_{\mathrm{d}}, n)$ be, respectively, the numbers of partitions of n into an even and an odd number of distinct parts. Then $\gamma(n) = p(\mathcal{E}_{\mathrm{d}}, n) - p(\mathcal{O}_{\mathrm{d}}, n)$, since a partition of n into an even number of distinct parts contributes 1 to $\gamma(n)$, while a partition of n into an odd number of distinct parts contributes -1 to $\gamma(n)$. The aim is now to show that there is a one-to-one correspondence between 'most' of the partitions of n of these two types; the number of left-over partitions gives $\gamma(n)$.

Consider a partition of n with distinct parts and represent it as a Ferrers diagram.[*] The lowest row of the diagram is called the *base*, the longest line of points starting at the top-rightmost and moving towards the bottom-left is called the *slope*; see Figure 13.3.

[*] ▸ p. 441

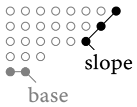
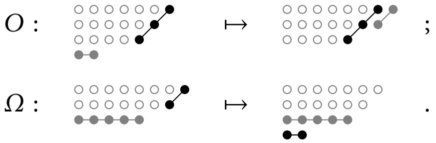


FIGURE 13.3
A Ferrers diagram, showing the base and the slope.

Define two partial transformations of partitions O and Ω in terms of Ferrers diagrams as follows. The partial transformation O consists of removing the base from the diagram and placing it parallel to and to the right of the slope, provided that the result is another Ferrers diagram with distinct parts; O is otherwise undefined. Similarly, Ω consists of removing the slope from the diagram and placing it parallel to and below the base, provided that the result is another Ferrers diagram of a partition with distinct parts; Ω is otherwise undefined. Diagrammatically,



Note that O and Ω are mutually inverse and are therefore partial bijections, and that both transform a permutation with an even number of parts to one with an odd number of parts and vice versa. Now consider three cases:

- The base is shorter than the slope. Then O is defined, while Ω is undefined (see the example in Figure 13.4(a)).
- The base and the slope have equal lengths. Then Ω is undefined, while O is defined (see Figure 13.4(b)) except when the base and the slope meet (see Figure 13.4(c)).
- The base is longer than the slope. Then O is undefined, while Ω is defined (see Figure 13.4(d)), except when the base and the slope

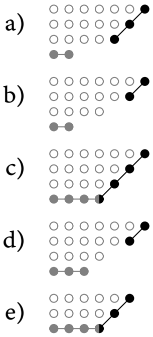


FIGURE 13.4
Examples showing the cases when the partial transformations O and Ω are defined or undefined.

meet and the base contains exactly one more point than the slope (see Figure 13.4(e)).

Thus there is a one-to-one correspondence between the two types of partition except for two situations where both O and Ω are undefined; that is, when the base and the slope meet and either they contain the same number of points or the base contains exactly one more point than the slope (these situations are shown in Figure 13.4(c, e)).

In the first situation, n is of the form

$$k + (k+1) + \ldots + (2k-1) = k(3k-1)/2,$$

where k is the number of parts, and there is an excess of one partition with an odd or even number of parts, depending on whether k is odd or even. In the second situation, n is of the form

$$(k+1) + (k+2) + \ldots + 2k = k(3k+1)/2,$$

where k is the number of parts, and the excess is the same. Hence $\gamma(n) = p_e(n) - p_o(n)$ is 0 unless $n = k(3k \pm 1)/2$, when it is $(-1)^k$. Thus $\gamma(n) = c_n$ as defined in the theorem. ☐

Does this proof possess Hardy's three 'purely aesthetic' qualities? Undoubtedly, the approach is unexpected: the result itself makes not the slightest mention of partitions or Ferrers diagrams. The result itself is a statement that one can interpret analytically, as a numerical equality that holds when one substitutes any x for which the two sides converge, or purely syntactically, as a formal equality that holds when one expands the two sides. There is nothing to suggest immediately that partitions should (or could) play a role in a proof. Even after the first paragraph has demonstrated that partitions govern the coefficients obtained by expanding the product on the left-hand side, there is a further unexpected step in considering the base and slope of their Ferrers diagrams and the two operations O and Ω.

The progress of the proof also seems to be inevitable: once the strategy is announced, there is no deviation from it. There is a division into cases, but these are swiftly dealt with and the flow of the proof is reunified. Furthermore, visualizing Ferrers diagrams helps the reader keep mental track of these cases in a unified way that might be more difficult for a purely syntactic argument.[1] This role for visual thinking also contributes to the economy of the proof, which is also evident in how an apparently complicated expansion of an infinite product is rendered tractable by using partitions and treated by the two very simple operations O and Ω. However, Hardy said that his own mathematical thinking was textual, not visual,[2] so perhaps for him the use of Ferrers diagrams was merely notational, as an easy way to describe the base and side.

It seems that there was only one occasion when Hardy compared aesthetically two different proofs of the same result, namely the theorem that *For any* $\varepsilon > 0$, *almost all natural numbers n have between*

[1] For the cognitive role of visualization, see, for example, Giaquinto, 'Visualizing in Mathematics'; Cain, 'Visual thinking and simplicity of proof'.

[2] Hardy, Review of *The Psychology of Invention*, p. 114.

$(1-\varepsilon)\log\log n$ *and* $(1+\varepsilon)\log\log n$ *prime factors.* He considered two proofs: the original proof that he and Ramanujan found, and a later proof by Paul Turán (1910–76). For Hardy, Turán's proof was 'very simple and elegant'; more so than original proof, which, for example, made use of Stirling's approximation. Hardy thought the original proof 'in some ways more suggestive', although he did not make clear what it was suggestive *of*.[1]

Beauty in theories

Beside beauty in theorems and proofs, Hardy also saw beauty in mathematical theories. In particular, he took clear pleasure in developments that allowed the reconstitution of theories into a more elegant form. An exemplary case was the work of Émile Borel (1871–1956) and then of Lebesgue, which led to the rewriting of the previously inelegant theory of integrals:

> 'Indeed the whole theory was æsthetically unsatisfying; it tended to be cumbrous, longwinded, and full of untidy exceptions; and the conservative mathematicians who detested it had a certain amount of excuse for their distaste.
>
> All this is changed; the theory is now one of the most beautiful in mathematics. It unfolds itself in a series of terse and comprehensive theorems, with all the smoothness and elegance of the best "classical" analysis. In particular, differentiation and integration dovetail harmoniously together.'[2]

He explained how the inelegance of a theory could lead to a reformulation as part of a larger, more general, theory. He illustrated his thesis with the examples of extending the rational numbers to the reals in analysis, and extending real to complex geometry. In both cases, the original theory had become 'honeycombed with exceptions and distinctions until it has become aesthetically intolerable',[3] so that one desired to banish the anomalies.

Hardy here admitted that aesthetic value had a role in choosing the direction of mathematical research; he also made this point more strongly: 'the mathematician's patterns, like the painter's or the poet's, *must* be beautiful'.[4] However, while he said nothing against beauty being a *conscious* guide in finding a proof, he was unconvinced by the suggestion made by Jacques Hadamard (1865–1963) that the role of the subconscious in mathematical problem-solving involves sorting among various combinations of ideas for those which are beautiful, since these are likely to lead to solutions.[5]* But Hardy agreed that the unconscious often had an important role in discovery.[6] He was simply sceptical that aesthetic selection explained how the subconscious worked to find solutions, and did not dismiss the idea that beauty could be a guide to truth. Indeed, his statement that certain results in Ramanujan's letter must be true because 'no one would have the imagination to invent them'[1] seems to come close to inferring truth from aesthetic value.

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[1] Hardy, *Ramanujan*, p. 51.

[2] Hardy, 'Prof. H. L. Lebesgue', p. 685; he expressed similar views in Hardy, Review of *The Theory of Functions*.

[3] Hardy, Review of *The Theory of the Imaginary in Geometry*, p. 78.

[4] Hardy, *Apology*, § 10, emphasis added.

[5] Hardy, Review of *The Psychology of Invention*, p. 113.

* ▸ pp. 662–4

[6] *Ibid.*, p. 112.

The motivation for mathematics

Hardy's motivation for doing mathematics was bound up with his appreciation of its beauty. His very definition of '*real* mathematics' depended on its beauty:

> 'I am including the whole body of mathematical knowledge which has permanent aesthetic value, as for example the best Greek mathematics has, the mathematics which is eternal because the best of it may, like the best literature, continue to cause intense emotional satisfaction to thousands of people after thousands of years. But I am not concerned with ballistics or aerodynamics [...]. That [...] is repulsively ugly and intolerably dull'.[2]

However, he did not limit 'real mathematics' to pure mathematics: he specifically admitted the mathematical physicists James Clerk Maxwell[*] (1831–79), Albert Einstein[†] (1879–1955), Arthur Eddington[‡] (1882–1944), and Paul Dirac[§] (1902–84) to the ranks of the 'real mathematicians'.[3] For him, the greatest advances in applied mathematics were in areas like relativity and quantum mechanics, which he saw as just as inapplicable as his own number theory. Those parts of both pure and applied mathematics that could be put to work happened to be 'dull and elementary'.[4]

Hardy, like Poincaré, imposed an aesthetic qualification: if one did not pursue, or at least appreciate, mathematical beauty, one could not be a 'real mathematician'.[5] Such appreciation was of course a necessary, not a sufficient, condition for being a 'real mathematician', for one could appreciate beauty in chess problems and puzzles, which, although genuine mathematics, were not 'real mathematics' in the Hardian sense. For Hardy, Hogben's support for the idea that mathematical beauty was only for the chosen few had a simple explanation:

> 'he has hardly any understanding of "real" mathematics (as anyone who reads what he says about Pythagoras's theorem, or about Euclid and Einstein, can tell at once), and still less sympathy with it (as he spares no pains to show). "Real" mathematics is to him merely an object of contemptuous pity.'[6]

Hardy's rhetoric here skirted around, though managed to avoid, the pit of circular reasoning: Hogben had no appreciation for mathematical beauty; therefore he had no understanding of 'real mathematics' and was not a 'real mathematician'; therefore he thought that aesthetic appreciation of mathematics had to be confined to an elect few.

Intellectual curiosity, professional pride, and ambition were the three motives for doing research that Hardy identified.[7] Hardy saw neither professional pride nor ambition as negative. Professional pride impelled one to maintain to a high standard, to avoid the shame of producing work that was deficient or inadequate. Ambition was likewise positive, and most improvements in the human condition had been due to the efforts of ambitious people. In particular, 'the noblest ambition is that of leaving behind one something of permanent value';[1] a goal which Hardy thought that most people were incapable of reaching.[2] And the permanence of real mathematics, in the last analysis, depended on its beauty: 'there is no permanent place in the world for ugly mathematics'.[3] [The word 'permanent' has often been omitted in quotation of this phrase, which obscures Hardy's meaning.[4]]

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[1] Hardy, *Ramanujan*, p. 9.

[2] Hardy, 'Mathematics in war-time', p. 5; see also Hardy, *Apology*, § 25.

[*] ▸ pp. 434–6

[†] ▸ pp. 886–92

[‡] ▸ pp. 893–4

[§] ▸ pp. 900–13

[3] Hardy, 'Mathematics in war-time', p. 5; Hardy, *Apology*, § 25

[4] Hardy, *Apology*, § 25.

[5] This observation is from Sinclair & Pimm, 'A Historical Gaze', pp. 13–14.

[6] Hardy, *Apology*, § 27.

[7] *Ibid.*, § 7.

For Hardy, ugly mathematics was *ephemeral*. Its 'place in the world' was dependent on some non-intrinsic value, such as a purpose it served, either internal or external to mathematics. For example, an ugly proof of a result would be forgotten when a new, beautiful proof was discovered. An ugly method with practical application would be used only until it was displaced by a better technique, or until a technological advance made it irrelevant, or until society ceased to think the application worthwhile. This transience is illustrated by the work Hardy's collaborator J. E. Littlewood did in ballistics, which was one of the examples Hardy gave of ugly mathematics.[5] During the First World War, Littlewood served with the Royal Garrison Artillery, and worked to devise methods to reduce the time required to calculate gunnery trajectories.[6] Such mathematical developments were valuable at the time for a nation at war, but their value did not last. Littlewood lived long enough to write (in 1971) that 'a computer can now calculate a trajectory as fast as the shell traverses it'.[7] His wartime work had not been permanent: electronics had made it obsolete practically, and, in Hardy's view, it had no beauty to give it enduring worth.

In contrast, the mathematics that was 'eternal' was the mathematics that 'permanent aesthetic value' and produced 'intense emotional satisfaction',[8] because it would continue to be read and admired regardless of utility other non-intrinsic value. Thus, for Hardy, the creation of 'something of permanent value' could not be achieved without the creation of beauty.

Given Hardy's insistence that mathematics was a creative, not contemplative, subject,[9] that he found it a 'melancholy experience'[10] to be writing about mathematics instead of adding to it, and that '[e]xposition, criticism, appreciation, is work for second-rate minds',[11] one might suppose that Hardy's mathematical career had been entirely focused on research — research with the single-minded aim of producing beautiful mathematics. And yet, as discussed above, throughout his life Hardy wrote a good deal about mathematics, including reviews and obituaries in which he evaluated the work of others, and in addition wrote the textbooks *A Course of Pure Mathematics* (which went through nine editions during Hardy's life), *An Introduction to the The-*

[1] Hardy, *Apology*, § 7.

[2] Hardy, *Some Famous Problems*, p. 5; quot. in Hardy, *Apology*, § 6.

[3] Hardy, *Apology*, § 10.

[4] See Cain, 'Legacy of the *Apology*'.

[5] Hardy, *Apology*, § 28; Hardy, 'Mathematics in war-time', p. 5.

[6] Burkill, 'John Edensor Littlewood', p. 328.

[7] Littlewood, 'Adventures in Ballistics, 1915–1918. I', p. 31.

[8] Hardy, 'Mathematics in war-time', p. 5; see also Hardy, *Apology*, § 25.

[9] Hardy, *Apology*, § 28; Hardy, 'Mathematics in war-time', p. 8.

[10] Hardy, *Apology*, § 1.

[11] *Loc. cit.*

ory of Numbers (with Wright; three editions), *Inequalities* (with J. E. Littlewood and George Pólya), the posthumously-published *Divergent Series*, and the book of lectures *Ramanujan*. Littlewood recalled, in his preface to Hardy’s *Divergent Series*, that Hardy thought that ‘the late years of a mathematician’s life were spent most profitably in writing books’[1] and that ‘all his books gave him some degree of pleasure’. Furthermore, whatever Hardy may have said in the *Apology*, Louis Mordell (1888–1972), who succeeded Hardy as Sadleirian Professor of Pure Mathematics at Cambridge, thought that Hardy’s research was not marked by an enduring quest for beauty, but was

> ‘distinguished more by his insight, his generality, and the power he displays in carrying out his ideas. [...] the proofs are often long and require concentrated attention, and this may blunt one’s feelings even if the ideas are beautiful.’[2]

George Pólya’s (1887–1985) view fitted with Mordell’s:

> ‘Hardy wrote very well and with great facility, but his papers [...] make no easy reading: The problems are very hard and the methods unavoidably very complex. He valued clarity, yet what he valued most in mathematics was not clarity but power, surmounting great obstacles that others abandoned in despair.’[3]

The solution to this contradiction may lie in Hardy’s cognizance of his own failing mathematical creativity. As Snow pointed out in his foreword to the *Apology*, it is

> ‘a book of haunting sadness. [...] a passionate lament for creative powers that used to be and that will never come again.’[4]

Hardy did not pity himself: rather, perception of his own intellectual decline doubtless coloured his memories and his evaluation of his own achievements. Furthermore, he had enjoyed good health and was physically vigorous until a coronary thrombosis in 1939, shortly before he wrote the *Apology*, which doubtless also darkened his outlook.[5] Looking back on his life, he quoted with approval Russell’s early view of mathematics as a refuge from the world. He did not deny that mathematics had been, for him, a route to a comfortable and happy life; a route that he chose because he judged it the one in which he was most likely to succeed.[6] What did he achieve with his ‘series of comfortable and “dignified” positions’[7] — what Cicero called ‘*cum dignitate otium*’?[8] By his own admission, nothing ‘useful’:

> ‘I have just one chance of escaping a verdict of complete triviality, that I may be judged to have created something worth creating. And that I have created something is undeniable: the question is about its value.’[9]

And the value of mathematics, for Hardy, was largely located in its beauty.

[1] Hardy, *Divergent Series*, Preface.

[2] Mordell, ‘Hardy’s “A Mathematician’s Apology”’, p. 834.

[3] Pólya, ‘Some mathematicians’, p. 751.

[4] Snow, ‘Foreword’, pp. 50–1.

[5] *Ibid.*, p. 50.

[6] Hardy, *Apology*, § 29.

[7] *Loc. cit.*

[8] Cicero, *Pro Sestio*, § xlv.

[9] Hardy, *Apology*, § 29.

Hardy and Russell

Hardy must have read 'The Study of Mathematics',[*] because he quoted from it in the *Apology* (although he only attributed the quotation to Russell, without giving the source).[1] Russell's essay is therefore the earliest work that was explicitly a precursor of the *Apology*, and there are clear parallels between them. These parallels do not immediately imply that Hardy's thinking was shaped by 'The Study of Mathematics', for Hardy and Russell had similar backgrounds mathematically and perhaps the commonalities are the result of drinking at the same founts of knowledge.[2] Both men shared a delight in mathematical demonstration.[3] Both suffered through the Cambridge Mathematical Tripos, which, at the time, tested its own particular brand of mathematical problem-solving, and was almost totally isolated from real mathematics — either in the Hardian or in almost any other conceivable sense.[4]

In philosophy of mathematics, Hardy was a follower of Russell[5] and kept abreast of at least the outlines of his work: he reviewed *Principles of Mathematics*[6] and the first volume of *Principia Mathematica*,[7] and some of his other book reviews contain references to Russell's philosophy.[8] Hardy's obituarist E. C. Titchmarsh (1899–1963), who studied under him and later succeeded him as Savilian Professor of Geometry at Oxford, went so far as to call him a 'disciple' of Russell.[9] In the *Apology* itself, there are passages that seem to have been influenced by Russell's language, although perhaps Hardy's view independently coincided with Russell's. For example, Russell's analysis of the aesthetic value of proofs[†] brings to mind Hardy's '"purely aesthetic" qualities' of theorems and proofs in the *Apology*.[10] Although Hardy did not use Russell's term *unity*, it was clearly implicit in his deprecation of enumeration of cases.

But Russell's and Hardy's views did not always coincide: although Russell also appreciated mathematical beauty, he did not see it as the *sole* valid motive for mathematics, and thus did not share Hardy's disdain for mathematics pursued for non-aesthetic reasons — for 'unreal mathematics', to negate Hardy's idiom.

[It is interesting — though of dubious significance — that the most positive contemporaneous reactions to 'The Study of Mathematics' and the *Apology* came from literary figures. Lytton Strachey (1880–1932), a former Apostle and a member of the Bloomsbury Group, praised Russell's essay highly: 'it's magnificent — one's carried upwards into sublime heights — perhaps the sublimest of all!'[11] The novelist Graham Greene (1904–91) reviewed the *Apology* in glowing terms: 'I know no writing — except perhaps Henry James's introductory essays — which conveys so clearly and with such an absence of fuss the excitement of the creative artist.'[12]]

However, the biggest difference between the two men was that Hardy held fast to mathematics justified as an aesthetic pursuit and mathematics as a refuge, while Russell retreated from these positions.

[*] ▸ pp. 479–85

[1] Hardy, *Apology*, § 28.

[2] Grattan-Guinness, 'Russell and G. H. Hardy'.

[3] Russell, 'Why I Took to Philosophy', p. 29.

[4] Becher, 'William Whewell and Cambridge Mathematics'; Hardy, 'The Case Against the Mathematical Tripos', pp. 64–5.

[5] Grattan-Guinness, 'The interest of G. H. Hardy', pp. 412–15.

[6] Hardy, Review of *The Principles of Mathematics*.

[7] Hardy, 'The New Symbolic Logic'.

[8] Hardy, Review of *A New Algebra*; Hardy, Review of *The Theory of the Imaginary in Geometry*.

[9] Titchmarsh, 'Godfrey Harold Hardy', p. 450.

[†] ▸ p. 481

[10] Hardy, *Apology*, § 18.

[11] Strachey, letter to Russell, dated 23 Oct. 1907, repr. in Russell, *Autobiography*, p. 190.

[12] Greene, 'The Austere Art'.

Although Russell retained an appreciation for mathematical beauty, he certainly ceased to view mathematics as a refuge. During and after the First World War, his work became dominated by its social and political aspects.

It is meaningless to ask for a specific reason why Hardy and Russell parted ways in this respect. It is possible to suggest some contributing factors, but the following paragraphs are highly speculative.

The first point is that when Russell and Hardy finished the Mathematical Tripos, their attitudes to mathematics were diametrically opposed. Hardy found out about 'real mathematics' while still an undergraduate by reading Jordan's *Cours d'analyse de l'École Polytechnique* on the advice of A. E. H. Love (1863–1940).[1] Russell was so disgusted by the lack of rigour and system in the mathematics he had been taught that after the Tripos he sold his books and vowed that he would never again read a mathematical book.[2] He then moved to philosophy and his return to mathematics was due to his interest in its philosophical underpinnings.[3] Hardy's interest in philosophy was 'top-down': he was always a mathematician with an interest in the philosophy of mathematics. Russell's interest was 'bottom-up': his focus remained on the foundations. Thus even if Hardy had been confronted with the same arguments that swayed Russell towards viewing mathematics as ultimately tautological (and Hardy did discuss philosophy with Wittgenstein[4]), his view of mathematics would not have been so readily changed. Similarly, Hardy's aesthetic views would have been less prone to change, even if he had, like Russell, read Santayana.

In a manner of speaking, there was little to 'push' Hardy away from his previous attitude to mathematics. There was, however, much to 'pull' Russell. For, whatever similarities one can find in their intellectual backgrounds, their social backgrounds were very different. Russell was scion of a noble family and grandson of a prime minister, accustomed to meeting dignitaries who visited the family home;[5] Hardy's father was a master at an independent school. Perhaps their different backgrounds shaped their attitudes toward the level of change they could aspire to effect.

Thus, although both were pacifists and opposed the First World War, it was Russell who stepped outside academia to champion publicly pacifist causes, who was dismissed from his Trinity College lectureship, and who was ultimately imprisoned for his activism. It was Russell who went on to campaign for social change, to analyse power structures, to become a public intellectual, and to be awarded the 1950 Nobel Prize in literature for his efforts to promote 'humanitarian ideals and freedom of thought'.[6] Russell's background made the world his scope: much later, during the Cuban Missile Crisis in 1962, he sent personal telegrams to President John F. Kennedy (1917–63) and Premier Nikita Khrushchev (1894–1971), in the full expectation that they would be read.[7] (And Khrushchev did indeed reply to Russell, in

[1] Hardy, *Apology*, § 29.

[2] Russell, 'Why I Took to Philosophy', pp. 29–30.

[3] See Griffin & Lewis, 'Bertrand Russell's Mathematical Education'.

[4] Grattan-Guinness, 'The interest of G. H. Hardy', p. 418.

[5] Russell, *Autobiography*, p. 46.

[6] [Swedish Academy], 'The Nobel Prize in Literature 1950'.

[7] Russell, *Unarmed Victory*, p. 37.

the form of an open letter.[1]) The First World War was the catalyst that made him realize he could no longer stay in the refuge of mathematics and logic.

This is not to criticize Hardy, who was unafraid to seek change within his own sphere of academia. He actively defended Russell when he was dismissed by the Trinity College council in 1916; promoted Russell's reinstatement in 1919; and his 1941 account of the affair in *Bertrand Russell & Trinity* was part of a campaign to bring Russell back again.[2] He called for reconciliation with German colleagues after the First World War and opposed attempts to 'punish' German scientists or exclude them from congresses and journals.[3] He helped recruit for the National Union of Scientific Workers,[4] later called the Association of Scientific Workers, and served as its president between 1924 and 1926.[5] In 1934, he excoriated Ludwig Bieberbach (1886–1982) for a lecture that argued that different mathematical styles were racially or nationally determined and which thus tried to provide a basis for anti-Jewish and anti-French remarks.[6] But Hardy did not have the aristocratic outlook that gave Russell an almost instinctive sense that he was *allowed* to work on a national or international scale, that he should be *listened to* by politicians and the public.

* * *

J. B. S. Haldane (1892–1964), writing shortly after Hardy's death, suggested that Hardy's politics were a direct consequence of his view that mathematics was worth pursuing for its own intrinsic value. He thought that Hardy, like anyone who believed in art for art's sake, felt he had to work towards a society where anyone who wanted to pursue art — or mathematics — would be able to do so, and therefore had to champion socialism.[7]

Uselessness and mischaracterization [8]

Hardy *did not* hold that beautiful mathematics must be useless, and he *did not* aim for or advocate uselessness. These points must be emphasized, for, as discussed below, they form the single most misunderstood aspect of the *Apology*.

For Hardy, pure mathematics was in some sense more useful than applied, since it was the vehicle through which mathematical technique was taught.[9] In terms of the actual *content* of mathematics, Hardy thought that the

> 'general conclusion must be that such mathematics is useful as is wanted by a superior engineer or a moderate physicist; and that is roughly the same thing as to say, such mathematics as has no particular aesthetic merit.'[10]

That is, for Hardy, useful mathematics had 'no particular aesthetic merit'. But Hardy did not argue, here or elsewhere in the *Apology*, that this was a *necessary* truth: the claim was that useful mathematics *happened to have* little aesthetic value. It does not follow that useful mathematics *had to have* little aesthetic value, or, consequently, that beautiful mathematics *had to be* useless. Hardy simply saw it as a *contingent* fact that 'real mathematics' was useless.[1] He made this contingency explicit: although it was 'the dull and elementary parts' of mathematics that were useful, '[t]ime may change all this'.[2] The contingency was also implicit in his remark that it was only 'very unlikely',[3] not impossible, that a use in war would be discovered for the theory of numbers or relativity. Further, Hardy did not object to applications happening to emerge for mathematics that had aesthetic value:

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[1] Russell, *Unarmed Victory*, pp. 40–5.

[2] Delany, 'Russell's dismissal from Trinity'.

[3] Dauben, 'Mathematicians and World War I'.

[4] Haldane, *Everything has a History*, p. 241.

[5] Grattan-Guinness, 'Russell and G. H. Hardy', p. 176.

[6] Hardy, 'The *J*-type'; for background, see Segal, *Mathematicians Under the Nazis*, ch. 7.

[7] Haldane, *Everything has a History*, pp. 241–2.

[8] This subsection is reprinted with minor changes from the author's essay 'Legacy of the *Apology*'.

[9] Hardy, *Apology*, § 26.

[10] *Loc. cit.*

> 'If the theory of numbers could be employed for any practical and obviously honourable purpose, [...] neither Gauss nor any other mathematician would have been so foolish as to decry or regret such applications.'[4]

Since a loss of beauty could reasonably be thought to cause regret, this statement implies that if an application arose for some piece of beautiful mathematics, it would not decrease its beauty. Therefore, for Hardy, beauty did not necessarily imply uselessness. [As a man with a hatred of war[5] witnessing the beginnings of a second global conflagration, Hardy may have hoped that beautiful mathematics and inapplicable mathematics would coincide (contingently, not necessarily). This idea would perhaps explain how he could give Maxwell as an example of a 'real' mathematician[6] when surely he could not have thought Maxwell's work useless.]

Thus, although Hardy thought that applications in war for number theory and other parts of 'real mathematics' would be unlikely to be found for many years,[7] his error here was only to be unduly optimistic (as a pacifist) about the probability of and time-scale for the emergence of such applications. His incorrectness on this point has no bearing on the views he expressed on aesthetic value.

Hardy explicitly rejected the notion that the uselessness of real mathematics — its lack of 'practical' applications — was a point of pride among mathematicians.[8] The comment that was quoted against him in this regard, that '[a] science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life'[9] (which was 'a conscious rhetorical flourish'[10]), was written during the First World War. It should be read as a pacifist's sarcastic remark on what *society* considered useful rather than a statement of what *Hardy* himself considered useful. Hardy's own view was given in the *Apology*:

> 'A science or an art may be said to be "useful" if its development increases, even indirectly, the material well-being and comfort

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[1] Hardy, *Apology*, § 26.

[2] *Ibid.*, § 25.

[3] *Ibid.*, § 28.

[4] *Ibid.*, § 21.

[5] Titchmarsh, 'Godfrey Harold Hardy', p. 451.

[6] Hardy, *Apology*, § 25.

[7] *Ibid.*, § 28.

[8] Hardy, *Apology*, § 21; Hardy, 'Mathematics in war-time'.

[9] Hardy, 'Prime Numbers', p. 350.

[10] Hardy, *Apology*, § 21, n. *; Hardy, 'Mathematics in war-time'.

> of men, if it promotes happiness, using that word in a crude and commonplace way.'[1]

As the quotation about the employment of mathematics for a 'practical and obviously honourable purpose' makes clear, no mathematician would oppose or regret the application of mathematics for good, but utility was never, and should never, be sought; the goal should be beauty and depth. Hardy did not advocate only the study of useless mathematics: for him, it was simply a fact that real mathematics had no applications in war,[2] and his own work in particular lacked *any* applications.[3] He did not celebrate these facts: he simply drew comfort in inferring that real mathematics could not, at least, be used for evil.[4]

Stressing these points is necessary because part of the legacy of the *Apology* is the conventional 'knowledge' that Hardy made uselessness a requirement for beauty, or even equated uselessness and beauty, or was proud of the uselessness of pure mathematics, or advocated the pursuit of useless mathematics:

1) 'Hardy argued that the utility of a given piece of mathematics is inversely related to its beauty (so that, say, the multiplication table — perhaps the most "useful" part of "mathematics" — is so devoid of beauty as hardly to deserve the name "mathematics").'[5]
2) 'Hardy went beyond the claim that mathematics is beautiful to also insist that applications of mathematical ideas to the physical world demean those ideas [...] — its usefulness detracts from its beauty.'[6]
3) 'Hardy [...] had a very odd view of what constitutes beauty in mathematics: to be beautiful, mathematics must be *useless*!'[7]
4) 'Hardy [...] argued that there is no place for "ugly" mathematics; important mathematics should always be beautiful. As a consequence, ugly mathematics is applied mathematics, useful mathematics.'[8]
5) 'Mr. HARDY takes great pride in being a "useless mathematician."'[9]
6) 'many, like Hardy, were proud that their work had no practical value.'[10]
7) '*MP*: With all the applications of elliptic curves to problems in cryptography, it seems that suddenly number theory has become a branch of applied mathematics.

   Selberg: It would have given great grief to Hardy!

   MP: That's right. Hardy would not have liked it at all. [...]

   [...]

   MP: [...] When he gets off into the part about his being so proud that he's never done anything that could be of use to humanity and so on, I wince a bit.'[11]
8) 'Hardy exults particularly in the uselessness of number theory'.[12]

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[1] Hardy, *Apology*, § 19.

[2] *Ibid.*, § 28.

[3] *Ibid.*, § 29.

[4] *Ibid.*, § 21.

[5] Netz, 'The Aesthetics of Mathematics', p. 253.

[6] Clawson, *Mathematical Mysteries*, p. 213.

[7] Nahin, *Dr. Euler's Fabulous Formula*, pp. 4–5, emphasis in orig.

[8] Landri, 'The Pragmatics of Passion', p. 425.

[9] Montagu, Review of *A Mathematician's Apology*.

[10] Stewart, *Letters to a Young Mathematician*, p. 135; see also Stewart, *Why Beauty is Truth*, p. 267.

[11] Albers & Alexanderson, *Fascinating Mathematical People*, pp. 265–6.

[12] Levinson, 'Coding Theory', p. 249.

9) 'Hardy, whose proudest boast was that he had never done anything useful.'[1]
10) 'what G. H. Hardy boastingly called the most useless of all mathematics.'[2]
11) 'G. H. Hardy even boasted that his work could never be used for practical purposes.'[3]
12) 'G. H. Hardy, who was proud that "the great bulk of higher mathematics is useless".'[4]
13) 'G. H. Hardy […] boasted that he had never done anything which was remotely useful — though he would be discomforted to know that bank security codes now use the prime numbers he delighted in.'[5]
14) 'G. H. Hardy who expressed the fervent hope that no result he had proven would ever be applied.'[6]
15) 'he fervently hoped that none of his work would ever be applied to anything.'[7]
16) 'He scorned the application of mathematics to anything at all and once expressed the hope that nothing he had ever discovered would have any practical use.'[8]
17) 'his advocacy of the uselessness of mathematics.'[9]
18) 'the *Apology*'s aestheticism and cult of uselessness.'[10]
19) '"Hardy syndrome" (the less useful the better).'[11]
20) 'one of my teachers, G. H. Hardy, justified his great life work on the ground that it could do no one the least harm — or the least good.'[12]
21) 'G. H. Hardy […] values number theory precisely for its lack of practical application.'[13]
22) 'Hardyism is the doctrine that one ought only to pursue useless mathematics. This doctrine is given as a purely personal credo in Hardy's *A Mathematicians's Apology*.'[14]

[Adam Pringle (1980–2018) seems to have been the only author to note this misconception;[15] the single instance he cited is but a symptom of a widespread disease.]

Quotations 1 to 4 represent Hardy as having equated ugliness with usefulness. Hardy simply held that mathematics that was useful happened to be ugly. (Quotation 4 also seems to contain a *non sequitur*.)

Quotations 5 to 13 suggest that Hardy expressed pride in the uselessness of number theory in general and his work in particular. Hardy expressed no pride; he only took solace from the fact that its uselessness meant that it could not be used for harm.

Contra quotations 14 to 16, Hardy expressed no 'hope', fervent or otherwise, about the enduring uselessness of his work. He simply stated as a fact that he had 'never done anything "useful"' and that, in

[1] Hammond, 'Mathematics', p. 22.

[2] Steen, *Mathematics Tomorrow*, p. 3.

[3] Mumford, 'The Synergy of Pure and Applied Mathematics', p. x.

[4] Goodman, 'Mathematics as natural science', p. 187; citation of the *Apology* omitted.

[5] Atiyah, 'Address of the President', p. 106.

[6] Harary, 'Conditional connectivity', p. 355; citation of the *Apology* omitted.

[7] Gardiner, 'Beauty in Mathematics', p. 80.

[8] Wells, *Games and Mathematics*, ch. 13.

[9] Grattan-Guinness, 'The interest of G. H. Hardy', p. 419.

[10] Harris, *Mathematics without Apologies*, ch. 10.

[11] Lucas, 'Growth and New Intuitions', p. 56.

[12] Bronowski, *Science and Human Values*, p. 11.

[13] Wiener, *Ex-Prodigy*, p. 189.

[14] Davis & Hersh, *Mathematical Experience*, p. 96.

[15] Pringle, 'A Hardian Theory', p. 5.

all likelihood, his work would never make 'the least difference' to the world.[1]

Finally, unlike the claims of quotations 16 to 22 and the suggestions in quotations 7 and 13 that he would have been aggrieved by later applications of number theory, Hardy in no way advocated uselessness or thought it a justification. As mentioned above, he drew some comfort from not having caused harm, but he held that mathematics should be pursued for aesthetic reasons, *independently of possible applications*.

[An uncharitable reader, thinking that mathematicians would understand that 'being useful is not a goal' does not imply 'not being useful is a goal', might consider as deliberate calumnies the contents of some of these quotations.]

The effects of this kind of mischaracterization are not limited to discussions of the value of mathematics. The principle in genetics now known as the Hardy–Weinberg law was published by Hardy in 1908 in a letter to *Science*.[2] The geneticist James F. Crow (1916–2012) suggested the following explanation for Hardy's choice of venue:

> 'It must have embarrassed him that his mathematically most trivial paper is not only far and away his most widely known, but has been of such distastefully practical value. He published this paper not in the obvious place, *Nature*, but across the Atlantic in *Science*. Why? It has been said that he didn't want to get embroiled in the bitter argument between the Mendelists and biometricians. I would like to think that he didn't want it to be seen by his mathematician colleagues.'[3]

Yet, given that Hardy did not advocate uselessness, there is little reason to believe Hardy would have found the law 'distastefully practical'. He did find the result trivial (he says as much in the letter: 'the very simple point which I wish to make'[4]). Would its triviality have embarrassed him? This reaction would depend on his *knowing* that his letter became widely known. Reginald Punnett (1875–1967), the geneticist who drew Hardy's attention to the problem the letter addressed, knew that Hardy had 'not the slightest interest in genetics' and thus phrased it as a mathematical problem,[5] which counts against Hardy having been aware of how famous his result became in genetics.

Applied mathematics in pure mathematics

It has become a cliché to note that the application of number theory in cryptography is a counterexample to Hardy's view that 'real mathematics' was useless. But there is a much stronger example that is almost contemporaneous with the *Apology*: the work of R. Leonard Brooks (1916–93), Cedric Smith (1917–2002), Arthur Stone (1916–2000) & William Tutte (1917–2002), published in 1940, on 'squaring the square'.[6] This problem asks whether a square, or more generally a rectangle, with integer side lengths can be dissected into

[1] Hardy, *Apology*, § 29.

[2] Hardy, 'Mendelian proportions in a mixed population'.

[3] Crow, 'Eighty Years Ago', p. 474.

[4] Hardy, 'Mendelian proportions in a mixed population', p. 49.

[5] Punnett, 'Early Days of Genetics', p. 9.

[6] Brooks et al., 'The dissection of rectangles into squares'.

smaller squares with integer side lengths, no two of which have equal size. (Figure 13.5 shows a later example.) Brooks et al. transformed the notion of a 'squared rectangle' into a directed graph: each horizontal line segment in a hypothetical squared rectangle corresponds to a node in the graph and a square between two such line segments corresponds to an edge from the upper line segment to the lower. They then viewed this directed graph as an electrical network, with the side-length of each square corresponding to current flowing in the edge (see Figure 13.6). This new viewpoint allowed them to draw on the mathematics of the theory of electrical networks to build up results on squared rectangles and to give examples of squared squares.

Here, therefore, is an instance of a problem in pure mathematics solved by techniques imported from applied mathematics. One cannot be sure that Hardy would have accepted it as 'real mathematics', but it certainly attracted the attention of real mathematicians like Max Dehn (1878–1952).[1] The problem perhaps lacks significance by Hardian standards, since it does not connect different mathematical ideas.[2] Nevertheless, the *solution* by Brooks et al. would be significant, since it links geometry and the theory of electrical networks.

Doubtless one could give a formal treatment of the theory Brooks et al. built which made no reference to electricity and instead just applied certain given rules. But this possibility is irrelevant, for Hardy's distinction between 'real' and 'applied' mathematics was grounded upon the *motivation* for their study: real mathematics was that 'which

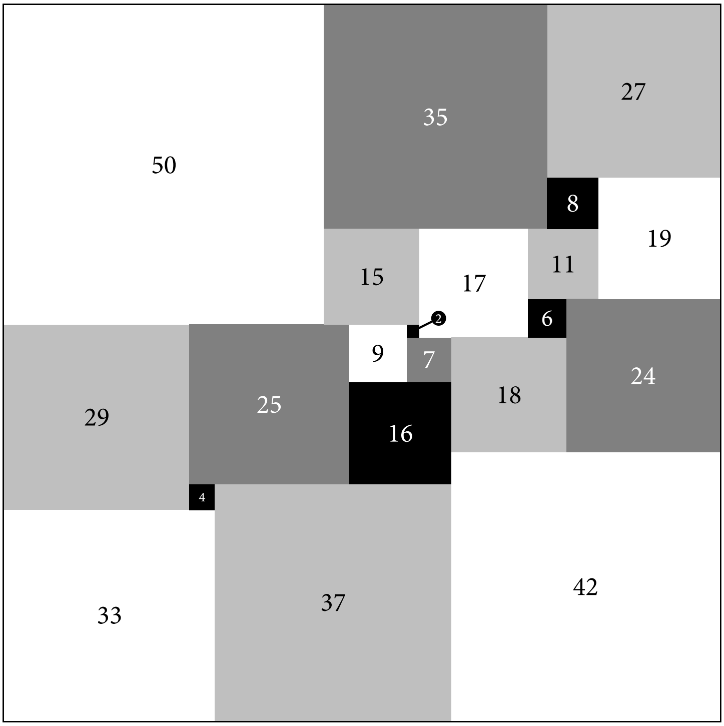

FIGURE 13.5
A square of side length 112, dissected into 21 squares of side lengths 2, 4, 6, 7, 8, 9, 11, 15, 16, 17, 18, 19, 24, 25, 27, 29, 33, 35, 37, 42, 50 (Bouwkamp & Duijvestijn, 'Catalogue of simple perfect squared squares', p. 1).

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[1] Dehn, 'Über Zerlegung von Rechtecken in Rechtecke'.

[2] Cf. Halmos, 'Applied Mathematics Is Bad Mathematics', pp. 10–11.

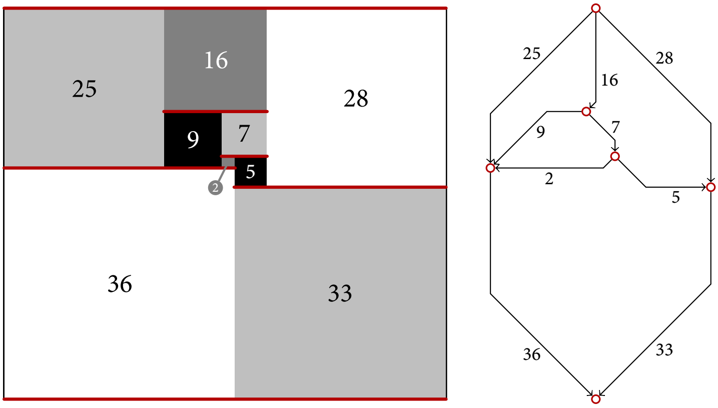

FIGURE 13.6
A 69 × 61 rectangle, dissected into 9 squares of side lengths 2, 5, 7, 9, 16, 25, 28, 33, 36, and the corresponding network (Brooks et al., 'The dissection of rectangles into squares', p. 314).

must be justified as art if it can be justified at all.'[1] The techniques which Brooks et al. applied from the theory of electrical networks, and in particular Ohm's and Kirchoff's laws, had no aesthetic justification.

Brooks et al. also gave a neat proof, appealing to spatial thinking, that there is no analogue of squared rectangles in higher dimensions: that is, there is no 'cubed cuboid'.[2] They argued as follows: suppose that a cubed cuboid does exist. The faces of the cubes that form the bottom face form a squared rectangle. Since the smallest square in a squared rectangle clearly cannot touch a side of the rectangle, the smallest of the cubes in the bottom layer must stand on an internal square. Its top face is enclosed by 'walls' formed by the larger surrounding cubes. Cubes must therefore stand on its top face and induce a squared square. This process continues indefinitely, which contradicts the side lengths being integers.

[Littlewood described this proof as 'elegant',[3] which is an indication that, unlike Hardy, he thought visually, not purely syntactically.]

Unseriousness and beauty

As an illustration of how the Hardian triad can be found in unserious mathematics, the author here presents an anachronistic example: a problem of retrograde analysis in chess invented by Raymond Smullyan (1919–2017). The question is to determine, given the board position shown in Figure 13.7 and the knowledge that it has been reached in a legal (if perhaps unlikely) game of chess, which piece stands on h4.[4] Smullyan's book *The Chess Mysteries of Sherlock Holmes* is framed as a series of vignettes involving the great detective in discussions of retrograde analysis — true deduction rather than his usual abductive logic. Confronted with this problem, he unfolds a *tour de force* that begins with the observation that the check on the black king from the rook on d7 must have been discovered by a pawn on c7 capturing on d8 and under-promoting to a rook, proves as a lemma that the one black pawn missing from the board must have under-promoted to a knight or bishop, uses this lemma to

[1] Hardy, *Apology*, § 28.

[2] Brooks et al., 'The dissection of rectangles into squares', § 10.4.

[3] Littlewood, *Miscellany*, p. 29.

[4] Smullyan, *The Chess Mysteries of Sherlock Holmes*, pp. 30–2.

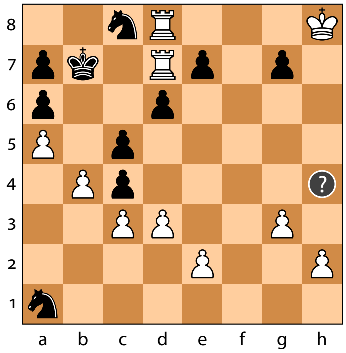

FIGURE 13.7
A problem of retrograde analysis in chess. This position has been reached in a legal chess game; the object is to identify the piece, black or white, on h4. (Smullyan, *The Chess Mysteries of Sherlock Holmes*, pp. 30–2)

argue that the unknown piece must be white, analyses which black pawn promoted, where it promoted, where it had to capture (for it promoted on a different file whence it started), and finally appeals to the colours of squares on which white pieces were captured.[1] Unexpectedness and inevitability and economy it surely has: Hardy would have conceded it beautiful, if unserious.[*]

Reviews of the Apology

Responses to the *Apology* by contemporaneous reviewers were varied, ranging from strong approbation to cutting dismissal. This section only surveys those reviews that specifically address Hardy's aesthetic views.[2]

E. T. Bell, 'Confessions of a Mathematician'

(*The Scientific Monthly*, January 1942)

Bell (1883–1960), mathematician and popular historian of mathematics, described the *Apology* as a 'sardonic confession'. The adjective is better applied to Bell's own short review, where he wrote that the *Apology* could be 'specially commended to solemn young men who believe they have a call to preach the higher arithmetic to mathematical infidels'.

In response to Hardy's assertion that '"[i]maginary" universes are so much more beautiful than this stupidly constructed "real" one',[3] Bell wrote: 'Well, God, not the mathematical physicist, must take the blame.' He dismissed Hardy's platonism as 'a museum piece from an incredibly credulous past'.

Bell's review was not detailed, but, read in conjunction with his own views on the balance between pure and applied mathematics, and his acceptance of aesthetic value as a motivation, it suggests that his fundamental objection was to Hardy's rejection of applications as a motivation. In addition, he may have objected to Hardy's rhetoric: he wrote elsewhere that '[m]athematics can not be made beautiful by spouting about its beauty'.[4]

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[1] Smullyan, *The Chess Mysteries of Sherlock Holmes*, pp. 33–7.

[*] ▸ p. 499

[2] For a full survey, see Cain, 'Reviews of the *Apology*'.

[3] Hardy, *Apology*, § 26.

[4] Bell, 'The Poetry of Mathematics', p. 562.

R. B. Braithwaite, Review of *A Mathematician's Apology* (*Mind*, October 1941)

The philosopher Braithwaite's (1900–90) review was generally sympathetic, but critical on certain points. He felt that Hardy should have chosen more example theorems from outside number theory.[1] He thought that Euclid's theorem on the infinitude of primes was much more beautiful than the result that $\sqrt{2}$ is irrational, for it is not purely negative: although Hardy gave it as a *reductio*, the proof of the infinitude of primes can be written in a positive form. The proof of the irrationality of $\sqrt{2}$ was a *reductio* proper, and the theories of proportions and real numbers, which can be traced to it, had, for Braithwaite, 'the æsthetic qualities which it itself lacks'.[2] He also gave what he considered to be immediate counterexamples to Hardy's thesis that it was the dull parts of mathematics that are useful.

Further, Braithwaite criticized Hardy's claim that 'Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not',[3] on the grounds that all thought, including mathematical thought, is dependent on a means of using symbols, and so 'mathematics will vanish with the rest of our intellectual heritage if we revert to our pre-linguistic apehood'.[4]

C. D. Broad, Review of *A Mathematician's Apology* (*Philosophy*, July 1941)

The philosopher of science and historian of philosophy Broad (1887–1971), along with C. P. Snow, read and commented on the manuscript of the *Apology*.[5] Broad's review was generally positive but criticized Hardy on individual points. He agreed with Hardy that mathematics is a form of artistic creation[6] and that its value derived from the discovery of patterns that were beautiful and significant. While he agreed with Hardy that '[h]e who demands some extrinsic justification for it betrays himself as a philistine',[7] Broad thought that Hardy's irritation with such philistinism led him to exaggerate his case when he claimed that it is the 'dull and elementary'[8] parts of mathematics, pure and applied, that work for good or ill:

> 'Surely it would be difficult to deny that Newton's theory of gravitation, Laplace's and Hamilton's reduction of the laws of dynamics to the Principle of Least Action, and Maxwell's theory of the electro-magnetic field are intrinsically beautiful and serious bits of mathematics. And surely they have had extremely important technical applications, both for good and for ill.'[9]

Broad agreed that the two examples of serious and beautiful theorems Hardy gave were 'quite obviously weighty and beautiful',[10] but argued that it would have been better if Hardy had distinguished between the theorem and the proof, for Broad felt that in these examples the beauty lies in the reasoning and the seriousness in the result. He did allow that in more complicated examples the theorem and the proof might share both seriousness and beauty. Similarly, with regard to the aesthetic qualities Hardy identified, he supposed that inevitability and economy lay in the proof and unexpectedness in the conclusion.[1]

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[1] Braithwaite, Review of *A Mathematician's Apology*, p. 420.

[2] *Loc. cit.*

[3] Hardy, *Apology*, § 8.

[4] Braithwaite, Review of *A Mathematician's Apology*, p. 421.

[5] See Hardy, *Apology*, Preface.

[6] Broad, Review of *A Mathematician's Apology*, p. 325.

[7] *Loc. cit.*

[8] Hardy, *Apology*, § 25.

[9] Broad, Review of *A Mathematician's Apology*, p. 326.

[10] *Ibid.*, p. 325.

Ananda K. Coomaraswamy, Review of *A Mathematician's Apology* (*The Art Bulletin*, December 1941)

The philosopher and art historian Coomaraswamy (1877–1947) engaged primarily with Hardy's discussion of beauty in mathematics. He praised Hardy's analysis, but criticized how Hardy rated mathematical beauty above that found in art. Hardy quoted lines from *Richard II* and thought their outward form was beautiful but the ideas they expressed were false and trite:[2]

> 'Not all the water in the rough rude sea
> Can wash the balm from an anointed King.'[3]

According to Coomaraswamy, what this example really demonstrated was not that beauty in art was independent of the validity of the ideas, but that beauty and validity were both relative to context: 'to the reviewer Shakespeare's words are beautiful *and* true, but they are not true for Professor Hardy or in any democratic context'. Coomaraswamy went on to suggest that Hardy should have applied his criteria of intelligibility and economy to art, and his avowed lack of qualifications in aesthetics[4] would have been no hindrance.

Finally, Coomaraswamy agreed that Hardy made the right decision to pursue mathematics, for it was good for his soul, as shown by the kind of monument he would have desired: a statue on a column so high that the features were visible.[5] Thus, thought Coomaraswamy, 'since it is man's first duty to work out his own salvation (from himself), no further defence is needed'.

E. H. Neville, Review of *A Mathematician's Apology* (*The Mathematical Gazette*, May 1941)

The mathematician Neville (1889–1961) accepted Hardy's argument that the justification for devoting oneself to mathematics is the intrinsic aesthetic value of the mathematics itself. 'Every mathematician believes this in his heart', he wrote, and complimented Hardy's explanation and defence of this view. But Neville pointed out a contradiction lurking in the *Apology*: although Hardy stated that most people can appreciate mathematics to some degree, and that the best mathematics gave pleasure after millennia, he nevertheless wrote that '[m]athematics is not a contemplative but a creative subject'.[6] For Neville, this statement indicated that Hardy could gain no satisfaction from reading mathematics produced by others, unless it led him to new mathematics of his own. Without his respect for Hardy being diminished, Neville did not believe him.

[1] Broad, Review of *A Mathematician's Apology*, p. 325.

[2] Hardy, *Apology*, § 10.

[3] Shakespeare, *Richard II*, Act 3, Scene 2.

[4] Hardy, *Apology*, § 11.

[5] *Ibid.*, Note.

[6] *Ibid.*, § 28.

Frederick Soddy, 'Qui s'accuse s'acquitte'
(*Nature*, January 1941)

Soddy (1877–1956), who was the 1921 Nobel laureate in chemistry, was mentioned in the *Apology* as an example of how people distinguished in their own fields can gain great pleasure in discovering a mathematical result, in his case regarding the 'hexlet'.[1]

[For any sphere A (shown in black in Figure 13.8) containing spheres B and C (shown in blue) such that A, B, and C are mutually tangent, there is a chain of six spheres $S_0, \dots, S_5$ (shown in red) such that each S_i is tangent to A, B, C, and to its neighbours S_{i-1} and S_{i+1} (taking subscripts modulo 6). These six spheres constitute a *hexlet*. Soddy's result, which he first published in verse form in June 1936, is that *The mean of the bends of any two neighbouring spheres in the hexlet equals the sum of the bends of the original three spheres.*[2] (The bend of a sphere is the inverse of its radius.) Frank Morley (1860–1937), of the eponymous trisector theorem,[*] gave a proof of the hexlet result, which Soddy thought was 'elegant'.[3] Soddy also thought beautiful various related results that emerged shortly afterwards.[4]]

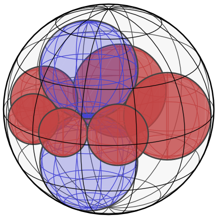

FIGURE 13.8
A hexlet: six spheres, each tangent to its neigbours and to three given spheres.

Soddy's caustic review begins 'This is a slight book. From such cloistral clowning the world sickens.'[5] The tone throughout is one of biting sarcasm. He disparaged Hardy's distinction of 'real' mathematics and the idea that 'trivial' mathematics is 'ugly in some sort of direct ratio to its usefulness'.[6] As for the assertion that '"[i]maginary" universes are so much more beautiful than this stupidly constructed "real" one',[7] Soddy's response was:

> 'Most scientists [...] believe that the saner outlook on Nature inaugurated by the experimental sciences does reveal the real universe, and that it is not stupidly constructed. Those of them duped by mathematical fantasy are to be put up with rather than pukkha'd.'[8]

Legacy of the Apology

A Mathematician's Apology has had a lasting influence: it is cited in discussions of ontology, epistemology, aesthetics, motivations, and creativity.[9] One reason for this is Hardy's beautiful, limpid, supremely quotable prose style; another is the *Apology*'s conciseness; another is the passion that is rare in philosophy of mathematics, at least in writing. It remains 'one of the most beautiful statements about the creative mind ever written or ever likely to be written.'[10] Many later discussions were inspired by it, and it compares well with them: as the writer David Foster Wallace (1961–2008) put it in 2000, the *Apology*

> 'is the unacknowledged father of most of the last decade's math-prose. There is very little that any of the recent books do that

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[1] Hardy, *Apology*, § 10.

[2] Soddy, 'The Kiss Precise'.

[*] ▸ p. 3

[3] Soddy, 'The Hexlet (Jan. 1937)'.

[4] Soddy, 'The Hexlet (Jan. 1937)'; Soddy, 'The Hexlet (Feb. 1937)'.

[5] Soddy, 'Qui s'accuse s'acquitte', p. 3.

[6] *Loc. cit.*

[7] Hardy, *Apology*, § 26.

[8] Soddy, 'Qui s'accuse s'acquitte', p. 3.

[9] For a full study, see Cain, 'Legacy of the *Apology*'.

[10] Snow, 'The Classical Mind', p. 812.

> Hardy's terse and beautiful *Apology* did not do first, and with rather less fuss.'[1]

The views that Hardy presented in the *Apology*, and in particular the lack of interest in applications, seem to have long held strong appeal for pure mathematicians. Michael Harris (1954–) related how reading the book as a teenager shaped his view of mathematics long afterwards.[2] Marcus du Sautoy (1965–) wrote when he was younger he 'was under the spell of G. H. Hardy's *A Mathematician's Apology*'.[3]

Even when not specifically linked to Hardy, it seems likely that the *Apology*'s defence of mathematics as an aesthetic pursuit was a major cause of the continuing support for this view amongst mathematicians. Recent Hardian echoes can be heard in statements such as: '[a]pplicability is not the reason we work, and plenty that is not applicable contributes to the beauty and magnificence of our subject'[4] and '[a]pplication is not the point […] "Beautiful intellectualizing, that is the satisfaction. […]"'.[5]

In the discourse on mathematical beauty, Hardy's *Apology* was a watershed: it shaped essentially all subsequent debate. Hardy's exposition and examination of mathematical beauty became a canonical statement that is taken as representative of mathematicians' thinking. His attitude to applications has often been criticized, though sometimes as the consequence of the misunderstandings discussed above. But in the debate since the *Apology* appeared, it has proved impossible to ignore; it is hard to find a recent work on mathematical beauty where it has not been cited, and harder still to find one that has not felt its influence. The aesthetics of mathematics became, and remained, *post-Hardian*.

RÓZSA PÉTER

Rózsa Péter (1905–77) was the founder of the theory of recursive functions, writing many of the early research articles as well as the first textbook. Despite her focus and influence on this area, her legacy has been less recognized than it should have been.[6] She made a similarly important but little-cited contribution to the discourse on aesthetics, utility, and the motivation for mathematics, which deserves to be set beside and contrasted with Hardy's.

Péter obtained her Ph.D. in 1935 and in 1937 became a member of the editorial board of the *Journal of Symbolic Logic*.[7] But she did not hold a permanent teaching appointment until after the end of World War II, and was dismissed from her temporary post in 1939 under fascist legislation.[8] She used her time to write her elegant little book *Playing with Infinity*, which grew out of letters to the writer Marcell Benedek (1885–1969), who lamented his inability to understand

[1] Wallace, 'Rhetoric and the Math Melodrama', p. 2267, n. 1.

[2] Harris, *Mathematics without Apologies*, ch. 10.

[3] du Sautoy, *Finding Moonshine*, ch. 10.

[4] Rowlett, 'The unplanned impact of mathematics', p. 166.

[5] Roberts, *Genius At Play*, ch. 16; the inner quotation is from John Conway and the remainder seems to be a paraphrase of his words.

[6] Morris & Harkleroad, 'Rózsa Péter'.

[7] Andrásfai, 'Rózsa Péter', p. 140; Morris & Harkleroad, 'Rózsa Péter', p. 60.

[8] Andrásfai, 'Rózsa Péter', p. 140.

mathematics and thought that this shortcoming was manifested in a weakening of his writing skills.[1]

Péter completed the book in late 1943, and it was published first in Hungarian in 1944, and subsequently translated into German, English, and other languages. Péter's declared aim was to make some of the beauty of mathematics accessible to non-mathematicians like Benedek:

> 'This book is written for intellectually minded people who are not mathematicians. It is written for men of literature, of art, of the humanities. I have received a great deal from the arts and I would now like in my turn to present mathematics and let everyone see that mathematics and the arts are not so different from each other. I love mathematics not only for its technical applications, but principally because it is beautiful.'[2]

Péter revisited the topic of mathematical beauty in a lecture delivered to East German high school teachers and students in 1963. She opened by declaring that her aim was to pass on her belief in the beauty of mathematics, and that, while she sometimes doubted her own worthiness for mathematics, she never doubted that mathematics was worthy of her devotion.[3] This lecture, which Péter admitted aimed to 'propagandize' for *Playing with Infinity*,[4] explicitly addressed the aesthetic motivation. To anyone who reduced mathematics to the monotony of calculation and asked how one could call beautiful 'the dull "two times two"', she said that she aimed to demonstrate that 'mathematics is not dull and not "two times two," but rather has much in common even with art'.[5]

Péter did not make any analysis of beauty in mathematics. She rather gave an informal discussion of various mathematical concepts, including number theory, combinatorics, and logic, which all contained beautiful theorems. An example of a beautiful theorem in number theory[6] was the result conjectured by Joseph Bertrand (1822–1900) and first proved by Pafnuty Lvovich Chebyshev (1821–94) that

For any $n \in \mathbb{N}$*, there is a prime number between* n *and* $2n$*.*

Some hints about Péter's notion of mathematical beauty can be gleaned from her discussion. Notably, Péter thought that mathematics could have different kinds of beauty. There was a beauty in overt regularity, but there was also a beauty in an apparent lack of organization being constrained by some hidden rule; Péter's example of this latter kind of beauty was the Prime Number Theorem,* where the seemingly chaotic distribution of the prime numbers was subject to a simple rule.[7]

While Péter saw the value of the applications of mathematics, when it came to areas such as number theory, she recognized that the motivations were mainly aesthetic: 'Here it is not utility that inspires research, nor convenience, but the beauty and the difficulty of the subject.'[8]

[1] Andrásfai, 'Rózsa Péter', pp. 141–2.

[2] Péter, *Playing with Infinity*, p. v.

[3] Péter, 'Mathematics Is Beautiful', p. 58.

[4] *Loc. cit.*

[5] *Loc. cit.*

[6] Péter, *Playing with Infinity*, p. 63.

* ▸ p. 407

[7] *Ibid.*, p. 65.

[8] *Loc. cit.*

For Péter, a particular parallel between mathematics and art is encountering the infinite, which she illustrated with a comparison to the opening of Thomas Mann's (1875–1955) *Joseph and his Brothers*:

> 'Deep is the well of the past. Should we not call it *bottomless*?
>
> […] And yet what happens is: the deeper we delve and the farther we press and grope into the underworld of the past, the more totally *unfathomable* become those first foundations of humankind, of its history and civilization, for again and again they retreat farther into the *bottomless* depths, no matter to what extravagant lengths we may unreel our temporal plumb line.'[1]

Although the 'well of the past' can seem frightening in its limitless depth, Péter thought that mathematics could serve as a way to confront and control the infinite.[2]

Péter's intention was to help those who thought mathematics was useful but dull — 'a necessary evil' — to see that it was useful and beautiful.[3] She pointed to her own work in the theory of recursive functions, which had emerged for reasons internal to pure mathematics, and which she 'would not have dreamed […] could also be applied practically'. And yet her book *Recursive Functions* was published in multiple editions and languages precisely because its subject-matter had become important in computer science:

> 'And so it goes, sooner or later, for all branches of so-called pure mathematics. […] one never has to worry that one is working on something useless, either.'[4]

Péter wrote *Playing with Infinity* during World War II. It is therefore natural to contrast this book with Hardy's *Apology*, and more generally to contrast Péter's and Hardy's positions. Both books praised mathematics for its beauty and proclaimed the aesthetic motivation for its study. But Péter did not draw a Hardy-esque dividing line that marked off as 'real mathematics' those areas pursued for aesthetic ends. While Hardy thought that it was very unlikely that any part of 'real mathematics' would find practical application, Péter rather had a faith that *any* branch of mathematics, even if pursued for aesthetic or other non-utilitarian motives, would in time find application.

Unlike Hardy, Péter did not exhibit the melancholy of a creative mathematician past their best. Perhaps, in the end, their contrasting attitudes were partly a consequence of the optimism and pessimism of the young and the old.

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[1] Mann, *Joseph and his Brothers*, p. 3, emphases added.

[2] Péter, 'Mathematics Is Beautiful', p. 64.

[3] *Loc. cit.*

[4] *Loc. cit.*

BEAUTY AND MOTIVATION, AFTER THE *APOLOGY*

This section is devoted to authors who commented on the aesthetic motivation for mathematics in a way that responded, explicitly or implicitly, to Hardy's *Apology*.

John von Neumann

In 1946, John von Neumann (1903–57) delivered a lecture entitled 'The Mathematician' as part of a series in which prominent persons were invited to discuss the creative fields in which they worked.[1] Von Neumann did not cite the *Apology* (nor any other work), but his lecture certainly suggests that he was aware of Hardy's views. Von Neumann's description of how mathematicians judge their own success was in agreement with Hardy: 'the mathematician's subjective criterion of success, of the worth-whileness of his effort, is very much self-contained and aesthetical and free (or nearly free) of empirical connections.'[2]

Furthermore, the features that von Neumann gave of beautiful theorems and theories were reminiscent of Hardy:

> 'One expects a mathematical theorem or a mathematical theory not only to describe and to classify in a simple and elegant way numerous and a priori disparate special cases. One also expects "elegance" in its "architectural," structural makeup. Ease in stating the problem, great difficulty in getting hold of it and in all attempts at approaching it, then again some very surprising twist by which the approach, or some part of the approach, becomes easy, etc. Also, if the deductions are lengthy or complicated, there should be some simple general principle involved, which "explains" the complications and detours, reduces the apparent arbitrariness to a few simple guiding motivations, etc.'[3]

Using other language, von Neumann described the Hardian qualities of unexpectedness ('some very surprising twist') and inevitability ('reduces the apparent arbitrariness').

But von Neumann was cautious about mathematics venturing too far away from applications. It is not bad to pursue a purely mathematical topic for its for own sake, *provided that* the field is guided either by its relationship to other areas that have a greater empirical component, or else by 'the influence of men with an exceptionally well-developed taste'.[4] Without this guidance, there is a 'danger of degeneration': that the subject 'will separate into a multitude of insignificant branches, and [...] will become a disorganized mass of details and complexities'.[5] In this situation, the only way for the discipline to recover is via an infusion of new ideas from empirical experience.

[1] See Heywood, *The Works of the Mind*.

[2] von Neumann, 'The Mathematician', pp. 6–7.

[3] *Ibid.*, pp. 8–9.

[4] *Ibid.*, p. 9.

[5] *Loc. cit.*

Helmut Hasse

In 1952, the algebraist and number theorist Helmut Hasse (1898–1979) gave a public lecture in Hamburg, expanding on his inaugural lecture at the University, in which he addressed his motivation for mathematics.[1]

He made a passing referencing to Hardy's *Apology*, but only to contrast his own position: he thought that mathematics needed to offer no humble apology. Rather, it should proudly declare:

> 'look with awe upon the deep insights into the realm of thought that I have won; admire my crystal-clear, living beauty and be amazed at the capabilities I put into your hands!'[2]

Thus he offered three motivations: epistemological, aesthetic, and utilitarian. But he admitted that the first, the desire to uncover the truth, was what attracted him to mathematics, for it was in mathematics, with its permanence and objectivity, that the drive to *know* was best fulfilled.[3]

But mathematics as a quest for knowledge was a specific kind of intellectual endeavour. Throughout his life, Hasse held fast to the thesis: 'Mathematics is one of the humanities [eine Geisteswissenschaft].'[4] But beyond this, he increasingly saw in mathematics the features of an *art*. For him, the parallel was primarily between mathematics and music, but he saw that it extended to other arts.[5] Formal correctness was necessary but not sufficient to qualify as 'mathematics in the true sense of the word';[6] what had to be added was the mathematical analogue of certain musical qualities: '*crystal-clear beauty* that lies above the whole and the *powerful dynamics* that permeate the whole thing.'[7] (Note the repetition of the idiom: 'crystal-clear beauty [kristallklare Schönheit]'.)

The mathematician, like the artist and unlike the natural scientist, was unconstrained by empirical adequacy, and can follow their inspiration and create freely.[8] In physics, by contrast, it had happened that an uglier theory had displaced a more beautiful one; fortunately, the uglier theory had usually turned out to be a step in the development of a new, more aesthetically pleasing one.[9]

Hasse identified three 'levels' or 'scales' in which beauty could be found. In music, from small to large, these were: the euphony of a note or chord; the beauty of a melody or chord progression; and the beauty of a movement or entire work.[10] There was a parallel division in mathematics: there is the beauty of a formula or single statement; the beauty of a proof; and the beauty of an entire theory.[11]

Hasse gave examples of the small-scale beauty of mathematics; one was a comparison of two formulae for the determinant of a 3 × 3 matrix, semantically identical, but with different symbols:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh;$$

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[1] Hasse, *Mathematik als Wissenschaft, Kunst und Macht*, p. 2.

[2] *Ibid.*, p. 4; trans. C.

[3] *Ibid.*, p. 8.

[4] *Ibid.*, p. 5; trans. C.

[5] *Ibid.*, p. 9.

[6] *Ibid.*, pp. 10–11.

[7] *Ibid.*, p. 10; trans. C., emphasis in orig.

[8] *Ibid.*, p. 11.

[9] *Loc. cit.*

[10] *Ibid.*, p. 12.

[11] *Ibid.*, pp. 12–16.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \left\{ \begin{matrix} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} \end{matrix} \right\}$$

The use of subscripts in the second formula made the organization of the terms clearer: in each term, the first numbers in the subscripts are in the sequence 1, 2, 3, while the second numbers correspond to one of the six permutations of {1, 2, 3}.[1] And greater clarity and conciseness were generally correlated with greater beauty.[2]

Medium-scale beauty, on the level of proofs, came about from the path being naturally determined and straightforward[3] (which may be akin to the Hardian quality of inevitability).

The large-scale beauty of a theory involved the clear formulation of each of the individual components, their mutual relationship, and how they formed a harmonious whole.[4] Hasse especially praised those who were behind such theories:

> 'Mathematicians who manage to conceive such a unified theory not only in rough outlines, but also to design it in its details, [...] are to be compared with the creators of monumental works of art, a BEETHOVEN, GOETHE, MICHELANGELO.'[5]

Hasse sounded a note of caution about elegance:

> 'Just as in art, however, in mathematics one must also beware of *exaggerated* elegance, which, although momentarily dazzling, overshadows the actual artistic or mathematical content.'[6]

He did not make clear in what elegance consisted. It clearly referred to something superficial to the underlying ideas, but, given Hasse's discussion of small-scale beauty, it presumably excluded changes of notation. Perhaps it simply referred only to the presentation of a proof or other medium-scale concept. In the other direction, he decried what he saw as a tendency for excessive formalization and axiomatization to displace normal-language mathematics.[7] He saw this as a particular problem in mathematical literature from North America, where he thought that 'the feeling for the *aesthetic* values of mathematics [...] is less developed there than in European countries'.[8]

Having discovered something beautiful, a 'real mathematician' would desire to divulge it. But the crude form of the new discovery was not satisfactory for this: it had to be recast into 'a form of *perfect* beauty'.[9] How mathematics is communicated can either veil or bring to light its beauty. For Hasse, this difference in expression was paralleled in art, where a beautiful idea might be hidden by an ugly form, or a beautiful piece of music might be disguised by a poor performance.[10] But there was a limit to how much the mathematician should strive towards perfecting their work,[11] and even unpolished work could be beautiful: 'some things are perhaps beautiful precisely because they have not yet reached the perfect but cold state of classical beauty'.[12]

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[1] Hasse, *Mathematik als Wissenschaft, Kunst und Macht*, p. 13.

[2] *Ibid.*, pp. 12–13.

[3] *Ibid.*, p. 15.

[4] *Ibid.*, pp. 15–16.

[5] *Ibid.*, p. 15; trans. C., emphasis in orig.

[6] *Ibid.*, p. 15; trans. C., emphasis in orig.

[7] *Ibid.*, pp. 14–15, 18–19.

[8] *Ibid.*, pp. 14–15; trans. C., emphasis in orig.

[9] *Ibid.*, pp. 16–17; trans. C., emphasis in orig.

[10] *Ibid.*, p. 17.

[11] *Ibid.*, pp. 17–19.

[12] *Ibid.*, p. 19.

Hasse said that the aesthetic values of mathematics should have *absolute* priority over the practical considerations.[1] But this was partly because experience shows that pure mathematicians were those who in the end contribute most to the natural sciences and technology, and the breadth of their study meant that, when called upon to work in a practical setting, they outdid those mathematicians with a specialized training.[2]

W. W. Sawyer

The mathematician W. W. Sawyer (1911–2008) gave a nuanced account of beauty and utility as motivations for mathematics in the first chapter of his book *Prelude to Mathematics*, first published in 1955. He described the scope of mathematics, which he defined for the purposes of his discussion as 'the classification and study of all possible patterns',[3] and how the concepts of mathematics find application in physics and engineering. But he then imagined a pure mathematician protesting:

> '"You are treating mathematics [...] as something useful. But mathematics is not a means to something else; it is an end in itself. Not the usefulness of mathematics is important but the beauty. Technical mathematics is the dullest part of mathematics. Look at the people working on the theory of numbers, which has no application at all; would you prefer them to be working at bookkeeping?"'[4]

Sawyer said that this view was 'sincerely held by many eminent and some great mathematicians'.[5] The protesting mathematician seems to have read Hardy's *Apology*, but perhaps without fully understanding it. Hardy might have agreed that mathematics is an end in itself, and would certainly have held that technical mathematics happened to be dull. But he would not have agreed that its usefulness is unimportant. Utility might not have been a value Hardy sought, but he recognized that it was important. (This is not to suggest that *Sawyer* did not understand the *Apology*; the errors of character and author are not to be conflated.)

Opposed to the position of the pure mathematician character was 'the utilitarian, bureaucratic view of mathematics', under which aesthetic motivations were to be shunned, and mathematics research directed by offialdom towards solving problems of immediate utility.[6]

Both positions were evidently deliberate caricatures, and Sawyer said that the motivation for mathematics should be located between them. Utility should be one of the motivations for mathematics, but any mathematical researcher who worked only towards utilitarian ends, without taking any joy in the research itself, could be expected to produce mediocre work. '[T]he beauty without the power is futile. [...] Power without beauty is liable to be impotent.'[7]

[1] Hasse, *Mathematik als Wissenschaft, Kunst und Macht*, p. 15.

[2] *Ibid.*, pp. 21–2.

[3] Sawyer, *Prelude to Mathematics*, p. 12.

[4] *Ibid.*, p. 16.

[5] *Loc. cit.*

[6] *Loc. cit.*

[7] *Loc. cit.*

C. Stanley Ogilvy

C. Stanley Ogilvy's (1913–2000) popular mathematics book *Through the Mathescope* (1956) contains an account of the motivation for mathematics that reads like a less assertive version of Hardy: pure mathematics is done for its own sake; applicability is not a motivation. The pure mathematician

> 'does mathematics for the same reasons that the artist paints: for the fascination of the subject itself; for the satisfaction of producing something that to himself and to his colleagues is beautiful; and, if he has a spark of real genius, because he can't help it. Mathematics, for all its scientific dress, is very like an art.'[1]

Indeed, Ogilvy explicitly compared his discussion with the *Apology*, although he said that Hardy 'sometimes overstates the case'.[2]

However, for Ogilvy, to have *any value*, a mathematical work must be 'serious' in the Hardian sense: it must have connections to established ideas, lest it resemble a work of modern art whose meaning escapes everyone, including the artist.[3]

Symposium

In a 1961 symposium on the place of applied mathematics in research and education, the mathematicians George F. Carrier (1918–2002), Richard Courant (1888–1972), Paul C. Rosenbloom (1920–2005), and the physicist Chen-Ning Yang (1922–2025), who was one of the 1957 Nobel laureates in physics, discussed matters such as usefulness, aesthetics, and motivation.[4] Hardy's *Apology* clearly sat in the background of the discussion, and two of the four speakers explicitly discussed it.

Of the four speakers, Courant was the most obviously opposed to Hardy's views. He dismissed as 'old blasphemous nonsense'[5] the idea of mathematics being ultimately justified on an aesthetic level, or indeed on any other level of purely intellectual satisfaction. He did not deny that mathematics can be beautiful, but simply that the pursuit of such beauty cannot justify mathematics. He further argued that '[m]athematics must not be allowed to split and to diverge towards a "pure" and an "applied" variety.'[6] That is, contra Hardy, he did not see any division between pure and applied mathematics in terms of their content. Rather, there was a only a difference in motivation between pure and applied *mathematicians*: the latter has 'a profound interest in the connection between mathematics and what may be called reality'. As a mathematical platonist, Hardy might have questioned this distinction, since he thought that research brought the mathematician into closer contact with reality than the physicist.*

Rosenbloom was more sympathetic to the role of aesthetic concerns in mathematics and science generally. He noted that there was only 'very inadequate evidence' for the general theory of relativity, but that it was accepted on grounds of elegance, simplicity, economy, and unity. (Today, there is much greater empirical evidence than there was at the time of the symposium.[1]) Thus to teach applied mathematics purely from the perspective of usefulness would be to elide a principal aspect of how judgements are actually made in applied mathematics.[2]

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[1] Ogilvy, *Through the Mathescope*, p. 6.

[2] *Ibid.*, p. 131.

[3] *Ibid.*, pp. 6–7.

[4] Carrier et al., 'Applied mathematics'.

[5] *Ibid.*, p. 298.

[6] *Loc. cit.*

* ▸ pp. 498–9

Rosenbloom also made an incisive observation which may go some way towards explaining some of the disdain in which some mathematicians held applications. In certain institutions, mathematics departments had to struggle to win for themselves an existence independent of servicing other departments. Their efforts may have contributed to a certain pride in developing mathematics for its own sake, rather than for its applications.[3] Rosenbloom further pointed out that most people who will have to teach applied mathematics became mathematicians for aesthetic reasons, and that therefore 'if you want to get applied mathematics into the curriculum of the college and the graduate school, you are going to have to present it in such a way as to emphasize that applied mathematics can also be beautiful, can also be deep.'[4] Carrier, emphasized that this was indeed possible: 'The use of mathematics in science and elsewhere can be as challenging, as esthetically pleasing, and as valuable to society, as the "pure" self-contained discipline.'[5]

Yang suggested that Hardy's training in pure mathematics was responsible for his lack of appreciation of the beauty found in the application of mathematics to the physical world; for Yang, this appreciation was vital for an applied mathematician.[6] Indeed, Hardy's views of applied mathematics were prescriptive of what mathematics should *not* be.[7]

Norman Levinson

In 1970, the mathematician Norman Levinson (1912–75) wrote an expositional article on the theory of error-correcting codes, which he considered to be an area of mathematics that was both beautiful and useful. The article is marred, however, by its stated aim of refuting a view of Hardy that includes the mischaracterization that Hardy was proud of the uselessness of his work.[*][8] It *does* successfully refute the idea that Hardy's 'real mathematics' lacked application. Although the theories of relativity and quantum mechanics had clearly become useful in the time since Hardy wrote and included them in 'real mathematics', Levinson made the observation that Hardy was not an expert in these fields. Thus a more secure refutation of this idea would use an example from pure mathematics. Levinson agreed with the aesthetic qualities Hardy identified in beautiful theorems and proofs and his aim was thus to show that these qualities were exhibited by the role of finite fields in coding theory; he also noted that the law of quadratic reciprocity, which Hardy thought beautiful,[9] enters

[1] See, for example, Will, 'The Confrontation between General Relativity and Experiment'.

[2] Carrier et al., 'Applied mathematics', p. 305.

[3] *Ibid.*, p. 307.

[4] *Loc. cit.*

[5] *Ibid.*, pp. 316–7.

[6] *Ibid.*, p. 310.

[7] *Ibid.*, p. 309.

[*] ▸ pp. 512–16

[8] Levinson, 'Coding Theory', p. 249.

[9] Hardy, *Apology*, § 12.

into coding theory.[1] While many works *mention* the applicability of Hardy's 'real mathematics' (in, for example, cryptography), Levinson's article appears to be the only work that actually makes the case at length. (Note, however, that the theory of error-correcting codes only began to develop after Hardy's death.)

Louis J. Mordell

Hardy was succeeded as Sadleirian Professor of Pure Mathematics at Cambridge by Louis J. Mordell (1888–1972), who wrote in 1970 a critique of the *Apology*, including the foreword by Snow in the 1967 reprint. Mordell made the important observation that Hardy often stated his views in absolute terms, without allowing for exceptions or limitations.[2]

To the Hardian triad of purely aesthetic qualities in theorems and proofs, namely unexpectedness, inevitability, and economy, Mordell added

> 'simplicity of enunciation. The meaning of the result and its significance should be grasped immediately by the reader, and these in themselves may make one think, what a pretty result this is. It is, however, the proof which counts. This should preferably be short, involve little detail and a minimum of calculations. It leaves the reader impressed with a sense of elegance and wondering how it is possible that so much can be done with so little.'[3]

The last two sentences seem to overlap with economy, but simplicity of enunciation clearly encompasses the concepts and notation used to express the result and proof; these are clearly distinct from Hardy's qualities.

Mordell also considered what would count as 'ugliness in mathematics', which Hardy mentioned in the *Apology* but did not define beyond describing ballistics and aerodynamics as 'repulsively ugly'.[4] For Mordell, ugly theorems and results included

> 'those involving considerable calculations to produce results of no particular interest or importance; those involving such a multiplicity of variables, constants, and indices, upper, lower, right, and left, making it very difficult to gather the import of the result; and undue generalization apparently for its own sake and producing results with little novelty. I might also mention work which places a heavy burden on the reader in the way of comprehension and verification unless the results are of great importance.'[5]

Mordell gave several counterexamples to the uselessness of 'real mathematics', including the application of conic sections to the orbits of the planets and of Riemannian geometry in relativity. He also noted

[1] Levinson, 'Coding Theory', p. 250.

[2] Mordell, 'Hardy's "A Mathematician's Apology"', p. 834.

[3] *Loc. cit.*

[4] Hardy, *Apology*, § 28.

[5] Mordell, 'Hardy's "A Mathematician's Apology"', p. 835.

that many new disciplines such as game theory and communications theory make increasing use of pure mathematics.

Finally, Mordell took issue with the view that mathematics is not a contemplative subject:

> 'Many people can derive a great deal of pleasure from the contemplation of mathematics, e.g., from the beauty of its proofs, the importance of its results, and the history of its development. But alas, apparently not Hardy.'[1]

However, what Hardy wrote was:

> 'Mathematics is not a contemplative but a creative subject; no one can draw much consolation from it when he has lost the power or the desire to create'.[2]

This statement is perhaps more nuanced than the position Mordell inferred: it suggests that Hardy could no longer enjoy the contemplation of mathematics *after* having lost his creative ability; contemplation would be a reminder of the passing of creative ability, leaving Hardy unable to enjoy it.

Seymour Papert

In a 1978 essay reflecting on mathematical cognition, the mathematician and educator Seymour Papert (1928–2016) discussed a study in which a number of participants attempted to prove that $\sqrt{2}$ is irrational while making their thinking explicit. Papert thought that this problem was particularly apposite to his discussion because the result had been chosen by Hardy as an exemplar of beauty in the *Apology*.[3]

Papert also compared the traditional proof — the one Hardy exhibited — to what he termed the '"flash" version of the proof':[4] deducing that if $p/q = \sqrt{2}$, then $p^2 = 2q^2$, which is impossible since the factorizations of p^2 and q^2 into primes each contain the factor 2 an even number of times, so there is an extra prime factor on the right-hand side of the equality, contradicting the Fundamental Theorem of Arithmetic.* Papert noted that many people were impressed by the brilliance of this proof, where the contradiction is essentially perceived instantly. But he thought that the traditional proof lost nothing for being sequential. Rather, the inexorable progress — what Hardy called inevitability — was a powerful feature, apparent even to those who did not know the Fundamental Theorem of Arithmetic.[5]

Papert suggested that inevitability did not necessarily contribute to beauty, because inevitability could be experienced in different ways: as a submission or surrender, or as exhilarating. And any of these experiences could be felt 'as beautiful, as ugly, as pleasurable, as repulsive, or as frightening'.[6]

[1] Mordell, 'Hardy's "A Mathematician's Apology"', p. 834.

[2] Hardy, *Apology*, § 28.

[3] Papert, 'The Mathematical Unconscious', p. 110; see Hardy, *Apology*, § 13.

[4] Papert, 'The Mathematical Unconscious', p. 114.

* ▸ p. 846

[5] *Ibid.*, pp. 114–5.

[6] *Ibid.*, p. 115.

Paul Halmos

The mathematician Paul Halmos (1916–2006) wrote in 1981 an essay that was provocatively titled 'Applied Mathematics Is Bad Mathematics'. Without explicitly citing the *Apology* (or indeed any other work), it clearly drew on Hardy's thought both in the general framing of its argument and in certain of its features (for example, using chess as an example of trivial mathematics[1]). Halmos argued that from one perspective, the difference between pure and applied mathematics was perhaps no better defined than the difference between pure and applied literature: there was a continuous spectrum ranging from one to another. Furthermore, aesthetic value could not serve as a distinction: applied mathematics can be beautiful too.[2]

There was a difference in motivations and attitudes between pure and applied mathematics:

> 'The motivation of the applied mathematician is to understand the world and perhaps to change it; the requisite attitude (or, in any event, a customary one) is one of sharp focus [...]. The motivation of the pure mathematician is frequently just curiosity; the attitude is more that of a wide-angle lens than a telescopic one'.[3]

Such fundamental differences were probable causes of the separate traditions in standards of exposition, aesthetics and 'perhaps even logical rigor'.[4] In particular, a pure mathematician would award another's work the highest praise by calling it 'beautiful'; an applied mathematician might prefer 'ingenious' or 'powerful'.[5]

Halmos ultimately explained his title with the aid of an artistic analogy:

> 'A portrait by Picasso is regarded as beautiful by some, and a police photograph of a wanted criminal can be useful, but the chances are that the Picasso is not a good likeness and the police photograph is not very inspiring to look at. Is it completely unfair to say that the portrait is a bad copy of nature and the photograph is bad art?'[6]

Halmos's point was that although the best discoveries of applied mathematics were great *applied mathematics* (mathematics as a mirror of nature) and deserved the highest respect and praise *as applied mathematics*, they were nevertheless bad *mathematics* (mathematics as an art).[7] Thus in the end Halmos's 'mathematics' was implicitly very close to the Hardian 'real mathematics'.

Nicholas Young

In the afterword to his 1988 textbook on Hilbert space, the mathematician Nicholas Young (1943–) engaged with Hardy's

[1] Halmos, 'Applied Mathematics Is Bad Mathematics', p. 17; cf. Hardy, *Apology*, §§ 10–11.

[2] Halmos, 'Applied Mathematics Is Bad Mathematics', pp. 12–13.

[3] *Ibid.*, p. 14.

[4] *Loc. cit.*

[5] *Ibid.*, p. 15.

[6] *Ibid.*, p. 20.

[7] Halmos, 'Applied Mathematics Is Bad Mathematics', p. 20; see also Albers & Alexanderson, *Mathematical People*, p. 127.

arguments regarding the value of mathematics.[1] He made the important point that Hardy justified *individuals* doing mathematics: the book was *A Mathematician's Apology*, not *An Apology for Mathematics*. Young suggested instead that mathematicians must justify themselves to society:

> 'This does not mean we must be crudely utilitarian, but it does mean we should be ready to give an account of the part played by mathematics as a whole in science, technology, industry, commerce and government, and that we should be subject to political judgement as to the resources it deserves.'[2]

He did not deny that the aesthetic value of mathematics exceeds the value of much other work that society supported, but he held that the argument for supporting mathematics research should be part of the argument for science and technology as a whole.[3] Thus his fundamental objection to the *Apology* was that it 'overemphasizes the individual':[4] a mathematician did not do their work alone but within a social structure. Science as a process was located in the activities of those who do it; '[i]t is a grand structure of which no one person can see more than a tiny part'.[5]

Although mathematicians might be driven by a desire for beauty, and perhaps in some cases were *only* motivated by beauty, it was impossible to ignore the service mathematics has rendered to the sciences generally, by supplying conceptual tools and by tackling new problems that were imported into mathematics from the other sciences.[6] Young gave the example of the emergence of functional analysis: when it became clear that there was a theory of great 'power and elegance' waiting to be established, and because the social circumstances were right, it attracted mathematicians, who built 'a splendid edifice' at least some parts of which found application.[7]

Peter Lax

The mathematician Peter Lax (1926–2025) wrote in a 1989 historical essay that in the middle of the twentieth century, American mathematicians considered their subject to be autonomous, governed by 'its own criteria of depth and beauty'.[8] Applications were then not a concern for mathematicians themselves, but rather for others who took up their ideas.[9] But times had changed: mathematicians and scientists were engaged with each other and ideas flowed in both directions. Not very long before, mathematicians might applaud attitudes such as '"applied mathematics is bad mathematics" or "the best applied mathematics is pure mathematics"', but not when Lax was writing.[10]

The first sentiment seems to refer to Halmos,* whom Lax discussed again in a 2005 interview. Lax recalled Halmos as having said that 'applied mathematics was, if not bad mathematics, at least ugly mathematics'. In response to this slightly mangled version of Halmos's position, Lax would point to the Abel Prize committee 'dwelling on the elegance' of his work.[1] To be precise, his Abel Prize citation used explicitly aesthetic language in one place, where it was noted that his work on scattering theory was

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[1] Young, *An introduction to Hilbert space*, Afterword.

[2] *Ibid.*, p. 230.

[3] *Ibid.*, p. 231.

[4] *Loc. cit.*

[5] *Loc. cit.*

[6] *Ibid.*, p. 232.

[7] *Ibid.*, pp. 232–3.

[8] Lax, 'The Flowering of Applied Mathematics in America', pp. 455–6.

[9] *Ibid.*, p. 456.

[10] *Loc. cit.*

* ▸ p. 534

> 'an unusual and very beautiful example of a framework built for applied mathematics leading to new insights within pure mathematics'.[2]

Lax himself saw mathematics as being an art akin to painting. In painting there was a tension between accuracy of representation and creating a pleasing image. In mathematics there was the same tension between studying the physical laws of nature and the desire 'to spin beautiful deductive patterns'.[3]

Ian Stewart

In 2007, the mathematician Ian Stewart (1945–) wrote in *Why Beauty is Truth*, one of his many popular mathematics books, that history showed that one should not dismiss a piece of beautiful mathematics just because it has no obvious utility.[4] Stewart gave two specific historical examples.[5] The first was Girolamo Cardano's passing examination of the problem of dividing 10 into two parts whose product is 40. Using standard techniques, Cardano showed that the two parts must be $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$, and dismissed the argument and solution as 'subtle [futile] and *useless* [inutile]'.[6] The importance of complex numbers throughout mathematics and science belies Cardano's judgement. Stewart's second example was Hardy's judgement of the lack of utility of number theory, though Stewart's account was marred by the claim that Hardy was 'pleased' by its uselessness.*

Stewart suggested — without citing examples — that there was a tendency among 'more "practical" people' to sneer at mathematical ideas, especially beautiful ones, that were developed for their own sake rather than for the sake of immediate utility.[7] The attitude that Stewart described was the mirror-image of the disdain for utility which, according to anecdote, seems to have emerged in the nineteenth century† and seems to have been prevalent between the wars.‡

Doron Zeilberger

In a 2017 essay whose tone and scattergun use of emphasis may have been deliberate signals that it should not be taken seriously, the combinatorialist Doron Zeilberger (1950–) argued against many facets of modern mathematics. When it came to the conception of mathematics as an aesthetic pursuit, he acknowledged the aesthetic motivation and thought that there was no harm in enjoying aesthetic value in mathematics.[8] But he thought it wrong to try to demarcate mathematics using aesthetic criteria. Rhetorically addressing Hardy, he repudiated the idea that '[*t*]*here is no place for ugly mathematics*',[1] which has a slightly different meaning from Hardy's 'there is no permanent place in the world for ugly mathematics'.[2] Zeilberger's rejection was grounded on his confidence that '[b]eauty is only skin-deep and is also in the *eyes of the beholder*'.[3]

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[1] Lax, quot. in Raussen & Skau, 'Interview with Peter D. Lax', p. 225.

[2] [Abel Committee], 'Abel Prisen: Peter D. Lax'.

[3] Lax, 'Mathematics and its Applications', p. 15.

[4] Stewart, *Why Beauty is Truth*, p. 267.

[5] *Loc. cit.*

[6] Cardanus, *Ars Magna*, ch. xvii, p. 287, emphasis added; trans. C.

* ▸ pp. 512–16

[7] Stewart, *Why Beauty is Truth*, p. 287.

† ▸ pp. 475–6

‡ ▸ pp. 496–7

[8] Zeilberger, 'What Is Mathematics and What Should It Be?', pp. 146–7.

He then went on to discuss the ranking of formulae by the subjects of Zeki et al.'s study of the neural activity accompanying the experience of mathematical beauty.[*] The highest-ranked formula was Euler's equation; the lowest was a formula for $1/\pi$ discovered by Ramanujan. For Zeilberger, this ranking showed that human mathematicians were superficial: 'In the "eyes of God," Ramanujan's formula is much prettier than Euler's':[4] Ramanujan's formula was a deep result, which was useful for rapidly computing π; Euler's equation was trivial. [Perhaps curiously, Zeilberger did not state Ramanujan's formula in the form used in Zeki et al.'s study, but in a different form that requires a specialized notation.] But the reference to God, if more than mere rhetoric, suggests that Zeilberger thought that there is some correct standard of beauty. This in turn implies that Zeilberger's remark about beauty being in 'the *eyes of the beholder*' should be interpreted as indicating a projectivist view of beauty rather than an assertion of subjectivity in aesthetic judgements.

Zeilberger characterized deep statements as those with long proofs (and thus no deep statement has what would traditionally be called a 'proof from The Book'[†]). He considered beautiful deep theorems that have short statements. Thus theorems that he thought beautiful included the four-colour theorem[‡] and the Kepler conjecture.[5][§] These characterizations of depth and beauty seems to imply that if a short proof were found for such a theorem, it would cease to be deep and thus cease to be beautiful. (That is, it would cease to be beautiful for Zeilberger: God presumably would already have known the short proof.)

But Zeilberger thus undermined his earlier assertion of the superficiality of beauty: if beauty involves depth, which in turn depends on the length of proof, then a judgement of beauty is grounded on knowledge of proofs, and perhaps on an estimation of the likelihood of a short proof being found, and is generally bound up with the subject's knowledge of how a theorem is embedded in its mathematical context.

BEAUTY AND MOTIVATION, WITHOUT *APOLOGY*

The debate on the role of beauty in motivating mathematics did not become totally dependent upon the *Apology*. This section is devoted to authors who engaged in this discourse without

[1] Zeilberger, 'What Is Mathematics and What Should It Be?', p. 146, emphasis in orig.

[2] Hardy, *Apology*, § 10.

[3] Zeilberger, 'What Is Mathematics and What Should It Be?', p. 146, emphasis in orig.

[*] ▸ pp. 837–8

[4] *Ibid.*, p. 147.

[†] ▸ p. 828

[‡] ▸ pp. 602

[5] *Loc. cit.*

[§] ▸ pp. 608–9

their contribution being a response to Hardy, although at least the later figures have doubtless been influenced, positively or negatively, directly or indirectly, by what he wrote.

L. E. J. Brouwer

Perhaps curiously, the aesthetic motivation for pure mathematics arose in the context of L. E. J. Brouwer's (1881–1966) opposition to the appointment of Bartel van der Waerden (1903–96) to a chair at the University of Amsterdam. Van der Waerden had been appointed professor at the University of Leipzig in 1931, and, though he protested the sacking of Jewish colleagues under the Nazis, had remained there until the end of the war. Consequently, after his return to the Netherlands, the queen, Wilhelmina (1880–1962; r. 1890–1948), refused her assent to his appointment to a chair at Amsterdam. Van der Waerden thus went to Johns Hopkins University as a visiting professor, and turned down a permanent appointment. In 1948 there was an attempt to establish an extraordinary chair in applied mathematics at the University of Amsterdam, funded by a foundation, with the intent that van der Waerden should occupy it.[1]

Brouwer, who by this time was a doyen of mathematics and philosophy in the Netherlands, argued in a letter to the Board of the University of Amsterdam that this motive — the appointment of van der Waerden to a position at Amsterdam in spite of the crown's wishes — was made evident by the professorship being in *applied* mathematics. His starting-point was the motivation for pure mathematics, which he declared an aesthetic pursuit:

> 'Mathematics is an introvert science and as such coalesces with philosophy, theology and reflective psychology, but it is constructive in a higher degree than these. And the mathematical creative urge is directed not only to inner enlightenment but also to beauty, a beauty related to that of architecture and music, but more immaterial.'[2]

Thus, however important and far-reaching the technological and scientific applications of mathematics, 'the mathematical state of mind is as a rule indifferent to natural science'.[3] A position in 'applied mathematics' only properly had a place in a technical university [technische hogeschool] that taught applied sciences; hence there had to be 'very special' circumstances motivating the establishment of such a position in a university. And hence, thought Brouwer, one was led back to the desire to keep a mathematician of van der Waerden's evident ability in the Netherlands.

[1] Soifer, *The Scholar and the State*, p. 351.

[2] Brouwer, letter to Board of Univ. Amsterdam, dated 2 Apr. 1948, repr. in Brouwer, *Selected Correspondence*, p. 433.

[3] Brouwer, letter to Board of Univ. Amsterdam, dated 2 Apr. 1948, repr. in *loc. cit.*

Alfréd Rényi's Dialogues on Mathematics

Starting in 1962,[1] the mathematician Alfréd Rényi (1921–70) published a series of three dialogues, collected in *Dialogues on Mathematics* (1967), in which were debated the nature of mathematical knowledge and its relation to theoretical and practical science. Each dialogue has a historical setting: classical Athens, Hellenistic Syracuse, Renaissance Rome. Rényi's writing of the dialogues, and particularly the first, was probably influenced by his friendship with the classical philologist and historian of science Árpád Szabó (1913–2001).[2] But, despite their narrative setting, the themes of each dialogue were timeless.

'A Socratic Dialogue on Mathematics'. The first dialogue was a discussion between Socrates (who was clearly meant to be the Platonic Socrates) and Hippocrates (whose youth suggests that he was probably not meant to be the mathematician Hippocrates of Chios (*fl.* late 5th century BCE)) on the nature of mathematics and of mathematical knowledge. Mathematics is held to be the study of objects that exist only in thought, and are thus the subject of the kind of exact knowledge that is impossible of things in the physical world. But the objects of mathematical thought are reflections of those in the physical world, and so mathematical knowledge is applicable.

'A Dialogue on the Applications of Mathematics'. In this second dialogue, Hiero (who, in the ahistorical setting of the dialogue, was ruler of Syracuse during the Roman siege) praises Archimedes for the war-machines he designed for the defence of the city. Archimedes says that his motivation was to inculcate the idea that theoretical mathematics could be usefully applied; Hiero expresses amazement that the repulse of the Roman attack was due to the application of the kind of mathematics that any educated man would have known. Archimedes points to the result that for any point P on a parabola, the line joining P to the focus F and the line through P parallel to the axis make equal angles with the parabola (see Figure 13.9).

> 'ARCHIMEDES Probably when you heard one of its ingenious proofs you understood it, and perhaps even admired its beauty and elegance, but that was all. Some mathematicians went one step further; they explored some of its purely geometrical consequences, or invented new proofs, but there they stopped. I simply went just one step further; I looked at its non-mathematical consequences also.'[3]

That is, while it was quite possible to admire the beauty of the purely theoretical proof, Archimedes went further and conceived

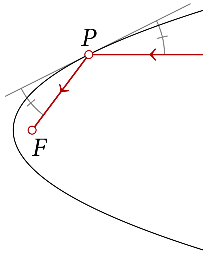

FIGURE 13.9
A ray parallel to the axis of a parabola is reflected to the focus.

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[1] Rényi, *Dialogues on Mathematics*, p. 91.

[2] Schmetterer, 'Alfréd Rényi', p. xxxviii.

[3] Rényi, *Dialogues on Mathematics*, p. 37.

of a parabolic mirror that could focus the Sun's rays. Rényi's dialogue here referred to the story, not attested by the earliest sources, that such mirrors, designed by Archimedes, had been used to set fire to the Roman fleet.[1]

Alfréd Rényi's Dialogues on Mathematics

'A Dialogue on the Language of the Book of Nature'. In this dialogue, the mathematician and physicist Evangelista Torricelli (1608–47) visits Galileo (1564–1642) in 1633 while he is under house arrest in the care of his friend Francesco Niccolini (1584–1650), ambassador to Rome of the Grand Duke of Tuscany. In the first part, Galileo and Torricelli discuss the role and process of science with occasional contributions by Niccolini's wife Caterina (1598–1676). After Torricelli leaves, Galileo and Caterina continue the discussion, with particular focus on the role of mathematics in the sciences.

Caterina admits to being a mathematical novice, but says that she can experience beauty in a proof:

> 'MRS. NICCOLINI [...] I also have found pleasure in the surprising and ingenious steps of the proof, such joy as is found in the most beautiful music.'[2]

Galileo agrees, saying that mathematics is not only useful, but beautiful. People who appreciate only the utility, not the beauty, have only a shallow appreciation for mathematics:

> 'GALILEO [...] think that the beauty of mathematics is not a subsidiary, accessory thing; it is one of its basic features. [...] Those who have barbarous notions about mathematics do not understand this: either they are blind to the beauty of mathematics, or if they see it, they are suspicious of it. They think that beauty is a luxury which is superfluous; and when they turn their back on her, they think they get nearer reality. They simper in the role of the practical man and arrogantly despise those who penetrate the real spirit of mathematics.'[3]

And thus teaching in mathematics should be founded upon making the student aware of its beauty:

> 'GALILEO [...] Until now I have spoken about the beauty of mathematics and about that joy which is very close to the joy the pure beauty of the arts produces in man, which real understanding produces and shining eyes signal. But this joy can be attained only by hard work. [...] But whoever has tasted the joy of pure knowledge, whoever has seen the beauty of mathematics will be willing to make the serious effort. The main aim of teaching mathematics must be to acquaint the pupil with this joy, and through this educate him in the dis-

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[1] For a survey of the historical sources, see Simms, 'Archimedes and the Burning Mirrors', § 1.

[2] Rényi, *Dialogues on Mathematics*, p. 70.

[3] *Ibid.*, p. 71.

ciplined, logical thinking which is indispensable to mathematics.'[1]

H. S. M. Coxeter

In the early 1960s, the communications engineer Gordon Lang (1922–) consulted the geometer H. S. M. Coxeter (1907–2003) about sphere packings, with the intention of applying them in communications, and ultimately developed a modem based on the sphere packing associated to the E_8 lattice. According to the computer scientist Robert Tennent (1944–), who said he was present at their conversation, in the 1970s Lang explained his work to Coxeter and asked if he was pleased that a practical application had been found for his work, and Coxeter responded that 'he was appalled that his beautiful theories had been sullied in this way'.[2]

This exchange is one of the few documented instances in the post-Hardian period of a mathematician seeming to mirror the caricature of Hardy that would have him think that application detracts from beauty.*

Mark Kac

The mathematician Mark Kac (1914–84) gave priority to the beauty and interest of the ideas as a motivation for mathematics. This view fits with the one result he described as a 'beautiful theorem'[3] in his autobiography, which he proved together with Paul Erdős (1913–96) in 1939:[4]

ERDŐS–KAC THEOREM. *If ω is an arbitrary real number and K_n denotes the number of those integers from 1 up to n with fewer than* $\log\log n + \omega\sqrt{2\log\log n}$ *prime divisors, then*

$$\lim_{n\to\infty} \frac{K_n}{n} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\omega} \mathrm{e}^{-x^2}\,\mathrm{d}x\,. \qquad \square$$

The number theorist Carl Pomerance (1944–), writing almost sixty years later, called this theorem 'one of the most beautiful and unexpected results in mathematics'.[5] Part of its interest is that it marked the entry of the normal distribution of statistics into number theory, and so led to the emergence of a new subfield.[6]

But Kac's desire for beauty and interest did not entail a restriction to pure mathematics. In a comparison that today's reader will doubtless wince at, he said: 'In mathematics, as in women, purity is a virtue. But it is much more important to be beautiful and exciting than pure.'[7] Indeed, Kac was eager to apply mathematics to the world.[8] But bare utility did not interest him; he would only work on a problem that demanded progress *in* mathematics: 'A mathematician must be motivated by intellectual curiosity'.[9] Nevertheless, the utility of beautiful ideas excited him.[10]

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[1] Rényi, *Dialogues on Mathematics*, p. 72.

[2] Roberts, *King of Infinite Space*, p. 198.

* ▸ pp. 512–16

[3] Kac, *Enigmas of Chance*, p. 91.

[4] See Erdös & Kac, 'The Gaussian Law of Errors'.

[5] Pomerance, 'Paul Erdős, Number Theorist Extraordinaire', p. 20.

[6] *Loc. cit.*

[7] Kac, quot. in Boehm, 'Mark Kac's Beautiful World of Mathematics', p. 6.

[8] *Ibid.*, pp. 6–8.

[9] Kac, quot. in *ibid.*, p. 10.

[10] *Ibid.*, p. 6.

Cyril Stanley Smith

In an essay first published in 1976, the historian of science Cyril Stanley Smith (1903–92), made a noteworthy observation about the motivation that leads to *any* kind of true progress. He was discussing technological advance in general and its relation to art, but what he said is apt to seem just as germane to mathematics:

> ‘Necessity is *not* the mother of invention — only of improvement. A man desperately in search of a weapon or food is in no mood for discovery; he can only exploit what is already known to exist. Discovery requires aesthetically motivated curiosity, not logic’.[1]

Alexander Grothendieck

Alexander Grothendieck (1928–2014) is generally accepted to have been one of the most prolific, influential, powerful, and original mathematicians of the twentieth century.[2] In 1970, he resigned his position at the Institut des Hautes Études Scientifiques for a complex of reasons. During the rest of his life, he produced mathematical, political, and reflective manuscripts that were circulated privately, including his autobiographical memoir *Récoltes et semailles*.

Récoltes et semailles makes clear that beauty was important to Grothendieck both inside and outside mathematics, though the views he expressed in different parts of the manuscript are sometimes difficult to reconcile. He was certainly motivated and directed by beauty:

> ‘My main guide in my work has been the constant search for a perfect coherence, a complete harmony that I divined behind the turbulent surface of things, and that I patiently endeavoured to free, without ever tiring of it. It was a keen sense of “beauty”, surely, which was my intuition and my compass.’[3]

Beauty was, for him, intimately connected with understanding:

> ‘The deepest and most fruitful work is that which attests to the most unrestrained sensitivity for apprehending the hidden beauty of things. Such a delicate sensitivity to beauty seems to be closely connected to […] “demand” (applied to oneself) or “rigour” (in the full sense of the term), which I described as an “attention to something delicate in ourselves”, an attention to a quality of comprehension of the thing probed. This quality of COMPREHENSION of a mathematical thing cannot be separated from a more or less intimate, more or less perfect, perception of the “beauty” particular to that thing.’[4]

He seems to have thought beauty also deeply connected with discovery; once he had ‘done’ a thing, it ceased to be beautiful for him.

[1] Smith, ‘On Art, Invention, and Technology’, p. 325.

[2] Artin et al., ‘Alexandre Grothendieck (part 1)’, p. 242.

[3] Grothendieck, *Récoltes et semailles*, pt 1, ch. VI, § 39; trans. C.

[4] *Ibid.*, pt 1, ch. VI, § 40 & n. 36; trans. C.; bold emphasis changed to small capitals.

He made a perhaps distasteful comparison with a man, who, having *known* a woman, no longer sees her beauty. When he came to write his *Récoltes et semailles*, he protested that in his own love life, he had been above such attitudes, while confessing to have not noticed that in his mathematical life, he had not been.[1]

Discovery, mathematical or otherwise, could only come about 'by the joint and inseparable action of the original drives of yin and yang in the same being'. In the fusion of the two lay the beauty of a work.[2] But the works of great mathematicians, even those who have had lasting influence, do not necessarily have this beauty; their work may not have the fusion and balance of yin and yang.[3]

The form or the effect of mathematical beauty could vary between individuals. Grothendieck thought that Jean Dieudonné (1906–92) delighted in 'seeing the beauty of things manifest itself in bright light', whereas he himself enjoyed 'pursuing it into the hidden secrets of mist and night'.[4] For Grothendieck, these alternative attitudes were the most profound difference between his and Dieudonné's approaches to mathematics. He thought that Dieudonné's perception of beauty seemed to manifest itself in a sense of wonder, while for Grothendieck it was 'less contemplative, more enterprising', less obviously emotional.[5]

Grothendieck suspected that his own sense of beauty might have been dulled during the 1960s by what he described as a growing complacency in his relationship with mathematics and other mathematicians.[6] He thought that, without some appreciation of beauty, he could not be a mathematician, and he thought no-one could do useful work in mathematics without having this sense, at least in a small way.[7] But, by his own arguments, his sense of beauty must have remained until the end of the 1960s, for it was then that he discovered what he considered the most hidden and mysterious mathematical concept of his career: the notion of *motives* in algebraic geometry.[8]

Ivar Ekeland

The title of Ivar Ekeland's (1944–) book *The Broken Dice* refers to the tale in *St Olaf's Saga* in which Olaf II of Norway (c. 995–1030) won the island of Hísing in a casting of lots when one of his dice split in half to give a score of *seven*.[9] In his book, Ekeland included copious quotations from and references to the Norse sagas, and in particular drew upon them to illustrate his own motivation for mathematics. He recalled[10] an episode in *St Olaf's Saga* where, on the morn before the Battle of Stiklestad, in which they would both die, Olaf called upon the poet Þormóðr to sing. Despite the toil and war that lay ahead, Olaf's men awoke encouraged.[11] Ekeland heard in Þormóðr's song a harmony higher than the chaos of battle, and a memory of the departed. 'Men die, only art survives.'[12] Just as Olaf's men were roused to follow him, so Ekeland was drawn to mathematics

[1] Grothendieck, *Récoltes et semailles*, pt 1, ch. VI, §§ 40; trans. C.

[2] *Ibid.*, pt 3, ch. XII, § 2.6; trans. C.

[3] *Ibid.*, pt 3, ch. XII, § 2.8.

[4] *Ibid.*, pt 1, ch. VI, § 39; trans. C.

[5] *Ibid.*, pt 1, ch. VI, § 39; trans. C.

[6] *Ibid.*, pt 1, ch. VI, §§ 39–40; trans. C.

[7] *Ibid.*, pt 1, ch. VI, §§ 40; trans. C.

[8] *Ibid.*, pt 1, ch. VI, §§ 40; trans. C.

[9] Sturluson, *Óláfs saga ins helga*, ch. 94 ⟦pp. 101–2⟧.

[10] Ekeland, *The Broken Dice*, pp. 170–1.

[11] Sturluson, *Óláfs saga ins helga*, ch. 208 ⟦pp. 241–2⟧.

[12] Ekeland, *The Broken Dice*, p. 172.

and science, despite their showing him that chance and probability rule the world:

> 'There is more to science than just chance [...]. Geometry, general relativity, the dynamics of conservative systems, the physics of elementary particles, are all theories of almost superhuman beauty, in which I find the same harmony expressing itself in the same mathematical form'.[1]

And even in the mathematics of probability, he saw beauty in results that in a sense show predictability in chance, such as the central limit theorem, which (loosely) asserts that the sum of a large number of independent identically distributed random variables with finite variance is close to a normally distributed random variable.[2]

Yehuda Rav

The philosopher Yehuda Rav (1930–) published in 1999 the oft-cited essay 'Why do we prove theorems?', which specifically addressed the motivation for seeking proofs as normally understood in mathematical discourse, as opposed to derivations, either automated or manual, within a formal system.[3] He pointed out that, as Henri Bergson (1859–1941) had said, humans are *homo faber*, makers of tools.[4] Mathematicians also make tools, for solving 'humanly meaningful problems',[5] but they are guided by 'a deep sense of beauty and search for harmony'.[6] Thus the selection of problems and the motivation for solving them is not to churn out as many theorems as possible: mathematicians are not deduction machines.[7]

Leone Burton

In 2004, Leone Burton (1936–2007) published a study of the practice of working mathematicians, based on interviews with seventy professionals at various career stages. Part of the study considered the function of aesthetic value in mathematical practice, and opened with a quotation from an unnamed participant, who said that their choice of an area of mathematics was based on its aesthetic appeal: 'the sort of excitement that you might get from looking at something structurally beautiful such as a beautiful picture'.[8]

In the responses of the interviewee, Burton detected a division between pure and applied mathematicians in terms of their attitudes to aesthetic value. Pure mathematicians seemed to view beauty as an intrinsic feature of mathematics; applied mathematicians thought it more closely connected with *pure* mathematics.[9] The views of two unnamed applied mathematicians, quoted in the study, suggest that while they and their colleagues might recognize elegance in mathematics, they considered inelegance to be a price worth paying for a working solution.[10] The division stems from different motivations;

[1] Ekeland, *The Broken Dice*, p. 173.

[2] *Ibid.*, p. 157.

[3] Rav, 'Why do we prove theorems?', p. 12.

[4] Rav, 'Why do we prove theorems?', p. 6; see Bergson, *Creative Evolution*, p. 139.

[5] Rav, 'Why do we prove theorems?', p. 17, n. 13.

[6] *Ibid.*, p. 6.

[7] *Ibid.*, p. 17, n. 13.

[8] Unnamed mathematician, quot. in Burton, *Mathematicians as Enquirers*, p. 63.

[9] *Loc. cit.*

[10] *Ibid.*, pp. 63–4.

the distinction is between being 'a pure mathematician driven by elegance and an applied mathematician driven by the problem'.[1]

Another participant in Burton's study said that beauty, while a cause of satisfaction, was not a criterion to determine the worth of a piece of mathematics. After all, a first approach might seem ugly because it was unfamiliar or because the wrong methods were being applied; thus one should not abandon it on aesthetic grounds: 'If one's treatment of something is ugly, too bad, it is an unfortunate fact of life.'[2]

One participant, a pure mathematician given the pseudonym 'Maria',[3] denied that her enjoyment of mathematics was aesthetic, though she admitted to having gained such enjoyment when younger. Rather, 'Maria' found doing mathematics to be relaxing and peaceful and to grant a sense of achievement.[4] She was even suspicious of the use of aesthetic terminology, saying 'words like beauty and elegance are over-used'.[5]

Bryan Birch

Bryan Birch (1931–) recalled that when he and Peter Swinnerton-Dyer (1927–2018) started work on elliptic curves, they enjoyed exploring an unstudied and unfashionable area of mathematics, because it was beautiful.[6] At least with the benefit of hindsight, he declared that the field was 'so beautiful that it was certain to be important'.[7] Subsequently, with the advent of cryptography using elliptic curves over finite fields, the area became what Birch dryly called 'horribly fashionable'.[8] (And, indeed, the eponymous Birch–Swinnerton-Dyer conjecture is now considered one of the great unsolved problems of mathematics,[9] and Maxim Kontsevich (1964–) & Don Zagier (1951–) called it 'one of the most beautiful and most intriguing open questions in number theory'.[10]) But Birch thought that this story pointed to how aesthetic judgement is the best way to evaluate research, even if one is aiming at practical applications.[11]

Terence Tao

In an essay entitled 'What is good mathematics?', published in November 2007, Terence Tao (1975–) listed twenty-one ways in which mathematics could be *good*, in the broad sense of being what mathematicians should aim to produce. There was good problem-solving, technique, theory, insight, discovery, application (to problems outside mathematics or in another field within mathematics), exposition, paedagogy, vision, taste (in the sense of choosing wise research goals), public relations, metamathematics; there was mathematics that is rigorous, beautiful, elegant, creative, useful (within mathematics itself), strong (such as sharpness and matching known counterexamples), deep, intuitive, definitive, 'etc., etc.'[12] Tao's list was arranged in no particular order, and the two items concerning aesthetic value were given no special prominence:

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[1] Unnamed applied mathematician, quot. in Burton, *Mathematicians as Enquirers*, p. 65.

[2] Unnamed mathematician, quot. in *ibid.*, p. 68.

[3] *Ibid.*, p. 95.

[4] *Ibid.*, p. 96.

[5] 'Maria', quot. in *loc. cit.*

[6] Cook, *Mathematicians*, p. 28.

[7] Birch, quot. in *loc. cit.*

[8] Birch, quot. in *loc. cit.*

[9] Gowers, 'The Birch–Swinnerton-Dyer Conjecture'.

[10] Kontsevich & Zagier, 'Periods', p. 797.

[11] Cook, *Mathematicians*, p. 28.

[12] Tao, 'What is good mathematics?', pp. 623–4.

> ‘(xiv) *Beautiful* mathematics (e.g. the amazing identities of Ramanujan,[*] results which are easy (and pretty) to state but not to prove);
>
> (xv) *Elegant* mathematics (e.g. Paul Erdős’s concept of “proofs from *The Book*”,[†] achieving a difficult result with a minimum of effort)’.[1]

Notice the distinction between beautiful *results* and elegant *proofs*, which, for Tao, seem to have been opposed: a beautiful result has no easy proof; an elegant proof minimizes the effort in establishing the result.

Tao thought that too strong a focus on just one or two of these values would cause a field to degenerate.[2] For instance — and these are the author’s examples, not Tao’s — a field that became devoted to metamathematics would doubtless be of only tenuous interest to most working mathematicians; a field that focused only on finding beautiful theorems might produce results that had no links to other areas and could only be admired as otherworldly and inaccessible art.

The very best mathematics fulfilled at least some of Tao’s criteria, but also, he maintained, was part of a story whose development produced yet more good mathematics.[3] Any effort to produce or debate over good mathematics had to be accompanied by an awareness of this broader setting.[4] Tao’s own example was a result which he seems to have consistently judged beautiful.[5] It was conjectured by Paul Erdős & Paul Turán (1910–76) in 1936 (in a different formulation and only explicitly in a special case),[6] and was proved by Endre Szemerédi (1940–) in 1975:[7]

SZEMERÉDI’S THEOREM. *Any subset of the natural numbers with positive upper density contains arbitrarily long arithmetic progressions.*[8] □

(The upper density of a set A of natural numbers is defined to be $\limsup_{n\to\infty} |A \cap [0,n)|/n$.[9]) There are various proofs of the result, intricate or deep, elementary or non-elementary: none is easy.[10] But the story in which the theorem sits — whence it emerged, the techniques and concepts from different areas that were applied to prove it, the new ideas it begot — illustrate what Tao thought was characteristic of the best mathematics.[11]

Karen Uhlenbeck

In 2009, Karen Uhlenbeck (1942–), who would be awarded the 2019 Abel Prize for her work in geometric analysis, not only preferred the aesthetic motivation, but questioned the utilitarian one:

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[*] ▸ pp. 5, 7, 837

[†] ▸ p. 828

[1] Tao, ‘What is good mathematics?’, p. 624, emphases in orig.

[2] *Ibid.*, p. 625.

[3] *Ibid.*, p. 632.

[4] *Ibid.*, p. 633.

[5] Tao & Vu, *Additive Combinatorics*, p. xiv; Tao, ‘What is good mathematics?’, p. 627.

[6] Erdös & Turán, ‘On Some Sequences of Integers’, pp. 263–4.

[7] Szemerédi, ‘On sets of integers containing no k elements in arithmetic progression’.

[8] Tao & Vu, *Additive Combinatorics*, Thm 10.1.

[9] *Ibid.*, Def. 1.21.

[10] Tao, ‘What is good mathematics?’, pp. 628–31.

[11] *Ibid.*, p. 632.

> ‘I am not so sure I am happy that mathematics is useful. The uses most likely do more harm than good […]. I am content with the aesthetic rewards.’[1]

This statement not only suggests a desire to pursue only useless mathematics, but points to a far more pessimistic view of the utility of mathematics than any discussed so far. Hardy’s thought that usefulness was what tended to increase well-being and comfort.[*] Even Coxeter’s reported dismay at the application of his work was not because of any putative worldly harm, but simply an aesthetic regret.[†] However, unlike Coxeter, Uhlenbeck did not hold that utility detracted from the aesthetic value.

Cédric Villani

The 2010 Fields medallist Cédric Villani (1973–), was reminded, when giving evidence to a committee of the French Senate in 2010, about an interview in which he compared mathematics and certain aspects of art. He was asked to develop this comparison, and responded by first noting that the view of mathematics as an artistic pursuit so obvious to mathematicians that it is scarcely noticed.[2] He continued by explaining the aesthetic motivation:

> ‘First, what generally drives a mathematician is the desire of producing something beautiful. It’s too bad if this is a caricature — although I do not think my colleagues will contradict me — but, when we learn of a result or a theorem, our first concern is to judge its beauty.’[3]

Applied to a mathematical result, the term *beauty* pointed to aesthetic shared with other arts: harmony between the parts, such as in linking two different areas, and surprise at something unforeseen. A result that had been foreseen lacked interest.[4]

BEAUTY AND MATHEMATICS AS A REFUGE

‘ Of all escapes from reality,
mathematics is the most successful ever. ’

— Gian-Carlo Rota, ‘The Lost Café’, p. 70.

In January 1937, the journalist Arthur Koestler (1905–83) was engaged to report on the Spanish Civil War. When the nationalist army captured Malaga in early February, he was arrested and imprisoned. He was initially held in solitary confinement and under the impression that he was soon to be executed.[5] In his account *Dialogue with Death*, written in the autumn of 1937 immediately after his release from prison,[6] and in *The Invisible Writing*, the second volume

[1] Uhlenbeck, quot. in Cook, *Mathematicians*, p. 44.

[*] ▸ pp. 513–14

[†] ▸ p. 541

[2] Office parl. sci. tech., ‘Débat sur les mathématiques en France’.

[3] Villani, quot. in *ibid.*, trans. C.

[4] *Ibid.*

[5] Koestler, *Dialogue with Death*, chs V–VII.

[6] Koestler, *The Invisible Writing*, p. 337.

Beauty and mathematics as a refuge

of his later autobiography, he described how, on the first day of his imprisonment, he started scratching mathematical formulae on the wall of his cell; this activity engrossed him to such a degree that he could forget he had not eaten for over a day.[1] He recognized that his absorption was in some degree aesthetic, recalling results from mathematics and in particular Euclid's proof of the infinitude of primes,[*] which, since his schooldays, had given him 'a deep satisfaction that was aesthetic rather than intellectual'.[2] He was enchanted by it to such an extent that it was only after some minutes that he recalled where he was: 'I noticed some slight mental discomfort nagging at the back of my mind — some trivial circumstance [...]: I was, of course, in prison and might be shot.'[3]

When André Weil (1906–98) was in prison in early 1940, accused of failing to report for military service when the Second World War began, he was able to pursue his mathematical researches, and was astonished at his productivity. Mathematics thus seemed to make imprisonment worthwhile or even desirable, and in a letter to his wife he humorously suggested that regular periods of imprisonment might be beneficial to his work. He was very pleased with the mathematics he produced, writing in the same letter: 'I am thrilled by the beauty of my theorems'.[4]

Paul Turán (1910–76) was in September 1940 drafted into the Hungarian labour service, which had been established for those deemed too unreliable to serve in the armed forces.[5] During his time there, mathematics helped him maintain his spirit. One of the officers at the camp knew Turán's work and assigned him to the lightest duties available, showing people where in a wood-yard logs of a given width could be found. Turán could therefore resume thinking about an extremal problem he had been considering. Towards the end of his life, he recalled this period:

> 'I cannot properly describe my feelings during the next few days. The pleasure of dealing with a quite unusual type of problem, the beauty of it, the gradual nearing of the solution, and finally the complete solution made these days really ecstatic. The feeling of some intellectual freedom and being, to a certain extent, spiritually free of oppression only added to this ecstasy.'[6]

It seems, therefore, that the aesthetic pleasure gained from the contemplation of a beautiful problem had contributed to his feeling of freedom.

Freeman Dyson (1923–2020) left his studies at Cambridge in 1943 and was assigned to operational research for the Royal Air Force, developing mathematical techniques to analyse and maximize the effectiveness of the British bombing campaign against Germany. During the winter of 1943–4, which he recalled as 'long, hard, grim',[7] he spent his evenings researching new kinds of Rogers–Ramanujan-type identities;[*] his favourite discovery was:

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[1] Koestler, *Dialogue with Death*, p. 66.

[*] ▸ p. 103

[2] Koestler, *The Invisible Writing*, p. 351.

[3] *Loc. cit.*

[4] André Weil, letter to Éveline Weil, dated 7 Apr. 1940, quot. in Weil, *The Apprenticeship of a Mathematician*, pp. 146–7.

[5] For background, see Braham, *The Wartime System of Labor Service in Hungary*.

[6] Turán, 'A Note of Welcome', p. 8.

[7] Dyson, 'A Walk Through Ramanujan's Garden', p. 15.

$$\sum_{n=0}^{\infty} x^{n^2+n} \frac{(1+x+x^2)(1+x^2+x^4)\cdots(1+x^n+x^{2n})}{(1-x)(1-x^2)\cdots(1-x^{2n-1})} = \prod_{n=1}^{\infty} \frac{1-x^{9n}}{1-x^n}.$$

Dyson said that working on 'these beautiful identities', visiting 'Ramanujan's garden', helped him maintain his sanity.[1] Thus, for Dyson, his pure research into a topic in which he clearly saw beauty served as a refuge from his work in a different field, which was laden with moral quandaries. Decades later, he still recommended 'Ramanujan's garden' to colleagues as 'therapy', a prescription that he followed himself when anger or depression afflicted him.[2] Indeed, when he simply needed 'relief from the serious business of physics', he would return to Ramanujan's papers and work on the unsolved problems. During such an excursion he found for the τ-function a formula that he described as 'so elegant that it is rather surprising that Ramanujan did not think of it himself'.[3] The τ-function is defined by

$$\sum_{n=1}^{\infty} \tau(n)x^n = \eta^{24}(n) = x\prod_{m=1}^{\infty}(1-x^m)^{24};$$

The formula that Dyson found for the τ-function is:[4]

$$\tau(n) = \sum_{\substack{a,b,c,d,e \equiv 1,2,3,4,5 \pmod 5,\\ a+b+c+d+e=0,\\ a^2+b^2+c^2+d^2+e^2=10n}} \frac{\begin{array}{c}(a-b)(a-c)(a-d)(a-e)\\ \cdot(b-c)(b-d)(b-e)(c-d)(c-e)(d-e)\end{array}}{1!\,2!\,3!\,4!}.$$

In his book *Regular Polytopes*, the final draft of which he submitted shortly after the end of the war,[5] Coxeter wrote that the principal motive for the study of regular polyhedra was aesthetic: 'their symmetrical shapes appeal to one's artistic sense'.[6] The higher-dimensional analogues are never perceptible by the senses, but the intellectual effort to comprehend them allows one to 'to peep through a chink in the wall of our physical limitations, into a new world of dazzling beauty'.[7] This glance offers a refuge: 'Such an escape from the turbulence of ordinary life will perhaps help to keep us sane.'[8]

* * *

A 2013 study of professionals involved in mathematics teaching and research offered some non-anecdotal evidence for the connection between aesthetic value and a sense of security that mathematics can engender. The interviews indicated that one can gain a feeling of security from mathematics not only through its precision and its

[*] ▸ p. 7

[1] Dyson, 'A Walk Through Ramanujan's Garden', p. 15.

[2] *Ibid.*, p. 25.

[3] Dyson, 'Missed Opportunities', p. 636.

[4] *Loc. cit.*

[5] Roberts, *King of Infinite Space*, p. 141.

[6] Coxeter, *Regular Polytopes*, p. ix.

[7] *Loc. cit.*

[8] *Loc. cit.*

verifiability,[1] but also through its aesthetic value. One participant said that '[m]athematics makes you feel secure because it reveals harmony and orderliness'.[2] The participants generally thought mathematics beautiful,[3] and one said that '[t]his beauty provides equilibrium in the chaos of an uncertain world'.[4]

BEAUTY AND IMPORTANCE

Hardy said that what distinguished the best mathematics from chess problems is *seriousness*,[*] a term that he preferred to *importance* and which indicates connections with many other mathematical ideas. Hardy was not quite clear on the relationship between beauty and seriousness; sometimes he seems to have thought them independent, sometimes connected. Some later authors engaged with this question.

In a dialogue published in 1979, the mathematician Lipman Bers (1914–93) said that the history of science had shown that our evolution as a species must have shaped us so that 'what is elegant and beautiful is what is useful and powerful'.[5] He did not offer support for his evolutionary hypothesis, but his contribution is notable as an early explicit suggestion that beauty was an indicator of importance in mathematics.

A similar point was made by Timothy Gowers (1963–) in a lecture delivered in May 2000. He said there was 'a remarkable correlation between mathematics that is beautiful, and mathematics that is important'.[6] This correlation was partly due to the way mathematics develops. The problem-solving mode of mathematics tended to yield techniques and results that were disparate and specialized. These products may be reformulated and unified through the discovery of some common underlying theory. Such unifications are important mathematics in the sense that Hardy described: they connect a variety of mathematical ideas. For Gowers, the explanation of a complex of different phenomena as the consequence of some simpler principle was an important part of aesthetic appreciation. Hence it is unsurprising that beauty and importance were associated.[7]

This connection between beauty and importance seems not to have received universal approbation. In 1996 and again in 2006 Freeman Dyson recalled his reaction a half-century earlier to hearing Mary L. Cartwright (1900–98) lecture on her work with Littlewood on the structure of solutions of non-linear differential equations used in radio engineering, work that was a precursor to the emergence of chaos theory two decades later:

> 'I could see the beauty of her work but I could not see its importance. I said to myself, "This is a lovely piece of work.

[1] Charalampous & Rowland, 'The Experience of Security in Mathematics', pp. 149–50.

[2] *Ibid.*, p. 149.

[3] *Ibid.*, p. 151.

[4] *Loc. cit.*

[*] ▸ pp. 499–501

[5] Bers, quot. in Hammond, 'Mathematics', p. 23.

[6] Gowers, 'The Importance of Mathematics', p. 22.

[7] *Loc. cit.*

> *Too bad it is only a practical wartime problem and not real mathematics*."'[1]

Looking back, Dyson thought that if only he had recognized that he was witnessing the origin of a new field of mathematics, he could have become one of its pioneers. He diagnosed the reason for having squandered the chance: 'I shared the tastes and prejudices of my contemporaries'.[2]

Beauty and importance

Notice that Dyson's recollected judgement was not based on the potential utility of Cartwright & Littlewood's work being detrimental to its beauty or worth. Rather, it seems to have been *irrelevant*: he could see the beauty, he could see the utility, but he could not see any *importance* (which doubtless means importance *within mathematics* or at least within theoretical science).

One of the mathematicians interviewed by Burton* clearly appreciated beauty but thought that it could not be used to judge importance:

> 'It is very satisfying to do things which are beautiful but I don't think that beauty is a criterion for whether or not something is worth doing.'[3]

Grothendieck, too, might have disagreed: he pointed out in *Récoltes et semailles* that many of his more brilliant colleagues had done beautiful work, but had made no lasting impact on mathematics, for their work was situated in a ready-made context that they did not seek to alter.[4] Thus, for Grothendieck, there seems to have been no correlation between beauty and importance.

◆

[1] Dyson, Review of *Nature's Numbers*, p. 612, emphasis added; cf. Dyson, 'Cartwright', p. 174.

[2] Dyson, Review of *Nature's Numbers*, p. 612; Dyson, 'Cartwright', p. 174.

* ▸ pp. 544–5

[3] Unnamed mathematician, quot. in Burton, *Mathematicians as Enquirers*, p. 68.

[4] Grothendieck, *Récoltes et semailles*, 'Présentation des Thèmes', ch. 11, 'L'importance d'être seul'.

14

All that Glisters is Not Gold

Beauty and the extreme and mean ratio

> ‘For lust and eagerness after gold and other things
> make men blind’
>
> — Agricola, *De Re Metallica*, bk I
> (trans. H. C. Hoover & L. H. Hoover).

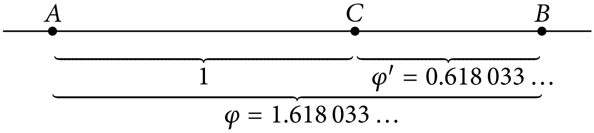
> ‘Dost thou laugh to see how fools are vex’d
> To add to golden numbers golden numbers?’
>
> — Thomas Dekker, ‘Sweet Content’, ll. 5–6.

‘A straight line is said to have been *cut in extreme and mean ratio* when, as the whole line is to the greater segment, so is the greater to the less.’[1] From this definition of Euclid (*fl.* c. 300 BCE) follows much that is important and beautiful in the *Elements* and later mathematics. But the concept of the extreme and mean — or ‘golden’ — ratio is also the nucleus of a miasma of exaggeration, misconception, and confusion regarding its aesthetic role.[2]

To unpick Euclid’s definition: a line segment AB is divided in extreme and mean ratio at a point C, nearer B than A, if $AB : AC :: AC : CB$. By elementary algebra this ratio must be given by the positive solution of the equation $x^2 - x - 1 = 0$, namely

$$\frac{1+\sqrt{5}}{2} = 1.618\,033\,988\,749\,894\,848\,204\,586\,834\ldots. \tag{E}$$

It has become customary to denote the quantity (E) by φ and the quantity $(-1+\sqrt{5})/2 = \varphi - 1$ by φ'. (The other solution to the equation $x^2 - x - 1 = 0$ is $-\varphi'$.) Thus, if AC has length 1, then the lengths of the segments are as follows:



In Proposition VI.30 of the *Elements*, Euclid described how to divide a given line AB thus: the key steps (see Figure 14.1) involve constructing a square $ABDE$ on AB, then drawing a rectangle $EFGH$

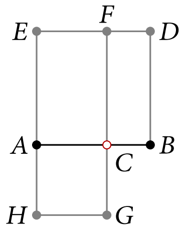


FIGURE 14.1
Euclid’s procedure for cutting in extreme and mean ratio (Euclid, *Elements*, Prop. VI.30).

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[1] Euclid, *Elements*, Def. VI.3.

[2] Markowsky, ‘Misconceptions about the Golden Ratio’.

of the same area as *ABDE* abutting the sides *AE* and *ED* such that the portion of the rectangle that lies outside of *ABDE* is a square (this construction is Proposition VI.29); it follows that this rectangle cuts *AB* at *C* in extreme and mean ratio. Euclid actually gave another construction earlier in the *Elements* (Proposition II.11), but without using the terminology 'extreme and mean ratio'. Probably the simplest construction is a variation of this earlier method due to the mathematician and engineer Hero of Alexandria (*fl.* 62 CE), shown in Figure 14.2: draw a perpendicular *BD* of half the length of the segment *AB*, draw an arc through *B*, centred at *D*, to its intersection *E* with the line *AD*, then draw an arc through *E*, centred at *A*, whose intersection *C* with the line *AB* will cut *AB* in extreme and mean ratio.[1]

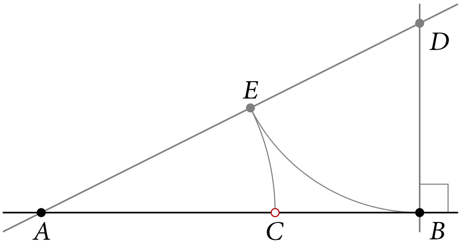

FIGURE 14.2
A simple construction for division in extreme and mean ratio. The segment *BD* is perpendicular to *AB* and half its length.

THE NAME OF GOLD

> ‘If confusion and misapprehension
> were confined to nomenclature,
> that would, it is evident, be bad enough;
> alas, more is to be described’
>
> — David Fowler, 'A Generalization of the Golden Section', p. 146.

The division in extreme and mean ratio, the ratio $\varphi : 1$, and the number φ itself have variously been called the 'divine proportion', 'divine ratio', 'divine section', 'golden mean',[2] 'golden number', 'golden ratio', 'golden proportion', 'golden section'. [The name 'golden mean' seems to stem from confusion with the philosophical 'doctrine of the mean',[3] called 'the golden mean' by Horace.[4]] Euclid used only 'extreme and mean ratio [*akron kai meson logon* [ἄ]κρον καὶ μέσον λόγον]',[5] which is the term used throughout this chapter; Proclus may have called it '*the* section [*tēn tomēn* τὴν τομὴν]'.[6]

The first ascription of the epithet 'divine' to the extreme and mean ratio was Luca Pacioli's (c. 1447–1517) book *Divina proportione*.[7][*] Pacioli's use of the term was not based upon aesthetic appreciation. Primarily it was grounded on a mystical identification of certain properties of the extreme and mean ratio with the attributes of God.[8] For instance, the incommensurability of the extreme and mean ratio corresponded to the indefinability and ineffability of God.[9] Secondarily,

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[1] Herz-Fischler, *Math. Hist. Golden Number*, p. 111.

[2] van Gerwen, 'Mathematical Beauty and Perceptual Presence', p. 13.

[3] See, for example, Kraut, 'Aristotle's Ethics', § 5.

[4] Horace, *Odes*, II.10, l. 5.

[5] Euclid, *Elements*, Def. VI.3 & notes.

[6] Proclus, *Commentary on Euclid*, FRI 67 & n. 41, emphasis added.

[7] Herz-Fischler, *Math. Hist. Golden Number*, pp. 149–51.

[*] ▸ pp. 247–9

[8] Pacioli, *Divina proportione*, ch. V, fols 3v–4r.

[9] *Ibid.*, ch. V, fol. 4r.

Pacioli justified his use of the term by citing the role of the extreme and mean ratio in the relationships between the five regular solids and their circumscribed spheres, but again there is no hint of an aesthetic appreciation.[1]

The earliest known application of the adjective ‘golden’ to the extreme and mean ratio dates to 1789, in the second volume of Johann Samuel Traugott Gehler’s (1751–95) general scientific dictionary *Physikalisches Wörterbuch*, where, in the entry ‘Freyer Fall der Körper’, a quantity is described as increasing ‘by means of the so-called golden section’: ‘durch den sogenannten güldnen Schnitt (media et extrema ratione, sectione aurea s[ive] divina)’.[2] The modern discoverer of this appearance thought that other uses of the term in the early nineteenth century derived from Gehler’s, and that possibly it was actually he who first used it, perhaps as a result of terminological confusion.[3] The earliest use in English of the adjective ‘golden’ for the extreme and mean ratio seems to have been in James Sully’s (1842–1923) article ‘Æsthetics’ in the ninth edition (1875) of the *Encyclopædia Britannica*,[4] where it is used in the context of Gustav Fechner’s experiments, which are discussed below.*

DISTINCT BEAUTIES

There is much confusion around the aesthetic value of the number φ and of the extreme and mean ratio. Two aesthetic aspects must be distinguished. On the one hand, there is the mathematical beauty of division in extreme and mean ratio, the ratio $\varphi : 1$, the number φ, and generally of mathematics that involves the concept. On the other hand, there is the sensory beauty of objects that embody the ratio $\varphi : 1$. It has often been contended that objects incorporating this ratio are aesthetically superior. This claim is bound up with assertions that the ratio is embedded into works of art and architecture: the Great Pyramid of Giza, the Parthenon, the United Nations building, the *Aeneid*, the paintings of artists from Leonardo da Vinci (1452–1519) to George-Pierre Seurat (1859–91). Such doctrines can be collectively called ‘golden numberism’. [The term was coined by Roger Herz-Fischler (1940–),[5] who illuminated much of the history of this topic, but is applied here in a slightly narrower sense than his.]

Begin with the actual history: the extreme and mean ratio enters the written mathematical record with Euclid’s *Elements* (c. 300 BCE). There is precisely no earlier evidence for awareness of it as a mathematical concept. This is not to say that Euclid defined it *ex nihilo*: it was almost certainly identified earlier. If Theaetetus’ (c. 417–369 BCE) discovery of the icosahedron (per Scholium 1 to book xiii of the *Elements*[6]) implies a *construction* of the regular icosahedron similar to Euclid’s, then it is probable that division in extreme and mean ratio

[1] Pacioli, *Divina proportione*, ch. v, fol. 4r.

[2] Gehler, *Physikalisches Wörterbuch*, vol. ii, p. 120.

[3] Lieberknecht, email to Herz-Fischler, dated 19 May 2018, available at `https://herz-fischler.ca/ZEISING/otfried-lieberknecht_2018.08.19.pdf`

[4] Fowler, ‘A Generalization of the Golden Section’, p. 146; see Sully, ‘Æsthetics’, p. 220.

* ▸ pp. 557–8

[5] Herz-Fischler, ‘The golden number’, p. 1580.

[6] Heath, hist. note. to Euclid, *Elements*, bk xiii, p. 438.

was known in the late fifth or early fourth centuries BCE. There are various theories connecting it with the pythagoreans,[1] but there is no reason to posit any earlier origin, whether Greek, Mesopotamian, or Egyptian.

Architecture and art

The earliest claim that the extreme and mean ratio was or should be applied to architecture and art seems to have been in the second edition (1799–1802) of Jean-Étienne Montucla's (1725–99) *Histoire des Mathématiques*.[2][*] Montucla wrote that a large part of Pacioli's *Divina proportione*

> 'is composed of plates, depicting the application of the divine proportion to architecture, to the design of capital letters that appear to me of such good taste, that I suspect that they are taken from ancient monuments'.[3]

This description is simply incorrect,[4] and this fundamental point must be emphasized: *before Montucla, there is no suggestion that the extreme and mean ratio was or should be applied in the plastic arts or architecture*. In particular, it makes no appearance in the extensive discussion of proportion by Vitruvius[†] (early 1st century–c. 25 BCE).[5]

The idea of applying the extreme and mean ratio in music had arisen earlier, in the eighteenth century, but G. W. Leibniz's (1646–1716) reaction makes it clear that the notion was neither customary nor, in his view, useful. An unnamed friend of Christian Goldbach (1690–1764) had proposed dividing the monochord in extreme and mean ratio, but Leibniz pointed out that the resulting intervals $\varphi : 1$ and $\varphi' : 1$ would closely approximate the minor sixth $8 : 5$ and the major sixth $5 : 3$. He thought that insofar as the division was pleasing at all, it would be due to the approximation to these consonances.[6]

One can also make here a cautious *argumentum ex silentio*: that Shaftesbury[‡] (1671–1713), Francis Hutcheson[§] (1694–1746), and Yves-Marie André[¶] (1675–1764) did not mention the extreme and mean ratio, despite the importance they assigned to proportion and mathematics, counts against it having been an established concept in art or architecture; that Johannes Kepler (1571–1630) did not consider it among the myriad harmonic proportions he extracted from polyhedra in *Harmonice Mundi*[♠] counts against it having had any accepted aesthetic significance.

Adolph Zeising and the beginnings of golden numberism

Golden numberism proper emerged in the middle of the nineteenth century with the publication of Adolph Zeising's (1810–76) book *New Theory of the Proportions of the Human Body* [*Neue Lehre von den Proportionen des menschlichen Körpers*] in 1854. Zeis-

[1] Herz-Fischler, *Math. Hist. Golden Number*, § 12.

[2] *Ibid.*, p. 150.

[*] ▸ pp. 346–51

[3] Montucla, *Histoire* New ed., vol. 1, p. 551; trans. C.

[4] Herz-Fischler, 'The golden number', pp. 1580–1.

[†] ▸ pp. 129–30

[5] Vitruvius, *Ten Books on Architecture*, *passim*.

[6] Leibniz, letter to Goldbach, dated 17 Apr. 1712, repr. in Leibnitius, 'Epistolæ Tres', p. 438.

[‡] ▸ pp. 353–6

[§] ▸ pp. 357–62

[¶] ▸ pp. 362–4

[♠] ▸ pp. 288–9

ing's book begins with an extensive historical survey covering ancient and modern philosophers, artists, anatomists, and physiologists. His position was that the theme of applying mathematical forms to art recurred throughout aesthetics in the western tradition. But no thinker had given an account of symmetry and proportion that was precise and concrete.[1] Zeising thought that every ancient and modern canon of proportion lacked a general axiom from which specifics could correctly be deduced. He accepted Hutcheson's criterion of 'uniformity amidst variety' in general terms, but criticized the inferences he drew from it, in particular the interpretations he made of uniformity and variety in his application of the criterion to polygons.[2][*] He concluded that Hutcheson had recognized the need for a law of proportion, but had not given one.[3] He thus set out his own law of proportion based on the extreme and mean ratio.[4] In the last and longest chapter of his book, he argued — apparently to his own satisfaction — that his law was in accordance with the proportions of many masterpieces of ancient and Renaissance art and architecture.[5]

Zeising's book was well-received, and it, together with other publications of his in the following years, led to the gathering of the school of golden numberists.[6] Friedrich Röber's short book *The Egyptian Pyramids in their Original Forms, with a Description of the Proportional Relationships in the Parthenon of Athens* [*Die aegyptischen Pyramiden in ihren ursprünglichen Bildungen, nebst einer Darstellung der proportionalen Verhältnisse im Parthenon zu Athen*] (1855) was the second work of golden numberism. Röber argued that the extreme and mean ratio had been used in the designs of almost all Egyptian pyramids, though *not* the Great Pyramid itself.[7] A modern reader must see the irony in this, for claims abound that the extreme and mean ratio is embodied in the dimensions of the Great Pyramid of Giza.[8] In fact, there is no evidence that the Egyptian scribes were aware of the extreme and mean ratio. Conceivably, they may have *approximated* φ using ratios of successive terms of the Fibonacci sequence.[9] However, it would be difficult to notice this convergence using their system of unit fractions.[†] Not quite impossible: a sufficiently ingenious scribe *could* have calculated unit fraction expressions for the successive terms.[10] But noticing the convergence presupposes some concept of limit and knowledge of the number φ. Such knowledge might have come from a geometrical concept of the extreme and mean ratio, but there is no evidence of awareness of the geometrical concept.[11] In the particular case of the Great Pyramid, simpler hypotheses explain its measured shape better than those involving the extreme and mean ratio, hypotheses that use mathematics the Egyptians are known to have had. For instance, one theory says that the cotangent of the angle of the sides of the pyramid is 22/28,[12] and the equivalent notion of *seked* [*sḳd* 𓋴𓐨𓂧𓏌𓏛] — the horizontal distance covered per vertical unit of slope — is found in surviving Egyptian mathematical texts, such as

[1] Brown, 'Adolph Zeising', p. 24.

[2] Zeising, *Neue Lehre von den Proportionen*, p. 103.

[*] ▸ p. 358

[3] *Ibid.*, p. 105.

[4] *Ibid.*, pp. 156–74.

[5] *Ibid.*, ch. IV.

[6] Herz-Fischler, *Adolph Zeising*, pp. 45–6.

[7] Herz-Fischler, 'The golden number', p. 1580.

[8] For the history of such claims, see Herz-Fischler, *The Shape of the Great Pyramid*, chs 11–13.

[9] Badawy, *Ancient Egyptian Architectural Design*, esp. 24–5.

[†] ▸ pp. 30–1

[10] Rossi & Tout, 'Were the Fibonacci series and the golden section known in ancient Egypt?', pp. 107–11.

[11] *Ibid.*, p. 111.

[12] Herz-Fischler, *The Shape of the Great Pyramid*, p. 28.

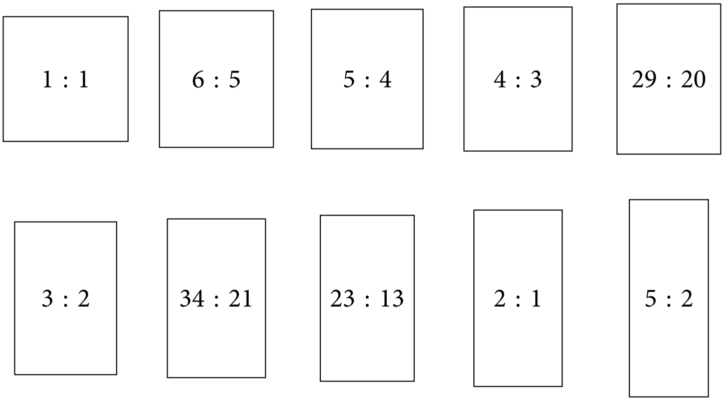


FIGURE 14.3
Rectangles in the proportions used by Fechner in his experiments.

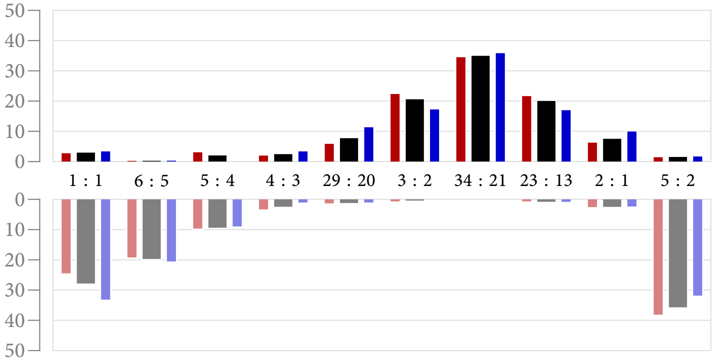


FIGURE 14.4
Percentage of participants in Fechner's experiments who selected a rectangle of each proportion as most pleasing (above) and least pleasing (below); percentage of male participants shown in red, female in blue (Fechner, *Vorschule der Aesthetik*, vol. 1, p. 195).

problem 56 in the Rhind mathematical papyrus.[1] Thus Occam's razor cuts away the extreme and mean ratio theory.

Gustav Fechner's experiments

The physicist and psychologist Gustav Fechner (1801–87) conducted experiments in which subjects were asked to choose, from a randomly-presented set of ten rectangles with varying proportions and equal areas, the ones they found most and least pleasing. If a participant chose several rectangles as most and least pleasing, their choice was counted fractionally for each. The proportions used were as shown in Figure 14.3.[2] The results (see Figure 14.4) showed that the most popular choice for most pleasing, chosen by 35.0 %, was the 34 : 21 rectangle, whose proportions *approximate* the extreme and mean ratio. An additional 20.6 % and 20.0 % preferred the choices of the 'neighbouring' 3 : 2 and 23 : 13 rectangles. No-one chose the 34 : 21 rectangle as least pleasing.

From this experiment and from others Fechner formulated certain conclusions. He thought that the $\varphi : 1$ rectangle, *and those close to it in proportion*, were aesthetically superior to other rectangles.[3] That is, there was a range of rectangles that subjects on average considered pleasing, and this range included the $\varphi : 1$ rectangle.[4]

[1] Imhausen, 'Egyptian Mathematics', p. 34.

[2] Fechner, *Vorschule der Aesthetik*, vol. 1, p. 193–4.

[3] *Ibid.*, vol. 1, p. 192, § d).

[4] Cf. Zusne, *Visual Perception of Form*, p. 399.

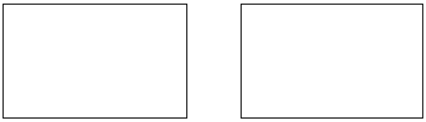
But even a slight deviation from being an exact rectangle detracted from the aesthetic value of a quadrilateral much more than the extreme and mean ratio contributed to it.[1] Divisions of horizontal or vertical lines in extreme and mean ratio were less pleasing than divisions in other proportions.[2] In the end, Fechner found that the extreme and mean ratio was aesthetically pleasing, but not overwhelming so:

> ‘I cannot help but find the aesthetic value of Zeising’s golden ratio to be overestimated, although I do not deny the interest and merit of Zeising’s discovery, that this ratio has a noteworthy aesthetic value’.[3]

[Many later accounts mis-report Fechner’s method or exaggerate his results.[4] There has been some dispute about whether the rectangles were presented in some fixed orientation, horizontal or vertical, or in random orientations,[5] although Fechner’s account seems clear that they were shown ‘criss-cross at various angular positions to each other [kreuz und quer in verschiedenster Winkelstellung zu einander]’.[6] And his conclusions were more cautious than a blunt assertion that that the $\varphi : 1$ rectangle was the most pleasing.]

Fechner was careful to distinguish the supposed visual aesthetic superiority of the extreme and mean ratio from its role in mathematics. Although he mistakenly believed that the name ‘golden section’ went back to antiquity, he thought that this terminology indicated only its special role in mathematics. He explicitly acknowledged that there was no extant historical evidence for it having had any aesthetic significance, and thought that *insofar* as the extreme and mean ratio could be found in works of ancient art, and *if* it had aesthetic superiority, then its presence must be due to an unconscious preference.[7]

Subsequent experimental research has left a confusing picture, and it is uncertain whether the extreme and mean ratio has any aesthetic superiority. Part of the problem is doubtless in designing an experiment that distinguishes aesthetic judgement of $\varphi : 1$ and $1.6 : 1$ (that is, $8 : 5$), which are the ratios of the following two rectangles:



A survey of this experimental work up to the early 1990s weakly favoured the claimed aesthetic effects of the extreme and mean ratio, suggesting that they were real but ‘fragile’.[8]

The spread and influence of golden numberism

Golden numberism seems to have been confined to Germany until the start of the twentieth century, when it started to spread and to be incorporated into theories of proportion by authors such as the artist Jay Hambidge (1867–1924) in *The Elements of Dynamic Symmetry* (1926) and the philosopher Matila Ghyka (1881–1965) in *Ésthetique des proportions dans la nature et dans les arts* (1927) and *Le Nombre d’or* (1931).[1] The architect Le Corbusier (Charles-Édouard Jeanneret; 1887–1965), influenced by Ghyka, started to use the extreme and mean ratio in his designs.[2] So in thrall was he to golden numberism that he tried to rewrite history by publishing drawings indicating that he had used the extreme and mean ratio in buildings he designed using different systems.[3] The futurist artist José de Almada Negreiros (1893–1970) used the extreme and mean ratio and the ratio 9 : 10, either exactly or in approximate geometrical constructions, in various artworks.[4] Most notably, geometrical diagrams and objects involving both ratios appear in his tapestry *The Number* [*O Número*].[5]

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[1] Fechner, *Vorschule der Aesthetik*, vol. 1, p. 192, § e).

[2] *Ibid.*, vol. 1, p. 192, §§ f), g).

[3] *Ibid.*, vol. 1, p. 192; trans. C.

[4] Green, ‘All that glitters’, pp. 942–4.

[5] *Ibid.*, p. 943.

[6] Fechner, *Vorschule der Aesthetik*, vol. 1, p. 194; trans. C.

[7] Fechner, ‘Zur experimentellen Aesthetik’, p. 590.

[8] Green, ‘All that glitters’, p. 966.

There has developed a self-reinforcing tendency to find the extreme and mean ratio in art and architecture, even when it is absent. It is illuminating that there is an analysis locating it in the work of the cubist artist Juan Gris (José Victoriano González-Pérez; 1887–1927), although he wrote that he *never* used it.[6] The moral is that if a golden numberist is convinced that the extreme and mean ratio is present in a work of art or architecture, they can probably find a way to extract it.[7] Indeed, there is an inherent danger in trying to deduce an underlying scheme of proportion from the structure itself:

> ‘If one takes the trouble to delve into some of the proportional analyses of the “poor old Parthenon” (to quote Theodore A. Cook) published from Penrose’s days on (1851), it will be seen that almost anything under the sun can be proved: that the design was based on the Golden Section (Zeising, 1854), on commensurable ratios (Pennethorne, 1878), on triangulation (Dehio, 1895), on the ratios of small whole numbers (Raymond, 1899), on root-five rectangles (Hambidge, 1924), on Greek modules (Moe, 1945), and so forth.’[8]

(One could add ‘on a system of ratios also present in a “perfect” human skeleton (Hay, 1846)’.[9][*])

Further difficulties abound. There can be uncertainty about the original dimensions of a ruined structure like the Parthenon; even the measurement of the largely-intact Great Pyramid is complicated by the loss of casing and summit stones.[10] To this one must add the fact that computing ratios can multiply errors in measurement.[11] Working from a small-scale drawing, the thickness of lines can hide discrepancies measured in metres.[12]

The twentieth-century Christian platonist philosopher Simone Weil (1909–43), whose appreciation of mathematical beauty was well-developed,[†] was one who stood against the golden numberist wind. Her France notebooks contain the following entry:

> ‘Golden number: ratio between a number situated between 3.20 and 3.25 and 2. It is also approximately the ratio between

[1] Herz-Fischler, ‘The golden number’, p. 1582; Grattan-Guinness, ‘Numerology and gematria’, p. 1586.

[2] Padovan, *Proportion*, §§ 15.7–12.

[3] Fischler, ‘The early relationship of Le Corbusier’.

[4] Trabucho de Campos, *O Número*, pp. 30, 36, 40–1, 95, 98.

[5] *Ibid.*, *passim*, esp. 75–8, 107–15.

[6] Fischler & Fischler, ‘Juan Gris’.

[7] Fischler, ‘How to Find the “Golden Number”’; Markowsky, ‘Misconceptions about the Golden Ratio’.

[8] Wittkower, ‘The Changing Concept of Proportion’, p. 209; endnote marks omitted.

[9] Hay, *First Principles of Symmetrical Beauty*, pp. 66–72.

[*] ▸ pp. 408–12

[10] Herz-Fischler, *The Shape of the Great Pyramid*, ch. 2.

[11] Markowsky, ‘Misconceptions about the Golden Ratio’, p. 5.

[12] Rossi, *Architecture and Mathematics in Ancient Egypt*, pp. 46–7.

[†] ▸ pp. 578–86

> the circle and the diameter. It seems probable that where one thinks one discerns the golden number, in architecture, sculpture and painting, it is rather a question of the latter ratio, which has a much more clearly defined symbolic meaning. Or is it an approximation of the two?'[1]

The second sentence is, as written, incorrect; presumably, Weil saw that π lay close to the range (3.2, 3.25) and wrote too hastily, meaning that the ratio $\pi : 2$ approximated the extreme and mean ratio. In any case, she thought that any apparent presence of the extreme and mean ratio $\varphi : 1$ in the plastic arts was more likely to be due to the (presumably unconscious) incorporation of the ratio $\pi : 2$, or possibly of an approximation of *both* ratios. Either possibility would count against aesthetic superiority of the extreme and mean ratio.

The original proposition that the extreme and mean ratio has some superior visual beauty is doubtful. On this shaky foundation golden numberists have erected an impressively large but rickety structure.

Posited connection between visual and mathematical beauty

Golden numberism seems to be reinforced by a connection that has been put forward between the beauty of mathematics involving the extreme and mean ratio and its alleged sensory aesthetic superiority.[2]

This con-fusion of sensory and mathematical aesthetic value is exemplified in the selective quotation of the famous statements by Kepler about the extreme and mean ratio. Kepler wrote that

> 'geometry possesses two treasures: the ratio of the hypotenuse in the right triangle to the legs, as well as division in the extreme and mean ratio. [Note 17; see below.] Out of this ratio comes the construction of the dodecahedron and the icosahedron. It is for this reason that the pyramid[3] can be inscribed into the cube, and the octahedron into both of these solids, so easily and simply.'[4]

Note 17 reads:

> 'Two propositions of unlimited use and consequently highly valuable; there exists, however, a big difference between them. The former, which states that in a right-triangle the sum of the legs squared is equal to the sum of the hypotenuse squared, I would like to compare to a gold-nugget; the other, relating to proportional division, I would like to call a gemstone. The one is nice in itself, but it is of no value without the other. When the former, after we have made the initial step with it, leaves us in the lurch, this one leads us further into science, it leads us to the calculation and construction of the side of the decagon and related quantities.'[5]

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[1] Weil, *The Notebooks*, p. 510.

[2] See, for instance, Posamentier & Lehmann, *The Glorious Golden Ratio*, Introduction; Huntley, *The Divine Proportion*, esp. pp. 50–1; Meisner, *The Golden Ratio*, esp. p. 58.

[3] The tetrahedron.

[4] Kepler, *Mysterium Cosmographicum* (1621), ch. XII, p. 68; trans. Herz-Fischler, *Math. Hist. Golden Number*, p. 175.

[5] Kepler, *Mysterium Cosmographicum* (1621), ch. XII, p. 74; trans. Herz-Fischler, *Math. Hist. Golden Number*, p. 175.

[These remarks occur only in the 1621 edition of *Mysterium Cosmographicum*, not the original 1596 edition.[1]] Quotations of this passage often include only the first sentence of the main text and the second sentence from the note ('The former [...] gemstone'), or even smaller fragments.[2] Such elision obscures the fact that Kepler explicitly wrote that the value of the extreme and mean ratio is bound up with that of the pythagorean theorem, and that their joint value derives from their usefulness in further geometry.

It is difficult to locate the first appearance of this supposed connection between abstract mathematical beauty and the putative sensory aesthetic superiority. There was a hint of it in 1931, when Ghyka wrote, in a book on proportion and geometrical canons in art and architecture, that Kepler had mentioned the extreme and mean ratio as 'one of two "jewels of geometry"'.[3] In 1970, H. E. Huntley (1892–1975) used the selective quotation of Kepler as an epigraph in his book *The Divine Proportion*,[4] in which, as discussed below, he suggested a deep connection between mathematical and sensory beauty.*

Arguments against a connection

Even if the extreme and mean ratio does possess sensory aesthetic superiority, to posit a connection to intellectual mathematical beauty is a *non sequitur*. There is much beautiful mathematics involving π and e, and yet there seems to be no sensory aesthetic preference for rectangles with proportions π : 1 or e : 1. Similarly, G. D. Birkhoff (1884–1944) pointed out that no special aesthetic value is assigned to the $\sqrt{2} : 1$ rectangle, which is made up of two rectangles with the same proportion (and so is used for the ISO 'A' paper sizes); the $\sqrt{3} : 1$ rectangle, made up of two $1 : \sqrt{3} : 2$ triangles, which Plato considered the fairest scalene triangle;† or the 2 : 1 rectangle, which is made up of two squares (see Figure 14.5).[5]

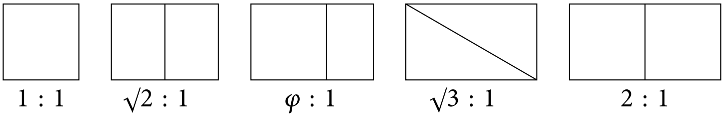

FIGURE 14.5
Rectangles used by Birkhoff to argue against a role for mathematics in the supposed aesthetic superiority of the $\varphi : 1$ rectangle.

In the other direction, Fechner's own survey of over ten thousand paintings[6] showed that the height-to-width ratio of vertically oriented paintings averaged around 5 : 4 and that of horizontally oriented paintings averaged around 3 : 4,[7] but one looks in vain for a collection of beautiful mathematics especially involving 5, 4, 5/4, 4/5 or 3, 4, 3/4, 4/3. So any connection between the mathematical and the supposed sensory aesthetic values of the extreme and mean ratio does not follow *a fortiori* from some general law about ratios.

What there seems to be is a kind of circularity: the supposed sensory beauty of the extreme and mean ratio has led to the inference of some connection with the mathematical beauty, and this connection has been used to support the supposition about sensory beauty.

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[1] Herz-Fischler, *Math. Hist. Golden Number*, p. 175.

[2] For example, Ghyka, *The Geometry of Art and Life*, p. 44; Livio, *The Golden Ratio*, p. 62; Posamentier & Lehmann, *The Glorious Golden Ratio*, Introduction.

[3] Ghyka, *Le Nombre d'or*, p. 50. The author has been unable to check Ghyka, *Ésthetique des proportions* (1927) for similar references.

[4] Huntley, *The Divine Proportion*, p. 23.

* ▸ pp. 563–71

† ▸ pp. 65–7

[5] Birkhoff, *Aesthetic Measure*, p. 28.

[6] Fechner, *Vorschule der Aesthetik*, vol. 2, p. 282, Tbl. III.

[7] *Ibid.*, vol. 2, p. 291, Tbl. VI.

DISCOURSE ON UBIQUITY

Part of the value of the extreme and mean ratio and the number φ is their ubiquity. As Kepler said, division in extreme and mean ratio is vital for the development of geometry. Its influence is felt throughout Euclid's *Elements*.[1] Some of the golden numberists seem to connect its ubiquity with its (mathematical) aesthetic appeal, possibly via the pleasant surprise when it crops up unexpectedly.[2] Unexpectedness has been recognized as an aspect of mathematical beauty both by some who accept golden numberism[3] and by others including, most famously, G. H. Hardy* (1877–1947).

But is the ubiquity of the extreme and mean ratio and the number φ unexpected on reflection? Division in extreme and mean ratio appears in some of the simplest geometric configurations: for example, the diagonals of a regular pentagon divide each other in extreme and mean ratio (see Figure 14.6). Consider also the continued fraction representation of φ:

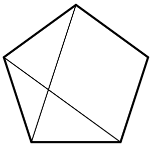

FIGURE 14.6
The diagonals of a regular pentagon divide each other in extreme and mean ratio.

$$\left.\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}\quad\right\} \quad (C)$$

G. H. Hardy & E. M. Wright (1906–2005) wrote that '[o]ne may say, again very roughly, that the structure of the continued fraction for [an irrational number] ξ affords a measure of the "simplicity" or "complexity" of ξ', and hence φ is the '"simplest" of all irrationals'.[4] Presumably, therefore, $\sqrt{2}$ is more complex than φ, since not all of its partial quotients are equal:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}}$$

Further, although the continued fraction representation (C) and the formula

$$\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}$$

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[1] Herz-Fischler, *Math. Hist. Golden Number*, § I.1.

[2] For example, Huntley, *The Divine Proportion*, p. 95; Livio, *The Golden Ratio*, p. 6; Posamentier & Lehmann, *The Glorious Golden Ratio*, 'Concluding thoughts'.

[3] Huntley, *The Divine Proportion*, p. 35.

* ▸ pp. 502

[4] Hardy & Wright, *Theory of Numbers*, p. 209.

are considered beautiful by some,[1] they follow from trivial manipulation of the equation $x^2 - x - 1 = 0$, of which φ is the positive solution. In fact, $x^2 - x - 1 = 0$ is the minimal polynomial for φ. Of the other irreducible polynomials of degree 2 with coefficients in $\{-1, 0, 1\}$, only one has real solutions: $x^2 + x - 1$, whose solutions are $(-1 \pm \sqrt{5})/2$, namely $-\varphi$ and $-\varphi'$. Thus also by the measure of minimal polynomials with small coefficients, φ is a very simple number.

If one accepts that, in some sense, simpler numbers are more likely to arise in mathematics than more complex ones, perhaps the very simplicity of φ explains its ubiquity.[2] Ultimately, though, the notion that φ is a relatively simple number is in part an aesthetic one.

H. E. HUNTLEY AND *THE DIVINE PROPORTION*

Herbert E. Huntley (1892–1975) was motivated to write *The Divine Proportion* (1970) by the challenge of communicating to students what he considered 'the best, and perhaps the only enduring motive for reading mathematics', *viz.*, 'the aesthetic motive'.[3] This slim book is primarily an anthology of beautiful mathematics and a spirited defence of the notion of beauty in mathematics, and only secondarily an attempt to analyse mathematical beauty. Most of Huntley's examples of beautiful mathematics are connected somehow with the extreme and mean ratio; hence the title of his book. Chronologically, Huntley's work sits in the late twentieth century, after the publication of G. H. Hardy's *A Mathematician's Apology* in 1940, which was a watershed in the discourse on mathematical beauty.[*] But Huntley's thought is something of an isolate in the post-Hardian period: though he nodded towards some of Hardy's positions,[4] and agreed with Jacques Hadamard[†] (1865–1963) that 'mathematical beauty serves a useful purpose as a *guide* to truth',[5] his arguments neither depended on nor seriously engaged with either; he drew rather on nineteenth-century golden numberism and on Jungian psychology.

The Divine Proportion, especially in its first half, is something of a mélange: there are frequent switches from examples of beautiful mathematics, to arguments about beauty in mathematics, to discussions about beauty generally, to putative parallels between beauty in mathematics and in music or visual art or chess. A degree of disentangling is required to make sense of Huntley's position.

On beauty

Huntley said that beauty in mathematics

> 'may mean the pleasure derived from the mental activity which the study of mathematics generates; or it may mean the aes-

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[1] Davis & Hersh, *Mathematical Experience*, p. 171; Chamberland, *Single Digits*, pp. 3–4.

[2] This observation is not original, but the author has forgotten where he read it.

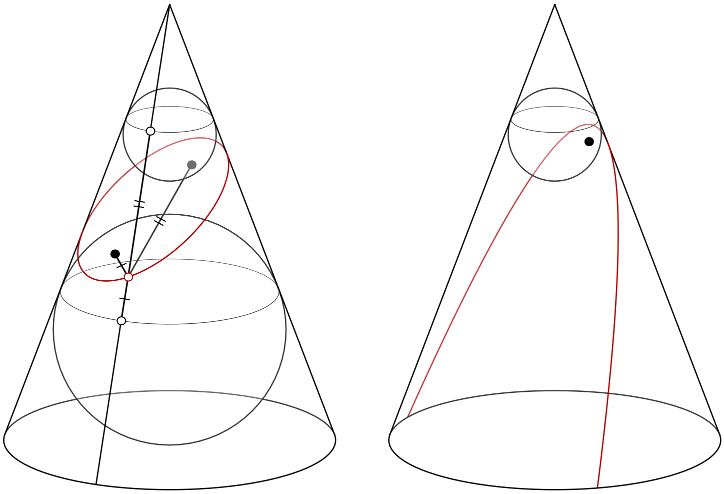
[3] Huntley, *The Divine Proportion*, p. vii.

[*] ▸ pp. 498–523

[4] For example, *ibid.*, pp. 131–2.

[†] ▸ pp. 662–4

[5] *Ibid.*, p. 75, emphasis in orig.



Dandelin spheres of an ellipse

Dandelin sphere of a parabola

FIGURE 14.7
An ellipse has two Dandelin spheres (*left*). As the angle of the intersecting plane increases, the ellipse becomes more eccentric and the larger Dandelin sphere increases in size. At the limit (*right*), the angle equals the slope of the cone, the conic section is a parabola, and there is only one Dandelin sphere (since in approaching the limit the radius of the larger sphere grows to infinity).

> thetic feeling evoked by (e.g.) a mathematical theorem, which, on account of this feeling, is regarded as a thing of beauty'[1]

but shortly afterwards wrote 'we are thinking of beauty that has an objective ingredient'.[2] Huntley seems to have thought of beauty in mathematics in at least three ways here: as a form of pleasure in the process doing mathematics, as a reaction to contemplation of mathematics, and as something objective about mathematics. Huntley also listed various 'ingredients' of beauty,[3] which are a mixture of criteria for and loci of beauty. Among the criteria, one can see links with Huntley's three ways of thinking of beauty:

Alternation of tension and relief. Huntley saw alternation in the guise of perplexity and illumination when reading mathematics, which fits with parallels he drew elsewhere between beauty in music and in mathematics, for the development of tension and its release is a common feature of music.[4]

Realization of expectation. This experience was perhaps connected with the alternation of tension and relief, for built into tension in music and mathematics was the anticipation that it will be released. But it also occurred when a result is conjectured from previous ones and turns out to be true. Huntley's example was how the configuration of two Dandelin spheres for an ellipse finds its limit in the single Dandelin sphere of a parabola (see Figure 14.7).[5] [Dandelin spheres give a beautiful[6] proof of the result that an ellipse is the set of points for which the sum of the distances to two fixed foci are equal to some constant.* The two spheres are tangent to the plane of the ellipse at its foci, and each are tangent to the cone at a circle. A straight line through the apex of the cone and a point on the ellipse lies on the surface of the cone and so meets each of the circles. The point where this line meets the upper

[1] Huntley, *The Divine Proportion*, p. 70.

[2] *Loc. cit.*

[3] *Ibid.*, pp. 81–8.

[4] *Ibid.*, p. 81.

[5] *Ibid.*, p. 82.

[6] Polster, *Q.E.D.*, pp. 36–7.

* ▸ p. 435

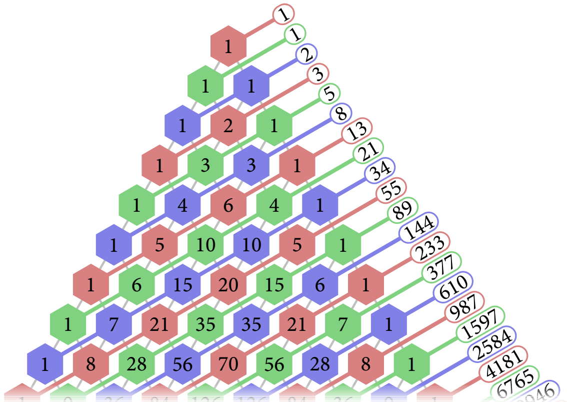


FIGURE 14.8
Pascal's triangle, where each entry is the sum of two entries above it, and where the i-th entry in the n-th row is $\binom{n}{i}$. Summing the diagonals yields the Fibonacci sequence. This appearance of the Fibonacci sequence is a consequence of each entry in the triangle being equal to the sum of the two entries above it, which lie in different diagonals.

circle, the point where it meets the ellipse, and the centre of the upper sphere define a triangle that is congruent to the triangle defined by the point on the ellipse, the centre of the upper sphere, and the focus where the upper sphere is tangent. So the distances from the point on ellipse to the focus and to the point where the line meets the upper circle are equal (see Figure 14.7 (left)). The same holds for the other focus and the lower circle. So sum of the distances to the foci is equal to the distances between the circles.]

Surprise at the unexpected. This experience was not exactly the opposite of realization of expectation; such an opposite would perhaps have been ugly for Huntley. Rather it referred to the discovery of an unanticipated fact. Huntley gave the example of the appearance in Pascal's triangle of the Fibonacci sequence $(F_n)_{n\geqslant 1}$, where $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ (see Figure 14.8).[1]

Perception of unsuspected relationships. Huntley stated this quality separately from 'surprise at the unexpected'. On the one hand, perception is presumably prior to any ensuing surprise. On the other hand, an 'unsuspected relationship' seems to have been a special case of 'the unexpected'. One of Huntley's examples was the relationship between the equation of a general conic (with axes rotated so that any xy term vanishes)

$$x^2 + y^2 + 2ax + 2by + c = 0$$

and the equation of its tangent at (x_1, y_1), which is

$$xx_1 + yy_1 + a(x + x_1) + b(y + y_1) + c = 0;$$

another was the possibility of finding the extreme and mean ratio in a $3 : 4 : 5$ triangle (see Figure 14.9).[2] [In a sense, the $3 : 4 : 5$ triangle is a distraction. Let ϑ be the angle $\angle ACB$, then $\cos\vartheta = 3/5$, so that $\sin\vartheta/2 = 1/\sqrt{5}$ by a standard trigonometric identity. Thus the bisector of $\angle ACB$ is the smallest angle of a $1 : 2 : \sqrt{5}$ triangle. And the extreme and mean ratio equals $\sqrt{5} + 1 : \sqrt{5} - 1$.]

[1] Huntley, *The Divine Proportion*, p. 82.

[2] *Ibid.*, pp. 82–3.

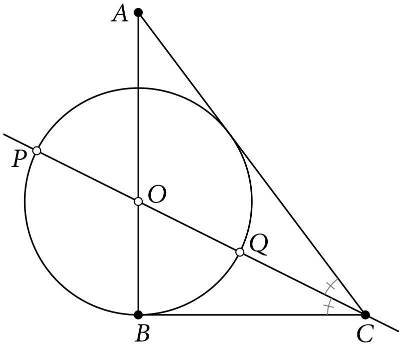


FIGURE 14.9
Huntley's illustration of a connection between the extreme and mean ratio and a 3 : 4 : 5 triangle. Let the bisector of $\angle ACB$ meet AB at O and describe a circle centred at O with radius OB. Let the circle meets the bisector at P and Q. Then Q divides CP in extreme and mean ratio (Huntley, *The Divine Proportion*, pp. 43–4).

Patterns. Amongst patterns Huntley placed chess problems (G. H. Hardy's 'hymn-tunes of mathematics'[1]), magic squares, the expressions for natural numbers as sums of polygonal numbers (whose existence is asserted by Fermat's polygonal number theorem[*]), and Pascal's triangle.[2]

'Brevity is the soul of wit'. For Huntley, this maxim referred to briefly stated theorems or plausible conjectures of great generality, such as Goldbach's conjecture, or short proofs, such as a proof of the pythagorean theorem[†] which goes back at least as far as Abū'l-Wafā' al-Būzjānī (940–77/8 CE), and for which Huntley accepted the widespread mistaken attribution to Bhāskara II (b. 1115).[3]

'Unity in variety'. Huntley ascribed this phrase to Samuel Taylor Coleridge (1772–1834); it was actually Jacob Bronowski's (1908–74) formulation[4] of Coleridge's 'general definition of Beauty [...] — Multëity in Unity',[5] which of course descended from Jean-Pierre de Crousaz[‡] (1663–1750) via Hutcheson.[§] Huntley's example was the cycloid,[¶] though he identified it primarily as the brachistochrone.[♠] This curve appears as the solution of Johann I Bernoulli's problem of a minimum-time descent path, as the path followed by a ray of light passing through a medium of varying density, and as the curve generated by a point on a rolling circle.[♥] Huntley saw unity in variety in the linking of mechanical, optical, and purely geometrical problems via the cycloid.[6]

Sensuous pleasure. Huntley thought that in geometry there was a visual pleasure in smooth curves such as circles, ellipses, cycloids, catenaries, the graphs of trigonometric functions, limaçons and in particular cardioids (see Figure 14.10), logarithmic and archimedean spirals. He suggested that such curves have a primitive psychological appeal that jagged lines do not.[7] Huntley wrote before fractal geometry had become well-known, and had doubtless been influenced by Oskar Lundholm's (1891–1955) results on the affective tone of lines, which he quoted in another part of the book.[8] [Lundholm's results may have a bearing on some of the reactions to the 'monsters' in analysis.[♦]]

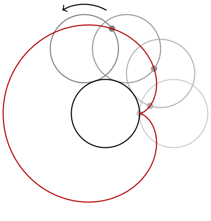


FIGURE 14.10
Generation of a cardioid.

[1] Hardy, *Apology*, § 10.

[*] ▸ p. 300

[2] Huntley, *The Divine Proportion*, p. 84, ch. IX.

[†] ▸ p. 224

[3] Huntley, *The Divine Proportion*, pp. 84–5; for Bhāskara's work, see Plofker, 'Mathematics in India', pp. 476–7.

[4] Bronowski, *Science and Human Values*, p. 27.

[5] Coleridge, 'Essays on the Principles of Genial Criticism', p. 372.

[‡] ▸ pp. 356–7

[§] ▸ pp. 357–62

[¶] ▸ pp. 296–7

[♠] ▸ pp. 331–3

[♥] ▸ pp. 331–3

[6] Huntley, *The Divine Proportion*, pp. 85–6.

[7] *Ibid.*, pp. 87–8.

[8] *Ibid.*, p. 167.

[♦] ▸ p. 655

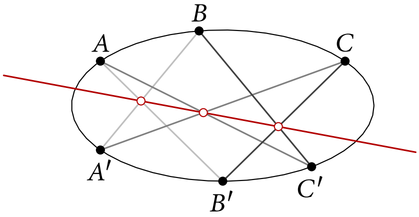


FIGURE 14.11
Pascal's theorem.

A sense of wonder, even of awe. Associated to this sense is 'the baffled sense of mystery produced by certain mathematical theorems, the beauty of which is accompanied by an initial feeling of inadequacy to explain such remarkable results'.[1] Huntley's discussion here seems to tiptoe around the word 'sublime'. He suggested that divergent series or the curve of a hyperbola could trigger the 'emotionally charged concept' of infinity.[2] The 'baffled sense of mystery' was produced, for example, by Pascal's theorem:[3]

PASCAL'S THEOREM. *Let A, B, C, A', B', C' be distinct points on a non-degenerate conic. Then the three intersection points of AB' and $A'B$, of AC' and $A'C$, and of BC' and $C'B$ are collinear.* (See Figure 14.11.) □

Taken together, these criteria suggest that Huntley's loci of mathematical beauty included proofs (which can cause tension and relief, realization of expectation, and surprise at the unexpected), results ('brevity is the soul of wit', 'unity in variety', a sense of wonder), and objects (patterns, sensuous pleasure).

Huntley gave another analysis of aesthetic feeling when he considered the aesthetic appeal of the result that for any Fibonacci sequence $(F_n)_{n\geqslant 1}$, where $F_{n+1} = F_n + F_{n-1}$, the ratio F_{n+1}/F_n approaches φ regardless of the initial terms. Huntley's view was that the appeal of such simple results 'appears to be compounded of a mixture of archaic emotions',[4] namely *surprise* at finding φ where one would not expect it, *curiosity* as to why this was so, and *wonder* at the further discoveries in the boundless and unexplored mathematical world.[5]

Sensory and mathematical beauty

In Huntley's list of 'ingredients', and especially under 'sensuous pleasure', there appears a theme that recurs throughout *The Divine Proportion*: a parallel between sensory beauty and mathematical beauty. Huntley held that appreciating the former is 'given' but that the ability to appreciate the latter must be 'acquired' by effort. His first example was that the regular solids have an immediate visual beauty in their outward appearance, but that mathematical training confers the beautiful knowledge that: there are precisely five; the cube is dual to the octahedron and the dodecahedron to the icosahedron; the twelve vertices of the icosahedron divide into three coplanar groups, each forming the vertices of a $\varphi : 1$ rectangle.[6]

[1] Huntley, *The Divine Proportion*, p. 87.

[2] *Loc. cit.*

[3] Pascal, 'Essay pour les Coniques'; trans. Pascal, 'Essay on Conics'.

[4] Huntley, *The Divine Proportion*, p. 36.

[5] *Ibid.*, pp. 36–7.

[6] *Ibid.*, pp. 31–4.

Again, he held the parabola had direct aesthetic appeal, but that that its source was hidden from those without mathematical training. This difference seems to have been somehow connected with the simplicity of the parabola described as the locus of points equidistant from a point and a straight line.[1] For Huntley, the mathematical aesthetic satisfaction arose from applying coördinate geometry to the parabola or viewing it as a section through a cone.[2]

But Huntley did not posit any actual close connection between the visual beauty and mathematical beauty in these examples, such as suggesting that one type of beauty supervened on the other. At other points, he seemed unclear about the distinction: his discussion of the brachistochrone curve seemed to mix the abstract beauty of the mathematical properties of this curve with its visual beauty.[3] Similarly, in discussing division in extreme and mean ratio (which he called the 'the golden section'), he wrote that

> 'The conclusion might well be that the claim to beauty by the golden section and cognate topics has a man-made, a *purely artificial* basis, that its appreciation is an acquired taste, that the properties of this remarkable mathematical constant may have an interest for the mathematically wise and prudent but can never revealed to babes. But [...] the golden section, appropriately displayed, appears to have an immediate artistic appeal which demands no preliminary mathematical education.'[4]

Huntley here seems to have assigned little importance to his mid-paragraph switch from mathematical beauty to visual beauty.

Mathematics, poetry, and music

This putative connection between sensory and mathematical beauty fits with a parallel Huntley saw between beauty in mathematics and in poetry and music. He quoted a statement by the classicist and poet A. E. Housman (1859–1936) with which he disagreed: 'I cannot satisfy myself that there are any such things as poetical ideas'.[5] Huntley preferred a Jungian view wherein poetry consisted in the ideas behind the words. Since mathematics, for Huntley, comprised 'nothing but ideas', his view indicated a compatibility between the aesthetic appreciation of mathematics and of poetry.[6] After a jarring segue from poetry to music, Huntley wrote that

> 'The accepted view that there is a definite connection between musical appreciation and mathematical taste is based not only on the observation that many gifted mathematicians have had a warm appreciation of music [...], but also on the similarity between the deep-seated structure of musical form and that of mathematical ideas. [...] It is possible for neither music nor mathematics to assume any arbitrary form if it is to be

---

[1] Huntley, *The Divine Proportion*, pp. 70–1.

[2] *Ibid.*, p. 72.

[3] *Ibid.*, pp. 85–6.

[4] *Ibid.*, p. 51, emphasis in orig.

[5] Housman, *The Name and Nature of Poetry*, p. 36.

[6] Huntley, *The Divine Proportion*, p. 77.

> intelligible to the mind. A fortuitous succession of notes makes as little sense as a haphazard chain of mathematical symbols. […] The fact remains that the forms of music and the shapes of mathematics which appeal to our minds are directed by a basic mental structure'.[1]

Huntley's argument seems to have been that the aesthetic structure of music and both aesthetic and logical structures of mathematics were ultimately dependent on the same mental frameworks. All aesthetic judgement was grounded in the Jungian collective unconscious,[2] which Huntley linked to an evolutionary shaping of the nervous system.[3] In this combined approach, he seemed to point to a Lamarckian inheritance of acquired characteristics that was manifested in the collective unconscious and thus in aesthetic judgement:

> 'Experiences of all the generations of a man's ancestors, repeated millions of times and recorded as memory structures in the brain, are scored ever more deeply as they are transmitted from generation to generation through the centuries […]. The mind of a newly born baby is no *tabula rasa*: […] he is equipped with these inherited memory structures.'[4]

Extreme and mean ratio and consonance

Huntley offered — in his own words, '[w]ith some hesitation' — what he termed a 'tentative' naturalistic explanation for the visual beauty of the $\varphi : 1$ rectangle, which he called by the common name of 'golden rectangle'.[5] He suggested that the perception of the length to width ratio of a rectangle proceeded via a subconscious comparison of the time taken for the focus of vision to move along the sides of the rectangle. This time was measured against some internal clock.[6] An analogous mechanism judged the relative frequencies of heard musical notes. Huntley then argued that the aesthetic evaluation of rectangles is in fact due to the same faculty that recognizes the pleasing harmony of musical intervals:

> 'There is a correspondence between musical intervals and pleasing geometrical shapes due to the reliance of both on the brain's clock. A square corresponds to unison, a double square to the octave, the golden ratio approximates the major sixth [*sic*].'[7]

What Huntley called the 'major sixth' is the interval $8 : 5$, which is conventionally labelled the 'minor sixth' and which approximates the extreme and mean ratio. Huntley's argument was that seeing the golden rectangle created a 'pleasurable echo' of the musical consonance of this interval.[8] The aesthetic superiority of the golden rectangle was thus dependent on that of this particular interval. Huntley asserted without citation or defence that

[1] Huntley, *The Divine Proportion*, p. 78.

[2] *Ibid.*, pp. 78–9.

[3] *Ibid.*, pp. 16–17.

[4] *Ibid.*, p. 79.

[5] *Ibid.*, p. 52.

[6] *Ibid.*, pp. 52–3.

[7] *Ibid.*, p. 55.

[8] *Ibid.*, pp. 55–6.

> 'it is in accord with observation and experiment that the musical interval which gives the greatest satisfaction to the greatest number is the major sixth [*sic*]'.[1]

However, this explanation for the aesthetic response to the golden rectangle is internally inconsistent. The results of Fechner's study,[2] which Huntley knew and quoted (albeit with the 29 : 20 and 34 : 21 ratios replaced respectively by 10 : 7 and $\varphi : 1$),[3] indicated that the frequencies with which the various rectangles were preferred form an approximate bell curve with its peak at 34 : 21 (see Figure 14.4 on page 557). If aesthetic responses to the rectangles were correlated with those to the corresponding musical intervals, one would expect higher frequencies at more consonant intervals and lower frequencies at others. At minimum, one would expect the frequency at the interval 23 : 13 to be lower than that at the octave 2 : 1 or the fourth 4 : 3.

Further, even if Huntley's explanation were valid, it would undermine the case for using the putative aesthetic superiority of the $\varphi : 1$ rectangle to reinforce arguments about the mathematical beauty of the extreme and mean ratio, for his hypothesis would give supreme visual beauty to the 8 : 5 rectangle, or at least to any rectangle approximating 8 : 5, which at best would accord no special distinction to the $\varphi : 1$ rectangle. [Huntley adopted the commonly-used name 'golden rectangle' for the $\varphi : 1$ rectangle,[4] but also referred to '[t]he golden rectangle, with sides in the ratio 8 : 5',[5] and elsewhere described the $\sqrt{2} : 1$ ratio of the ISO 'A' paper sizes as 'not seriously different in shape from the golden rectangle'.[6]]

[Huntley also seems to have been too eager to connect purely mathematical beauty to the extreme and mean ratio. He never made a clear distinction between the generic logarithmic spiral that Jakob Bernoulli so admired (see Figure 9.12 on page 329) and particular logarithmic spirals that are connected with the extreme and mean ratio, such as those with equation $r = ae^{k\vartheta}$ where $k = (3/5\pi)\log\varphi$, in which can be placed nested isosceles triangles each of which has base and side lengths in ratio $1 : \varphi$ so that the spiral passes through their apices (see Figure 14.12). There is only one value of k yielding such a logarithmic spiral.]

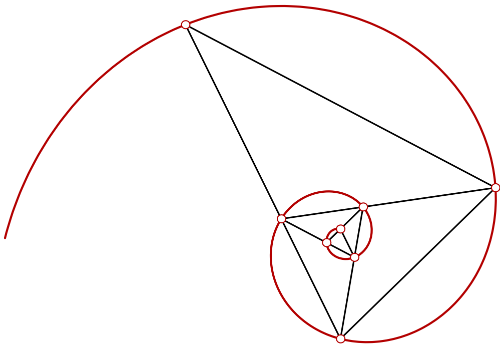


FIGURE 14.12
A logarithmic spiral with equation $r = ae^{k\vartheta}$ where $k = (3/5\pi)\log\varphi$, in which can be placed nested isosceles triangles each of which has base and side lengths in ratio $1 : \varphi$.

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[1] Huntley, *The Divine Proportion*, p. 55.

[2] Fechner, *Vorschule der Aesthetik*, vol. 1, p. 193–5.

[3] Huntley, *The Divine Proportion*, p. 64.

[4] *Ibid.*, pp. 30, 53.

[5] *Ibid.*, p. 56.

[6] *Ibid.*, p. 66.

Legacy

Huntley deserves applause for his effort to make mathematical beauty accessible, and *The Divine Proportion* remains a fascinating work to dip into. But, despite the book's subtitle '*A Study in Mathematical Beauty*', Huntley's aim was primarily mathematical propaganda (using that word in a non-pejorative sense), rather than a coherent examination of mathematical beauty. His book is a kind of stream-of-consciousness of thoughts and ideas and connections about mathematical beauty, and is valuable in its own way. But although it contains many gems, it was too inchoate to be an important contribution to the discourse on mathematical beauty, and seems to have had little further influence.

POST-HUNTLEY

Later accounts of the extreme and mean ratio usually cited Huntley's *The Divine Proportion*, but none truly engaged with his arguments or made a comparable attempt to present a integrated explanation of both its mathematical beauty and its supposed visual beauty. But it is in order to survey briefly later works on the extreme and mean ratio that discuss mathematical beauty.

The English version of Hans Walser's (1944–) book *The Golden Section*, originally published in German in 1989 (2nd ed. 1996, English trans. 2001), has a deceptive cover: the *Mona Lisa* with $\varphi : 1$ rectangles superimposed around the face, the shoulders, and the arms. Perhaps the cover designer saw some significance in their placement; perhaps they chose a placement of no significance in order to mock the idea the $\varphi : 1$ ratio is to be found in works of art. The text itself, aside from a brief nod to the supposed aesthetic value of the ratio in the preface,[1] is exclusively a survey of its mathematical properties and does not explicitly discuss mathematical beauty.

Mario Livio's (1945–) book *The Golden Ratio* (2002), a popular history of φ, is a strange beast. While Livio evinced scepticism about the ratio $\varphi : 1$ being a canon of beauty,[2] he stopped short of stating outright that there was no evidence for its incorporation into the design of the Parthenon.[3] In his explicit discussions of mathematical beauty, Livio gave examples that were in some way connected to the ratio. For instance, he constructed the pentagonal Sierpiński gasket, which he called 'extremely beautiful', by removing from a regular pentagon isosceles triangles whose side to base ratio is $\varphi : 1$, and iteratively combining copies of the resulting shape (see Figure 14.13).[4] But he never quite connected the putative visual beauty of the extreme and mean ratio with mathematical beauty, and did not offer a theory of mathematical beauty.

[1] Walser, *The Golden Section*, pp. vii–viii.

[2] Livio, *The Golden Ratio*, pp. 162, 201.

[3] *Ibid.*, pp. 74–5.

[4] *Ibid.*, pp. 64–5.

Richard Dunlap's (1952–) book *The Golden Ratio and Fibonacci Numbers* (2003) is primarily devoted to surveying mathematics involving the extreme and mean ratio. In an introductory chapter, he surveyed some of the claims about the putative aesthetic value of the ratio. He was sceptical, although he made some historiographical errors, such as his positive assertions that the extreme and mean ratio was studied by the sculptor Phidias (*fl.* c. 465–c. 425 CE), who was famed for creating the statues of Zeus at Olympia and of Athena at the Acropolis of Athens,[1] and that the supposed aesthetic superiority of the $\varphi : 1$ rectangle was accepted in ancient Greece.[2]

Importantly, Dunlap noted the distinction between the mathematical properties of the extreme and mean ratio and the visual aesthetic value of the $\varphi : 1$ rectangle:

> 'It is [...] not clear that the particular properties of [φ] as an irrational number are of any fundamental importance to its role in the pleasing shape of the golden rectangle.
>
> [...]
>
> [...] The possibility of the occurrence of [φ] in the dimensions of man-made objects [...] is primarily of interest from an archeological, social or psychological point of view and does little to provide information concerning the interesting properties of [φ] itself.'[3]

Explicitly maintaining this distinction makes Dunlap almost unique among recent writers. He never mentioned mathematical beauty *per se*, though he did consider 'more elegant' the short-cut to calculating the n-th term of the Fibonacci sequence that is provided by the formula

$$F_n = \frac{1}{\sqrt{5}}(\varphi^n - (-\varphi)^n).$$

The 2012 book *The Glorious Golden Ratio* by Alfred S. Posamentier (1942–) & Ingmar Lehmann (1946–) is also a survey of the extreme and mean ratio that mentions the beauty of mathematics and of the visual proportion. The authors were uncritical about the supposed aesthetic superiority of the $\varphi : 1$ rectangle, saying that it was 'supported through numerous psychological studies in a variety of cultures',[4]

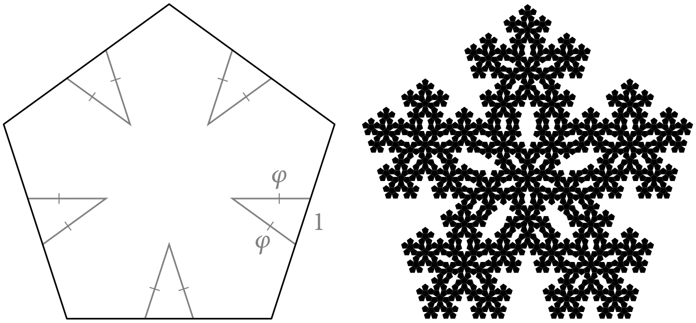

FIGURE 14.13
Removing five isosceles triangles whose side to base ratio is $\varphi : 1$ from a regular pentagon and combining 6^3 suitably scaled copies gives the fourth iteration of the construction of a Sierpiński gasket.

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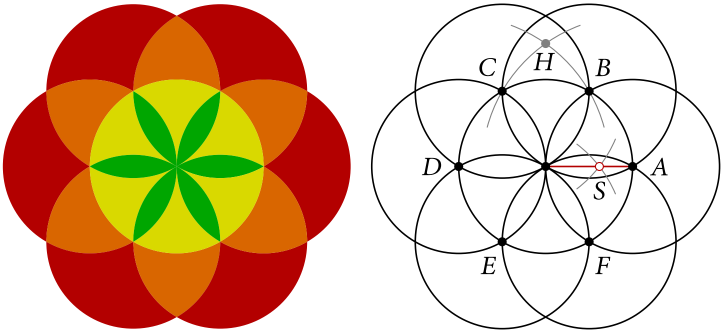
[1] Stewart, 'Phidias'.

[2] Dunlap, *The Golden Ratio and Fibonacci Numbers*, pp. 1, 2.

[3] *Ibid.*, pp. 2, 5.

[4] Posamentier & Lehmann, *The Glorious Golden Ratio*, Introduction.



A floral design.

The construction of the extreme and mean ratio dividing the segment OA.

FIGURE 14.14 Example from Posamentier & Lehmann, *The Glorious Golden Ratio*, ch. 5 showing how to extract the extreme and mean ratio from a floral design. The circle centred at D through B and the circle centred at A through C intersect at H. The circles centred at E and C with radius OH intersect at S, which divides OA in the extreme and mean ratio. No feature of the design is used beyond $ABCDEF$ being a regular hexagon.

and claiming that it was used by Phidias (and transferring his statue of Zeus from Olympia to the Parthenon).[1] They paid little heed to the distinction between visual and mathematical beauty, and often seemed to link them. For instance, they wrote that a floral design such as is shown in Figure 14.14 (left) had 'optical beauty' and also a 'subtle beauty owing to its *reliance* on the golden section'.[2] But the construction with which they illustrated this 'reliance' (see Figure 14.14 (right)) serves to annul their claim: it shows how to divide the radius of a regular hexagon in extreme and mean ratio, and nothing more.

Posamentier & Lehmann's book, despite its value as a mathematical tour of the extreme and mean ratio, is perhaps evidence of a kind of overmastering enthusiasm that forces the ratio into places where it does not arise. Gary Meisner's lavishly-designed book *The Golden Ratio* (2018) is evidence for a similarly powerful desire to seek the extreme and mean ratio in art, architecture, and physics. Some of the examples seem to suggest a kind of numerology or mysticism, which would be worthy of study in a different work. But despite its subtitle *The Divine Beauty of Mathematics*, it does not delve into mathematics beyond pointing out the presence of the ratio in certain geometric configurations, and so is not examined further here.

* * *

Doubtless other books will be written on the extreme and mean ratio. Perhaps one will advance a convincing argument connecting the beauty of mathematics involving the ratio to the supposed visual beauty of objects or artworks that incorporate it. Perhaps someone will devise a convincing test that shows a clear aesthetic preference for $\varphi : 1$. Perhaps some author will present plausible evidence for its use in the arts and architecture of antiquity. *Perhaps.*

[1] Posamentier & Lehmann, *The Glorious Golden Ratio*, ch. 2.

[2] *Ibid.*, ch. 5, emphasis added.

Inaccessible Islets

15

Mathematical aptitude and the perceptibility of beauty

> ‘ you can […] perhaps catch a reflection of the “mysterious beauty”
> (as a non-mathematician friend wrote to me)
> of the world of mathematical objects,
> rising like so many “strange inaccessible islets”
> in the vast moving waters of thought… ’
>
> — Alexander Grothendieck,
> *Récoltes et semailles*, ‘Présentation des Thèmes’, ch. 1,
> trans. C.

> ‘ for the sake of this remote and inaccessible beauty,
> and not for its few useful formulae ’
>
> — Michael Polanyi, *Personal Knowledge*, p. 182.

Henri Poincaré (1854–1912) thought that only a few people have the ability to perceive beauty in mathematics; it was a ‘special sensibility that all mathematicians know but of which laymen are so ignorant that they are often tempted to smile at it’.[1] Poincaré seems to have imagined that there would be amusement or even mockery from mathematical laity confronted by the notion of mathematical beauty. There was perhaps a reflected harshness in Poincaré’s own position, which suggested that the possession or lack of a ‘special sensibility’ was innate, forever dividing those who had an aptitude for mathematics from those who did not.

Some of the thinkers studied in Chapter 13 commented on the accessibility of mathematical beauty. Lancelot Hogben (1895–1975) thought mathematics had no aesthetic appeal for the vast majority.[*] J. W. N. Sullivan (1886–1937) thought the beauty of chess was accessible to the public in a way that the beauty of mathematics was not.[†] On the other hand, G. H. Hardy (1877–1947) wrote in *A Mathematician’s Apology* that beauty in mathematics was accessible to essentially everyone, precisely in the form of chess problems, which he counted as genuine but non-serious mathematics.[‡] Years before, Hardy seems to have taken a more optimistic or expansive view of the audience for mathematical beauty. At the end of his 1922 presidential address to the British Association for the Advancement of Science, he observed that while much in mathematics must remain out of reach of the amateur, ‘however beautiful and important it may be’, number theory was special: there was a great deal that could be published in, and

[1] Poincaré, ‘Mathematical Discovery’.

[*] ▸ pp. 495–6

[†] ▸ p. 488

[‡] ▸ p. 499

would attract new readers to, newspapers of even the most moderate quality.[1]

Martin Gardner (1914–2010) would doubtless have agreed with Hardy. In his celebrated 'Mathematical Games' column, which ran from 1957 to 1980 in *Scientific American*, he called all kinds of mathematics beautiful.[2] His appreciation of mathematical beauty is significant because he had no formal university-level mathematical training, and he admitted that his knowledge extended only as far as the calculus.[3] Thus he, as a self-proclaimed mathematical tyro, found mathematical beauty accessible.

This question of accessibility was an important aspect of the discourse on mathematical beauty, especially in the post-Hardian era. The breadth of the audience to which the beauty of a mathematical artefact is accessible is important for the motivation for and justification of mathematics, for it bears upon the degree to which the subject can be defended as what Bertrand Russell (1872–1970) distinguished as a creative, social pursuit.

HERBERT SPENCER

The philosopher, biologist, and political theorist Herbert Spencer (1820–1903) has a minor but curious role in the history of mathematical beauty, and in particular in the history of its accessibility, because of comments he made about Monge's theorem:

MONGE'S THEOREM. *For any three circles in a plane, none contained within another, the intersections of the outside tangents of the three pairs of circles are collinear.* (See Figure 15.1.)

Spencer called this result

> 'a truth which I never contemplate without being struck by its beauty at the same time that it excites feelings of wonder and of awe: the fact that apparently unrelated circles should in every case be held together by this plexus of relations, seeming so utterly incomprehensible.'[4]

However, Spencer's reaction of wonder and of awe may ultimately have been born of mathematical *ignorance*. Spencer expressed enthusiasm for mathematics, and was proud enough of a rather slender theorem he proved in 1840 to reprint the paper[5] in his autobiography.[6] (The result had actually already been discovered, in the mid-eighteenth century.[7]) His geometrical knowledge did not extend much beyond the plane geometry contained in the first six books of Euclid's *Elements*.[8] The actual statement he gave of Monge's theorem is carelessly imprecise,[9] and one point in particular betrays the limits of his geometrical experience: he thought the circles had to be *unequal*.

[1] Hardy, 'The Theory of Numbers', p. 24.

[2] Taking a single volume of collected columns, see, for example, Gardner, *The Second Scientific American Book of Mathematical Puzzles & Diversions*, pp. 15, 36, 106, 205, 208.

[3] Gardner, *Undiluted Hocus-Pocus*, ch. 15.

[4] Spencer, *An Autobiography*, vol. 1, p. 187.

[5] Spencer, 'Geometrical Theorem'.

[6] Spencer, *An Autobiography*, vol. 1, Appendix B.

[7] Mackay, 'Herbert Spencer and Mathematics', p. 99.

[8] *Ibid.*, p. 100.

[9] *Loc. cit.*

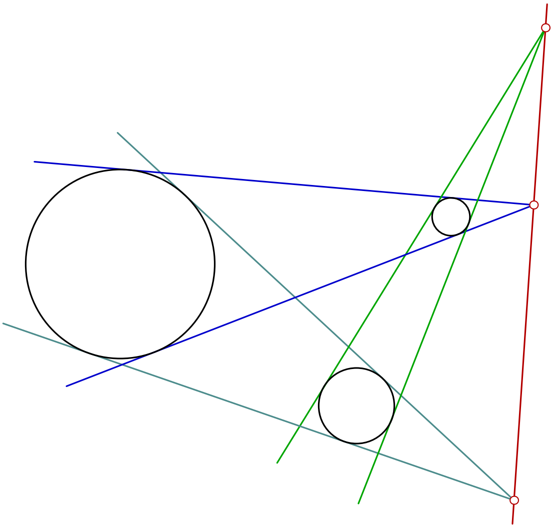


FIGURE 15.1
Monge's theorem: for any three circles in the plane, none completely inside another, the three intersection points of the outer tangents lines of each pair of circles are collinear.

This inequality is necessary for the theorem to hold in the euclidean plane, for with two equal circles there will be a pair of parallel outside tangents, which do not intersect. But the theorem holds even for equal circles in the projective plane.

Further, Gaspard Monge's (1746–1818) original proof[1] is simple and *explains* why the 'plexus of relations' holds, and would perhaps have eliminated much of the incomprehensibility Spencer felt. The proof also works in euclidean geometry if the circles are unequal.

Proof. Imagine the circles are great circles of three spheres. Then there are two planes that are tangent with all three spheres. (See Figure 15.2.) These planes are also in contact with the cones defined by each pair of spheres, and the tangent lines are where these cones pass through the original plane. Thus both these planes contain the apices of the three cones, which are the intersections of pairs of the tangents. That is, the intersection points lie in the intersection of the two planes, which is a straight line. ☐

If Spencer's admiration of Monge's theorem arose from a paucity of mathematical knowledge or ability, then he serves as a useful admonition for the historian: greater knowledge of mathematics does not necessarily correspond to greater appreciation of its beauty.

AUBREY KEMPNER

In a 1928 lecture, the mathematician Aubrey Kempner (1880–1973) argued that the 'marvelous beauty' of mathematics is an aspect of its cultural value, for this value must be ascribed to anything

[1] Monge, *Géométrie Descriptive*, § 44.

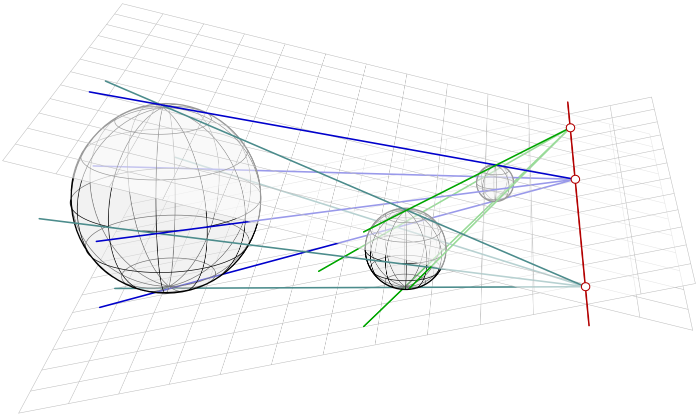


FIGURE 15.2
The proof of Monge's theorem. The two planes are tangent to each sphere and the cone defined by each pair of spheres, in which lie the tangents to the original three circles, which are viewed as 'equatorial' great circles of spheres.

that 'enriches our lives' via beauty.[1] He thought that the beauty of mathematics was not limited to those who can discover new mathematics, or even follow the proof of a result.[2] He drew parallels with the beauty to be found in nature, in music, and in poetry:

> 'We respond to a Beethoven Sonata even though we be not able to read notes; we love a beautiful flower even though botany be to us a sealed mystery; the majestic swing of the Iliad thrills us even though we know nothing of the technical rules of rhythm.'[3]

In mathematics, thought Kempner, the same holds true: theorems could be admired in the same way, for 'their sheer aesthetic beauty, their simplicity of statement and richness of content'.[4] There could be an appreciation of the theorem detached from the proof and more generally from everything but the immediate context required to understand its statement.

For Kempner, an example of such a 'beautiful' and 'thrilling' result was the following theorem:[5]

LAGRANGE'S FOUR-SQUARE THEOREM. *Every natural number can be expressed as a sum of at most four square numbers.* □

This result was first stated by Claude-Gaspar Bachet (1581–1638) in his 1621 edition and translation of Diophantus' *Arithmetica*; Bachet suggested that Diophantus assumed the result.[6] It is a special case of the polygonal number theorem, of which Fermat claimed to have a proof and which he thought beautiful.* But the first known proof of the four-square theorem was by Joseph-Louis Lagrange (1736–1813), who published it in 1770.[7]

Kempner implicitly made an important distinction about *degrees* of accessibility. A person with enough education in mathematics to understand the statement of the four-square theorem can aesthetically appreciate it. Someone with more mathematical training will have an enhanced appreciation of the theorem itself if they are familiar

[1] Kempner, 'The Cultural Value of Mathematics', p. 145.

[2] *Ibid.*, p. 142.

[3] *Loc. cit.*

[4] *Loc. cit.*

[5] *Ibid.*, p. 143.

[6] Dickson, *Hist. Theory of Numbers*, vol. II, p. 275.

* ▸ p. 300

[7] *Ibid.*, vol. II, p. 279.

with related results, such as Fermat's two-square theorem,[*] even if the proof of the four-square theorem itself is beyond them. A still higher form of appreciation is available to those who have enough mathematical background to understand the theorem, its proof, and to place it in the broad context of number theory.

SIMONE WEIL

Without doubt, the most fascinating and singular platonist of the twentieth century was Simone Weil (1909–43). Her philosophy, like Plato's, was closely connected to and influenced by mathematical ideas, and was infused by a combination of mysticism and Christian faith that had echoes both ancient and mediaeval.

Weil's parents and upbringing were wholly secular, but she considered herself to have always professed a creed of love for one's neighbour[1] and so to have always been Christian.[2] She was influenced in her development by Alain (Émile-Auguste Chartier; 1868–1951), who was one of her teachers, and who especially emphasized the need to read deeply and to confront intellectual history.[3]

In studying Weil, a key resource is the series of notebooks she kept, starting c. 1933, of which her France notebooks of 1940–2 and her New York notebook of 1942 are the most relevant here.[4] While these notebooks give insight into her philosophical positions, a degree of caution is required, since the entries represent an embryonic and experimental stage of her thought, which might later be revised. Counterbalancing this concern is that her writing usually reflects precisely what she meant, as is indicated by the lack of erasures in her manuscripts.[5]

Her notebooks show that, despite a lack of specialized training, she had an advanced knowledge of mathematics.[6] In her France notebooks one can find Goldbach's conjecture that *Every even number greater than* 2 *can be expressed as a sum of two prime numbers*, which she mistakenly attributed to the philosopher Paul-Henri d'Holbach (1723–89),[7] and Carl Friedrich Gauss's (1777–1855) characterization of the regular polygons that can be constructed with straightedge and compass.[8][†] Her mathematical learning was undoubtedly aided by her close relationship with her older brother André (1906–98), who was a mathematical prodigy and who would go on to become one of the preëminent mathematicians of the twentieth century. [An anecdote has him naming Carl Ludwig Siegel (1896–1981) as the greatest mathematician of the twentieth century, and smilingly refusing to answer when pressed for the name of the *second* greatest.[9]]

But although Weil clearly had mathematical ability above the average, she was conscious that she was not on André's level. She later wrote of the distress that this caused her:

[*] ▸ p. 7

[1] Fiori, *Simone Weil*, pp. 23–4.

[2] Weil, letter to Perrin, dated c. 15 May 1942, repr. in Weil, *Waiting for God*, p. 65.

[3] Fiori, *Simone Weil*, pp. 31–3.

[4] See Weil, *The Notebooks*; Weil, *First and Last Notebooks*, § III.

[5] Dunaway & Springsted, *The Beauty That Saves*, p. 3.

[6] Doering & Springsted, *The Christian Platonism of Simone Weil*, p. 10.

[7] Weil, *The Notebooks*, p. 185.

[8] *Ibid.*, p. 617.

[†] ▸ p. 395

[9] Krantz, *Mathematical Apocrypha*, p. 186.

> ‘The exceptional gifts of my brother […] brought my own inferiority home to me. I did not mind having no visible successes, but what did grieve me was the idea of being excluded from that transcendent kingdom to which only the truly great have access and wherein truth abides. […]
>
> Under the name of truth I also included beauty, virtue, and every kind of goodness’.[1]

Such is the special frustration of talent confronted by genius: to have enough knowledge and skill to be *truly* conscious of what lies out of reach. Perhaps Weil’s awareness of her inability to access ‘that transcendent kingdom’ helped shape her philosophy. She certainly recognized mathematical beauty, and her interpretations of Plato’s examination of beauty, particularly in the *Symposium*, tended to emphasize the mathematical.

And while Weil had the greatest respect for mathematics as a tool to understand the world, she also considered it to be a ‘reflection of divine truths’,[2] and boldly pointed to its mystical side:

> ‘One does double harm to mathematics when one regards it only as a rational and abstract speculation. It is that, but it is also the very science of nature, a science totally concrete, and it is also a mysticism, those three together and inseparably.’[3]

Mathematics

For Weil, ‘[e]ternal mathematics’ was ‘the stuff of which the order of the world is woven’.[4] The world has a mathematical order such as was emplaced by the demiurge in the *Timaeus*;* furthermore, ‘[t]he order of the world is the same as the beauty of the world’.[5] Take these points together with the statement in Weil’s 1943 London notebook that ‘[t]he world’s beauty is indistinguishable from the world’s reality’:[6] they imply the *identity* of the world, of its beauty, of its order, and of the mathematics that constitutes that order. Individually, these three assertions could be called platonic; combined, they point to this more pythagorean conclusion.

But the mathematical order and beauty of the world had to be distinguished from the practice of mathematics. Weil suspected that modern mathematics had moved away from the Greek mode. It had become more a game than an art, and so no longer put humans in contact with the universe; rather, it impeded this contact.[7] In her France notebooks, and probably around this time, she wondered what modern mathematicians aimed at and suggested that they built theories without knowing how to make use of them.[8]

Beauty

Weil wrote that ‘beauty is the only value that is universally recognized’,[9] and, according to her France notebooks, *joy* was the emotional response that, at least in its pure form, was special to the beautiful.[1] But beauty could not be defined in psychological terms, because the aesthetic experience was so all-encompassing as to deny the possibility of introspection.[2] But, for metaphysical reasons, seeing beauty was a special kind of perceptual experience. It was the one way in which the senses could get a glimpse into the intelligible world,[3] into reality itself.[4] Thus her assertion that '[j]oy [...] is the feeling of reality' fits with her connection of beauty to the real.[5] Her New York notebook contains a very concise formulation:

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[1] Weil, letter to Perrin, dated c. 15 May 1942, repr. in Weil, *Waiting for God*, p. 64.

[2] Weil, ‘The Pythagorean Doctrine’, pp. 192–3.

[3] *Ibid.*, p. 191.

[4] Weil, *The Need for Roots*, p. 287.

* ▸ pp. 61–70

[5] *Ibid.*, p. 288.

[6] Weil, *First and Last Notebooks*, p. 341.

[7] Weil, letter to André Weil, dated Jan.–Apr. 1940, extr. in Weil, *Seventy Letters*, p. 118.

[8] Weil, *The Notebooks*, p. 35.

[9] Weil, ‘The Pythagorean Doctrine’, p. 103.

> 'The beautiful = manifest presence of the real. το ὄν [*to on*]. Joy, sensation of the real.'[6]

There was in fact a unique beauty in creation, of which any other beauty was a reflection, and it was the beauty of the world.[7] The reflected beauty could be found works of painting, sculpture, and music. Beauty was something that could hold our attention;[8] indeed, we could desire to maintain this contemplation indefinitely. Beautiful things were eternal and inexhaustible in a way that transcended the limits of our ability to maintain attention and contemplation.[9] But this characterization of beautiful things prompts a question:

> 'by what mystery does he who makes them, making them in a time that is limited, with an attention that is finite, manage to put into them the inexhaustible?'[10]

Weil suggested three answers. '— Love (Lucretius) — [Mathematics, in what way?] — Death.'[11]

Mathematics and beauty

Immersion in the beauty of mathematics or the beauty of the world were alike and were a special kind of experience:

> 'When joy is a total and pure adherence of the soul to the beauty of the world, it is a sacrament (the sacrament of St. Francis).
> (In the same way, mathematical beauty.)'[12]

Leave aside for the moment the spiritual component of the experience. The beauty of the world was for Weil the only beauty in creation, and other beauties were reflections of this beauty. Thus '[t]he world is beautiful in the way in which a work of art is beautiful.'[13] It is thus unsurprising that Weil should conclude that

> 'The theory of beauty in the arts and the contemplation of beauty in the sciences — these two things must coincide through some hitherto unexplored path.'[14]

Weil thus thought that the theoretical examination of beauty in art had not been, but should be, extended to science. After all, Weil thought that 'the true definition of science' was 'the study of the beauty of the

[1] Weil, *The Notebooks*, p. 360.

[2] *Ibid.*, p. 240.

[3] *Ibid.*, pp. 320–1.

[4] *Ibid.*, pp. 319, 360, 387.

[5] *Ibid.*, p. 360.

[6] Weil, *First and Last Notebooks*, p. 72.

[7] Weil, 'The Pythagorean Doctrine', p. 190.

[8] Weil, *The Notebooks*, p. 81.

[9] *Ibid.*, p. 100.

[10] *Loc. cit.*

[11] *Ibid.*, p. 100, brackets in orig.

[12] Weil, *First and Last Notebooks*, p. 83.

[13] Weil, *The Notebooks*, p. 526.

[14] *Ibid.*, p. 440.

world',[1] and noted that the object of science was beauty, not truth, which was rather the object of philosophy.[2]

Mathematics was not a part of creation, but was embodied in its order. And so mathematical beauty was identical with the order and beauty of the world. And hence all beauties in the created world were connected to mathematical beauty. But there was something *different* about mathematical beauty. Mathematics was not like art; it was not of human artifice:

> 'Mathematics are beautiful in the way in which nothing that is human is.'[3]

Entries in the latter part of her France notebooks give some indications as to what Weil found beautiful in mathematics. At one point, she wrote that what demonstrated that mathematics was not of human origin was also what made it beautiful, and 'that thing is *contradiction*'.[4] Although Weil particularly associated contradiction with *reductio ad absurdum* proofs,[5] she seems not to have meant here logical contradiction *per se*, but rather reasonings that lead to unexpected conclusions,[6] that shatter preconceptions, that force the mind to open itself to new vistas:[7]

> 'The point where one runs up against an impossibility and a new notion has to be developed. Such points are the points of harmony, the points of beauty.'[8]

The discovery that some magnitudes were incommensurable was the earliest such discovery:

> 'The beauty in mathematics lies in contradiction. Incommensurability, λόγοι ἄλογοι [*logoi alogoi*], was the first radiance of beauty manifested in mathematics.'[9]

But this phenomenon, and the mental states that accompany it, were consequences of the iron laws of logical necessity. As Weil put it, the things of mathematics were absolutely 'docile'. They obeyed laws. They might resist our efforts to understand, but this resistance was an outcome of their submission to laws.[10] 'Mathematics invites the exercise of the intuition and at the same time offers it a stony resistance.'[11]

So the shifting of perception and opening of the mind did not come about by any change in the mathematics, but a change in the person doing mathematics. Mathematics was thus *special*: 'Mathematics alone make us feel the limits of our intelligence.'[12] Logical necessity did not just generate this special intellectual effect of mathematics: it also grounded mathematical beauty.[13] Necessity was not limited to logical consequence, but to the mathematical laws that determined the behaviour of matter. And although there were 'mysterious concordances' in mathematics itself and in connection to the world and to 'transcendent truths', these concordances were subject to the same logical strictures.[14] And so,

[1] Weil, *The Need for Roots*, p. 255.

[2] Weil, *The Notebooks*, p. 548.

[3] *Ibid.*, p. 526.

[4] *Ibid.*, pp. 387–8, emphasis added.

[5] *Ibid.*, p. 408.

[6] *Ibid.*, p. 453.

[7] Springsted, 'Contradiction, Mystery and the Use of Words in Simone Weil', p. 3.

[8] Weil, *The Notebooks*, p. 516.

[9] *Ibid.*, p. 387.

[10] *Ibid.*, p. 599.

[11] *Ibid.*, p. 509.

[12] *Ibid.*, p. 511.

[13] Weil, 'The Pythagorean Doctrine', p. 191; Weil, *The Notebooks*, p. 513.

[14] Weil, *The Notebooks*, p. 526.

> ‘through the medium of mathematics (and otherwise …) transcendent truths have their symbols in the actual forms of mechanical necessity which govern matter. Thus mathematical beauty leads to the notion of Order of the World.’[1]

There was a related and more specific beauty in proofs, in which the beauty ‘lies in the union between the Same and the Other’.[2] From a particular logical perspective, a proof was a deduction that started from premisses and, via unyielding necessity, produced a result that was not more than tautological; this tautology constituted the Same. From a mathematical perspective — that of mathematics as an art — writing a proof was a creative act that produced novelty in the newly-established result; this novelty was the Other.[3]

Mathematics and God

According to entries in her France notebooks, Weil thought that the access from the sensible world to the intelligible, through beauty, was evidence for the *possibility* (but not the *actuality*) of the Incarnation.[4] But for Weil there were deep and extensive connections between mathematical beauty, the beauty of the world, and Christian faith.

Other entries in her France notebooks show that Weil considered the beauty of the world to be either the unique ‘trace of the divine mercy’ or one of four ‘tokens of the divine mercy’ that could be found in the created universe.[5] This beauty proved the existence of God; the order of the world indicated its author. But the physical laws of the world were governed by mathematics, and the necessity of mathematics was a higher order that had no author.[6] The Creator *chose* to shape the world to obey the beauty of this higher mathematical order. In her New York notebook, she pointed to this choice as a reason to love the beauty of the world, which is a mathematical order: it was a sign of ‘love between the Creator and the creation’.[7]

Mathematics, thought Weil, was a *metaxy* [μεταξύ] that could take one toward God.[8] The Greek word *metaxy* is an adverb or preposition indicating *between*;[9] Weil preferred the Greek word and used it as a noun signifying *intermediary*, and developed a theory of *metaxy* as bridges between the rational and the mystical, between the human and the transcendent.[10] What is important here is that some intermediary was necessary to access the higher realm of thought that was normally beyond our intellectual reach. Mathematics was the archetype of such a *metaxy*.[11]

In the context of Christian faith, Weil saw at least an analogy between *metaxy* and the Incarnation as the intermediary between God and the created world. But such an imprecise analogy was not enough for Weil. She therefore looked to Plato’s *Timaeus*, quoting the reason for the geometric mean being the ‘fairest of bonds’:* it ‘most perfectly unites into one both itself and the things which it binds

[1] Weil, *The Notebooks*, p. 526; ellipsis in orig.

[2] *Ibid.*, p. 560.

[3] *Loc. cit.*

[4] *Ibid.*, p. 440.

[5] *Ibid.*, pp. 449, 450.

[6] *Ibid.*, p. 241.

[7] Weil, *First and Last Notebooks*, p. 139.

[8] Weil, *The Notebooks*, p. 241.

[9] LSJ, ‘μεταξύ’.

[10] Springsted, *Christus Mediator*, pp. 196–219.

[11] Weil, *The Notebooks*, p. 62.

* ▸ pp. 64–5

together'.[1] Weil thought that this statement was not *uniquely* about mathematics. The interpretation that she favoured, and which she thought was its true realization, was 'when not only the first term but also the bond itself is one; that is to say, is God'.[2]

To explicate this interpretation of Plato's 'fairest of bonds', Weil built on accounts of the pythagoreans and took *harmony* to denote a union of opposites;[3] she doubtless drew in particular on the fragment of Philolaus asserting that it was necessary for unlike and unrelated things to be 'bonded together by harmony'.[*] If one viewed this harmony as proportion, or the geometric mean, then one had 'exactly the idea of the Mediation of the Word'.[4] Weil thought that one should always translate *Logos* [Λόγος] not as *Word*, but as *Mediation* [*Médiation*].[5] Thus, for Weil, the 'least inexact translation' of the opening of the Gospel of John[6] would perhaps be:

> 'A l'origine était la Médiation, et la Médiation était auprès de Dieu, et la Médiation était Dieu.'[7]

> 'In the beginning was the Mediation, and the Mediation was with God, and the Mediation was God.'[8]

Weil used the French word *médiation* in two senses that correspond in English to the word *mediation* and to the mathematical terms *mean* and *median*.[9] Thus there was always a mathematical subtext to her use of the term.

The first and archetypal pair of opposites comprised the *Creator* and the *creature*, and what bound them together was the Incarnation,[10] which Weil identified with the geometric mean:

> 'The Son is the unity of these contraries, the geometrical mean which establishes a proportion between them: He is the mediator.'[11]

She made this identification even more explicit in the form of a equation in her notebook:

> 'The *mean* (μεταξύ) is the perfectly just man —

$$\frac{\text{God}}{\text{God-man}} = \frac{\text{God-man}}{\text{man}}\text{'.}^{12}$$

And she made the same kind of equational assertion just a few pages later, appending an important observation:

> 'Number is a mediation between one and indefinitude.

$$\frac{1}{\text{number}}\ [=]\ \frac{\text{number}}{\text{indefinitude}},$$

$$\text{just as } \frac{1}{\sqrt{\text{number}}} = \frac{\sqrt{\text{number}}}{\text{number}}.$$

> *Relation is always analogy*'.[13]

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[1] Plato, *Timaeus*, STE 31C.

[2] Weil, 'The Pythagorean Doctrine', p. 160.

[3] Weil, 'Divine Love in Creation', p. 95.

[*] ▸ pp. 49–50

[4] Weil, *The Notebooks*, p. 587.

[5] Weil, 'Notes on Cleanthes', p. 139; Weil, 'Notes sur Cléanthe', p. 152.

[6] John 1:1.

[7] Weil, 'Notes sur Cléanthe', p. 157.

[8] Weil, 'Notes on Cleanthes', p. 143.

[9] Geissbuhler, trans. note. to Weil, 'The Pythagorean Doctrine', p. 161, n. 1.

[10] Weil, 'Divine Love in Creation', p. 95; cf. Weil, *The Notebooks*, p. 460.

[11] Weil, 'Divine Love in Creation', p. 95.

[12] Weil, *The Notebooks*, p. 385.

[13] *Ibid.*, p. 389, insertion in ed., emphasis added.

The bond between the opposites of Creator and creature was (literally) 'the mean proportional, the Christ'.[1] This bond was not directly between God and his creatures, but between the God–Christ and the Christ–man relationships.[2]

The parallel mediatory roles of the Incarnation on one hand and the geometric mean on the other was at most an allegory for Augustine of Hippo[*] (354–430 CE). For Weil, it was far more fundamental. There was what is indeterminate and there was a determining principle: the former was what is created; the latter was the Creator. The harmony and the bond between them was divine *Logos*, which was both the Word and the geometric mean, and which was the mathematical order and beauty of the world.[3] This identification of the geometric mean and the *Logos* was a kind of pythagoreanism, but of a novel kind: it was not the numbers, or the ratio of numbers, that were identified with the *Logos*, but the very *concept* of the geometric mean. And the identification was not with the things of the world, but with the Incarnation. Weil had taken the pythagorean identification of numbers and things and applied it at a higher level of abstraction on the one hand and at a higher metaphysical level at the other.

While the fundamental role of the Incarnation gave this line of thought a wholly Christian character, Weil thought it had the aura of more ancient platonism and pythagoreanism, which in turn had been influenced by the entirely original character of Greek mathematics. For Weil, the key difference between Greek and Mesopotamian mathematics was *ethical*. Greek mathematics was an art and a pursuit of beauty, informed, thought Weil, by a tradition that was certainly *communicated* by Plato and Pythagoras and may have stretched back to the Orphic or Egyptian mysteries or beyond.[4] [This view could have been an instance of a kind of nostalgia Weil exhibited for the intellectual life of earlier times.[5]]

In Weil's interpretation of Plato's *Symposium*, the ascent to 'the beautiful itself' described by Diotima to Socrates[†] concerns the contemplation of *mathematical beauty*.[6] Beauty in any natural science was beauty in the order of the world, which was beauty in the mathematics that described that order, which was grounded in the necessity of mathematical proof.[7] Therefore pursuing science would ultimately lead, via the last stage of contemplating mathematical beauty, to seeing the Form of the Beautiful.[8]

The sign that by tradition forbade those ignorant of geometry to enter Plato's Academy[‡] was, for Weil, a reference by Plato to the same truth as expressed by Jesus' 'No one comes to the Father except through me'.[9] And the platonic phrase 'God is always doing geometry'[§] pointed both to the mathematical order and beauty of the world and to the mediation of the Word.[10] In the probably pseudo-Platonic dialogue *Epinomis* there is a passage that assigns a divine origin to the use of geometry for finding a ratio between incommensurable magnitudes (see Figure 15.3).[11] On this passage, Weil wrote in her

[1] Weil, 'The Pythagorean Doctrine', p. 170; trans. mod.

[2] *Ibid.*, p. 170.

[*] ▸ pp. 165–6

[3] Weil, *The Notebooks*, p. 460.

[4] Weil, letter to André Weil, dated Jan.–Apr. 1940, extr. in Weil, *Seventy Letters*, pp. 116–7.

[5] Weil, letter to Roché, dated 23 Jan. 1941, extr. in *ibid.*, p. 131.

[†] ▸ p. 77

[6] Weil, *The Notebooks*, p. 361.

[7] Weil, 'The Pythagorean Doctrine', p. 148.

[8] *Loc. cit.*

[‡] ▸ p. 59

[9] John 14:6.

[§] ▸ pp. 135–6

[10] *Ibid.*, p. 171.

[11] ps.-Plato, *Epinomis*, STE 990.

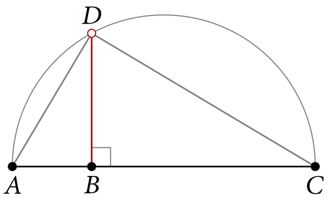

FIGURE 15.3 The construction of the geometric mean BD of AB and BC. Since the angle in the semicircle is right, the triangles are similar and so $AB : BD :: BD : BC$.

France notebooks the rhetorical question: 'doesn't this show that the Greeks saw in the geometrical mean an image of the Incarnation?'[1]

Thus, for Weil, geometry was a means towards and an indicator of faith. The *act* of geometrizing was grounded upon the acceptance of the mathematically beautiful order of the world: 'geometry, and consequently the whole of modern science, were born of faith in the Incarnation'.[2] Weil put it even more succinctly: 'All geometry proceeds from the Cross'.[3]

[Occasionally Weil seemed to have been reaching in her attempts to link mathematics and faith: a parenthetical question in her France notebooks asked whether crucifixes were designed so that the extremities lie on a circle.[4] To which the answer is simply: not as a general rule.]

Beauty and goodness

Aesthetics, for Weil, was not a specialist or isolated discipline, but 'the key to supernatural truths'.[5] Beauty was not just connected to goodness, but the only way to come to knowledge of it: 'It is impossible to conceive Good without going via Beauty.'[6] And although goodness was more than beauty, it was not *beyond* beauty: rather, it was the limit of beauty, just like the point at the end of a line segment.[7] Thus mathematics did not just serve as an intermediary between the Creator and the creature, but the practice of mathematics allowed access to the beauty of the world — the true beauty in creation — and leads to goodness. This role for mathematics was why Weil quoted in full the passage from the *Gorgias* where Socrates reprimands Callicles for his inattention to geometry and the deleterious effect on his character.[8]* As Weil saw it, in Greek geometry and philosophy,

> 'the idea of proportion had been the theme of a meditation which was one of the chief methods, and perhaps the chief method, of purifying the soul.'[9]

To purify the soul, one had to imitate God, and this could be achieved through studying mathematics and the mathematical structure of the world, for thus became 'the geometer an imitator of the supreme lawgiver'.[10] Weil seemed to lament the shift of mathematics away from the ethical goal, writing to her brother:

[1] Weil, *The Notebooks*, p. 439.

[2] *Ibid.*, p. 441.

[3] Weil, *First and Last Notebooks*, p. 98.

[4] Weil, *The Notebooks*, p. 560.

[5] *Ibid.*, p. 627.

[6] *Ibid.*, p. 293.

[7] *Ibid.*, p. 605.

[8] Weil, 'The Pythagorean Doctrine', p. 155.

* ▸ p. 75

[9] Weil, letter to André Weil, dated Jan.–Apr. 1940, extr. in Weil, *Seventy Letters*, p. 117.

[10] Weil, letter to André Weil, dated Jan.–Apr. 1940, extr. in *ibid.*, pp. 117–8.

> 'I wonder how many mathematicians today regard mathematics as a method for purifying the soul and "imitating God?"'[1]

GEORGE STEINER

In his book *Grammars of Creation*, based on his 1990 Gifford lectures, the philosopher George Steiner (1929–2020) lamented his inability, as someone lacking in mathematical ability, to perceive mathematical beauty. He had a theoretical, second-hand, grasp of what it meant: he cited with approval Poincaré's distinction of mathematicians as those with a special aesthetic sense,[*] and Hardy's characterization of the mathematician as 'a maker of patterns'.[2][3] Steiner recognized that the term *beauty* was not applied to mathematics as some 'vague analogue borrowed from the arts'.[4] It was linked to truth. If a proof had beauty,

> '[i]t possesses economies of sequence, transparencies of demonstration, potentialities for ramification, which give to "beauty" a substantive, though untranslatable meaning (as in music).'[5]

But Steiner confessed that this beauty was inaccessible to him, writing:

> 'I find nothing more frustrating, more humbling, than my incapacity as a mathematical innumerate, to grasp this lucent realm of "truth-beauty".'[6]

He could only glimpse what he considered a shadow of the beauty of real mathematics:

> 'I am only able to simulate indistinctly by trying to experience, by trying to analyse, the presence of beauty in the style of certain chess masters, in the configuration and solution of certain end-games and chess problems. But here the beauty of the truth is, in some complicated ways, also trivial.'[7]

Here Steiner was clearly following Hardy in considering chess as a kind of mathematics that is legitimate but not serious.[†] But he both went further than Hardy had imagined and implicitly disagreed with him. First, Steiner did not simply enjoy the beauty of chess for its own sake, but tried to *use* it to gain an idea of the beauty of serious mathematics. Second, Steiner seemed to consider that there was some great difference, either in kind or in degree, between the beauty of chess and the beauty of what Hardy called 'real mathematics'.

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[1] Weil, letter to André Weil, dated Jan.–Apr. 1940, extr. in Weil, *Seventy Letters*, p. 118.

[*] ▸ p. 574

[2] Hardy, *Apology*, § 10.

[3] Steiner, *Grammars of Creation*, § IV.1.

[4] *Ibid.*, ch. IV.1.

[5] *Loc. cit.*

[6] *Loc. cit.*

[7] *Loc. cit.*

[†] ▸ p. 499

RUSH RHEES

The philosopher Rush Rhees (1905–89) admired Simone Weil's works, though he mainly valued her as a religious writer and often criticized her as a philosopher.[1] When it came to mathematics, for instance, he complained that she did not distinguish properly between the logical necessity of mathematical reasoning and necessity such as material objects moving in obedience to the laws of mechanics.[2]

Rhees was always generous to Weil, and exhibited the same kind of intellectual courage and humility as Steiner, wondering if his own shortcomings prevented him from seeing as Weil had.[3] His integrity is apparent in his discussion of Weil's appreciation of mathematical beauty. He noted that Weil

> 'speaks of the "beauty" of mathematical proofs, of the beauty of a natural science, of the beauty of a dramatic tragedy, of the beauty of a piece of music — as though anyone could see that one means the same in each case.'[4]

Such statements seemed to baffle Rhees. Although he recognized that one could find 'aesthetic delight in the symmetry, the neatness or economy and the elegance' of a proof, what he termed 'an impression of *perfection*', he seemed to balk at the idea of there being anything *beautiful*, 'which comes from the heart or goes to one's heart'.[5] He confessed that he was 'confused when she speaks of "the beauty of mathematics" side by side with the beauty of a stretch of mountain scenery',[6] and pointed out that he did not contemplate a proof as he would a landscape, or listen to it like music.[7] This argument is curious: one does not contemplate a drama as one does a landscape, yet Rhees recognized that either could be beautiful. So there being a different mode of aesthetic engagement with a proof does not exclude it from being beautiful.

Rhees was by training a philosopher, and, despite his knowledge of the foundations of mathematics, perhaps had too little experience in mathematics itself to have encountered real beauty in a proof. If the reader will forgive a possibly uncharitable speculation: perhaps the fundamental problem — his inability to engage with Weil's platonic–pythagorean metaphysics, which was alien to his own[8] — predisposed him to question her views on the aesthetics of mathematics.

But Rhees never criticized Weil too harshly: as with his metaphysical disagreements, he seems to have thought that there was some perspective, inaccessible to him, from which he could have understood Weil:[9] he wrote that Weil 'sees in geometry what most of us do not and cannot see there [...] although it may be that Pythagoras did'.[10] And if he did not quite castigate himself for this blindness, he hoped that someone might be able to show him the light.[11]

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[1] Mounce, 'On the Differences between Rush Rhees and Simone Weil', p. 71.

[2] Rhees, *Discussions of Simone Weil*, pp. 51–2.

[3] *Ibid.*, p. 64.

[4] *Ibid.*, p. 90.

[5] *Ibid.*, p. 65.

[6] *Ibid.*, p. 61.

[7] *Ibid.*, p. 56.

[8] Morgan, *Weaving the World*, p. 88.

[9] *Ibid.*, pp. 83–4.

[10] Rhees, *Discussions of Simone Weil*, p. 67.

[11] Morgan, *Weaving the World*, pp. 84–5.

VERN S. POYTHRESS

The theologian and philosopher Vern S. Poythress (1946–) wrote in a 1976 essay that the mathematician should be motivated by love of God to seek to understand the mathematical laws that the Creator had decreed for the world and by love of neighbour to seek to apply mathematics to the physical and economic sciences.[1] For Poythress, God knew all the truths of mathematics,[2] and a theistic view of mathematics was the only one that ontologically and epistemologically secured mathematics.[3] Furthermore, 'anti-theistic mathematics has no satisfactory ethical foundations':[4]

> 'The mathematician should work for the glory of God. He should praise God for the beauty and usefulness that he finds in mathematics, for the incomprehensible nature of God which it displays, for the human mind which God has enabled to understand mathematics'.[5]

Implicit here was the notion that *only* mathematicians should praise God thus, for it is only they who are motivated by the beauty that they find there.

Poythress's 2015 book *Redeeming Mathematics* explained further what he meant by mathematical beauty, although the range of his beautiful mathematics was so wide as to verge on collapsing into vacuity. Simple and elegant physical laws such as $F = ma$ and $E = mc^2$ were examples of beauty;[6] the simplicity of axiom systems could be beautiful;[7] symmetry seems to have been connected to rectitude and a notion of 'fittingness' that could be a component of mathematical beauty;[8] and the theory of complex functions contained many beauties.[9] Such examples are perhaps uncontroversial. But for Poythress, even the simplest arithmetical statements have mathematical beauty:

> 'Is $2 + 2 = 4$ beautiful? I think so. But not everyone is good at seeing the beauty in mathematics. I think there is beauty in the simplicity of $2+2 = 4$. It is in harmony with the world. It is beautiful that its truth is displayed repeatedly, in four apples, four pencils, and four chairs. It is beautiful in its harmony with other arithmetical truths, with which it can be combined.
>
> The beauty in arithmetic shows the beauty of God himself.'[10]

It is doubtful that any professional mathematician would see beauty in '$2+2 = 4$'. In her lecture 'Mathematics Is Beautiful', Rózsa Peter (1905–77) came very close to denying beauty in precisely this statement when she referred to 'the dull "two times two"'[11] and declared her aim of showing that such calculation was not representative of mathematics.* Furthermore, there seems to be no room for such 'beauty' in even the broadest interpretation of the ancient pythagorean association of small number ratios with concordant ratios or of the aesthetic aspects of ancient or mediaeval number symbolism.

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[1] Poythress, 'A Biblical View of Mathematics', pp. 187–8.

[2] *Ibid.*, pp. 162, 173.

[3] *Ibid.*, pp. 168 sqq.

[4] *Ibid.*, p. 175.

[5] *Ibid.*, p. 188.

[6] Poythress, *Redeeming Mathematics*, p. 22.

[7] *Ibid.*, p. 88.

[8] *Ibid.*, p. 98.

[9] *Ibid.*, p. 127.

[10] *Ibid.*, p. 22.

[11] Péter, 'Mathematics Is Beautiful', p. 58.

* ▸ p. 524

Look back to the quotation of Poythress's view of '2 + 2 = 4'. He noted that 'not everyone is good at seeing the beauty in mathematics'. Statements such as this normally indicate that the beauty of the mathematics is not accessible to many people because understanding the mathematics is dependent on technical training and ability. But Poythress seems to have meant something entirely different. In this case, the mathematics can be understood by a child, but the beauty that Poythress sees there seems absent to working mathematicians, including the present author. Someone untrained in mathematics might accept that a piece of mathematics beyond their comprehension could be beautiful to those who did understand it. But here the apparent absence of beauty is not due to any lack of comprehension of the mathematics.

For Poythress, the beauty in 2 + 2 = 4 was an image of the beauty of God:

> 'The beauty of God himself is reflected in what he has made. We are accustomed to seeing beauty in particular objects within creation [...]. But beauty is also displayed in the simple, elegant form of some of the most basic physical laws [...]. *The same goes for the simple beauties in arithmetic* and the more profound beauties that mathematicians discover in advanced mathematics.'[1]

Similarly, the beautiful properties of complex numbers 'come from God, who is beautiful',[2] as does the beauty in the simplicity of the Peano axioms,[3] and the beauty that comes from symmetry.[4] Poythress thought that all scientific laws, and in particular the laws of arithmetic, were decreed by the Creator: 'Law implies a law-giver'.[5] Therefore all the beauty of mathematics is in the end a reflection of the beauty of God.

But Poythress's position seems to imply that all mathematics derives from God and so should be beautiful. It leaves no space for ugly mathematics. An ugly number-theoretical result is just as rational as a beautiful one, and just as securely grounded in the laws of arithmetic. Why is the ugly result not also a reflection of the Creator? Poythress did not say.

CONNOISSEURS

> ‘ CONNOISSEUR, *n.* A specialist who knows
> everything about something
> and nothing about anything else. ’
>
> — Ambrose Bierce, *The Devil's Dictionary*.

As the French root of the word suggests, a *connoisseur* is one who has sufficient knowledge of and experience in a field to understand deeply a work and relate it to its context. It usually names

[1] Poythress, *Redeeming Mathematics*, p. 22, emphasis added.

[2] *Ibid.*, p. 128.

[3] *Ibid.*, p. 88.

[4] *Ibid.*, p. 98.

[5] *Ibid.*, pp. 18–19.

one who appreciates art, music, food, or drink, but several authors have drawn pointed out the parallel between the background required for an advanced understanding of art or music and the knowledge required to access mathematical beauty.

Helmut Hasse. The mathematician Helmut Hasse* (1898–1979) admitted in a 1952 public lecture that mathematical beauty would be out of reach of most of his audience, and asked them to believe him.[1] This inaccessibility was due to the level of training required to rise above the technical details, which he thought was a prerequisite for perceiving beauty in mathematics.[2] But the same could be said of certain works of art: Beethoven's last sonatas and quartets, he thought, could only be appreciated as beautiful by someone who had an advanced education in and experience of music.[3]

Morris Kline. The celebrated historian of mathematics Morris Kline (1908–92) wrote that 'the question of whether mathematics possesses beauty can be answered only by those who have studied the subject'.[4] Kline himself recognized mathematical beauty, and knew that seeking it was a powerful motivator:[5] 'Mathematics is an art, and as such affords the pleasures which all the arts afford'.[6] A person who did not recognize the position of mathematics had not conceived art properly. An effort and an education were required to truly appreciate any art, and mathematics was no exception.[7]

Arthur Koestler. In his 1964 book *The Act of Creation*, a study of discovery, invention, and creativity, Arthur Koestler (1905–83) wrote that both logic and emotion, both reason and intuition, enter into scientific or mathematical discovery on the one hand and into artistic creation on the other.[8] The aesthetic appreciation of intellectual elegance or of scientific achievement, such as in a mathematical proof, chess problem, cosmological theory, or neurological study, may equal the appreciation of art. But the appreciation of either art or science beyond a basic level required connoisseurship.[9] (Koestler seems to have been the first to use this term with reference to intellectual beauty.)

Armand Borel. Armand Borel (1923–2003) said in a lecture delivered in 1981 and published in English in a slightly modified form in 1983:

> 'For the lay person it is often surprising to learn that one can speak of aesthetic criteria in so grim a discipline as mathematics. But this feeling is very strong for the mathematician'.[10]

The reason that most people could not share in the intellectual beauty of mathematics was that it was like poetry written in a highly specialized, unique, and untranslatable language.[11] The implication was that mathematical beauty could not be *made* accessible: rather, one must be educated to appreciate it. But, thought Borel, this conclusion

* ▸ pp. 527–9

[1] Hasse, *Mathematik als Wissenschaft, Kunst und Macht*, p. 11.

[2] *Loc. cit.*

[3] *Loc. cit.*

[4] Kline, *Mathematics in Western Culture*, p. 469.

[5] Kline, *Mathematics: A Cultural Approach*, pp. 7–8.

[6] *Ibid.*, p. 8.

[7] *Loc. cit.*

[8] Koestler, *The Act of Creation*, p. 264.

[9] *Loc. cit.*

[10] Borel, 'Mathematics', pp. 11–12.

[11] *Ibid.*, p. 15.

strengthened the parallel between mathematics and art, for one also had to gain 'a certain education' to appreciate music or painting.[1]

W. S. R. Anglin. In 1987, W. S. R. Anglin (1949–2004) considered how Francis Hutcheson's* (1694–1746) criterion of 'uniformity amidst variety' applied to the evaluation of solutions in mathematics, specifically to solving Diophantine equations. He pointed out that even considering the degree to which this criterion applies could be a subjective judgement. A mathematician of the calibre of Alan Baker (1939–2018), who was a 1970 Fields medallist, might find certain applications of his results in solving Diophantine equations to be too simple to fulfil the criterion and perhaps to lack beauty.[2] Anglin's hypothesis was a negation of the usual conception of the accessibility of mathematical beauty: having greater mathematical knowledge and ability could nullify mathematical beauty perceived by others.

David Ruelle. In his 2007 book on the way mathematicians think, David Ruelle (1935–) wrote that while a layperson could find aesthetic enjoyment in mathematics, the professional mathematician's aesthetic enjoyment differed from theirs not just in degree, but in kind, just as the average listener's enjoyment of music differed in kind from that of the composer.[3] To enjoy mathematics as a professional mathematician and to enjoy music as a composer required ability, training, perseverance, and luck.[4] The aesthetic standards of mathematics were historically contingent, since, at a given time, mathematicians made use of a standard tool-kit of theorems and techniques. It was in terms of this tool-kit that new results were stated and proved.[5] Thus to gain aesthetic enjoyment of contemporaneous developments, one had to have sufficient training and ability to master that tool-kit.

Robert Langlands. At a January 2010 interdisciplinary conference on beauty, the mathematician Robert Langlands (1936–) gave a talk entitled 'Is there beauty in mathematical theories?' He shied away from offering a direct answer to the question. There was certainly an experience that mathematicians *called* 'beauty', but he was ambivalent about whether it was an aesthetic experience such as great art or architecture might evoke.†

But whatever this term *beauty* described in mathematics, it encompassed but extends beyond 'delight in simple, elegant solutions', which Langlands admitted to be 'a genuine pleasure certainly not to be scorned'.[6] The part of beauty that extended beyond such delight was, however, 'often surrounded by a gangue' that was neither elegant nor appealing in itself.[7] What Langlands seems to have referred to is an effort required to gain access to the beauty. There are intellectual obstacles to surmount, such as mastery of concepts and results that have no intrinsic appeal, but which are necessary to understand the beautiful mathematics.

[1] Borel, 'Mathematics', p. 15.

* ▸ pp. 357–62

[2] Anglin, 'The Nature of Solutions in Mathematics', p. 7.10.

[3] Ruelle, *The Mathematician's Brain*, p. 78.

[4] *Ibid.*, pp. 78–9.

[5] *Ibid.*, p. 89.

† ▸ pp. 758–60

[6] Langlands, 'Is there beauty in mathematical theories?', § 1, emphasis added.

[7] *Ibid.*, § 1, emphasis added.

Langlands raised the question of whether beauty, whether in mathematics or in other fields such as music or literature, depends upon a building up concepts and details. Such an accumulation, if necessary, would create an obstacle to the appreciation of beauty. But Langlands thought that in each case — in mathematics and elsewhere — it was *not* necessary, and there would be little disagreement about this claim.[1]

In mathematics, at least the 'simple mathematical pleasures of arithmetic and geometry' faced no such obstacle.[2] There was no reason a non-mathematician could not appreciate the beauty of a proof such as was found in Euclid's *Elements*.[3] However, Langlands maintained that the mastery of a greater number of deeper concepts opened the way to a beauty of 'a different nature', though he conceded that this point could be debated.[4]

A 'TWO CULTURES' DIVIDE?

> ‘ EINSTEIN: […] I can think of nothing more objectionable
> than the idea of science for the scientists.
> It is almost as bad as art for the artists
> and religion for the priests. ’
>
> — Max Planck, Albert Einstein & James Murphy,
> 'A Socratic Dialogue', p. 209.

Some authors detected a culturally-ingrained *denial* of the possibility of recognizing beauty in mathematics and in the sciences generally, which was analoguous to or perhaps even a component of the divide between the humanities and the sciences famously set out by C. P. Snow in his 1959 lecture *The Two Cultures*.

Morris Kline. At least by 1962, Kline seems to have become despondent about communicating the *possibility* of finding aesthetic value in mathematics to someone who has not studied the subject. In his book *Mathematics: A Cultural Approach*, he wrote that his experience with liberal arts students made him prefer to show the relevance of mathematics by connecting it to the broader culture, because 'any appeal to intellectual discipline […] or to the beauty of mathematics, or to the pleasures of mental activity is met with condescending scorn'.[5] If a student's starting-point was scorn, they were unlikely to be disposed to make an effort to uncover mathematical beauty.

Arthur Koestler. For Koestler, two related aspects of 'cultural prejudice'[6] explained why aesthetic appreciation of mathematics and the sciences was less recognized, or at least less widely expressed. First, society seemed to feel that mathematicians and scientists were '*not supposed* to appeal to emotion'.[7] Second, there was the 'two cultures' divide, which Koestler thought 'absurd' and responsible for the 'paradoxical phenomenon' that an educated person hesitated to admit that a work of art was inaccessible to them, but would happily and even pridefully admit ignorance of the scientific principles that grounded nature and supported the technological society in which they lived.[1]

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[1] Langlands, 'Is there beauty in mathematical theories?', § A.4, n. 9.

[2] *Ibid.*, § A.4, n. 9.

[3] Dundas & Skau, 'Interview with Abel Laureate Robert P. Langlands', p. 19.

[4] Langlands, 'Is there beauty in mathematical theories?', § A.4, n. 9.

[5] Kline, *Mathematics: A Cultural Approach*, p. viii.

[6] Koestler, *The Act of Creation*, p. 267.

[7] *Ibid.*, p. 264, emphasis added.

Seymour Papert. Seymour Papert (1928–2016) wrote in 1978 that the popular conception of mathematics emphasized its purely logical aspects and minimizes all others. This perspective informed the teaching of mathematics in schools, and so perpetuated the idea that mathematics was isolated from human experiences such as pleasure and beauty.[2] Papert thought that the idea that mathematical beauty is accessible only to a (possibly minuscule) minority was 'deeply embedded in our culture'.[3] He seems to have blamed Poincaré, if not for originating this idea, then at least for turning it into a principle that marked off from the general population an intellectual elite who have an innate aesthetic sense for mathematics and are thus capable of becoming creative mathematics.[4]*

Boyd Tonkin. In 2023 the literary critic Boyd Tonkin published an essay addressing the question 'Why won't our artistic-literary establishment recognise that there's mystery, beauty and humanity in maths?' He suggested that the literary establishment, at least in the United Kingdom, derided the idea that mathematics was part of culture and that it was perceived as 'boring sums':[5]

> 'Our knee-jerk literary maths-haters still can't see over their self-erected fences to relish the beauty, and wonder, on the other side. That's their loss, and their shame.'[6]

At the same time, he argued, more people than ever appreciated mathematics either through works of popular mathematics or through literary treatments.[7]

But *was* there such a literary rejection of the general idea of mathematics as part of culture and the particular notion of mathematical beauty? Tonkin cited no-one, so it is difficult to judge. Those philosophers who would argue against mathematical beauty do so with care and nuance, and Tonkin seems to have in mind some rather cruder and more reflexive rejection.

EDUCATORS

If many people, even current or former students of mathematics, are unaware of its beauty, where does the blame lie? Some authors have place responsibility on the way mathematics has been taught, at both school and university level.

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[1] Koestler, *The Act of Creation*, p. 264.

[2] Papert, 'The Mathematical Unconscious', p. 104.

[3] *Ibid.*, p. 105.

[4] *Loc. cit.*

* ▸ p. 574

[5] Tonkin, 'The joy of sets'.

[6] *Ibid.*

[7] *Ibid.*

Seymour Papert. Papert himself disagreed with Poincaré's principle: the problem was not that few had the required innate aesthetic sense to pursue mathematics, but that the education system and culture generally utterly failed to recognize and nurture such aesthetic sense.[1] He thought that this failure was a historical accident. A certain amount of skill in elementary arithmetic, instilled by rote learning, was necessary to function in society before the advent of electronic calculators.[2] But this utilitarian motive was only part of the story: Papert thought a major part of school mathematics was selected, not because it formed an intellectually coherent whole, but because it was suitable to be taught in a classroom equipped with blackboard, chalk, pencil, and paper.[3]

Tommy Dreyfus & Theodore Eisenberg. In a study published in 1986, Tommy Dreyfus (1946–) & Theodore Eisenberg (1942–) presented university mathematics students at both undergraduate and graduate levels with a set of problems, each of which admitted more than one solution, one of which experts considered to be more elegant. The students only indicated a desire to find a solution, not necessarily an elegant one. Moreover, even when shown an elegant solution, they showed no aesthetic appreciation and did not value it over an inelegant one.[4] Dreyfus & Eisenberg concluded that the students had no aesthetic appreciation for mathematics, and they attributed this lack to flawed teaching: 'something is terribly amiss in the mathematics curriculum'.[5] To rectify this problem, they thought that mathematics educators should look to the formation of aesthetic appreciation in artistic fields.[6]

Gian-Carlo Rota. Gian-Carlo Rota (1932–99), on the other hand, was sceptical of the idea of teaching mathematics so as to emphasize its beauty; his views, however, were doubtless shaped by his reinterpretation of mathematical beauty as enlightenment.* In 1997, Rota said that

> 'courses in "art appreciation" are fairly common; it is however unthinkable to find any "mathematical beauty appreciation" courses.'[7]

But the analogy in Rota's argument was flawed: the parallel should not have been between art and mathematical beauty, but between art and mathematics. Beauty need be no part of art, nor of mathematics; the artefacts of both *may* possess beauty or other kinds of value. And the idea of courses in 'mathematics appreciation' has had its approbators: witness William Dunham's (1947–) proposed 'Great Theorems' course[8] and his 1991 book *Journey Through Genius*.

Maryam Mirzakhani. In a 2008 interview, Maryam Mirzakhani (1977–2017), who would be awarded the Fields Medal in 2014, said that many students did not give themselves an opportunity to be engaged by mathematics: they made no effort, and so it remained uninteresting to them. She thought that '[t]he beauty of mathematics only shows itself to more patient followers'.[1] After she was awarded the Fields Medal, she said explicitly that '[y]ou have to spend some energy and effort to see the beauty of math'.[2] *Prima facie*, her comments place the responsibility directly on the student, but one could argue that students were not prepared to be engaged by mathematics because of earlier failures in their education system, and that it would be the system's responsibility to try to rectify this failure or at least mitigate its consequences.

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[1] Papert, 'The Mathematical Unconscious', p. 106.

[2] Papert, *Mindstorms*, p. 51.

[3] *Ibid.*, pp. 52–3.

[4] Dreyfus & Eisenberg, 'On the Aesthetics of Mathematical Thought', p. 7.

[5] *Ibid.*, p. 9.

[6] *Loc. cit.*

* ▸ pp. 742–5

[7] Rota, 'Phenomenology of Mathematical Beauty', p. 171.

[8] Dunham, 'A "Great Theorems" Course in Mathematics', p. 810.

Francis Su. Francis Su, who was formerly president of the Mathematical Association of America, wrote his 2020 book *Mathematics for Human Flourishing* with the aim of encouraging engagement with mathematics in a way that offered self-cultivation. His book was not concerned solely with beauty, though that was one of the values it examined, being addressed in part to 'the artist who never thought math was beautiful'.[3]

Su made the important observation that trying to *describe* mathematical beauty cannot convey the *experience* of mathematical beauty. Su's analogy was that 'listing characteristics of beauty for the uninitiated is a lot like explaining the allure of good sushi by citing its temperature, texture, or acidity'.[4] Rather, the experience should be prompted by the way in which mathematics is taught. No student would be inspired if they were taught mathematics as a set of meaningless mechanical rules to handle problems that required neither insight nor creativity.[5] Problems should push the student to create solutions that are elegant and beautiful and so to come to appreciate that elegance and beauty.[6] But the duty was not just the teacher's: everyone should *demand* the opportunity to experience mathematical beauty.[7]

EXPOSITORS

Various authors commented on why the general populace, or even scholars outside of mathematics and the sciences, might be unaware of even the possibility of beauty in mathematics, and how they might be shown at least a part of that beauty.

Hermann Weyl. In a 1946 address, Hermann Weyl (1885–1955) contrasted the few people who read mathematical papers with the many who can appreciate music, and said that this difference pointed to a failure: 'If the aesthetic side of mathematics is essential, our endeavors look somewhat futile.'[8] But Weyl thought that this failure was due to a general decline in expositional abilities since the start of the twentieth century. At that time, a half-century before Weyl was speaking, writers had made a greater effort to address a general audience, avoiding a surfeit of technical language in expressing their ideas and drawing connections with other fields.[1]

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[1] Mirzakhani, quot. in [Anonymous], 'Interview with Maryam Mirzakhani', p. 13.

[2] Mirzakhani, quot. in Klarreich, 'A Tenacious Explorer of Abstract Surfaces'.

[3] Su, *Mathematics for Human Flourishing*, ch. 1.

[4] *Ibid.*, ch. 5.

[5] *Loc. cit.*

[6] *Loc. cit.*

[7] *Ibid.*, ch. 1.

[8] Weyl, 'Address', pp. 170–1 [§ 20].

Emil Artin. Emil Artin (1898–1962) wrote in 1953 that, although the lecturer strove to exhibit the beauty of mathematics, they always failed.[2] The fundamental problem was that the logic of mathematics was linear (and thus suited for transmission in the lecture-hall or text-book), but the 'the whole of it, the real piece of art,' where the aesthetic value is seen, was not: rather, it should be instantaneously grasped as a whole.[3] This structural difference in between communication and aesthetic perception suggest that there could be a fundamental difficulty in communicating the beauty of mathematics even to specialist students.

Jerry King. Jerry King (1935–2020) wrote in 1981 that the beauty of mathematics should be used as a means of attracting humanists to study it.[4] Even professional mathematicians, he suggested, only rarely set out to become mathematicians: rather, they studied mathematics for its applications in engineering or physics until at some point they underwent a Pauline conversion wherein, without warning, they saw the aesthetic value of the subject.[5] Humanists would not be interested in the applications of mathematics, and so this route was unavailable; indeed, forcing them to take the road to Damascus would discourage them. Instead, mathematics should be taught without reference to its applications, 'as a classical and intellectual art'.[6]

In 1992, King wrote *The Art of Mathematics*, addressed to humanists and with the aim of communicating to them the aesthetic value of mathematics.[7] But King suggested that at least part of the reason for the aesthetic side of mathematics being inaccessible to humanists was that 'the *right view* of mathematics has been kept hidden from them'.[8] His language was almost accusatory, as if there had been a deliberate effort to not to disseminate the correct perspective beyond 'the closed aristocracy of mathematicians'.[9] But King did not posit any actual conspiracy, offering instead a different diagnosis: that mathematicians want to do and to write mathematics, not to write *about* mathematics.[10] This complaint was one that Hardy had made in his *Apology*:[*] writing *about* mathematics, he maintained, was 'a melancholy experience', an admission of defeat, a recognition that he was no longer intellectually first-rate.[11]

Herbert Hauptman. The mathematician Herbert Hauptman (1917–2011), who shared in the 1985 Nobel Prize in chemistry for his work in mathematical methods for crystallography, asked in his 2008 book *On the Beauty of Science* whether educators should try to make accessible to a larger group the beauty of science generally. The examples he

[1] Weyl, 'Address', pp. 171 [§ 20].

[2] Artin, Review of *Éléments de mathématique* Book II, p. 475.

[3] *Loc. cit.*

[4] King, 'The Unexpected Art of Mathematics', p. 36.

[5] *Ibid.*, pp. 30–1.

[6] *Ibid.*, p. 36.

[7] King, *The Art of Mathematics*, pp. 2–3.

[8] *Ibid.*, p. 3, emphasis in orig.

[9] *Ibid.*, p. 3.

[10] *Ibid.*, pp. 3–4.

[*] ▸ pp. 498–523

[11] Hardy, *Apology*, § 1; cf. King, *The Art of Mathematics*, pp. 4–5.

put forward were focused on mathematics. He thought that some of the most beautiful theorems were learned only by a few hundred people each year.[1] Most people would never hear of a result of 'such beauty and such fundamental importance' as the law of quadratic reciprocity,[*] nor would they read any of the many proofs motivated by its beauty and importance.[2] Trying to communicate that beauty to someone who knew nothing about number theory 'would be like showing a chimpanzee a Cézanne'.[3] Hauptman was in favour of attempts to popularize science, and of educators extending their remit to science *appreciation*, but his desire was both to expand to science the range of pleasures a large number of people could enjoy, and to promote the social end of opposing 'the rising tide of religious belief'.[4]

Eduard Frenkel. In his 2013 book *Love and Math*, Edward Frenkel (1968–) extolled what he termed the 'inherent democracy' of mathematics: its results could not be owned or patented or otherwise restricted (which is the legal position in the United States[5]). For Frenkel, this position was grounded in formulae such as $E = mc^2$ expressing immutable truths about the universe. 'Nothing in this world is so profound and elegant, and yet so available to all.'[6] But he implicitly acknowledged that mathematics and its beauty are *not* accessible to everyone: 'we, mathematicians, need to do a better job of unlocking the power and beauty of our subject to a wider audience'.[7]

John L. Bell. John L. Bell (1945–) wrote in 2015 that the unpopularity of mathematics was largely due to its beauty being difficult to access. He drew the comparison with music, where experts such as composers and performers can enjoy the intellectual and sensory beauty, but where the general listener can only appreciate the sensory beauty. But he thought that it would be possible to convey some of the intellectual beauty of mathematics to a general audience, if only an effort were made in teaching elementary mathematics. One could present beautiful theorems that did not require deep knowledge to understand, such as there being exactly five regular solids,[†] Euler's polyhedron formula,[‡] Lagrange's four-square theorem,[§] or the four-color theorem.[8][¶]

[1] Hauptman, *On the Beauty of Science*, p. 51.

[*] ▸ p. 405

[2] *Loc. cit.*

[3] *Loc. cit.*

[4] *Loc. cit.*

[5] Gottschalk v. Benson, 409 U.S. 63 (1972).

[6] Frenkel, *Love and Math*, ch. 18.

[7] *Ibid.*, ch. 17.

[†] ▸ pp. 26–8

[‡] ▸ p. 683

[§] ▸ p. 577

[8] Bell, 'Reflections on Mathematics and Aesthetics', p. 178.

[¶] ▸ pp. 602–8

BEAUTY CONVEYED VISUALLY

> ‘ The responses to our pictures quite often express the hope
> that this unity will become more easily visible,
> not just as a remote kind of beauty accessible to a few initiates ’
>
> — Heinz-Otto Peitgen & Peter H. Richter,
> *The Beauty of Fractals*, p. 4.

Lipman Bers

In a conversation published in 1979, Lipman Bers (1914–93) noted that the enjoyment of art forms such as painting and music was easily accessible to many people, but he thought that previous attempts to convey the beauty of mathematics to a wide audience had been abject failures.[1] He suggested that it could be successfully communicated by methods such as showing on a screen a approximation of a curve converging to a space-filling curve (such as a Peano curve; see Figure 15.4), which could grant a viewer ‘an intense feeling of geometrical joy’, without their having to understand formal definitions.[2]

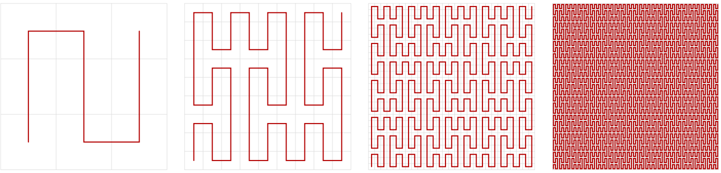

FIGURE 15.4
The first four iterations in the generation of a Peano space-filling curve.

Helman Ferguson’s mathematical art

The sculptor Helaman R. P. Ferguson (1940–), who was formerly a mathematician, is noted for creating works that embody some piece of mathematics. For instance, his work *Umbilic Torus NC* is a rendering in bronze of the mathematical surface shown in Figure 15.6, formed by rotating a hypocycloid with three cusps around the z-axis while adding a turn of $2\pi/3$ about its own centre. (A *hypocycloid* is generated by a point on a circle rolling on the inside of a larger circle. Three cusps are produced when the ratio of the radii is 3; see Figure 15.5.) Ferguson found the umbilic torus fascinating because of its connection with the representation theory of $GL(2, \mathbb{R})$.[3]

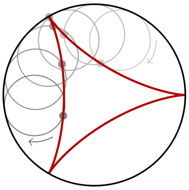

FIGURE 15.5
Generation of a 3-cusp hypocycloid.

Ferguson believed that mathematicians did not make enough effort to inform laypeople about their subject,[4] and in particular that the motivation of aesthetic pleasure was often not made clear.[5] He thought that reifying abstract mathematics in sculpture could communicate it, through the artistic medium, to a general audience, and that such sculptures could make visible the beauty of mathematics.[6] In this connection, he found very apt G. N. Watson’s (1886–1965) comparison of the beauty of Michelangelo’s statues in the Sagrestia Nuova

[1] Hammond, ‘Mathematics’, p. 33.

[2] *Loc. cit.*

[3] Ferguson & Ferguson, ‘Celebrating Mathematics in Stone and Bronze’, p. 841.

[4] Cannon, ‘Mathematics in Marble and Bronze’, p. 36.

[5] Peterson, ‘Equations in Stone’, p. 152.

[6] Ferguson, ‘Two Theorems’, p. 590.

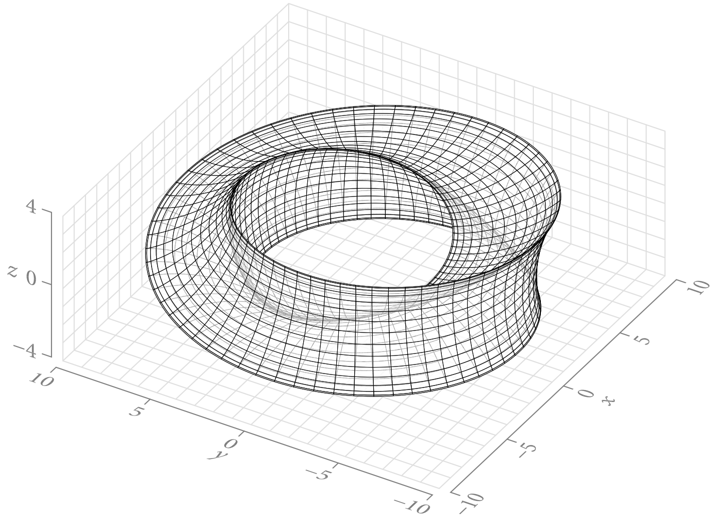

FIGURE 15.6
The umbilic torus, formed by rotating a 3-cusp hypocycloid around the z-axis while adding a one-third turn about its own centre. This surface is defined parametrically by the equations

$$\begin{aligned} x &= \big(7 + \cos(u/3 - 2v) \\ &\qquad + 2\cos(u/3 + v)\big) \\ &\qquad \cdot \sin u, \\ y &= \big(7 + \cos(u/3 - 2v) \\ &\qquad + 2\cos(u/3 + v)\big) \\ &\qquad \cdot \cos u, \\ z &= \sin(u/3 - 2v) \\ &\qquad + 2\sin(u/3 + v), \end{aligned}$$

where $u, v \in [0, 2\pi)$.

to that of Ramanujan's equations.[1]* With a modicum of rhetorical flair, he described what he thought he had achieved as a sculptor:

> 'Beauty and truth: these my sculpture celebrates and incorporates. Sculptural beauty that moves souls and mathematical truth that moves minds. That is my work.'[2]

> 'Each of my sculptures involves a circle of beautiful mathematical theorems.'[3]

And he reported having witnessed people change their views of mathematics and of art in response to viewing his works.[4]

The engineer Clem Padin suggested that Ferguson's ambition was laudable but misdirected. In response to a September 1990 article on Ferguson's sculptures in *Science News*, Padin wrote that visual geometrical forms were the wrong medium to communicate the abstract aesthetic value of mathematics, and that he wearied of what he saw as a singular focus on such static representations. Padin held instead that 'mathematical aesthetics seems more closely aligned with dance than with any time-invariant media'.[5]

In 1991, James W. Cannon (1943–), a topologist and group theorist, concurred with Ferguson's aim of expressing beautiful mathematics in a way that everyone could appreciate.[6] Even as a professional mathematician, he could be affected by Ferguson's work: he recalled his own response to a sculpture based on the Alexander horned sphere, which is a topological object which, together with its interior, is simply connected, but whose complement — its exterior — is not simply connected. As a mathematical object, it is the limit of an iterative construction, which starts with a sphere with a pair of 'horns'. At each step, each horn splits into a pair of sub-horns that *almost* interlink with the sub-horns growing from the other horn of the pair (see Figure 15.7).[7] Ferguson's sculpture can only depict a few iterations, but

[1] Ferguson & Ferguson, 'Eightfold Way', pp. 138–9.

* ▸ p. 5

[2] Ferguson, quot. in Cannon, 'Mathematics in Marble and Bronze', p. 36.

[3] Ferguson & Ferguson, 'Celebrating Mathematics in Stone and Bronze', p. 840.

[4] Cannon, 'Mathematics in Marble and Bronze', p. 36.

[5] Padin, 'The mathematical aesthetic'.

[6] Cannon, 'Mathematics in Marble and Bronze', pp. 38–9.

[7] See Alexander, 'An example of a simply connected surface bounding a region which is not simply connected'.

Beauty conveyed visually

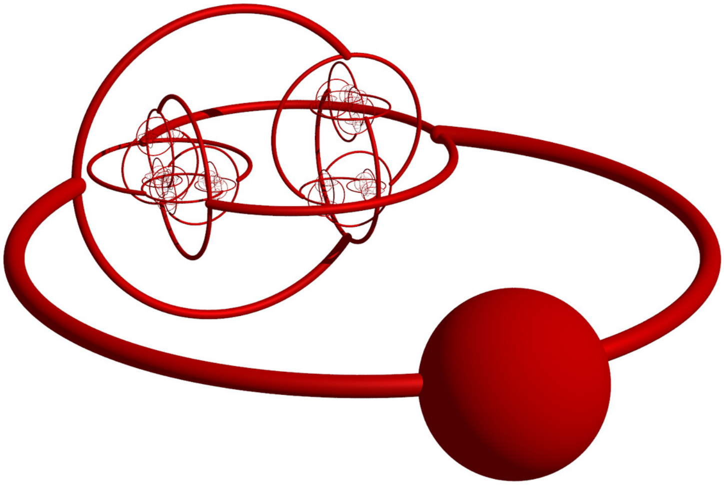

FIGURE 15.7
An Alexander horned sphere.

Cannon wrote that the sculpture had opened his eyes to the beauty of the Alexander horned sphere. He had studied the mathematical object over many years, and thought he understood what it looked like. But he had not seen 'some of the essentially 3-dimensional aspects of this beautiful object, some of its potential beauty'.[1]

Keith Devlin

The mathematician Keith Devlin (1947–), who wrote a number of popular mathematics books, observed in 1994 that the development of computer graphics had made mathematical beauty accessible — partially and in a limited way — to those untrained in mathematics. Devlin had in mind visualizations such as images of fractal sets. Such a 'performance' of mathematics could convey *something* of the aesthetic experience of the mathematician, although much of the beauty remains hidden.[2]

Devlin admitted that large parts of mathematics are too abstract to be amenable to graphical presentation and so had to remain inaccessible to non-mathematicians. But, thought Devlin, those who had ability and training in mathematics should not shy from *trying* to communicate their aesthetic experience to others, and give them 'some sense of the simplicity, the precision, the purity, and the elegance that give the patterns of mathematics their very considerable aesthetic value'.[3]

◆

[1] Cannon, 'Mathematics in Marble and Bronze', p. 36.

[2] Devlin, *Mathematics*, pp. 4–6.

[3] *Ibid.*, p. 5.

Machine Aesthetics

16

Beauty, computers, and proof

> ‘as classical philosophy was concerned with
> the true, the good, and the beautiful,
> so also should the philosophy of computation be concerned with
> true computation, good computation, and beautiful computation.’
>
> — Philip J. Davis & Reuben Hersh, *Descartes' Dream*, p. xiv.

> ‘THE PURPOSE OF COMPUTING IS INSIGHT, NOT NUMBERS’
>
> — Richard Hamming,
> *Numerical Methods for Scientists and Engineers*, p. v.

The age of the computer proof dawned in 1966, when around twenty hours of time on the G-21 mainframe at the Carnegie Institute of Technology was used to check that the minimum number of comparators in a sorting network for 7 elements is 16 (see Figure 16.1). The algorithm generated sets of networks with up to 16 comparators that, by a notion of ‘permutation equivalence’, sufficed to establish whether any network with the same number of comparators could sort all sequences; none with fewer than 16 comparators did.[1] Since then, computers have been used to assist in proving results, to search for conventional proofs, and to check formalized proofs. The philosophical debate on computer proofs began with the celebrated and controversial proof of the four-colour theorem by Wolfgang Haken (1928–2022) & Kenneth Appel (1932–2013) in 1976, and centred on whether computer-assisted proofs should actually count as proofs and whether they should be believed in the same way as conventional proofs. Distinct from — but connected to — these discussions were aesthetic questions. Could computer-assisted proofs be beautiful? Might they one day be seen as beautiful?* Have beautiful conventional proofs been found by a computer?

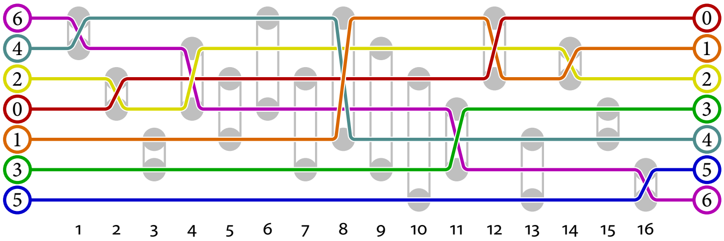

FIGURE 16.1
A sorting network for 7 elements with the minimum possible number of comparators, shown sorting $(6, 4, 2, 0, 1, 3, 5)$. Each comparator interchanges two elements if they are not ordered correctly.

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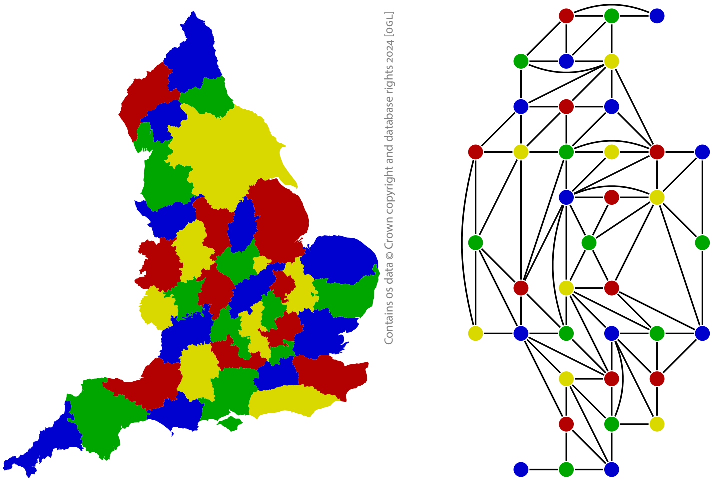
[1] Floyd & Knuth, ‘The Bose-Nelson Sorting Problem’, Thm 5.

* ▸ pp. 794–5



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A map of the historical counties of England coloured using 4 colours.

The corresponding graph with the same 4-colouring.

FIGURE 16.2
Colouring a map of the historical counties of England using 4 colours so that no adjacent regions receive the same colour; it cannot be coloured with fewer than 4 colours.

FOUR COLOURS SUFFICE

Appel & Haken's solution of the four-colour problem was the first great triumph of the computer proof. The question arose from an observation of Francis Guthrie (1831–99) who noticed that it was possible to colour a map of England using four colours so that no neighbouring regions had the same colour (see Figure 16.2 (left)), and wondered if four colours were enough for any map. He asked his younger brother Frederick Guthrie (1833–86), who in turn asked his professor, Augustus De Morgan (1806–71), on 23 October 1852. De Morgan realized the subtlety of the problem and wrote a still-extant letter about it that same day to William Rowan Hamilton[*] (1805–65).[1] Serious mathematical attention seems to date to 1878–9, when Arthur Cayley (1821–96) asked the London Mathematical Society if anyone had solved the problem[2] and pointed out the absence of a proof to the Royal Geographical Society.[3]

The question can be re-stated in terms of graph theory: a map is turned into a planar graph by replacing regions with vertices and joining neighbouring regions by edges. The four-colour conjecture then asserts that for any planar graph, four colours can be assigned to vertices so that no adjacent vertices receive the same colour (see Figure 16.2 (right)). Alfred Bray Kempe (1849–1922) claimed a proof in 1879,[4] but P. J. Heawood (1861–1955) pointed out a gap in 1890, noting that the essence of Kempe's method still applied to give a proof that five colours sufficed.[5] The essential idea of the Kempe 'proof' can be reformulated as a *reductio ad absurdum*: any graph that requires at least five colours contains at least one unavoidable configuration, and each of these configurations is reducible in the sense that it yields a graph with fewer vertices that also requires five colours; thus there is no smallest graph that requires five colours.[6] Kempe had found three

[*] ▸ pp. 412–21

[1] Fritsch & Fritsch, *The Four-Color Theorem*, pp. 5, 7–8.

[2] Smith, 'Proceedings, 13 Jun. 1878'.

[3] Cayley, 'On the Colouring of Maps'.

[4] Kempe, 'On the Geographical Problem of the Four Colours'.

[5] Heawood, 'Map-Colour Theorem'.

[6] MacKenzie, *Mechanizing Proof*, p. 119.

unavoidable configurations and thought them all reducible; the flaw was that one case was not reducible.

Heinrich Heesch (1906–95) became convinced that Kempe's approach could work with a larger set of perhaps 10 000 configurations. Appel & Haken developed and refined algorithms to find unavoidable configurations and show them reducible.[1] The interaction with the computer investigations led to a 450-page manual proof that there was set of 1936 unavoidable reducible configurations, which they whittled down to 1405 by eliminating redundancies and simplifying parts of the argument.[2] The reducibility of the unavoidable configurations was checked by computer.[3] Haken announced the proof in August 1976.

In 1977, Frank Allaire announced another proof, which used the same overall strategy but used a different set of unavoidable reducible configurations and an independently-developed and more powerful reducibility testing program.[4]

Another proof by Neil Robertson (1938–), Daniel P. Sanders (1969–), Paul Seymour (1950–) & Robin Thomas (1962–2020) was announced in 1996 using a set of only 633 unavoidable reducible configurations.[5] George Gonthier (1962–) subsequently formalized this entire proof using the Coq theorem prover.[6]

Reactions: philosophers

A 1979 essay by Thomas Tymoczko (1943–96) began the philosophical debate on the implications of the Appel–Haken proof. Tymoczko's position was that

> 'if we accept the [four-colour theorem] as a theorem, then we are committed to changing the sense of "theorem", or, more to the point, to changing the sense of the underlying concept of "proof."'[7]

That is, the claimed proof was not a proof as previously construed. Resorting to computer checking of reducibility left a gap in the proof — a gap that, for Tymoczko, was filled by empirical evidence obtained by the 'experiment' of running a computer program.[8] This perspective meant that the four-colour theorem was known *a posteriori* and raised the question of how to delineate the boundary between mathematics and natural science.[9]

Tymoczko's argument was based around the notions of *surveyability* and *formalizability*, which were two of the essential characteristics of proofs that helped explain why they were convincing.[10] Surveyability meant that the proof could be looked over and checked by a mathematician.[11] Formalizability meant that the proof could be embedded into an appropriate formal language and 'filled out' to be rigorous.[12] (The notion of surveyability long pre-dates the proof of the four-colour theorem, although discussion of the proof granted it a new prominence.[13])

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[1] MacKenzie, *Mechanizing Proof*, pp. 130–4.

[2] Appel & Haken, 'Every Planar Map is Four Colorable Part I'; MacKenzie, *Mechanizing Proof*, p. 136.

[3] Appel, Haken & Koch, 'Every Planar Map is Four Colorable Part II'.

[4] Allaire, 'Another Proof of the Four-Color Theorem I'.

[5] Robertson et al., 'A New Proof of the Four-Colour Theorem'; Robertson et al., 'The Four-Colour Theorem'.

[6] Gonthier, 'Formal Proof'.

[7] Tymoczko, 'The Four-Color Problem', p. 58.

[8] *Loc. cit.*

[9] *Loc. cit.*

[10] *Ibid.*, p. 59.

[11] *Ibid.*, pp. 59–60.

[12] *Ibid.*, p. 60.

[13] Bassler, 'The Surveyability of Mathematical Proof', p. 99.

Tymoczko thought that most mathematicians would agree that all surveyable proofs are formalizable.[1] But the Appel–Haken proof 'does drive a wedge between the criteria of surveyability and formalizability'.[2] It could be formalized, but it was too long ever to be surveyed. Thus to accept this proof was tacitly to modify what was considered to be a proof.[3] Tymoczko had no objection to mathematicians accepting such computer proofs (with a duly adjusted notion of proof), provided that they were accepted for the right reasons and with clear recognition that a computer proof was a probabilistic argument for correctness, albeit one which could grant a high enough probability of correctness to justify belief in the truth of the result.[4]

'Tymoczko had touched on a raw nerve.'[5] Many philosophers disagreed with his position.

In a paper published in August 1980, Israel Krakowski disagreed with Tymoczko, writing that the Appel–Haken proof raised no new important issues.[6] He thought that Tymoczko had not shown that surveyability extended beyond convincingness and formalizability.[7] The *computer* had surveyed the proof; it was 'chauvinism' to suggest otherwise. The potential for an error in software or hardware renders the computer fallible, but no more so than a human.[8]

In 1980, Paul Teller (1943–) published a response to Tymoczko in which he argued that surveyability was not an essential characteristic of a proof, but simply that, before the advent of computers, surveyability had been necessary to check the correctness of a proof.[9] The computer was simply a new tool for surveying, like pencil and paper, tables, or calculator.[10] Thus the fundamental sense of proof had not been affected.[11] Margarita Levin made a very similar point, that computer-checking differed only in degree, not in kind, from using symbols.[12]

Michael Detlefson (1948–2019) & Mark Luker (1947–), on the other hand, agreed with Tymoczko's view that empirical evidence had been used in proving the four-colour theorem, but that this recourse was not a novelty.[13] They pointed to the Feit–Thompson theorem, which states that *All finite groups of odd order are soluble.*[14] The original proof of this result occupies about 250 pages — an entire issue of the journal — and in it several errors had been found and fixed, which suggested that a belief in the correctness of theorem depends at least partly on the empirical evidence that no 'serious' error had been found thus far.[15] The mathematician Daniel Gorenstein (1923–92) made essentially the same point, without stressing philosophical implications, in the more general setting of the programme of classifying the finite simple groups. In 1979, before the classification was complete, he wrote:

> 'it seems beyond human capacity to present a closely reasoned several hundred page argument with absolute accuracy. I am

[1] Tymoczko, 'The Four-Color Problem', p. 60.

[2] *Ibid.*, p. 63.

[3] *Ibid.*, p. 78.

[4] Tymoczko, 'Computer Use to Computer Proof', p. 124.

[5] Shanker, *Wittgenstein*, p. 130.

[6] Krakowski, 'The Four Color Problem Reconsidered', p. 95.

[7] *Ibid.*, p. 91.

[8] *Ibid.*, p. 94.

[9] Teller, 'Computer Proof', p. 798.

[10] *Loc. cit.*

[11] *Ibid.*, p. 801.

[12] Levin, 'On Tymoczko's Argument for Mathematical Empiricism'.

[13] Detlefsen & Luker, 'The Four-Color Theorem', p. 804.

[14] Feit & Thompson, 'Solvability of Groups of Odd Order'.

[15] Detlefsen & Luker, 'The Four-Color Theorem', p. 811.

> not speaking of the inevitable typographical errors, or the overall conceptual basis for the proof, but of "local" arguments that are not quite right — a misstatement, a gap, what have you. They can almost always be patched up on the spot, but the existence of such "temporary" errors is disconcerting to say the least. […] how can one guarantee that the "sieve" has not let slip a configuration which leads to yet another simple group? […] there is a prevalent feeling that, with so many individuals working on simple groups over the past fifteen years, and often from such different perspectives, every significant configuration will loom into view sufficiently often and so cannot remain unnoticed for long.'[1]

On the other hand, Stuart G. Shanker (1952–) went further than Tymoczko: he held that Appel & Haken offered, not a proof, but rather an experimental procedure to find an unavoidable reducible set of configurations. He did not think that the notion of proof had to be adjusted, but simply that *the four-colour conjecture had not been proved*.[2]

Reactions: mathematicians

Debate among mathematicians on Appel & Haken's work seems to have been concerned mainly with its correctness and its value, not whether (if correct) it counted as a proof or as an empirical result or as something else.

The initial announcement of the proof garnered only polite applause.[3] There were early concerns about its correctness: mathematicians who were more accustomed to computers doubted whether the long manual proof of the set of unavoidable configurations was valid; others doubted the computer checking of reducibility.[4] W. T. Tutte (1917–2002), an eminent graph theorist who had previously indicated scepticism about the strategy descending from Kempe's work,[5] believed the proof valid, which was important for its general acceptance.[6]

Making a point related to Levin's, the combinatorialist Edward Swart thought that using a computer to check cases was simply extending a range: from entirely mental case-checking, to case-checking aided by pencil and paper, to case-checking with pencil and paper and immense dedication over thousands of hours of work, to case-checking with a computer.[7]

Today, there seems to be little doubt about the validity of the proof:

> 'There is nothing in the Appel–Haken proof that is any less *convincing* than, say, the classification of simple groups; or indeed of the hosts of theorems published every day that are outside a particular mathematician's speciality — which he will never check himself. It is also much less likely that a computer will

[1] Gorenstein, 'The Classification of Finite Simple Groups I', p. 52.

[2] Shanker, *Wittgenstein*, p. 157.

[3] Albers, 'Polite Applause'.

[4] MacKenzie, *Mechanizing Proof*, p. 140; cf. Swart, 'The Philosophical Implications', p. 700.

[5] Whitney & Tutte, 'Kempe Chains and the Four Colour Problem'.

[6] MacKenzie, *Mechanizing Proof*, pp. 137–8.

[7] Swart, 'The Philosophical Implications', p. 699.

Four colours suffice

> make an error than a human will. Appel and Haken's proof strategy makes good logical sense and is sufficiently standard that no fallacy is likely there; the unavoidable set was in any case obtained by hand; and there seems little reason to doubt the accuracy of the program used to check reducibility. Random "spot tests" and, more recently, repetitions of the computations using different programs, have found nothing amiss. I know of no mathematician who currently doubts the correctness of the proof.'[1]

And indeed, as a consequence of its formalization in the Coq automated theorem prover,[*] the proof by Robertson et al. is 'one of the most meticulously verified proofs in history'.[2]

However, mathematicians seem to have been consistent in making negative evaluations of the proof as regards qualities other than correctness. An early reaction, pre-dating the completion of the proof, was recalled by Appel & Haken themselves:

> 'A combinatorialist friend of ours, when told in 1975 that we thought we could prove the theorem by a method involving computer proofs of reducibility exclaimed in horror, "God would never permit the best proof of such a beautiful theorem to be so ugly." When faced with such a proof even the fairest minded mathematician can be forgiven for wishing that it would just go away rather than being forced to think about the fact that an "elegant" proof may never appear and thus our Eden is defiled.'[3]

(The 'combinatorialist friend' had the same attitude as C. P. Snow (1905–80) reported G. H. Hardy (1877–1947) as having towards a hypothetical ugly proof of Goldbach's conjecture.[†]) Haken also related how another mathematician complained that the proof used 'totally inappropriate means', but since it had been proved, the best mathematicians would no longer work on it, since the motivation of proving it first had gone. Thus it could be a long time before a 'good, satisfactory proof' emerged.[4] In contrast, Frank R. Bernhart (1945–2016), who also worked in combinatorics and graph theory, wrote in a 1977 expositional article about the proof that he hoped it would lead to 'a short elegant proof'.[5]

In 1981, Philip J. Davis (1923–2018) or Reuben Hersh (1927–2020) or both described his or their reaction:

> 'When I heard that the four-color theorem had been proved, my first reaction was "Wonderful! How did they do it?" I expected some brilliant new insight, a proof which had in its kernel an idea whose beauty would transform my day. But when I received the answer, "They did it by breaking it down into thousands of cases, and then running them all on the computer, one after the other," I felt disheartened. My reaction

---

[1] Stewart, *The Problems of Mathematics*, pp. 153–4.

[*] ▸ p. 603

[2] Hales, 'Formal Proof', p. 1372.

[3] Appel & Haken, 'The Four Color Proof Suffices', p. 12.

[†] ▸ p. 501

[4] Haken, quot. in MacKenzie, *Mechanizing Proof*, p. 138.

[5] Bernhart, 'A Digest of the Four Color Theorem', p. 207.

> then was, “So it just goes to show, it wasn’t a good problem after all.”’[1]

In 1982, the mathematician Frank Bonsall (1920–2011) seemed broadly to agree: the four-colour problem had ‘little intrinsic importance’. Although a proof using some new idea would have been ‘magnificent’, an exhausting application of known methods was not.[2]

[Haken actually considered the lack of new ideas to be a strength of the proof: ‘In a purely mathematical and ingenious proof [...] something relatively simple may have been overlooked, and that may destroy everything’.[3]]

The mathematician Paul Halmos (1916–2006) said in 1982 that he had been convinced to the extent that he would not seek a counterexample to the four-colour theorem. But he hoped that in the future, a manual proof would appear in a journal, followed in the more distant future by a canonical four-page proof, using new concepts, and this proof would ‘belong to the grand, glorious, architectural structure of mathematics’.[4]

[Halmos’s hope can perhaps be illustrated with the history of the proof of the Bieberbach conjecture, which concerns holomorphic injective functions from the unit disc to the complex plane with Taylor series of the form

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots .$$

The conjecture asserts that for all such functions, $|a_n| < n$. In 1984, Louis de Branges (1932–) announced a proof, which took the form of a 385-page book manuscript; such a proof seems either unsurveyable or at best barely surveyable. De Branges was reported to have been disappointed by the failure of other mathematicians to read and verify his proof.[5] By dint of considerable effort by de Branges and collaborators, some errors were corrected and a 16-page proof[6] distilled.[7] A further simplification and shortening, using only the calculus, was obtained by Lenard Weinstein in 1991,[8] and then Doron Zeilberger (1950–) found an even more elementary proof in 1994.[9]]

Halmos also complained that Appel–Haken proof offered no insight or explanation: ‘The present proof relies in effect on an Oracle, and I say down with Oracles! They are not mathematics.’[10] This lack of insight has been a reason commonly given for why the proof was not beautiful: it did not *explain* the four-colour theorem.[11] Paul Erdős (1913–96) was explicit on this point:

> ‘I’m not an expert on the four-color problem, but I assume the proof is true. However, it’s not beautiful. I’d prefer to see a proof that gives insight into why four colors are sufficient.’[12]

Similarly, William P. Thurston (1946–2012) thought that there was little doubt about the correctness of the proof, but that the controversy was because the proof did not offer understanding.[13] And Marcus

[1] Davis & Hersh, *Mathematical Experience*, p. 384.

[2] Bonsall, ‘A down-to-earth view of mathematics’, p. 13.

[3] Haken, quot. in MacKenzie, *Mechanizing Proof*, p. 141.

[4] Albers, ‘Paul Halmos’, pp. 239–40.

[5] Korevaar, ‘Ludwig Bieberbach’s Conjecture’, § 5.

[6] de Branges, ‘A proof of the Bieberbach conjecture’.

[7] Korevaar, ‘Ludwig Bieberbach’s Conjecture’, § 5.

[8] Weinstein, ‘The Bieberbach Conjecture’.

[9] Ekhad & Zeilberger, ‘A [...] Proof of the Bieberbach Conjecture’. The proof is due to Zeilberger; Ekhad is a nickname for his computer.

[10] Halmos, remarks to the 1990 meeting of the Mathematical Association of America, quot. in Hersh, *What is Mathematics, Really?*, p. 54; see also MacKenzie, *Mechanizing Proof*, p. 363, n. 2 to p. 102.

[11] See, for example, Stewart, *Concepts of Modern Mathematics*, app. A, emphasis in orig.

[12] Erdős, quot. in Hoffman, *The Man Who Loved Only Numbers*, p. 44.

[13] Thurston, ‘On proof and progress in mathematics’, p. 162.

du Sautoy (1965–), like Erdős, thought that that the proof was not beautiful and did not give an 'aha!' moment of insight.[1]

Jerry King (1935–2020) found it unpleasant to contemplate a future for mathematics in which it was commonplace to discover proofs that relied on computers checking countless cases:

> 'Appel and Haken indicate this might come about.[2] And I gather it is a future they welcome.
>
> But I do not. For it is a future without elegance, a world of disfigured mathematics. Truth may choose to live in that world but beauty will not. There, the *art* of mathematics will, like Prospero's spirits, vanish into thin air. In this brave new world the magician will snap his fingers and mathematicians will be changed from poets into cabinetmakers.'[3]

Note that King had no doubts about the validity of theorems proved by computers: his concern was wholly aesthetic.

But might perhaps negative aesthetic judgements of the proof influence epistemic judgements of its correctness? A serious error was found in the proof and fixed in 1982; a less serious error was found in 1985.[4] In 1986, Appel & Haken thought that people who found the proof ugly would be relieved to hear that the proof might be invalid.[5]

Anecdote suggests that the Appel–Haken proof is subject to *universal* aesthetic damnation among mathematicians. Paul J. Nahin (1940–) asserted in 2006 that the Appel–Haken proof has become the canonical recourse of mathematicians asked for an instance of ugly mathematics.[6] Chandler Davis (1926–2022), Majorie Senechal (1939–) & Jan Zwicky (1955–) wrote in 2008 that while all mathematicians think that Euclid's proof of the infinitude of the primes is the kind of elegant proof found in The Book,* '*no mathematician* doubts that computer-generated proofs, the kind that methodically check case after case, are not'.[7]

LATER COMPUTER PROOFS

Although the Appel–Haken proof of the four-colour theorem was the earliest and still seems to be the most famous computer proof, other noteworthy results have been proved using computers. There seems to have been little aesthetic commentary on them, but some of them are aesthetically suggestive.

Kepler conjecture

The Kepler conjecture asserts that *There is no denser packing of congruent spheres in three-dimensional euclidean space than the face-centred cubic packing.* The face-centred cubic packing

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[1] du Sautoy, 'Exploring the mathematical library of Babel', p. 24.

[2] See Appel & Haken, 'The Four Color Problem', esp. pp. 178–9.

[3] King, *The Art of Mathematics*, pp. 92–3, emphasis in orig.

[4] MacKenzie, *Mechanizing Proof*, p. 140.

[5] Appel & Haken, 'The Four Color Proof Suffices', p. 12.

[6] Nahin, *Dr. Euler's Fabulous Formula*, p. 5.

* ▸ p. 828

[7] Davis, Senechal & Zwicky, *The Shape of Content*, p. xiii, emphasis added.

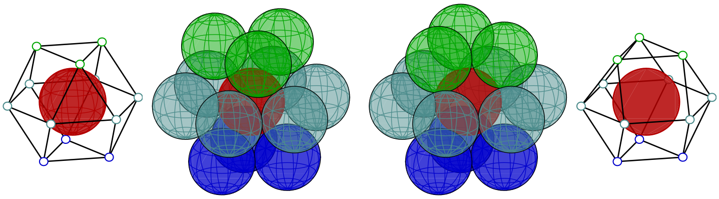

Face-centred cubic packing   Hexagonal close packing

FIGURE 16.3
The two optimal sphere packings. The two packings differ only in the relative positions of the top and bottom illustrated layers. In the face-centred cubic packing, the centres of the twelve neighbours of each sphere form a cuboctahedron.

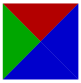
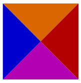
and the hexagonal close packing, shown in Figure 16.3, both comprise layers of spheres arranged hexagonally, but differ in the relative positioning of alternating layers. Both have a density of $\pi/\sqrt{18}$. Johannes Kepler* (1571–1630) made the claim, without giving a proof, in his 1611 pamphlet *The Six-Cornered Snowflake*.[1] Although this work seems to have remained obscure until its translation into German in 1907,[2] the conjecture itself formed part of David Hilbert's (1862–1943) eighteenth problem,[3] and there were numerous attempts to prove it.[4]

In 1998, Thomas Hales (1958–) announced a proof which involves solving a large number of nonlinear optimization problems by computer.[5] There seems to have been no aesthetic commentary on this proof, although it seems to entail both the consideration of many cases (like the proof of the four-colour theorem) and extensive — though automated — calculation.

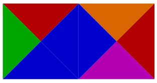
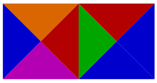
[Originally, *Annals of Mathematics* intended to publish Hales's paper with a caveat to the effect that the journal could not be certain of its correctness.[6] No caveat appears in the published version.]

Minimal aperiodic set of Wang tiles

Wang tiles are unit square tiles divided by the two diagonals into four quadrants, each of which is coloured. A valid tiling using a set of Wang tiles, which are not permitted to be rotated or reflected, must obey the rule that two tiles adjoin only along complete edges of same-coloured quadrants. For instance,

 and 

are Wang tiles. They can be located together in a valid tiling

like  but not like .[7]

There exist *aperiodic* sets of Wang tiles, which can tile the entire plane only aperiodically: that is, they do not form a tiling in which some $m \times n$ rectangle of tiles repeats horizontally and vertically. The first known aperiodic set of Wang tiles, found in 1966 by Robert Berger (1938–), contained 20 426 tiles; Berger later found a smaller aperiodic

* ▸ pp. 279–94

[1] Kepler, *The Six-Cornered Snowflake*, p. 15.

[2] Whyte, foreword to *ibid.*, p. v.

[3] Hilbert, 'Mathematical problems', § 18 [p. 467].

[4] Szpiro, *Kepler's Conjecture*, *passim.*

[5] Hales, 'A proof of the Kepler conjecture', esp. § 12; see also Hales, 'Cannonballs and Honeycombs'.

[6] Krantz, *The Proof is in the Pudding*, p. 169.

[7] Grünbaum & Shephard, *Tilings and Patterns*, p. 584.

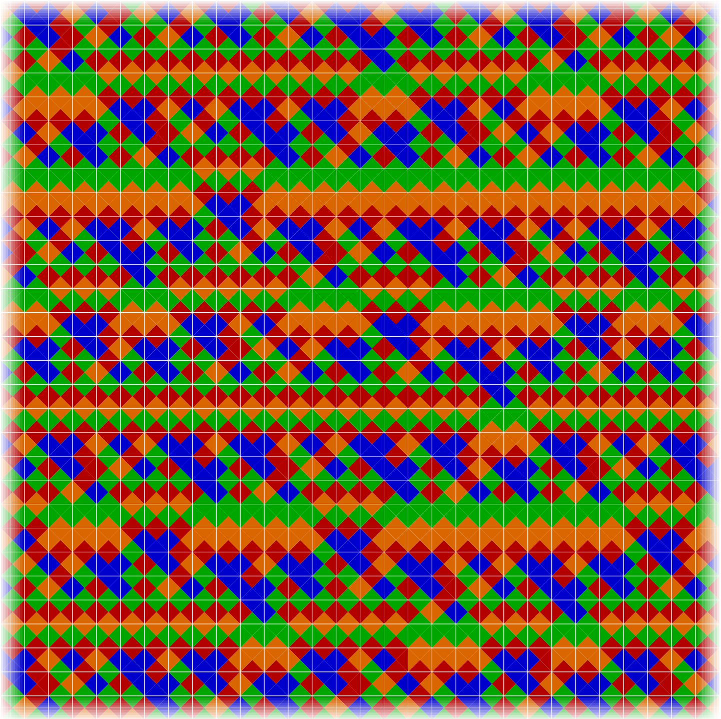


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FIGURE 16.4
Part of a tiling using an aperiodic set of 11 Wang tiles found by Jeandel & Rao using a computer search.

set of 104 Wang tiles.[1] The search for a minimal possible aperiodic set of Wang tiles ended in 2015, when Emmanuel Jeandel & Michaël Rao proved that there is an aperiodic set of 11 Wang tiles, and there is no smaller set.[2] Computer searching and checking showed that almost all sets of Wang tiles with 10 or fewer tiles were not aperiodic; one intractable case required a manual proof.[3] The same searching yielded candidates for aperiodic sets of 11 Wang tiles, and Jeandel & Rao were able to prove that three such sets were in fact aperiodic.[4] The proofs that these sets can tile the plane (see Figure 16.4), and only aperiodically, are long and technical.

By comparison, the idea behind the aperiodic set of 14 Wang tiles found by Jarkko Kari (1964–) allows a proof of aperiodicity called 'elegant' by Karel Culik (1934–), who modified Kari's idea to give an aperiodic set of 13 Wang tiles, at the cost of slightly more technical proofs.[5] Kari devised a set of tiles so that the colours along the (horizontal) boundary between two vertically-adjacent infinite rows of tiles corresponded to terms in a bi-infinite sequence

$$\big(\lfloor i\alpha\rfloor - \lfloor (i+1)\alpha\rfloor\big)_{i\in\mathbb{Z}}$$

for an irrational number α in the interval $I = [\frac{2}{3}, 2]$.[6] Such a sequence uniquely determines α, for the average of n consecutive terms of the sequence differs from α by less than $1/n$. Kari's tiles (see Figure 16.5) fall into two groups that multiply (going downwards) or divide (going

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[1] Grünbaum & Shephard, *Tilings and Patterns*, p. 584.

[2] Jeandel & Rao, 'An aperiodic set of 11 Wang tiles'.

[3] *Ibid.*, § 3.

[4] *Ibid.*, §§ 4–7.

[5] Culik, 'An aperiodic set of 13 Wang tiles'.

[6] Kari, 'A small aperiodic set of Wang tiles', esp. § 3.

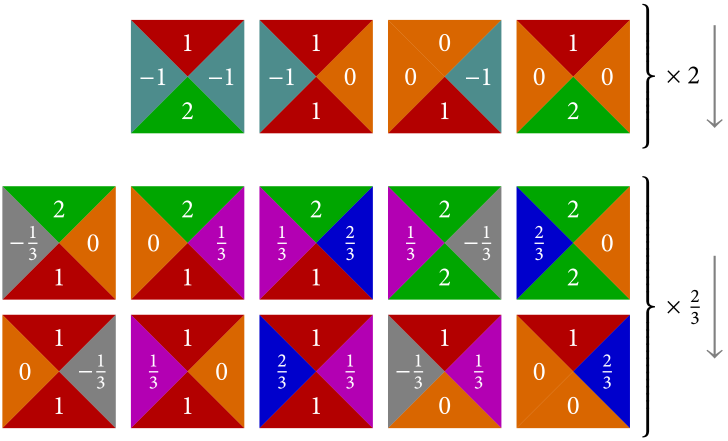

FIGURE 16.5
Kari's set of 14 aperiodic Wang tiles.

upwards) the number α by 2 and by $\frac{2}{3}$. A suitable choice of multiplication and division can always yield another number in the interval I, so, starting from an infinite horizontal row of tiles from one of the two groups, a tiling can be extended indefinitely vertically upwards and downwards. That no such tiling is periodic follows quickly from the fact that no non-empty product made up of multiplicands 2 and $\frac{2}{3}$ can equal 1.[1] As well as being elegant, this proof, which uses only elementary number theory, is explanatory, while the computer search is not.

Pentagonal tiling classification

Another tiling problem that was solved by computer checking of cases was the classification of the convex pentagonal tilings of the plane. By 2015, fifteen families of convex pentagons capable of tiling the plane were known.[2] Michaël Rao first imposed restrictions on the angles of a pentagon that can tile the plane, based on how vertices could meet. He then used a computer search to obtain 371 cases satisfying these restrictions.[3] A computer program was then used to check whether, in each case, a pentagon of that species could tile the plane.[4] Twenty-four cases passed the test: five were degenerate and another four were special cases of the previously-known fifteen families.[5]

Non-existence of a finite projective plane of order 10

A *finite projective plane* of order n is an abstract set of $n^2 + n + 1$ *points* and $n^2 + n + 1$ sets of points called *lines*, such that each line contains exactly $n + 1$ points and each point is contained in exactly $n + 1$ lines. (See Figure 16.6.)

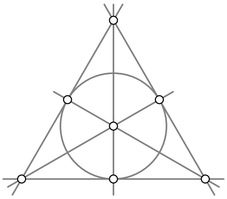

FIGURE 16.6
Diagram of the unique finite projective plane of order 2.

For some orders there exists a unique finite projective plane, for others many, and for others none. The non-existence of a finite projective plane of order 6 was proved in 1938 when Raj Chandra Bose

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[1] Kari, 'A small aperiodic set of Wang tiles', Prop. 1.

[2] Rao, 'Exhaustive search of convex pentagons which tile the plane', p. 1.

[3] *Ibid.*, § 3.

[4] *Ibid.*, § 4.

[5] *Ibid.*, pp. 14–15.

(1901–87) showed that there exists a finite projective plane of order $n \geqslant 3$ if and only if there exists a set of $n - 1$ mutually orthogonal $n \times n$ Latin squares.[1] [An $n \times n$ Latin square is an array whose entries are drawn from a set of n symbols, such that each symbol appears exactly once in each row and each column. Two Latin squares are mutually orthogonal if, when superimposed, each pair of symbols appears exactly once as a combined entry; see Figure 16.7.] Leonhard Euler (1707–83) conjectured that there did not exist a pair of mutually original $n \times n$ Latin squares when $n \equiv 2 \pmod{4}$. This conjecture is trivial to prove for $n = 2$, and Gaston Tarry (1843–1913), by dint of systematic manual search, showed in 1900 that there do not exist mutually orthogonal 6×6 Latin squares.[2] But Bose, S. S. Shrikhande (1917–2020) and E. T. Parker (1926–91) showed that Euler's conjecture fails for all $n = 6 + 4k > 6$: there exist (at least) two mutually orthogonal Latin squares for each such n.[3] Their work left open the question of the existence of a finite projective plane of order 10, which was equivalent to the existence of *nine* mutually orthogonal Latin squares of order 10, and this problem was seen as having beauty.[4] C. W. H. Lam, Larry Thiel, and Stanley Swiercz used a Cray supercomputer to check that there was no such finite projective plane.[5] They did not view their results as a proof, but only as experimental evidence, and discussed the possibilities of hardware or software errors.[6]

αa	βb	γc	δd
βc	αd	δa	γb
γd	δc	αb	βa
δb	γa	βd	αc

FIGURE 16.7
Two mutually orthogonal Latin squares of order 4.

◆

Note that the proof that there is no aperiodic set of fewer than 11 Wang tiles, the classification of convex pentagonal tilings, and the proof of the non-existence of a finite projective plane of order 10 are all examples of computerized 'proofs by searching cases'. And all of these, like the Appel–Haken proof, seemingly fail to offer an explanation.

Geometry

Shang-Ching Chou (1946–), Xiao-Shan Gao & Jing-Zhong Zhang (1936–) developed a method to allow a computer to prove theorems in elementary geometry. These authors wrote that 'many proofs produced by the program are very beautiful'.[7] The computer scientist Robert S. Boyer (1946–) agreed, writing in his foreword to their book the proofs had 'stunning beauty' and 'stunning brevity and beauty'.[8]

Their method used ratios of lengths along a given segment and signed areas of oriented triangles (and, by extension, quadrilaterals). They built up basic properties of these ratios, such as the result that, for collinear points A, B, C, and D and a point P not on this line,

$$\frac{AB}{CD} = \frac{S_{PAB}}{S_{PCD}},$$

---

[1] Bose, 'On the Application of the Properties of Galois Fields'.

[2] Tarry, 'Le problème (1)'; Tarry, 'Le problème (2)'.

[3] Bose, Shrikhande & Parker, 'Further Results'.

[4] Lam, 'The Search for a Finite Projective Plane of Order 10', p. 305.

[5] Lam, Thiel & Swiercz, 'The Non-Existence of Finite Projective Planes of Order 10'.

[6] *Ibid.*, § 4.

[7] Chou, Gao & Zhang, *Machine Proofs in Geometry*, p. 255.

[8] Boyer, foreword to *ibid.*, p. viii.

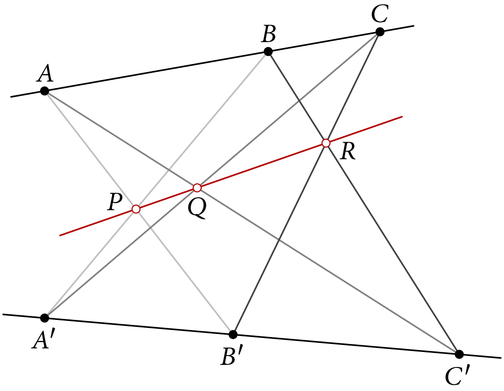

FIGURE 16.8
Pappus' theorem.

where in the division AB and CD are interpreted as the lengths of these segments and S_{PAB} and S_{PCD} are the signed areas of the triangles PAB and PCD.[1] (However, Chou et al. used an abstract axiomatic treatment that does not require these interpretations.[2])

Neither Chou et al. nor Boyer singled out specific proofs as beautiful. But as an example of the kind of proofs that their program found, consider the following proof of Pappus' theorem.[3] Only the main proof is given; the lemmata were proved by Chou et al. in setting up their method.

PAPPUS' THEOREM. *For collinear points A, B, C and collinear points A', B', C', the points of intersection of AB' and $A'B$, of AC' and $A'C$, and of BC' and $B'C$ are collinear.* (See Figure 16.8.)

Proof. Let S be the intersection of BC' and the line PQ. The aim is to prove that $BR/C'R = BS/C'S$:

$$\begin{aligned}
&\left(\frac{BR}{C'R}\right)\Big/\left(\frac{BS}{C'S}\right)\\
&= \frac{S_{C'PQ}}{S_{BPQ}}\cdot\frac{BR}{C'R} && \text{[Lemma]}\\
&= \frac{(-S_{B'BC})\cdot S_{C'PQ}}{S_{BPQ}\cdot(-S_{B'C'C})} && \text{[Lemma]}\\
&= \frac{S_{B'BC}\cdot S_{AC'P}\cdot S_{A'C'C}\cdot S_{A'ACC'}}{(-S_{BCP}\cdot S_{A'AC'})\cdot S_{B'C'C}\cdot(-S_{A'ACC'})} && \text{[Lemma]}\\
&= \frac{S_{B'BC}\cdot S_{AC'P}\cdot S_{A'C'C}}{S_{BCP}\cdot S_{A'AC'}\cdot S_{B'C'C}} && \text{[Simplification]}\\
&= \frac{S_{B'BC}\cdot S_{AB'C'}\cdot S_{A'AB}\cdot S_{A'C'C}\cdot S_{A'ABB'}}{(-S_{AB'B}\cdot S_{A'BC})\cdot S_{A'AC'}\cdot S_{B'C'C}\cdot(-S_{A'ABB'})} && \text{[Lemma]}\\
&= \frac{S_{B'BC}\cdot S_{AB'C'}\cdot S_{A'AB}\cdot S_{A'C'C}}{S_{AB'B}\cdot S_{A'BC}\cdot S_{A'AC'}\cdot S_{B'C'C}} && \text{[Simplification]}
\end{aligned}$$

[1] Chou, Gao & Zhang, *Machine Proofs in Geometry*, Prop. 2.7.

[2] *Ibid.*, § 2.2.1.

[3] *Ibid.*, Ex 2.42 [p. 80].

$$
\begin{aligned}
&= \frac{\begin{array}{l} BC \cdot OB' \cdot \beta \cdot (-B'C' \cdot OA \cdot \beta) \\ \quad \cdot AB \cdot OA' \cdot \beta \cdot (-A'C' \cdot OC \cdot \beta)((2))^4 \end{array}}{\begin{array}{l} (-AB \cdot OB' \cdot \beta) \cdot BC \cdot OA' \cdot \beta \\ \quad \cdot A'C' \cdot OA' \cdot \beta \cdot (-B'C' \cdot OC \cdot \beta)((2))^4 \end{array}} && \text{[Lemma]} \\
&= 1. && \text{[Simplification]}
\end{aligned}
$$

Hence $BR/C'R = BS/C'S$ and therefore R and S coincide and so P, Q, and R are collinear. ☐

Is this proof beautiful? For the present author, it is not, though it does arguably possess all three properties in the Hardian triad:[*] the approach is perhaps unexpected, the single calculation gives it a certain inevitability, and the few lemmata used are so straightforward that it has at least some economy. But the proof is effectively just a calculation with an overabundance of symbols.

[*] ▸ pp. 502

Nevertheless, Chou et al. and Boyer at least thought that similar proofs were beautiful, which seems to be one of the few instances where proofs found by computer are so judged.

Robbins problem

The Robbins problem, named for the mathematician Herbert Robbins (1915–2001) asked whether any algebra satisfying the three equations

$$
\left.
\begin{aligned}
x \vee y &= y \vee x \\
(x \vee y) \vee z &= x \vee (y \vee z) \\
\neg(\neg(x \vee y) \vee \neg(x \vee \neg y)) &= x \qquad \text{[Robbins equation]}
\end{aligned}
\right\} \quad \text{(R)}
$$

is a Boolean algebra. It is easy to see that Boolean algebras satisfy the equations (R). Steven Winker proved — in the traditional way — that any algebra satisfying (R) and

$$(\exists c)(\exists d)(c \vee d = c)$$

is a Boolean algebra.[1] In 1996 William McCune (1953–2011) used the automated theorem prover EQP (Equational Prover) to show that the equations (R) imply the existence of elements c and d such that $c \vee d = c$; hence, by Winker's result, any algebra satisfying (R) is indeed a Boolean algebra.[2] The proof is a short but intricate sequence of deductions using paramodulation starting from the Robbins equation and ending with an equation of the form $c \vee d = c$.[3]

The logician and computer scientist Louis Kauffman (1945–) analysed the EQP proof of the Robbins conjecture and translated it into a box notation that is visually easier to comprehend than the nested parentheses, sometimes six levels deep, of the original proof.[4] Of the proof, he wrote

> 'EQP's proof is much more than a calculation. The proof depends upon a successful search among a realm of possibilities

[1] McCune, 'Solution of the Robbins Problem', Lem. 1.

[2] *Ibid.*, Thm on p. 265.

[3] *Ibid.*, Proof of Lem. 3.

[4] Kauffman, 'The Robbins problem', esp. § 5.

> and the skilful application of pattern recognition and the application of axioms. This is very close to the work of a human mathematician. EQP, being able to handle many possibilities and great depths of parentheses has an advantage over her human colleagues. I understood EQP's proof with an enjoyment that was very much the same as the enjoyment that I get from a proof produced by a human being. Understanding a proof is like listening to a piece of music — or like improvising a piece of music from an incomplete score.'[1]

Although no explicit aesthetic terminology is used here, there is an implicit aesthetic aspect to the pleasure Kauffman wrote of, given the immediately following comparison to music.

Boolean pythagorean triples problem

A pythagorean triple comprises three natural numbers x, y, and z such that $x^2+y^2 = z^2$. The *boolean pythagorean triples problem* asked whether the set of natural numbers can be partitioned into two sets, neither containing a pythagorean triple. This question is one of a natural class of problems in Ramsey theory, concerning whether natural numbers can be partitioned into a certain number of sets so that none of them contains a solution to a given equation or set of equations.

The question was precisely answered in 2015 by Marijn Heule (1979–), Oliver Kullman, and Victor Marek (1943–), who used a constraint satisfaction solver to find an explicit partition of $\{1, 2, \dots, 7824\}$ into two sets, neither of which contains a solution to $x^2 + y^2 = z^2$ (see Figure 16.9), and to show that it is impossible to so partition $\{1, 2, \dots, 7825\}$.[2]

An important difference between this demonstration of impossibility and that of, say, the non-existence of a finite projective plane of order 10 is that the solver generated a formal proof of the result.[3] This proof can be checked using an automated theorem prover, but, at over 200 TB in size, it is beyond any human capacity to read or survey.

Nevertheless, the philosopher Jeremy Avigad (1968–) saw aesthetic value in the *theorem*: he pointed out that the proof was not simply a mechanical check of a result that everyone already thought true, but had established 'a new theorem of mathematics, a natural Ramsey-like result, and a very pretty one at that'.[4]

CRITIQUE

John Ewing (1944–) wrote in 1984 that computer-aided experiments were worthwhile, but he argued that computations

[1] Kauffman, 'The Robbins problem', p. 729.

[2] Heule, Kullman & Marek, 'Solving and Verifying', Thm 1.

[3] *Ibid.*, p. 239.

[4] Avigad, 'The Mechanization of Mathematics', p. 687.

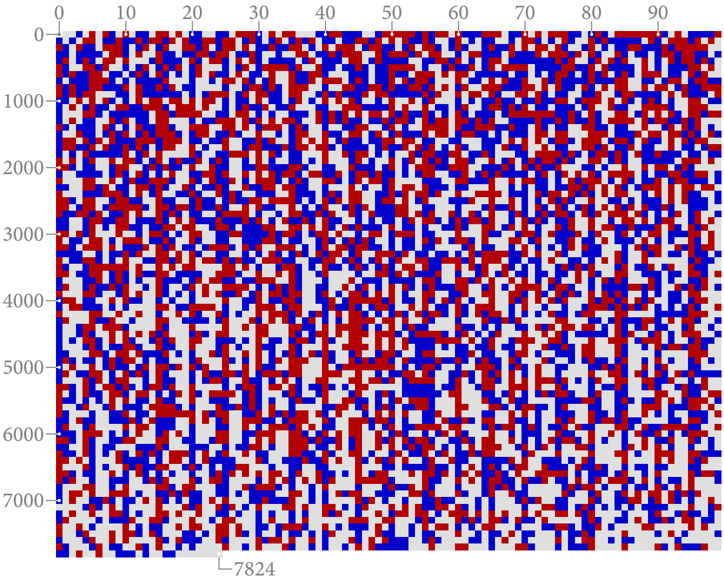

FIGURE 16.9
Partitioning the set $\{1,\ldots,7824\}$ into two sets, neither of which contains any pythagorean triples. Numbers coloured red are placed in one set, numbers coloured blue in the other; the remaining numbers can be placed in either set.

are such bad art that anyone who delivered a lecture on them — which would be boring — was guilty of bad manners.[1] They should have better taste:

> 'For generations mathematicians have used taste to select those results that are worthy of public scrutiny. [...] Without taste, mathematics ceases to be Art. Like scientists, we should continue to perform experiments, by any means available. But like artists, sculptors, and composers, we must exercise judgement about what should be placed on public display.'[2]

◆

Daniel I. A. Cohen (1946–) compared learning what a proof is to learning an artistic subject:

> 'I push you onto the ice — you'll fall a few times, but eventually you can skate. [...] We're learning to skate — sensitivity, taste, aesthetics, we're learning to paint, we're not learning to type, we're not learning how to do long division. For long division, I can give you the set of rules. But anything you can learn by a complete set of rules isn't worth knowing, leave it to androids.'[3]

Cohen here set out a strict criterion for what mathematics was worth pursuing: it could not be purely mechanical, and it had to involve creativity, which in turn required developing aesthetic taste.

[1] Ewing, 'A Breach of Etiquette', p. 4.

[2] *Loc. cit.*

[3] Cohen, quot. in MacKenzie, *Mechanizing Proof*, p. 318.

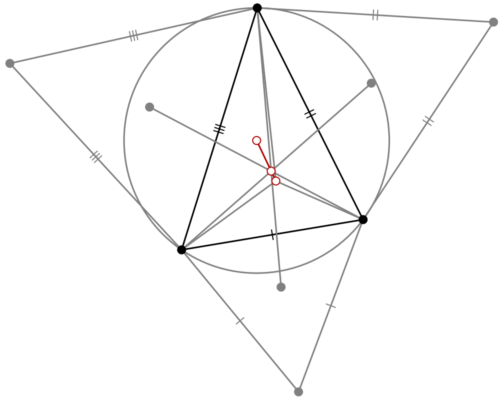

FIGURE 16.10
For any triangle, the outer Napoleon point (the meeting-point of the lines that join a vertex of the triangle to the centre of the equilateral triangle erected on the outside of the opposite side), the Fermat point (where the sum of the distances to the vertices is minimum), and the circumcentre are collinear.

CONJECTURES FOUND BY COMPUTER

In a 1992 essay on the process of discovery in geometry, the cognitive scientist Douglas Hofstadter (1945–) commented on an investigation made by Clark Kimberling (1942–) on the relationships that existed between 91 special points of a triangle, including the incentre X_1, the barycentre X_2, the circumcentre X_3, the orthocentre X_4, the centre of the nine-point circle X_5, ….[1] Kimberling's investigation used computer programs to examine what relations held between these points in a collection of 10 740 systematically generated triangles and thus to suggest conjectures for general results. Particular conjectures were then also proved by computer. Among the results conjectured and proved was the theorem asserting that *The circumcentre, the outer Napoleon point, and the Fermat point of a triangle are collinear* (see Figure 16.10).[2] Here, the *outer Napoleon point* is the meeting-point of the three lines that join a vertex of the triangle to the centre of the equilateral triangle erected on the outside of the opposite side. The *Fermat point* is where the sum of the distances to the three vertices of the triangle is minimum.

Hofstadter called this result 'indisputably elegant',[3] but thought that Kimberling would have had to sift through many uninteresting results to find one that was really beautiful. The computer did not propose particular conjectures as beautiful; Kimberling made the choice.[4] Thus, for Hofstadter, Kimberling's aim was not to 'automate the doing of mathematics', but to 'automate the discovery of fresh new theorems of geometry'.[5] The difference was aesthetic: a computer program can only churn out conjectures by brute force; it cannot make a selection by ranking conjectures in terms of importance or elegance or unexpectedness.[6] And discovery without the aesthetic aim was, quite simply, not 'the doing of mathematics',[7] for Hofstadter characterized mathematics as 'the study of the beauty of the interrelationships of patterns',[8] and it was the difficulty of mechanizing aesthetic judgement that lay behind the difficulty of automating mathematical discovery.[9]

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[1] Kimberling, 'Point-to-line Distances in the Plane of a Triangle'.

[2] Hofstadter, 'From Euler to Ulam', p. 17.

[3] *Ibid.*, p. 28.

[4] *Ibid.*, p. 29.

[5] *Loc. cit.*

[6] *Ibid.*, p. 30.

[7] *Ibid.*, p. 29.

[8] *Ibid.*, p. 32.

[9] *Loc. cit.*

HUMAN/MACHINE SEPARATION

> ‘ to some, collaboration between mathematician and machine is not ugly but beautiful. ’
>
> — Reuben Hersh, ‘Proving is convincing and explaining’, p. 394.

The computer scientist Steven Seiden argued in a 2001 article that it was *possible* for a computer proof to be elegant. A computer proof could allow a very clean separation of routine verifications from the truly important parts of the proof. It could also lead to a very neat enumeration of cases using `for` loops or recursive functions. In a computer proof, ‘the inelegant part of a proof’ was handed over to the machine.[1] Further, if these calculations or cases were treated using an elegant algorithm, the proof as a whole could be considered elegant.[2]

That is, excising the *actual* calculations or *actual* enumeration of cases from the part of the proof that a human mathematician would read, in favour of an elegant algorithm describing *how* they are performed, could yield an elegant proof. Presumably this separation was a necessary but not a sufficient condition: there could still be inelegance in the remaining part of the proof or in how the manual and computer components meshed together.

Seiden’s idea is an intriguing one, but seems never to have been examined further. But if such proofs, separated into human and machine components, were to be accepted as elegant, it would seem to necessitate a re-evaluation of the connection between the aesthetic value of a proof and it supplying an explanation or granting enlightenment, or perhaps a new conception of explanation. Such a conception would have to accept as explanatory proofs with lacunae that are known to be fillable, but which in actuality remain unfilled.

[1] Seiden, ‘Can a computer proof be elegant?’, p. 112.

[2] *Ibid.*, p. 113.

17

The Chaos of Appearances

Fractals, dynamical systems, and beauty, visual and mathematical

> ‘Beauty had been born,
> not, as we so often conceive it nowadays, as an ideal of humanity,
> but as *measure*, as the reduction of the chaos of appearances
> to the precision of linear symbols.
> Symmetry, balance, harmonic division,
> mated and mensurated intervals
> — such were its abstract characteristics.’
>
> — Herbert Read, *Icon and Idea*, p. 75.

> ‘Number helps more than anything else
> to bring order into the chaos of appearances.’
>
> — C. G. Jung, ‘Synchronicity’, § 870
> (trans. R. F. C. Hull).

That branch of mathematics that involves the concept of *chaos*, and its visual artefacts, have attracted aesthetic approbation[1] and disdain, and have produced many questions and few answers on the relationship between visual and mathematical beauty.

The first glimmering of the behaviour that would later come to be called *chaotic* appeared in an 1879 article by Arthur Cayley* (1821–96) on applying Newton's method to calculate approximations to the solutions of a suitable function $g(z)$ with complex coefficients. Let $f(z) = z - g(z)/g'(z)$. Take some complex number z_0 and compute the sequence $(z_i)_{i\geqslant 0}$ via $z_{i+1} = f(z_i)$. If this sequence converges, its limit is one of the solutions of $g(z) = 0$; which solution depends on the choice of the initial term z_0. Cayley pointed out that when $g(z)$ is quadratic, there is an ‘easy and elegant’ way to determine which regions of the complex plane contain points converging to each solution, but that there seemed to be ‘considerable difficulty’ in the next case of cubic equations.[2] Only the application of computers to the problem in the late twentieth century could illustrate the true complexity that lay behind Cayley's impression of ‘considerable difficulty’.

Henri Poincaré's (1854–1912) work on the three-body problem — that is, the question of how a system of three planets or other bodies would behave under the laws of Newtonian physics — further pointed to chaotic behaviour. His study revealed infinitely many doubly asymp-

[1] Kemp, ‘Attractive attractors’.

* ▸ pp. 446–54

[2] Cayley, ‘The Newton-Fourier imaginary problem’, p. 406.

totic solutions;[1] in particular, there are solutions that do not asymptotically converge to a periodic solution. He drew no especial attention to this result, but admitted in correspondence that it caused him distress.[2] What Poincaré saw, in later terms, was the sensitivity to initial conditions which is one of the characteristics of chaotic systems.

The first steps in the modern study of chaos, only recognized in hindsight, were galvanized by a call made by the United Kingdom's Department of Scientific and Industrial Research in January 1938 for the expert assistance of pure mathematicians in understanding non-linear differential equations that were used in the field of radio engineering,[3] and which would have vital implications for technologies important in wartime, such as radar.[4] The problem attracted the attention of Mary L. Cartwright (1900–98) and J. E. Littlewood (1885–1977), who discovered the extremely complex structure of the periodic solutions of certain of these equations. Only in retrospect was their work seen as presaging chaos. [The physicist Freeman Dyson (1923–2020) later wrote that, when he heard Cartwright lecture on their work in 1942, he had been 'delighted with the beauty of her results',[5] but did not recognize their mathematical importance beyond the engineering applications.*]

STRANGE ATTRACTORS

The emergence proper of chaos began with the work of Edward Lorenz (1917–2008) on the numerical integration of a simple mathematical model comprising the ordinary differential equations[6]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z. \qquad \text{(L)}$$

This model — the *Lorenz system* — was originally devised for convection,[7] although the equations (L) have been found to be accurate models in other situations. In the context of convection, t is time and x, y, and z are physical properties of the convective system; the constants σ, ρ, and β are positive real parameters. When these parameters lie within certain ranges, the system exhibits chaotic behaviour. In his initial investigation, Lorenz used a computer to calculate an approximation of how the values x, y, and z would evolve from chosen initial values. Lorenz noticed that when he re-started a computation using values output part-way through a previous run, the values calculated during the new run diverged from those in the previous run, becoming wildly different. The small difference between the rounded output values he had re-started from and the actual values the program had been using at that point led to ever-larger differences in the trajectory.

The evolution of the Lorenz system (L) can be visualized as a path in three-dimensional space (see Figure 17.1).

[1] Poincaré, *The Three-Body Problem and the Equations of Dynamics*, p. 202.

[2] Barrow-Green, 'Poincaré and the Three Body Problem', pp. 153–4.

[3] Hayman, 'Cartwright', p. 30.

[4] Dyson, 'Cartwright', pp. 169–70.

[5] Dyson, Review of *Nature's Numbers*, p. 612; Dyson, 'Cartwright', p. 174.

* ▸ pp. 550–1

[6] Lorenz, 'Deterministic Nonperiodic Flow', Eqs (25)–(27).

[7] *Ibid.*, § 5.

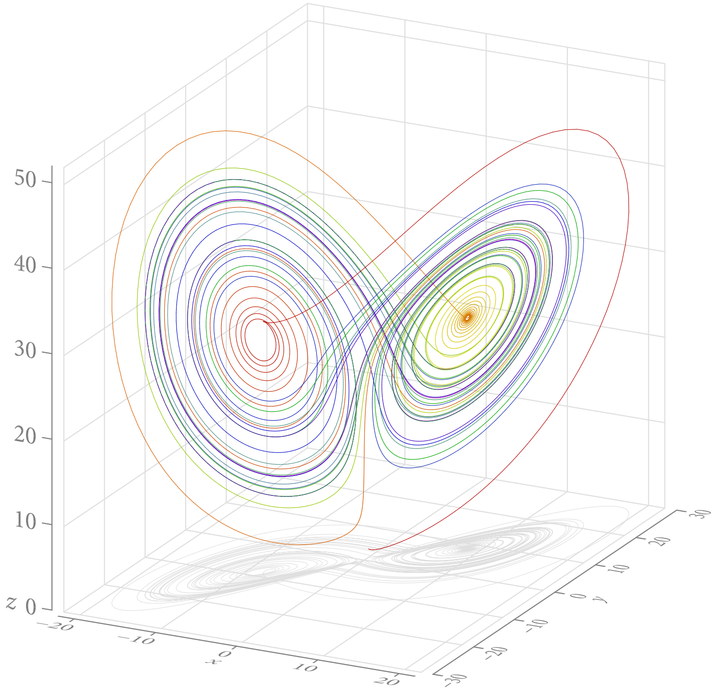

FIGURE 17.1
Numerical solution to the Lorenz system with parameters $\sigma = 10$, $\rho = 28$, $\beta = 8/3$, starting value $(0, 1, 1)$, and time step $\mathrm{d}t = 0.005$, together with the projection of the solution to the xy-plane.

The behaviour of the Lorenz system marks it as a *strange attractor*. Informally, there is a particular subset A of the space of points and a neighbourhood U of A such that the trajectory from any starting point in U remains in U; the trajectories are *attracted* to A. Further, the trajectories from starting points in U are sensitive to the initial conditions; the attractor is *strange*. Lastly, the trajectory from any starting point in A will eventually pass arbitrarily close to any other point in A; the attractor thus cannot be decomposed into two separate attractors.[1] The term 'strange attractor' seems to have first been used in print by David Ruelle (1935–) & Floris Takens (1940–2010), though whether either actually coined it is not clear.[2] However, Ruelle thought the name beautiful and appropriate.[3]

Computer-calculated and -plotted numerical solutions of strange attractors aid in understanding them.[4] Ruelle himself thought beautiful the resulting diagrams.[5] His remarks on the 'esthetic appeal of strange attractors' seem to suggest that it was connected both to the visual diagram and to the underlying mathematical structure: 'These systems of curves, these clouds of points suggest sometimes fireworks or galaxies, sometimes strange and disquieting vegetal proliferations.'[6] The cognitive scientist Douglas Hofstadter (1945–) also thought that strange attractors could be beautiful; the context suggests that he found beautiful both the mathematical structure and the diagrams.[7] The physicist Heinz R. Pagels (1939–88), writing about the role of the

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[1] Ruelle, 'Strange Attractors', p. 131.

[2] *Loc. cit.*

[3] *Loc. cit.*

[4] *Ibid.*, p. 126.

[5] *Ibid.*, p. 133, Fig. 3.

[6] *Ibid.*, p. 137.

[7] Hofstadter, *Metamagical Themas*, p. 386.

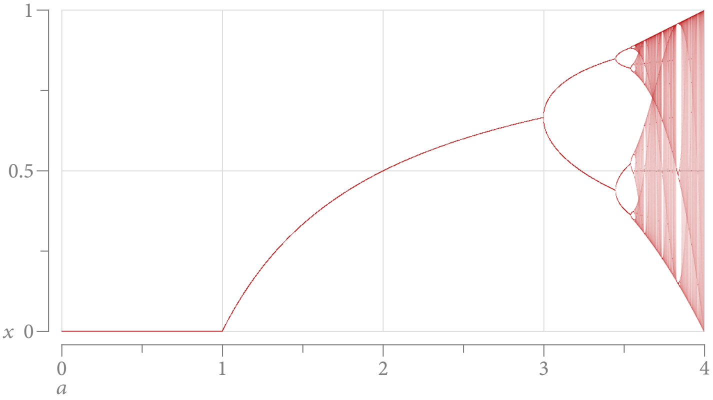

FIGURE 17.2
Bifurcation diagram for the logistic equation.

computer in chaos theory, called them 'rather beautiful geometrical objects'.[1]

DYNAMICAL SYSTEMS

The term *chaos* itself first appeared in the 1975 article 'Period Three Implies Chaos' by Tien-Yien Li (1945–2020) & James A. Yorke (1941–), which showed that if a dynamical system generated by a continuous map on an interval J had a point with period 3, then there were points with period k for every $k \in \mathbb{N}$, and, furthermore, that there was an uncountable subset S of J such that (loosely) every pair of distinct points in S would become arbitrarily close at some iteration of the system, but would never remain arbitrarily close, and that no point in S would remain arbitrarily close to a periodic point.[2]

One of the simplest dynamical systems in which chaotic behaviour arises is the logistic map

$$f(x) = ax(1 - x),$$

which gained attention after a 1976 article by Robert M. May (1936–2020).[3] Iteration of this map models a population x_t scaled to the range $[0, 1]$. The constant $a \in [0, 4]$ is a parameter on which the behaviour of the model depends. For $a < 1$, the population tends to 0. For $a \in [1, 3]$, the population is attracted to $1 - 1/a$. As a is increased beyond 3, the population is attracted to an oscillation between two values, then to an orbit of period 4, then period 8. For $a > 3.569\,945\,671\ldots$,[4] there is chaotic behaviour, although for certain values of a stable orbits emerge again. It is possible to plot the behaviour for all a in a *bifurcation diagram*, which shows the doubling of orbits (see Figures 17.2 and 17.3).

In fact, the initial bifurcations — or doubling of periods — obey the rule that the ratio between the values of a where successive bifurcations occur tends to the limit $4.669\,201\,609\ldots$.[5] This fact is a

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[1] Pagels, *The Dreams of Reason*, p. 79.

[2] Li & Yorke, 'Period Three Implies Chaos', Thm 1.

[3] May, 'Simple mathematical models with very complicated dynamics', See.

[4] OEIS, A098587.

[5] OEIS, A006890.

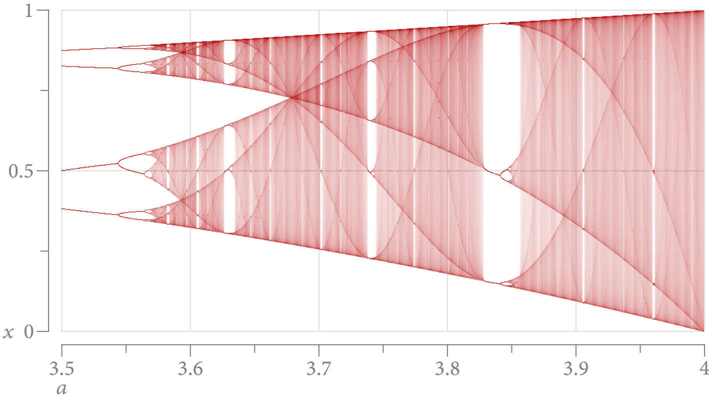

FIGURE 17.3
Bifurcation diagram for the logistic equation, for the restricted range $a \in [3.5, 4]$.

particular instance of a result of Mitchell Feigenbaum (1944–2019), which states that if the n-th doubling of the period occurs at $a = a_n$, then

$$\lim_{n\to\infty} \frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}} = 4.669\,201\,609\ldots;$$

this limit is the same for a broad family of maps.[1]

In a text originally published in November 1981, Hofstadter said that the bifurcation diagram showed 'unexpectedly beautiful fine structure'[2] and was 'elegant'.[3] In 1989, Ian Stewart (1945–) said that this bifurcation diagram has 'a beautiful tree structure'.[4] In 1991 David Ruelle wrote that the Feigenbaum period-doubling cascade stood out for 'particular beauty and significance'.[5]

FRACTALS

Benoit Mandelbrot (1924–2010) coined the term *fractal* in 1975 to denote the geometry of chaotic shapes found both in nature and in mathematics.[6]

Some fractals had been known in mathematics long before the term itself. The Koch curve (see Figure 19.2 on page 673) was introduced in 1904 by Helge von Koch (1870–1924) as a curve that had no tangent at any point and whose construction was elementary.[7] François le Lionnais (1901–84) thought it had — in his terms — classical and romantic beauty.* After the fractal concept was introduced, the Koch curve became a standard example. Richard F. Voss (1948–), who investigated the appearances of fractals in nature, thought it beautiful.[8]

Take a suitable complex function $f\colon \mathbb{C} \to \mathbb{C}$. A point $z \in \mathbb{C}$ is a *stable point* for f if it has a neighbourhood for which there is an infinite sequence of iterates of f that converges uniformly on every compact subset of that neighbourhood. The *Julia set* of f is the complement

* ▸ pp. 672–3

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[1] Alligood, Sauer & Yorke, *Chaos*, pp. 501–2.

[2] Hofstadter, *Metamagical Themas*, p. 374.

[3] *Ibid.*, p. 375.

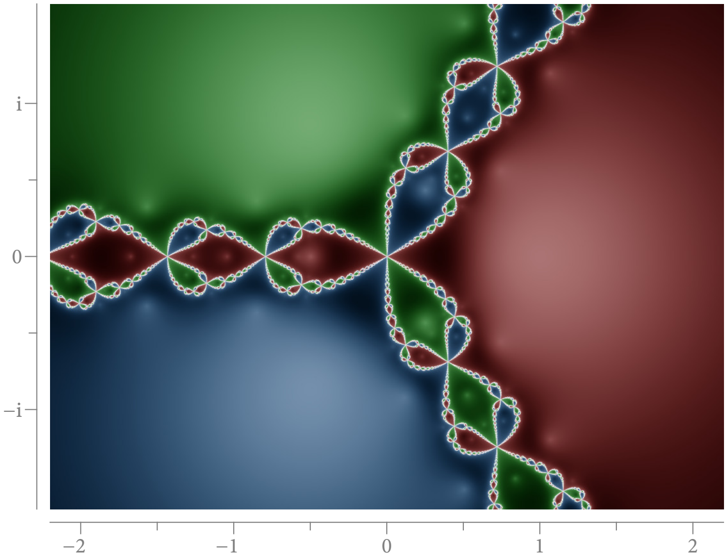
[4] Stewart, *Does God Play Dice?*, p. 173.

[5] Ruelle, *Chance and Chaos*, p. 67.

[6] Mandelbrot, *The Fractalist*, ch. 26.

[7] Koch, 'Sur une courbe continue sans tangente'.

[8] Voss, 'Fractals in nature', p. 26.



Georg-Johann Lay. [Public domain]

FIGURE 17.5
The Newton fractal of $g(z) = z^3 - 1$. Points that converge to the solution 1 are shaded red; to the solution $e^{2i\pi/3}$, green; to the solution $e^{4i\pi/3}$, blue. The Julia set of the function $f(z) = z - (z^3 - 1)/(3z^2)$, used in applying Newton's method, is white.

of the set of stable points. Loosely, the Julia set comprises the points whose iterates under f neither converge nor go to infinity.

When Gaston Julia (1893–1978) and Pierre Fatou (1878–1929) first investigated these sets, they must have drawn diagrams as they attempted to understand them. Some of Julia's manuscripts contain very clear diagrams that did not appear in the published versions.[1] They understood the complexity of the sets, and Julia pointed out that in general an infinity of operations would be required to construct them.[2]

The first published visualization of a Julia set was by Hubert Cremer (1897–1983) in 1925, and was approximately as shown in Figure 17.4.[3]

Approximations such as this, and even more detailed approximations to fractals such as the Koch curve, can be drawn by hand. But true renderings of fractals arising from iteration of functions, such as Julia sets, would have required too many person-years of work.[4] Cayley's problem of investigating the convergence to solutions of a cubic equation using Newton's method aptly illustrates this: the sets that converge to each solution are nested within each other in a unlimited, self-similar, way (see Figure 17.5).

The canonical example of a fractal arising from iteration of a function is the *Mandelbrot set*, which comprises the points $c \in \mathbb{C}$ such that the iterates of 0 under the function $f_c(z) = z^2 + c$ do not go to infinity, which are precisely the values c for which the Julia set of this function f_c is connected. (See Figure 17.6.)

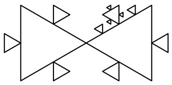


FIGURE 17.4
Visualization of a Julia set similar to Cremer's.

[1] Audin, *Fatou, Julia, Montel*, pp. 98–9.

[2] Julia, 'Mémoire sur l'itération des fonctions rationnelles', p. 50.

[3] Cremer, 'Über die Iteration', p. 208.

[4] Mandelbrot, 'Fractals and an Art for the Sake of Science', p. 21.

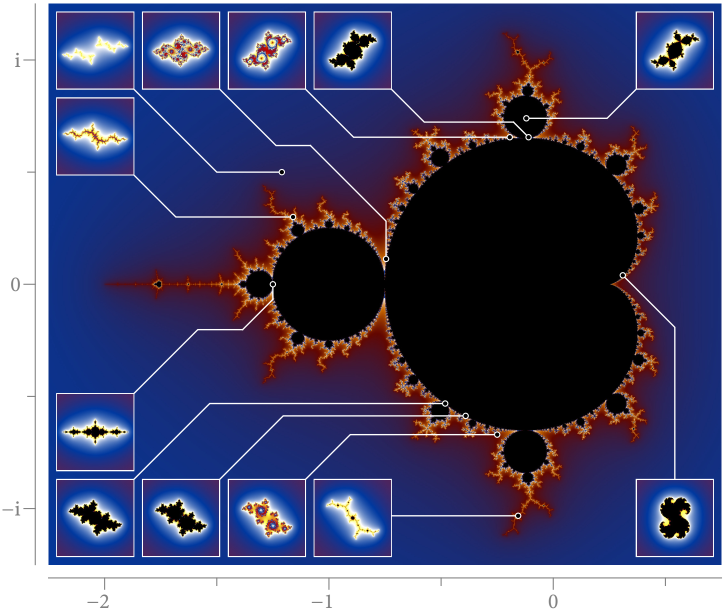


FIGURE 17.6
The Mandelbrot set, namely the set of points $c \in \mathbb{C}$ such that the iterates of 0 under $f_c(z) = z^2 + c$ do not go to infinity, and, for twelve values of c, the Julia sets of the function f_c, comprising the points whose iterates under f_c neither converge nor go to infinity. The Julia set of f_c is connected if and only if c lies in the Mandelbrot set.

VISUAL AND MATHEMATICAL BEAUTY

> ‘ I look at these things because they’re beautiful in their own right. ’
>
> — John Milnor,
> quot. in Regis, *Who Got Einstein’s Office?*, pp. 75, 92.

There is little doubt that laypeople and mathematicians alike see beauty in fractals. But it is difficult to sift statements about the visual beauty of renderings of fractals and those about the intellectual beauty of the mathematical objects and theorems about them. Matters are further complicated by the division between exact fractals and statistical fractals. An exact fractal is a mathematical object, an approximate image of which can be generated by a deterministic algorithm which mirrors the construction of the mathematical object. A statistical fractal is generated by a process that involves a degree of randomness or non-determinism. This process may occur in the physical world, such as via the geological shifts and the erosion by wind and sea that shape a coastline or landscape, or it may occur in a computer, such as via a piece of software that generates an image of a coastline or landscape.

In short, exact fractals are mathematical objects; statistical fractals are, at most, mathematical processes. There seems to be a potentially close connection between the aesthetic properties of an image of an exact fractal and the aesthetic properties of the fractal itself (*qua* mathematical object) or of results about it. The possibility of a connection between the aesthetic properties of the statistical fractal and those (if any) of the generative process seem at best tenuous.

Visual and mathematical beauty

For instance, Mandelbrot recalled that the beauty of the very first fractal images, produced in the 1960s,[1] was an unexpected bonus, but that 'by 1981 their beauty had matured and it deserved respect'.[2] But Mandelbrot's statement concerned *statistical* fractals, for he only turned to fractals in pure mathematics during 1978–80.[3] This new direction led him to the set now known as the Mandelbrot set, which he said later had 'infinite beauty'.[4] This statement was *prima facie* about the mathematical object, but he shortly afterwards said that 'zooming in on a point on this set's boundary fascinates all eyes, young and old',[5] which points rather to the statement being about the visible beauty of the computer renderings.

When A. K. Dewdney (1941–2024) wrote about the beauty of the Mandelbrot set, he seemed to refer primarily, although perhaps not solely, to the visual beauty of its graphical depictions. In 1985, in discussing how a computer program can be used to render and zoom in on these images, he referred to the 'infinite regress of detail that astonishes us with its variety, its complexity and its strange beauty' and how the zooming process yields ever-new views, 'infinitely various and frighteningly lovely'.[6] In 1987, he said that the Mandelbrot set was

> 'the newest and brightest star in the firmament of popular mathematics. It is both beautiful and profound. Indeed, the beauty of the Mandelbrot set is only a veil over its significance: the casual observer sees a miniature riot of filaments and curlicues near the set's boundary without suspecting that such patterns encode the various forms of chaos and order'.[7]

The phrase 'only a veil' suggests that Dewdney saw a division between the visual beauty, which was accessible to all, and the mathematical properties of the set itself. His statement that 'Julia sets have an intrinsic fractal beauty of their own'[8] could have referred either to visual or to intellectual beauty.

In 1986, in their book *The Beauty of Fractals*, Heinz-Otto Peitgen (1945–) & Peter Richter (1945–2015) suggested that '[p]erhaps the most convincing argument in favor of the study of fractals is their sheer beauty'.[9] This statement seems specifically to have referred to the visual beauty of the images, for it opened their description of how the idea of a public exhibition of images of dynamical systems, including fractals, arose from their research. The initial inspiration led to a series of exhibitions entitled *Harmonie in Chaos und Kosmos* (January 1984), *Morphologie komplexer Grenzen*, and *Schönheit im Chaos/Frontiers of Chaos*.[10] (Note that the German title of the last exhibition translates as '*Beauty in Chaos*'.) Peitgen & Richter thought that the 'aesthetic appeal of the pictures themselves would be sufficient' for the public, but the audiences wanted to understand the mathematical context.[11]

[The artist Friedrich Meckseper (1936–2019) wondered why the pictures were necessary if one had the formula from which they were generated.[12] An art critic for *Die Zeit* thought the pictures lacked an element of choice and so could not be art.[13]]

[1] Mandelbrot, *The Fractalist*, ch. 21.

[2] Mandelbrot, 'People and events behind the "Science of Fractal Images"', p. 7.

[3] Mandelbrot, *The Fractalist*, ch. 21, 25.

[4] *Ibid.*, ch. 25.

[5] *Loc. cit.*

[6] Dewdney, 'A computer microscope', p. 16.

[7] Dewdney, 'Beauty and profundity', p. 140.

[8] *Loc. cit.*

[9] Peitgen & Richter, *The Beauty of Fractals*, p. VI.

[10] *Ibid.*, pp. VI sqq.

[11] *Ibid.*, p. 20.

[12] *Loc. cit.*

[13] *Loc. cit.*

Visual and mathematical beauty

In the same volume, Mandelbrot seems to have located the beauty of fractals in the visual image, not in the underlying mathematics:

> ‘The beauty of many fractals is the more extraordinary for its having been wholly unexpected: they were meant to be mathematical diagrams drawn to make a scholarly point, and one might have expected them to be dull and dry.’[1]

He also thought Julia sets were ‘extraordinarily beautiful’.[2]

In another chapter, Adrien Douady (1935–2006) wrote that ‘Julia sets are among the most beautiful fractals.’[3] But, unlike Mandelbrot, he then supported this assertion by appeal to the self-similarity of most Julia sets: essentially, a view of them does not depend on the location or the level of zoom; the boundary of the Mandelbrot set, by contrast, varies depending on both.[4] Thus for Douady, the beauty seems have been located in, or at least to have been directly dependent on, the mathematics.

Several of the contributions to the 1988 book *The Science of Fractal Images* mention both the visual beauty and the intellectual beauty of fractals. In his chapter, Robert L. Devaney (1948–) wrote that the mathematics of fractals was ‘quite beautiful’,[5] while the images it produced makes the field accessible to a wide audience.[6] He also noted that ‘[t]he computer graphics which accompanies the study of entire functions reveals pictures of incredible beauty as well as some very startling mathematics’.[7] Note the distinction here: the pictures have beauty; the mathematics is surprising; one cannot conclude that he saw a link between the visual beauty of the pictures and the abstract beauty of the mathematics.

Peitgen twice referred to the ‘beautiful theory’ of iteration of mappings in the complex plane,[8] and to beautiful results within the theory.[9] More importantly, he connected the visual aesthetic value of images of exact fractals to the underlying mathematics: ‘Their aesthetic appeal stems from structures which at the same time are beyond imagination and yet look extremely realistic.’[10] But this statement stops frustratingly short of making an explicit connection with the aesthetic value of the mathematical structures.

The process of generation of exact fractals can have aesthetic appeal. Dietmar Saupe (1954–) wrote in 1988 that rewriting systems were an ‘elegant’ method of producing fractal curves.[11] For instance, the rewriting system — → _∧_, applied iteratively, generates the Koch curve (see Figure 19.2 on page 673). Similarly, the rewriting system — → ∨ generates the *Lévy C curve* (see Figure 17.7). Applying this rule and its oppositely oriented equivalent — → ∧ iteratively to alternate line segments produces *dragon curves* (see Figure 17.9, rows 3–5 on page 631).

Yuval Fisher called beautiful[12] a 1982 result of Douady & John H. Hubbard (1945–), in which they proved that the Mandelbrot set is connected, by constructing an analytic homeomorphism from the complement of the closed unit disc in ℂ to the complement of the

[1] Mandelbrot, ‘Fractals and the Rebirth of Iteration Theory’, p. 151.

[2] *Ibid.*, p. 153.

[3] Douady, ‘Julia Sets and the Mandelbrot Set’, p. 162.

[4] *Loc. cit.*

[5] Devaney, ‘Fractal patterns arising in chaotic dynamical systems’, p. 138.

[6] *Ibid.*, p. 137.

[7] *Ibid.*, p. 159.

[8] Peitgen, ‘Fantastic deterministic fractals’, pp. 172, 206.

[9] *Ibid.*, p. 182.

[10] *Ibid.*, p. 169.

[11] Saupe, ‘A unified approach to fractal curves and plants’, p. 273.

[12] Fisher, ‘Exploring the Mandelbrot set’, p. 288.

Visual and mathematical beauty

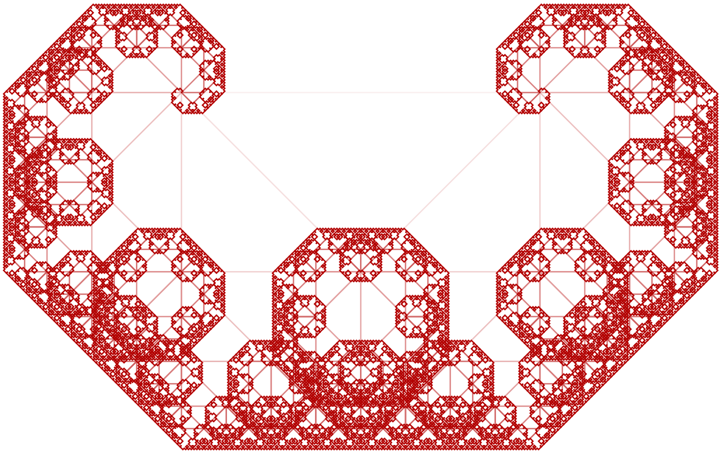


FIGURE 17.7
The sixteenth iteration of the generation of the Lévy C curve, with early iterations shown in a lighter shade.

Mandelbrot set in ℂ.[1] Douady & Hubbard noted that Mandelbrot had obtained by computer a 'very beautiful'[2] picture of the set, and that this picture showed that the set included little 'islands' apparently detached from the 'mainland'. Douady & Hubbard's result showed that in fact the 'islands' and 'mainland' were connected by filaments.

In 1989, in a review of *The Beauty of Fractals*, Steven Krantz (1951–) complained that fractal geometry seemed to have become an investigation of the images, having detached from the original mathematics. Such praise as mathematicians gave to fractal geometry was actually indirect, via praising the work of Douady, Hubbard, and William Thurston (1946–2012).[3] But, said Krantz, 'these mathematicians don't study fractals — they prove beautiful theorems'.[4]

Kenneth Falconer (1952–), who authored one of the standard mathematical textbooks on fractal geometry, wrote in the foreword to its first edition 1989:

> 'In the past few years, fractals have become enormously popular as an art form, [...]. Whilst it is possible in some ways to appreciate fractals with little or no knowledge of their mathematics, an understanding of the mathematics that can be applied to such a diversity of objects certainly enhances one's appreciation. The phrase "the beauty of fractals" is often heard — it is the author's belief that much of their beauty is to be found in their mathematics.'[5]

This declaration seems to have been the first explicit statement that the visual beauty of the images was connected to the beauty of the underlying mathematics. In 2013, he wrote that Julia sets and the Mandelbrot set possess both 'aesthetic and actual beauty';[6] the term 'aesthetic' here must denote a connection to the senses, while the term 'actual' suggests a near-platonic regard for a higher beauty beyond the sensory realm.

[1] Douady & Hubbard, 'Itération des polynômes quadratiques complexes'.

[2] *Ibid.*, 123; trans. C.

[3] Krantz, 'Fractal Geometry', p. 16.

[4] *Loc. cit.*

[5] Falconer, *Fractal Geometry*, p. x.

[6] Falconer, *Fractals*, ch. 4.

The physicist Freeman Dyson (1923–2020), however, characterized the field of chaos as having 'an abundance of quantitative data, an unending supply of beautiful pictures, and a shortage of rigorous theorems'.[1] The only rigorous result that Dyson knew was Li & Yorke's 'Period Three Implies Chaos', which he thought 'one of the immortal gems in the literature of mathematics', and which he found more illuminating than innumerable beautiful pictures.[2]

In 2015, Frank Wilczek (1951–), the 1994 Nobel laureate in physics, wrote that a fractal image could be admired in purely visual terms as a beautiful work of abstract art, but that for him, there was a two-way connection between the aesthetic appeal of the image and the mathematics that produced it. First, the knowledge that the image was encoded in a simple piece of mathematics added to its beauty. Second, the beauty of the image added to the beauty of the mathematics: reading the algorithm that produced the image was only mildly interesting, but seeing the output made understanding of the algorithm into 'a spiritual quest, reaching for the sublime'.[3]

PHYSIOLOGICAL AND BEHAVIOURAL STUDIES

In nature, statistical fractals prevail. Thus most studies of aesthetic appreciation have focused on statistical fractals rather than exact fractals.[4] The results varied for different kinds of fractals, but all indicated an aesthetic preference for statistical fractals of low dimension, ranging from 1.1 to 1.5.[5]

Furthermore, a 2015 study by Hägerhall et al. showed that there were measurable differences in the brain activity arising from viewing statistical and exact fractals. In this study, electroencephalography (EEG) was used to record the brain-wave responses of volunteers exposed to the visual stimuli of nine variant Koch curves, both exact and statistical. In the generation of a Koch curve, each line segment is iteratively replaced by four line segments according to the rule — → _/_. By varying the length of the four new segments in the rule, one obtains Koch curves of different fractal dimensions (see Figure 17.8, row 1). A random element is added by specifying a probability p for a given segment to be replaced, not according to the usual rule, but via the rule with opposite orientation — → ‾\/‾. Regardless of the value of p, the resulting curves have the same fractal dimension as the exact Koch curve (which corresponds to $p = 0$), but do not possess its symmetry or precise self-similarity (see Figure 17.8, rows 2–3).

Hägerhäll et al.'s study used as stimuli fixed Koch curves of fractal dimensions 1.1, 1.3, and 1.5, generated with the values 0, 0.25, and 0.5 for p. Analysis showed that the alpha wave response, which is characteristic of a state of relaxation, increased as p increased, for

[1] Dyson, 'Birds and Frogs', p. 221.

[2] *Loc. cit.*

[3] Wilczek, *A Beautiful Question*, ch. 'Newton III'.

[4] Bies et al., 'Aesthetic Responses', p. 2.

[5] See, for instance, Sprott, 'Automatic Generation of Strange Attractors'; Aks & Sprott, 'Quantifying Aesthetic Preference'; Spehar, Clifford et al., 'Universal aesthetic of fractals'; Spehar & Taylor, 'Fractals in art and nature'; Taylor et al., 'Perceptual and Physiological Responses'.

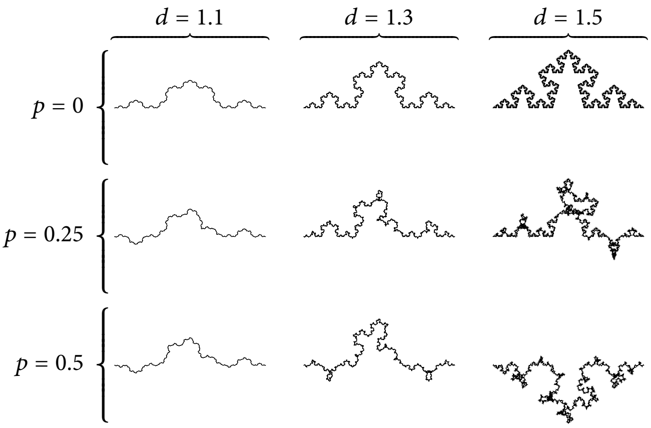


FIGURE 17.8

Koch curves of fractal dimension d for $d \in \{1.1, 1.3, 1.5\}$, with each replacement switched to the opposite orientation with probability p for $p \in \{0, 0.25, 0.5\}$. Fractals with the same parameters were used in Hägerhäll et al.'s study. (Cf. Hägerhäll et al., 'Human Physiological Benefits of Viewing Nature', Fig. 1)

all dimensions.[1] This fits with the idea that the response would be greater for statistical fractals, which are found in nature, than for exact fractals, which are artificial constructs.[2]

Since the work of Hägerhäll et al. suggested that the aesthetic response to exact fractals might differ from the response to statistical fractals, in 2016, Bies et al. made the first systematic study of behavioural responses to exact fractals.[3] [In 1996, Clifford Pickover (1957–) had reported anecdotally that 'viewers often prefer' recursive lattice patterns with a high fractal dimension of around 1.8,[4] but gave no details.] Bies et al.'s study comprised two experiments. In both, participants were shown images of fractals of varying dimension and asked to rate their appeal on a scale from 1 (low) to 7 (high).[5]

In the first experiment, the only fractals used were exact midpoint displacement fractals, which are essentially flattened images obtained from an exactly generated fractal landscape (see Figure 17.9, row 1). The results showed an overall increase in participants' preference ratings with dimension, but cluster analysis indicated two groups of participants: for the majority (76 %), preference ratings increased with dimension; for the minority (24 %), preference ratings varied little as dimension increased 1.1 to 1.5, then fell sharply as dimension increased to 1.6, then continued to decline. [The present author speculates that trypophobia — the aversion to clusters of holes — may play some role in the existence of the minority group who preferred lower dimensional fractals.[6]]

Bies et al. noticed the difference with statistical fractals, where the preference had its maximum at low dimension. For exact fractals, preference seems to have continued to increase with dimension. But they noted that this first experiment had not considered differing levels of recursion or spatial symmetry.[7] This they addressed in their second experiment, which used several types of fractals:

Sierpiński gaskets. These fractals are generated by iteratively removing a certain portion of the image and repeating this in each region not so removed. By varying the removed portion, gaskets

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[1] Hägerhäll et al., 'Human Physiological Benefits of Viewing Nature', p. 7.

[2] *Ibid.*, p. 4.

[3] Bies et al., 'Aesthetic Responses', p. 2.

[4] Pickover, *Keys to Infinity*, p. 206.

[5] Bies et al., 'Aesthetic Responses', pp. 4, 8.

[6] See Cole & Wilkins, 'Fear of Holes'.

[7] Bies et al., 'Aesthetic Responses', p. 6.

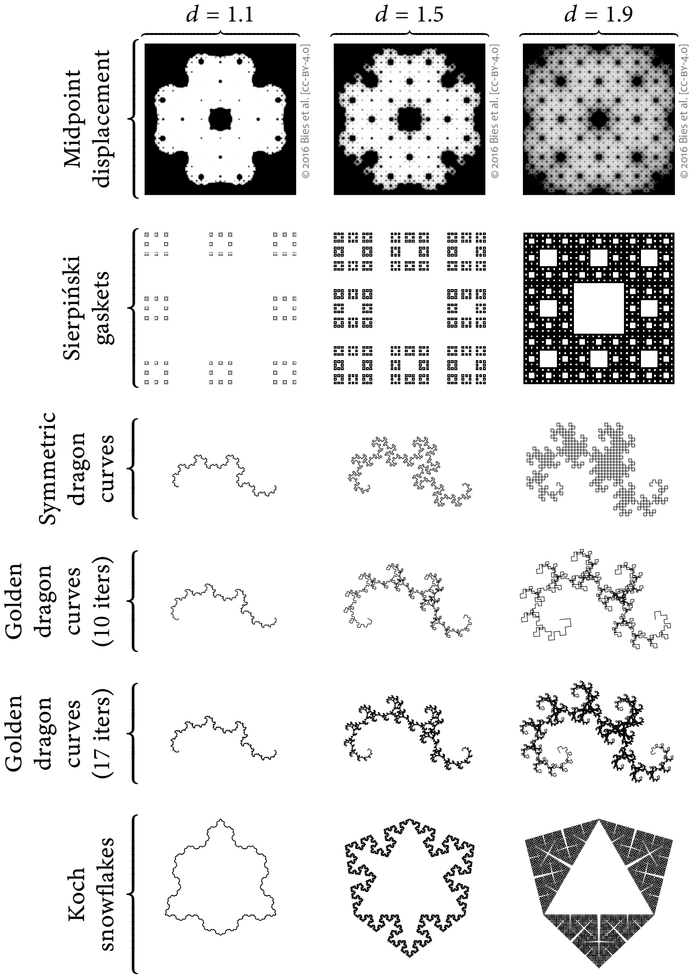
d = 1.1
d = 1.5
d = 1.9
Midpoint displacement
Sierpiński gaskets
Symmetric dragon curves
Golden dragon curves (10 iters)
Golden dragon curves (17 iters)
Koch snowflakes
© 2016 Bies et al. [CC-BY-4.0]
© 2016 Bies et al. [CC-BY-4.0]
© 2016 Bies et al. [CC-BY-4.0]

FIGURE 17.9
Some of the exact fractals of dimension d for $d \in \{1.1, 1.5, 1.9\}$ used in Bies et al.'s experiments. (Cf. Bies et al., 'Aesthetic Responses', Figs 1, 4)

of different fractal dimension are obtained (see Figure 17.9, row 2). The results showed that the mean preference for Sierpiński gaskets increased with dimension. Unlike for midpoint displacement fractals, there was no subgroup with radically different views.

Dragon curves. These curves are formed by replacing line segments by an angled segment according to the rules — → ∨ and — → ∧, applied alternately to the replacements at each iteration. By varying the angle of the replacement, curves of different fractal dimensions are obtained (see Figure 17.9, row 3). It is also possible to vary the lengths of the replacement segments so as to obtain different shapes of curves with the same fractal dimension. When the two replacement segments have the same length, the *symmetric* dragon curve is generated; when they stand in the ratio $1/\varphi^{1/\varphi} : 1$, where $\varphi : 1$ is the extreme and mean (or 'golden') ratio (so that $\varphi = (1 + \sqrt{5})/2$), the *golden* dragon curve is generated (see Figure 17.9, rows 4–5.).

The results showed that the mean preference for symmetric dragon fractals generally increased with dimension, but the increase was not consistent: local decreases occurred both on average and within subgroups of participants.[1] For the golden dragon curves, there was a major difference in the mean preference ratings for the curves produced by 10 and by 17 iterations. The former had low mean preference ratings, with a small increase up to 2.3 at dimension 1.4, then roughly steady; the latter had much higher ratings, with an increase up to 5.7 at dimension 1.6. This could be due to the linear angularity still being very visible after only 10 iterations.

Koch snowflakes. These closed curves comprise three Koch curves between the vertices of an equilateral triangle. (See Figure 17.9, row 6.) The results showed that the mean preference ratings increased with dimension, reaching their peak at dimension 1.7, followed by slight decreases at dimension 1.8, then a slight increase at dimension 1.9.

Regardless of the details of the results, they stand in contrast to those of earlier studies on statistical fractals: they indicate that the preference for exact fractals was maximal at notably higher dimensions than for statistical fractals.[2]

Symmetric dragon curves had a higher mean preference rating at each dimension than golden dragon curves, with a bigger difference at higher dimensions. Perhaps curiously, Bies et al. concluded that this result conformed to a preference for radial symmetry, which is more pronounced at higher dimensions.[3] The symmetric dragon curve does not actually have radial symmetry in a mathematical sense, but perhaps it bears a visual resemblance to structures that do. On the other hand, for a minority (33 %), the Koch snowflake, which is clearly radially symmetrical, had substantially lower appeal than the golden dragon.[4]

CONCLUSION

Bies et al., while allowing that there could be cultural variation in the visual aesthetic response to fractals, speculated that exact fractals might be seen *by some* as beautiful via a type of perceptual processing that was independent of culture, being grounded in properties such as dimension and recursion and symmetry.[5] If visual aesthetic responses to exact fractals were focused on these properties, then it would suggest that the perceptual processing parallels — in a perhaps very limited and approximate way — the cognitive processing of the underlying mathematical object.

[1] Bies et al., 'Aesthetic Responses', pp. 9–12.

[2] *Ibid.*, p. 13.

[3] *Ibid.*, pp. 11–12.

[4] *Ibid.*, Fig. 10B.

[5] *Ibid.*, p. 16.

Such a parallel, if it exists, would at least partly explain the connection many mathematicians seem to have discerned between the visual and mathematical beauty of fractals. But, at present, there simply seems to be insufficient evidence for more than speculation about this supposed parallel.

18

Literary Theory

Beauty in poetry, prose, drama, and mathematics

> ‘In the great majority of mathematical text-books
> there is a total lack of unity in method
> and of systematic development [...].
> But in the greatest works,
> unity and inevitability are felt as in the unfolding of a drama’
>
> — Bertrand Russell, ‘The Study of Mathematics’, p. 66.

> ‘Reading your proof had a similarly pleasant effect on me
> as seeing a really good play.’
>
> — Kurt Gödel, letter to Paul Cohen, dated 5 Jun. 1963, quot. in Gödel, *Collected Works*, vol. 4, p. 378.

‘Every mathematical argument tells a story.’[1] Many works have drawn parallels between mathematics and various literary forms.[2] The very notion of mathematical proof may owe its emergence in ancient Greece at least in part to a cultural propensity to question everything,[3] to the rhetoric by which orators sought to inform and persuade the citizens in the assembly, and to the scrutiny of the rhetors’ arguments by the philosophers.[4]

Explorations of parallels between literature and mathematics can be focused upon different aspects such as logic or ontology,[5] or style and symbolism.[6] Here the concern is aesthetics. There are scattered observations relating the appreciation of literary art and of mathematics in the twentieth century. For instance, Gustaf Mittag-Leffler (1846–1927) wrote in his 1907 biography of Niels Henrik Abel (1802–29):

> ‘Abel’s best works are true lyrical poems of sublime beauty, where the perfection of the form makes clear the depth of the thought, at the same time as it fills the imagination with dream paintings taken from a remote world of ideas, more elevated above the banality of life and more directly emanated from the soul than could be produced by any poet in the ordinary sense of the word.’[7]

Jacob Bronowski (1908–74) wrote in 1956 that both in poetry and in mathematics or science, the experience of appreciation was a re-enactment of the experience of creation or discovery: the pleasure in the appreciation is thus an echo of the pleasure felt by the original creator.[8] In 1970, Kenneth R. Conklin (1942–) agreed that at least

[1] Galison, ‘Structure of Crystal’, § 1.

[2] See, for example, the collection Doxiadis & Mazur, *Circles Disturbed*.

[3] Lloyd, *Magic, Reason, and Experience*, pp. 248–9.

[4] Lloyd, *Demystifying Mentalities*, ch. 3; Doxiadis, ‘A Streetcar Named Proof’, esp. § 5.

[5] Thomas, ‘The Comparison of Mathematics with Narrative’; Thomas, ‘Mathematics and Narrative’.

[6] Pycior, ‘Mathematics and prose literature’, *passim*; Fauvel, ‘Mathematics and poetry’, esp. § 6.

[7] Mittag-Leffler, *Niels Henrik Abel*, p. 40; trans. C.

[8] Bronowski, *Science and Human Values*, p. 31.

some of the aesthetic experience of discovery was felt by the person appreciating the discovery, and compared studying a single proof or a whole branch of mathematics to reading a novel and enjoying the development of the story.[1] In his 1995 book *Le Ton Beau de Marot*, on language, poetry, translation, and aesthetics, Douglas Hofstadter (1945–) noted that there was such a thing as appreciation of how mathematics is *performed*, just as drama or music is performed:

> 'one of the most memorable experiences of my entire college career was what I would call a "performance" of a proof of a beautiful theorem about Fourier series in an advanced mathematics class I was taking on the topic of Real Analysis. [...] The moment the bell rang, he got down to business, [...] writing the statement down in a beautiful, crisp, angular European cursive on the board, and then spending the next 49 minutes moving in an impeccable, inexorable, spellbinding rhythm towards the climax, which was delivered in such an exquisitely timed manner that just as he put the last period on the final sentence on that board and wiped the chalk dust off his hands, the closing bell rang. [...] that lecture was one of the greatest performances I have ever seen, not just of some mathematics, but of *anything*.'[2]

But each of these observations stands in isolation, being embedded in a work with a different focus. Extended and detailed study of parallels between mathematical and literary aesthetics seems to have begun only in the twenty-first century.

FREEDOM AND NECESSITY

In a 2005 study of the aesthetics of mathematics, Reviel Netz (1968–) wrote that mathematical beauty occurs on three levels: mathematical states of mind; mathematical texts; and mathematical objects.[3] He focused on texts, particularly since there was both a readily-available corpus of material to study and an existing theory, namely poetics, with which to study it.[4] Within texts, beauty could be perceived on the large scale, when contemplating a whole treatise, or on the small scale, when experiencing the unfolding text. The large-scale structure was analogous to narrative; the small-scale structure to prosody.[5]

For an example of a superb narrative structure, Netz pointed to Archimedes' first book *On the Sphere and the Cylinder*.* Archimedes presented his argument in a very dramatic format, first setting a goal and then taking a plot-twist route to it; Netz compared Archimedes' chosen form to a narrative with an important revelation, such as *Jane Eyre*.[6] Netz saw the first book of Euclid's *Elements* as having a very

[1] Conklin, 'The Aesthetic Dimension of Education', pp. 28–9.

[2] Hofstadter, *Le Ton Beau de Marot*, pp. 363–4, emphasis in orig.

[3] Netz, 'The Aesthetics of Mathematics', p. 254.

[4] *Loc. cit.*

[5] *Ibid.*, p. 255.

* ▸ pp. 107–8

[6] *Ibid.*, pp. 256–7, 259.

different narrative structure, beginning *in medias res* with definitions, postulates, and common notions. The subject of the book was not announced, but was implicitly seen to be the triangle. From a starting point of basic results and constructions, there was a gradual build-up towards general statements on triangles. The subject-matter was then extended to quadrilaterals, reaching towards what might be seen as the goal of the book: *To construct, in a given rectilineal angle, a parallelogram equal to a given rectilinear figure.*[1] The pythagorean theorem and its converse[2] served as a coda.[3] As Netz pointed out, Euclid never wrote *about* the theorems or their progression. The direction of the mathematics was implicit and had to be inferred by the reader.[4] He compared the structure of book I of the *Elements* to a narrative that has a steady progression, such as *Anna Karenina*.[5]

Similar narrative differences could occur on a smaller scale. Netz analysed the structure of individual proofs by numbering their statements and drawing diagrams showing their logical dependence.[6] He pointed out the contrast between the straightforward linear structure of the proof of Proposition II.5 of Euclid's *Elements* (see Figure 18.1) and the much more complex structure of the proof of Proposition 1 of Archimedes' *The Method* (see Figure 18.2). Netz argued that Archimedes made a deliberate choice of such a structure, intending to delay and so to emphasize the surprise of his master-stroke of identifying an area with a sum of lines.[7]

Furthermore, both Euclid and Archimedes chose to scatter 'starting' statements — those that are not consequences of others and so do not sit above another statement in the diagrams (such as 1, 2, 5, 7, 10, 12, 13, 15, 16 in Figure 18.1). Logically, the proof could begin with these statements, but this approach would be both cognitively and aesthetically awkward.[8] Netz thought the relationship between deduced statements, starting statements, and arguments — deduction steps — to be analogous to the relationship in poetry between stressed syllables, unstressed syllables, and feet.[9] This analogy explains why there could be a sense of a proof moving faster or slower.[10] Finally, there were various devices by which the structure of a proof could be brought out.[11]

Thus, for Netz, narrative style existed in mathematics and narrative could be a source of beauty in mathematics. Although a mathematical text had to obey the dictates of logic, its author could choose

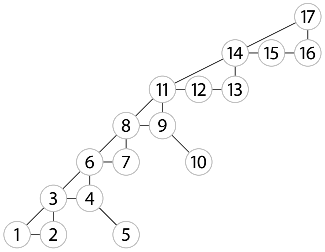

FIGURE 18.1
Diagram of the structure of Euclid, *Elements*, Prop. II.5 (adapted from Netz, *The Shaping of Deduction*, Fig. 5.18).

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[1] Euclid, *Elements*, Prop. I.45.

[2] *Ibid.*, Props I.47–8.

[3] Netz, 'The Aesthetics of Mathematics', pp. 257–8.

[4] Euclid, *Elements*, p. 258.

[5] *Ibid.*, p. 259.

[6] Netz, *The Shaping of Deduction*, pp. 168–9.

[7] Netz, 'The Aesthetics of Mathematics', pp. 260, 266–7; note that Fig. 5 is incorrect; see Netz, *The Shaping of Deduction*, p. 212, Fig. 5.23.

[8] Netz, 'The Aesthetics of Mathematics', pp. 266–7.

[9] *Ibid.*, p. 267.

[10] *Ibid.*, pp. 267–8.

[11] *Ibid.*, pp. 268–9.

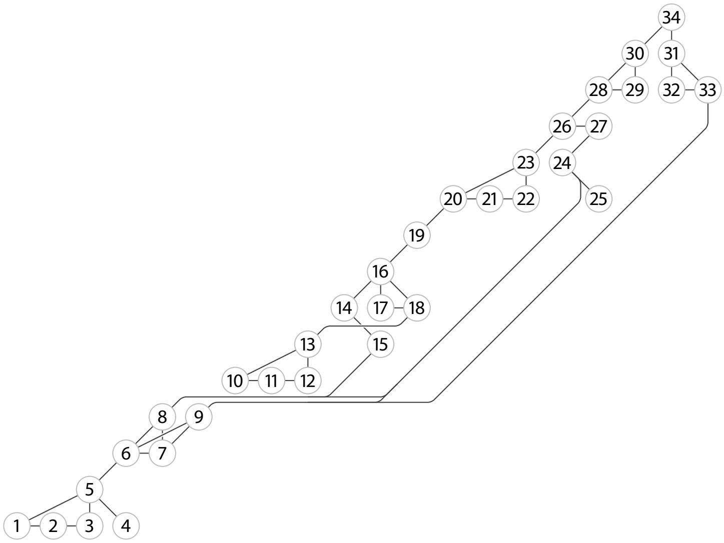


FIGURE 18.2
Diagram of the structure of Archimedes, *The Method*, Prop. 1, (adapted from Netz, *The Shaping of Deduction*, Fig. 5.23).

which premisses to assert, which consequences to pursue, and how to present them. For Netz, the resulting 'dialectic of freedom and necessity is [...] often at the root of the beauty of mathematical narratives';[1] in a mathematical narrative one perceived both the freedom of the author and the necessity of the arguments. Both aspects had parallels in narrative in literature, where the freedom of the author was expressed though the choice of the direction of the plot, and the necessity in the constraints of believable actions of the persons of the story.[2] For Netz, this freedom and necessity seem to have been linked to surprise and inevitability, which together made a mathematical statement beautiful.[3]

Netz also made an important observation about how aesthetic properties of mathematics interact with translation. When an ancient Greek proof is rendered in modern notation, the aesthetic properties are *enhanced*, for they become more readily perceived by a modern reader.[4] (Netz considered that without such perception there was no aesthetic object; here he took a projectivist position.[5]) Such enhancement set mathematics apart from literary narrative or — especially — poetry, where the most important aesthetic properties were seen in the specific language used, and were diminished under translation.[6]

DEUS EX MACHINA

Caveat lector: here the present author enters his own history. He makes no pretence at neutrality in confronting his past arguments, but will at least attempt to be fair to his critics.

* * *

[1] Netz, 'The Aesthetics of Mathematics', p. 261.

[2] *Ibid.*, p. 262.

[3] *Ibid.*, p. 283.

[4] *Ibid.*, p. 284.

[5] *Loc. cit.*

[6] *Loc. cit.*

In 2010, the present author argued that the literary concept of *deus ex machina* could be applied to proof. His starting-point was the observation that there is a *prima facie* clash between unexpectedness and inevitability, two of the three 'purely aesthetic' qualities that make up the Hardian triad.[1][*] The author held that, like a narrative, a proof could feature both 'good' surprises and *dei ex machina*.

A *deus ex machina* is a plot-device that serves to resolve a seemingly intractable situation, and which does not fit with the plot. The term (literally, 'god from the machine') is derived from ancient Greek drama, where such a resolution might be brought about by divine intervention, with the entrance of the actor playing the deity effected by stage-machinery. Aristotle complained of the unsatisfactory nature of such a resolution, giving the example of how in Euripides' (c. 480–407/6 BCE) play *Medea*, the title character escapes Jason's vengeance in the chariot of the Sun-god Helios;[2] no divinity has a role in the play before this point.[3]

Not every surprise in a narrative is a *deus ex machina*. A 'good' surprise is an unexpected happening that arises naturally from the story so far and reveals a previously unrecognized pattern that fits the plot.

The analogy made by the present author was as follows. In a narrative, a reader built up from the portrayed events a mental conception of the setting, characters, background, and motivations. Whether a new development in the story was seen as expected, as a good surprise, or as a *deus ex machina* depended on the reader's evaluation of it within the mental conception thus far built up. Similarly, in a proof, a reader built from the reasoning a mental conception of how the objects behave. Whether a new step in the proof was seen as expected, as a good surprise, or as a *deus ex machina* depended on the reader's evaluation of it within the mental conception thus far built up.[4]

Indeed, it has been noted that the *author* of a narrative could be equally directed by their mental conception of the world they have devised. The critic Northrop Frye (1912–91) put it in explicitly mathematical terms:

> 'The poet, like the pure mathematician, depends, not on descriptive truth, but on conformity to his hypothetical postulates. [...] Mathematics, like literature, proceeds hypothetically and by internal consistency'.[5]

That is, the initial creative freedom of the author gave rise to restrictions on how the story developed.[6] As the mathematician Armand Borel (1923–2003) once put it, authors sometimes find their creatures acting on their own, with the authors themselves observing and narrating as passive observers.[7] The mediaevalist, philosopher, and novelist Umberto Eco (1932–2016) made precisely this point when he reflected on writing *The Name of the Rose*:

[1] Cain, 'Deus ex machina', p. 7.

[*] ▸ pp. 502

[2] Euripides, *Medea*, ll. 1314 sqq.

[3] Aristotle, *Poetics*, BEK 1454a33–b8.

[4] Cain, 'Deus ex machina', pp. 7–8.

[5] Frye, *Anatomy of Criticism*, pp. 69–70, 86.

[6] Cf. Margolin, 'Mathematics and Narrative', § 5.1.

[7] Borel, 'On the Place of Mathematics', p. 148.

> ‘The constructed world will then tell us how the story must proceed. […] when I put Jorge in the library I did not yet know he was the murderer. He acted on his own, so to speak. […] The fact is that the characters are obliged to act according to the laws of the world in which they live. In other words, *the narrator is the prisoner of his own premises*.’[1]

In a narrative, *deus ex machina* spoiled any attempt to build tension, since any subsequent predicament can be solved by another *deus ex machina*. Aesthetically, it was also displeasing, for it runs counter to the natural development of the story. There was no reason *internal to the story* why a *deus ex machina* should intervene. Thus it was difficult for the reader to incorporate the *deus ex machina* into their mental conception of the narrative world.[2]

In the present author’s view, a *deus ex machina* in a proof took the form of an unexplained construction or a calculation of elements or a definition of a function that simply ‘happened to work’. A reader could not see a reason *internal to the proof* for such a step. Thus it was difficult for the reader to incorporate the *deus ex machina* into their mental conception of the proof.[3]

The idea that some kind of *deus ex machina* could appear in proof has a long pedigree, particularly with regard to exposition. Thomas Tucker (1945–) explicitly recommended that, when teaching the calculus, one should not use ‘deus-ex-machina auxiliary functions’.[4] The idea, if not the term, seems to have been present when Timothy Chow (1969–) wrote that ‘every step should be motivated and clear’[5] and followed Donald J. Newman (1930–2007) in declaring that proofs should be ‘natural’ in ‘not having any ad hoc constructions or *brilliancies*’.[6] There appears to have been no discussion of *deus ex machina* in connection with the aesthetics of mathematics prior to the present author’s 2010 essay, although George Pólya (1887–1985) tiptoed past it when, in discussing exposition,[7] he wrote that ‘[t]he most conspicuous blemish of an otherwise acceptable presentation is the “deus ex machina.”’[8]

The author held that taking ‘inevitability’ in proof to mean ‘avoidance of *deus ex machina*’ allowed one to understand how inevitability and unexpectedness could both occur in a beautiful proof. In proof, just as in narrative, unexpectedness did not spoil beauty provided it fitted its structure and setting: that is, provided it was not a *deus ex machina*.[9]

Notice the difference from Netz’s view: Netz saw surprise and inevitability in a mathematical text as deriving respectively from the choices that the writer of a text made and the necessity of logic. The present author located both in a reader’s perception of the progression of an argument.

In 2012, Viktor Blåsjö (1982–) independently gave a *definition* of beauty in proof that, except for the terminology used, was very

[1] Eco, ‘Postscript’, § ‘The Novel as Cosmological Event’, emphasis added.

[2] Cain, ‘Deus ex machina’, p. 8.

[3] *Loc. cit.*

[4] Tucker, ‘Rethinking rigor in calculus’, pp. 239–40.

[5] Chow, ‘A beginner’s guide to forcing’, p. 1.

[6] Newman, *Analytic Number Theory*, 59, emphasis in orig.

[7] Polya, *Mathematics and Plausible Reasoning*, [vol. II] §§ XVI.4–6.

[8] *Ibid.*, [vol. II] § XVI.4.

[9] Cain, ‘Deus ex machina’, p. 8.

close to the present author's idea of applying the concept of *deus ex machina*:

> 'A beautiful proof is one which the mind can play its way through with a natural grace, as if it were created for this very purpose. We grasp a beautiful proof as a whole, yet see the role of every detail; it is vivid and transparent [...]. I call this type of proof *cognisable* for short.
>
> An ugly proof resorts to computations, algorithms, symbolic manipulation, ad hoc steps, trial-and-error, enumeration of cases, and various other forms of technicalities. The mind can neither predict the course nor grasp the whole; it is forced to cope with extra-cognitive contingencies. The mind's task is menial: it can only grasp one step at a time, checking it for logical adequacy.'[1]

In both Blåsjö's and the present author's formulations, there was a hint of Gian-Carlo Rota's (1932–99) suggestion that 'mathematical beauty' is a term mathematicians use to avoid describing a piece of mathematics as enlightening.[*] It is reasonable to think that enlightenment somehow depends on the theorem fitting naturally into the reader's mental conception, and so a proof that involved a *deus ex machina* would be less likely to be perceived as enlightening.[2]

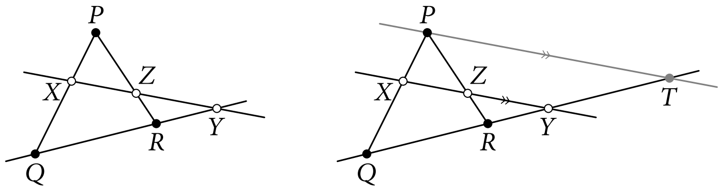
An example of a deus ex machina

In correspondence with his friend the psychologist Max Wertheimer (1880–1943), Albert Einstein[†] (1879–1955) gave an example of a *deus ex machina* construction, though not in those terms. He compared two proofs of Menelaus' theorem:[3]

MENELAUS' THEOREM. *Let PQR be a triangle; if a line intersects the sides PQ, QR, and RP at X, Y, and Z, respectively, then*

$$\frac{PX}{XQ} \cdot \frac{QY}{YR} \cdot \frac{RZ}{ZP} = 1.$$

(See Figure 18.3 (left).)



An instance of Menelaus' theorem.

The proof that Einstein found ugly.

FIGURE 18.3
Illustrating an instance of Menelaus' theorem, in which

$$\frac{PX}{XQ} \cdot \frac{QY}{YR} \cdot \frac{RZ}{ZP} = 1,$$

and the construction used in a proof that Einstein found ugly.

In this result, terms such as PX are *unsigned* lengths. There is a stronger version of Menelaus' theorem that uses signed lengths.[‡]

The first proof Einstein perceived as ugly:

[1] Blåsjö, 'A Definition of Mathematical Beauty and Its History', p. 93.

[*] ▸ pp. 742–5

[2] Cain, 'Deus ex machina', pp. 10–11.

[†] ▸ pp. 886–92

[3] Einstein, letter to Wertheimer, dated Autumn 1937?, repr. in Luchins & Luchins, 'The Einstein-Wertheimer correspondence', p. 38–9.

[‡] ▸ p. 776

Proof. Draw a line though P parallel to XY intersecting QR at T (see Figure 18.3 (right)). By the similarity of $\triangle PQT$ and $\triangle XQY$ and the similarity of $\triangle PRT$ and $\triangle ZRY$,

$$PX : XQ = TY : YQ \text{ and } PZ : ZR = TY : YR.$$

Eliminating TY gives the result. $\square$

The second proof he perceived as beautiful:[1]

Proof. The areas of two triangles with a common angle or supplementary angles have the same ratio as the two products of the adjacent sides. [This statement holds because a triangle with two sides a and b adjacent to a common angle ϑ has area $\frac{1}{2}ab \sin \vartheta$; that the same holds when the angles are supplementary follows from $\sin(\pi - \vartheta) = \sin \vartheta$.] Therefore

$$\frac{\triangle PZX}{\triangle QXY} = \frac{PX \cdot ZX}{QX \cdot XY}.$$

Multiplying this equation with the analogous equations involving $\triangle QXY/\triangle RYZ$ and $\triangle RYZ/\triangle PZX$ yields the result. $\square$

Einstein explained the reasons for his differing aesthetic judgements of the two proofs:

> 'Although the first proof is somewhat simpler, it is not satisfying. For it uses *an auxiliary line that has nothing to do with the content of the proposition to be proved*, and the proof favors, *for no reason*, the vertex P, although the proposition is symmetrical in relation to P, Q, and R. The second proof, however, is symmetrical, and can be read off directly from the figure.'[2]

In the present author's terms, the addition of the auxiliary line and the favouring of the vertex P form a *deus ex machina*, for nothing in the proof seems to compel them. Rather, the symmetry is broken by an arbitrary choice, and is later restored by the elimination step.

*Examples of non-*deus ex machina *surprises*

An example has already been mentioned of unexpectedness that does not constitute a *deus ex machina*: Archimedes' move in his first book *On the Sphere and Cylinder* of rotating polygons around an axis, giving three-dimensional analogues to two-dimensional results and revealing their applicability to the results he announced at the start.* This step is unexpected, but not a *deus ex machina*, for it fits with the internal structure of the proof and illuminates the preliminary results.

Another is the original proof of Ptolemy's theorem,[3] about which Roger Penrose (1931–) wrote that he 'was amazed by the beauty and simplicity of the argument, which just involved drawing one unexpected construction line'.[1] But the 'unexpected construction line' is not a *deus ex machina*, because the reason for it is immediately apparent in the similarity arguments.

---

[1] The translation in Luchins & Luchins, 'The Einstein-Wertheimer correspondence', p. 38 has 'elegant'; the original German is 'schöner'.

[2] Einstein, letter to Wertheimer, dated Autumn 1937?, repr. in *ibid.*, p. 38, emphases added.

* ▸ pp. 107–8

[3] Ptolemy, *Almagest*, HEI 36–7.

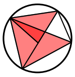
PTOLEMY'S THEOREM. *For a quadrilateral ABCD inscribed in a circle,*

$$AC \cdot BD = AB \cdot DC + AD \cdot BC.$$

Proof. Choose E on AC such that $\angle ABE = \angle DBC$. Then adding $\angle EBD$ to both shows that $\angle ABD = \angle EBC$. (See Figure 18.4.)

But $\angle BDA = \angle BCE$ also, since they subtend the same segment.

$\therefore$ triangle $ABD \mathrel{|||}$ triangle BCE.



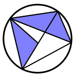
$$\therefore \quad BC : CE = BD : DA.$$

$$\therefore \quad BC \cdot AD = BD \cdot CE.$$

Again, since $\angle ABE = \angle DBC$,

and $\angle BAE = \angle BDC$,

triangle $ABE \mathrel{|||}$ triangle BCD.



$$\therefore \quad BA : AE = BD : DC.$$

$$\therefore \quad BA \cdot DC = BD \cdot AE.$$

But it was shown that $BC \cdot AD = BD \cdot CE$.

Therefore, by addition, $AC \cdot BD = AB \cdot DC + AD \cdot BC$. ☐

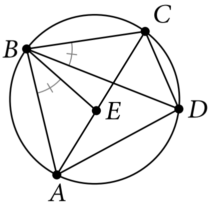


FIGURE 18.4
Proof of Ptolemy's theorem

Exceptional possibilities; possible exceptions

There is perhaps a kind of *deus ex machina* that is so outrageous that it makes a narrative, if not beautiful, then at least impressive: an example might be Satoshi Kon's (1963–2010) film *Tokyo Godfathers*, which is quite self-consciously and unapologetically driven forward by a series of *dei ex machina*.

A mathematical example of this phenomenon is Don Zagier's (1951–) 'one-sentence proof' of Fermat's two-square theorem, which asserts that every prime $p \equiv 1 \pmod 4$ is a sum of two squares.[2]* This proof has an unexpected starting point, considering the finite set

$$S = \{\, (x, y, z) \in \mathbb{N}^3 : x^2 + 4yz = p \,\}$$

and the involution defined on it by

$$(x, y, z) \mapsto \begin{cases} (x + 2x, \quad z \quad , y - x - z) & \text{if } x < y - z, \\ (2y - x, \quad y \quad , x - y + z) & \text{if } y - z < x < 2y, \\ (x - 2y, x - y + z, \quad y \quad) & \text{if } x > 2y; \end{cases}$$

---

[1] Penrose, 'The Rôle of Aesthetics', p. 268.

[2] Zagier, 'A one-sentence proof'.

* ▸ p. 7

since this involution has exactly one fixed point, S contains an odd number of elements and so the involution $(x, y, z) \mapsto (x, z, y)$ also has a fixed point, which gives the expression of p as a sum of two squares. The single sentence is a very compressed outline of the proof, which requires unpacking. In the first edition of *Proofs from THE BOOK*, Martin Aigner (1942–2023) & Günter Ziegler (1963–) took two pages to check that Zagier's map is indeed a well-defined involution, and that, as claimed, it has exactly one fixed point.[1]

The present author no longer agrees with his own earlier asseveration that Zagier's proof is a non-*deus ex machina* surprise.[2] Rather, he considers the one sentence to be a *deus ex machina* whose sheer brazenness can prompt a kind of appreciation that is perhaps very different from an aesthetic response.

Examples outside mathematics

The distinction of *dei ex machina* from 'good' surprises can also be applied to other forms of abstract beauty, as the following examples from the game of Go illustrate. The first is from the 1938 'retirement game' played between Hon'inbō Shūsai (1874–1940), playing white, and Minoru Kitani (1909–75), playing black.[3] This game was played over many days with a rule whereby whichever player wished to adjourn would be required to make a 'sealed move': he would decide his move and seal it in an envelope to be left with the adjudicator; the idea was that neither player would be able to think about how to respond to his opponent's last move during an adjournment. At move 121, Kitani made a sealed move that was not in the natural 'flow' of the game. Rather, his move, marked ▲ in Figure 18.5, threatened the group of white stones at the top of the board, forcing Shūsai into an immediate defence of this group. Normally, moves like this would be played in the endgame or used as *ko* threats. By sealing a forcing move, Kitani could spend the adjournment thinking about how to respond to Shūsai in the 'main game'. Yasunari Kawabata (1899–1972), who reported on the match as a journalist and who became the 1968 Nobel laureate in literature, wrote a semi-fictional account of it in his 1951 novel *The Master of Go*. In the novel, in which Kitani is renamed Otaké, Shūsai describes his feeling of disgust:

> 'The match is over. Mr. Otaké ruined it with that sealed play. It was like smearing ink over the picture we had painted. The minute I saw it I felt like forfeiting the match.'[4]

Kitani's sealed move can be considered a *deus ex machina* because it did not fit with the natural development of the game: it arose though something external to the formal rules of Go: the timing of the adjournment and Kitani's desire to think carefully about his next 'real' move.

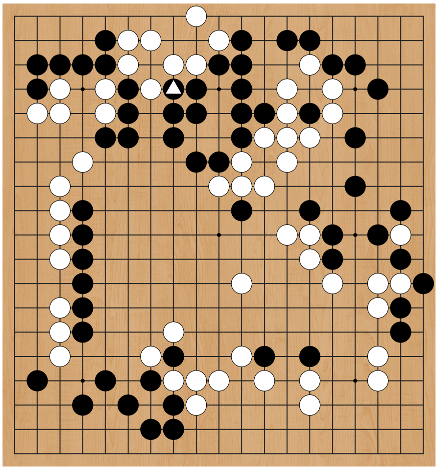
The second example is from the 1846 game played between Hon'inbō Shūsaku (1829–62), playing black, and Gennan Inseki (1798–1859),

[1] Aigner & Ziegler, *Proofs from THE BOOK* (1st edn), pp. 19–20.

[2] See Cain, 'Deus ex machina', p. 10.

[3] Shūsai, *Dago senshū*, § 173.

[4] Kawabata, *The Master of Go*, ch. 38.



'jessicuak' (Pixabay user). [Public domain]

FIGURE 18.5
Move 121 (marked ▲) in the 1938 game between Hon'inbō Shūsai (white) and Minoru Kitani (black). This move was sealed before an adjournment and forced Shūsai to defend the group of white stones at the top, allowing Kitani to think about the 'main game' during the adjournment. It did not fit with the natural progression of the game and can thus be thought of as a *deus ex machina*.

playing white.[1] Until this move, Gennan seemed to be leading. He then played the stone marked ① in Figure 18.6. This was a 'locally strong' move that simultaneously threatened the four black stones to its right and defended against a potential play by black at the position marked a, which would have forced Gennan to defend the group of white stones at the bottom and allowed Shūsaku to connect the four black stones to the larger black group to the left. Shūsaku's answer, marked ▲, had 'global' implications. It simultaneously expanded the influence of the black stones above, reduced the influence of the white stones to the right, strengthened the four black stones below, and aimed towards the white-dominated left. Such bold moves in the middle of the board are unusual: conventionally, play progresses gradually from the sides to the middle. This move was a surprise, and not a pleasant one for Gennan: a doctor who watched the game noticed that Gennan's ears turned red when this move was played, indicating that he was upset.[2] However, the unexpected move fitted with the natural flow of the game and was thus not a *deus ex machina*.

Deus est machina

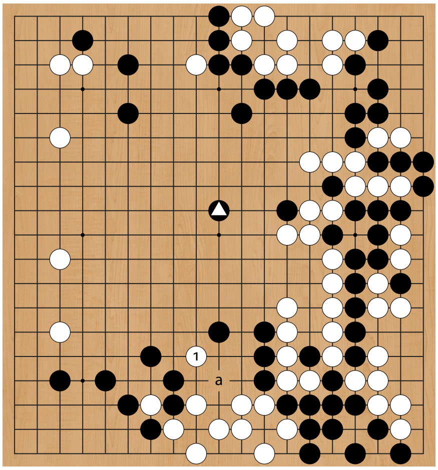
The aesthetic distaste that many mathematicians exhibited for the Appel–Haken proof of the four colour theorem* can also be explained with reference to *deus ex machina*. A computer proof is a *deus ex machina*: it is a black box of logic that a reader cannot incorporate into their mental conception. In narrative, the parallel would be a leap ahead in a story from being confronted with a problem to having solved it, with no inkling of the solution. As the number theorist Marcus du Sautoy (1965–) put it:

> 'We don't read mathematical mystery stories just to find whodunnit. The pleasure comes from seeing how the convolutions of the plot unfold themselves in the build-up to the moment

[1] Power, *Invincible*, pp. 99–110.

[2] *Ibid.*, p. 106.

* ▸ pp. 605–8



'jessicuak' (Pixabay user). [Public domain]

FIGURE 18.6
Move 127 (marked △), known as the 'ear-reddening move', in the 1846 game between Hon'inbō Shūsaku (black) and Gennan Inseki (white). Shūsaku's move expanded black influence from above, reduced white influence from the right, strengthened the four black stones below, and aimed towards the left. This move revealed a previously unseen pattern; it was a surprise but not a *deus ex machina*.

> of revelation. Appel and Haken's proof of the Four-Colour Problem had deprived us of the "Aha — now I get it!" feeling that we crave when we read mathematics.'[1]

In his 2010 essay, the present author illustrated the problem by means of a hypothetical variation of book IX of the *Odyssey*. In the real epic, Odysseus, at the court of Alcinous, tells of his travels and in particular of how he and his men had been trapped by the Cyclops Polyphemus, but how they escaped through plying him with strong wine, blinding him, and tricking both him and the other Cyclopes. In the imagined variant, Odysseus jumps ahead to their leaving the island, saying that his cunning allowed their escape. His tale would be plausible, given his prior success with stratagems and ruses. But it would be a *deus ex machina*, for a reader could not modify their mental conception to go from Odysseus and his men being in peril to their setting sail.[2]

Example in dispute

The only direct replies to the author's essay 'Deus ex machina and the aesthetics of proof' did not confront his arguments on whether *deus ex machina* was applicable to proofs, or whether it clarified inevitability, or whether it bore upon aesthetics at all. Rather, the replies disagreed with his view that there was a *deus ex machina* in one of his key examples, namely John Conway's (1937–2020) proof of Morley's trisector theorem. [This proof has been in circulation since at least 24 November 1997, when Conway posted it to the `geometry.puzzles` newsgroup,[3] but it only appeared in print in 2014.[4]]

MORLEY'S TRISECTOR THEOREM. *In any triangle, the points of intersection of the trisectors of adjacent angles are the vertices of an equilateral triangle.* (See Figure 18.7.)

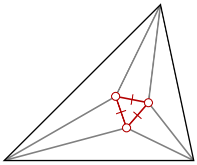


FIGURE 18.7
Morley's trisector theorem.

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[1] du Sautoy, *The Music of the Primes*, p. 212.

[2] Cain, 'Deus ex machina', p. 10.

[3] *Ibid.*, p. 11.

[4] Conway, 'On Morley's trisector theorem'.

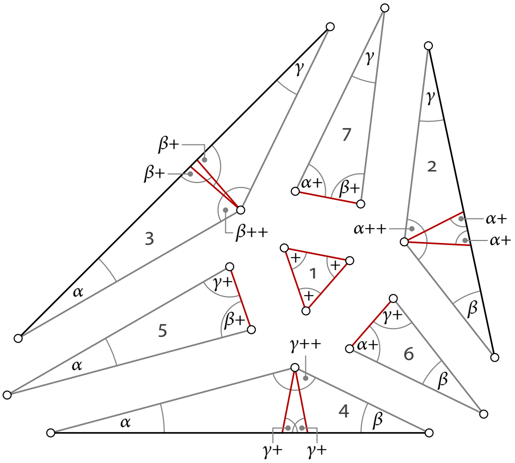

FIGURE 18.8
Conway's proof of Morley's trisector theorem

Proof (*Conway*). Suppose the given triangle has angles 3α, 3β, 3γ. For any angle ϑ, let $\vartheta+$ denote $\vartheta + \pi/3$; formally, $+$ denotes $\pi/3$. As shown in Figure 18.8, consider seven triangles 1, 2, …, 7, with angles as follows:

$$\begin{array}{lll} & 2:\ \alpha{+}{+}, \beta, \gamma; & 5:\ \alpha, \beta{+}, \gamma{+}; \\ 1:\ +, +, +; & 3:\ \alpha, \beta{+}{+}, \gamma; & 6:\ \alpha{+}, \beta, \gamma{+}; \\ & 4:\ \alpha, \beta, \gamma{+}{+}; & 7:\ \alpha{+}, \beta{+}, \gamma. \end{array}$$

These triangles are currently specified only up to similarity. In triangles 2, 3, and 4, inscribe isosceles triangles whose bases are segments of the longest edges, whose apices are the opposite vertices, and whose legs make angles $\alpha+$, $\beta+$, and $\gamma+$ respectively with the longest edges. (Note that these angles may be internal or external; there is no difference in the subsequent reasoning.) The legs of these isosceles triangles are the red lines inside triangles 2, 3, and 4 in Figure 18.8.

Fix the scale of the seven triangles so that all the red line segments have the same length. Arranged as in Figure 18.8, the seven triangles can be pushed together to form a single triangle similar to the original one. The corresponding sides are either both red and so are equal by the choice of scale, or else are equal using the angle–side–angle congruence condition; the 'side' always being a red line segment. At the points where four triangles meet, their angles sum to 2π. The three angles of the resulting triangle are, as desired, 3α, 3β, 3γ, and the triangle 1 is equilateral. ☐

This proof is extremely simple. Its most complicated idea is the angle–side–angle congruence condition. Conway called his proof 'the indisputably simplest';[1] others agreed.[2] The present author held — and still does — that it contains a *deus ex machina* in that the initial specification of the seven triangles is produced out of a hat.[3]

In a 2014 paper, O. A. S. Karamzadeh argued against the presence of a *deus ex machina* in Conway's proof.[4] But he seems to have held

[1] Conway, 'On Morley's trisector theorem'.

[2] Erickson, *Beautiful Mathematics*, § 4.2; Hofstadter, quot. in Roberts, *Genius At Play*, ch. 14.

[3] Cain, 'Deus ex machina', p. 9.

[4] Karamzadeh, 'Is John Conway's Proof …?'

to a different conception of *deus ex machina*. Notably, he thought that a certain short proof of the existence of irrational numbers a and b such that a^b is rational, which will be examined later,[*] was a *deus ex machina*.[1] This view suggests that, for Karamzadeh, *deus ex machina* was something closer to 'lack of enlightenment' or 'something not explainable from the proof', and thus that proofs lacking *deus ex machina* were (*ceterus paribus*) more enlightening or explanatory. Karamzadeh made a detailed explanation of the angle specifications in Conway's proof and concluded that the specifications were natural.[2] Thus, *in Karamzadeh's sense*, Conway's proof was free from *deus ex machina*. But Karamzadeh's argument does not mean that *in the present author's sense* it was free from *deus ex machina*. Quite the opposite: Karamzadeh's argument actually reinforced the present author's position. Karamzadeh filled in the reasoning that was not present (or needed) *in the proof*, in a way analogous to filling in the lacuna in the hypothetical variation of the *Odyssey* mentioned above. But, because Karamzadeh's reasoning is unnecessary for establishing the truth of the theorem, it would simply bloat the proof if it were added. A variation of Conway's proof that Karamzadeh published with co-authors in 2015,[3] using a lemma proved in the earlier paper, certainly avoids the *deus ex machina* of Conway's own proof, but is less economical. To gain one side of the Hardian triad — inevitability — the price was the sacrifice of another — economy.

In 2016 Michael Neubrand (1947–) made an important clarification: a particular move in a proof could be a *deus ex machina* if it was seen as a one-off trick, but it would not be a *deus ex machina* if it was part of a standard toolkit, known to be useful and ready to be applied.[4] A parallel distinction in narrative would be that a use of magic is a *deus ex machina* in a 'real-world' setting, but is *not* a *deus ex machina* in a fantasy setting where magic is commonplace.

Etsi deus non daretur

Ulianov Montano (1973–) attempted to develop a comprehensive theory of mathematical beauty,[†] and part of his work explored parallels between proof and narrative. He touched upon this analogy in his 2010 doctoral thesis,[5] but only elaborated it in a 2012 essay[6] and then incorporated it into his 2014 book *Explaining Beauty in Mathematics*.[7]

In his essay, he independently almost duplicated the present author's *deus ex machina* analogy in the context of computer proofs and why they are seen as ugly.

Montano argued that a reader enjoyed the structure of a narrative though the *way* in which events lead to one other. Unexpected but consistent developments were pleasing. (In the present author's terms, such developments would be surprises but not *dei ex machina*.) In general, the 'quality of *storytelling* (how events develop and resolve, how a situation leads to another, how fitting is the ending, etc.)' was important in a reader's enjoyment.[1]

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[*] ▸ p. 847

[1] Karamzadeh, 'Is John Conway's Proof …?', p. 4.

[2] *Ibid.*, pp. 5–6.

[3] Gorjian, Karamzadeh & Namdari, 'Morley's theorem is no longer mysterious!'

[4] Neubrand, 'Conway's Nonperpendiculars as a Tool'.

[†] ▸ pp. 795–802

[5] See, for example, Montano, 'Beauty in Mathematics', pp. 133, 204–5, 225.

[6] Montaño, 'Ugly mathematics'.

[7] Montano, *Explaining Beauty in Mathematics*, ch. 3.

This interpretation of the enjoyment of narrative explained why computer proofs are ugly: '*the assistance of the computer impairs the proof's storytelling*'.[2] Montano compared such impaired storytelling to a narrative in which, at the dénouement, the reader is informed that certain events are too complicated to be related in their entirety, but that there is good reason to believe they lead to the happy ending. The reader would feel frustrated or even cheated, which would give rise to 'an expression of aesthetic aversion'.[3] Even if the computer proof was accepted as *epistemologically* sound, it is not *aesthetically* sound, for it would disrupt the act of following the proof.[4] There is a close parallel here with the present author's hypothetical variation on the *Odyssey*.[*]

[Montano became aware of the present author's work while writing his book *Explaining Beauty in Mathematics*, and briefly discussed it.[5]]

Montano also made the important distinction that in considering parallels between proof and narrative, one treats proof as an *intentional* object.[6] Proof as a mathematical object is a static thing, existing — depending on one's ontological commitments — either as a sequence of symbols or in no spatio-temporal location. For Montano, proof, in parallel with narrative, was experienced by taking a certain attitude towards it, which gave rise to two mental states in which aesthetic experience played a role: *performative*, which occurs over time as the reasoning unfolds mentally, and *contemplative*, which happens when the proof has been understood wholly and can be thought of as a whole.[7] Montano held that previous authors had mostly considered the contemplative perspective, and he wanted to shift attention to the performative aspect.[8]

CROSSED STRUCTURES

Apostolos Doxiadis (1953–) co-edited a 2012 collection of essays about connections between mathematics and narrative, and his own contribution considered connections with beauty. He discussed two literary forms that he saw in proof. The first, *chiasmus*, is a structure in which there is sequence $A_1 A_2 \cdots A_{n-1} A_n$ $A'_n A'_{n-1} \cdots A'_2 A'_1$, where each A_i and A'_i are linked somehow. The other, a *ring-composition*, is an analogous structure $A_1 A_2 \cdots A_{n-1} A_n B$ $A'_n A'_{n-1} \cdots A'_2 A'_1$ where there is a central isolated 'pivot'. Such structures exist in the earliest literature.[9] Chiasmi and ring-compositions can be nested and combined.

[1] Montaño, 'Ugly mathematics', p. 24, emphasis in orig.

[2] *Ibid.*, p. 25, emphasis in orig.

[3] *Ibid.*, p. 25.

[4] *Loc. cit.*

[*] ▸ pp. 644–5

[5] Montano, *Explaining Beauty in Mathematics*, p. 34.

[6] Montaño, 'Ugly mathematics', p. 26.

[7] *Ibid.*, pp. 26–7.

[8] *Ibid.*, p. 27.

[9] See, for example, Reece, 'The Three Circuits of the Suitors'.

Doxiadis saw chiamus and ring-compositions in proof. At the most fundamental level, he argued that *modus ponens* and *modus tollens* have the structure of basic ring-compositions:[1]

$$\begin{array}{ll} P \Rightarrow Q & P \Rightarrow Q \\ \quad P & \quad \neg Q \\ Q & \neg P \end{array}$$

He proceeded to outline a typology of chiasmus/ring-composition in Greek mathematics. The type relevant here is what he termed the *long rule-composition*, where the ring-composition was made up of the whole theorem and proof. His first example, which had this structure in a very pure form, was the first result from Euclid's *Elements*:

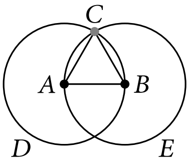
PROPOSITION I.1. '*On a given finite straight line to construct an equilateral triangle.*

[*Proof of I.1.*]

Let AB be the given finite straight line.

Thus it is required to construct an equilateral triangle on the straight line AB.

With centre A and distance AB let the circle BCD be described; [Post. 3] again, with centre B and distance BA let the circle ACE be described; [Post. 3] and from the point C, in which the circles cut one another, to the points A, B let the straight lines CA, CB be joined. [Post. 1]



Now, since the point A is the centre of the circle CDB, AC is equal to AB. [Def. 15] Again, since the point B is the centre of the circle CAE, BC is equal to BA. [Def. 15] But CA was also proved equal to AB;

therefore each of the straight lines CA, CB is equal to AB.

And things which are equal to the same thing are also equal to one another; [*C.N.* 1]

therefore CA is also equal to CB.

Therefore the three straight lines CA, AB, BC are equal to one another.

Therefore the triangle ABC is equilateral;

and it has been constructed on the given finite straight line AB.

(Being) what it was required to do.'[2] I.1

Doxiadis wrote that the 'formal beauty of proposition I.1, as a perfect long [ring-composition], derives from the fact that it involves a single argument'.[3] As he noted, it was rare to find such a structure: most theorem–proof assemblies are more complicated.[4] Though Doxiadis did not make the comparison, the ring-composition structure recalls Proclus' point about the symmetry present in the 'cyclic' nature of proofs, where the conclusion reaches back to the starting-point.*

[1] Doxiadis, 'A Streetcar Named Proof', § 5.4.2.

[2] Euclid, *Elements*, Prop. I.1; structural division from Doxiadis, 'A Streetcar Named Proof', § 5.5.2.a.

[3] *Ibid.*, § 5.5.2.a.

[4] *Ibid.*, § 5.5.2.a.

* ▸ pp. 151–2

Although a ring-composition or chiasmus is partly generated by the semantic content of the proof — for the connections between the components are linguistic, notational, and semantic — the ring-composition or chiasmus itself is *purely* a structural property. Thus an entirely different proof with the same structure would derive beauty in the same degree from that structure. Doxiadis did not claim that the beauty of the proof derives *solely* from the structure, so there is room for different degrees of beauty in proofs that nevertheless have the same structure.

◆

19

This Philosophy of Goodness

Aesthetic value as a guide in mathematics

> ‘I have this philosophy of goodness.
> Mathematics should contain goodness. […]
> It’s a rather crude philosophy but one can always take it as a starting point. […] Most mathematicians do mathematics from an aesthetic point of view and that philosophy of goodness comes from my aesthetic viewpoint.’
>
> — Gorō Shimura,
> quot. in Singh, *Fermat’s Last Theorem*, ch. 5.

> ‘Intuition is glorious,
> but the heaven of mathematics requires much more.’
>
> — Saunders Mac Lane,
> ‘Response to “Theoretical mathematics”’, p. 191.

Towards the end of the nineteenth century, there began to be expressed openly the idea that aesthetic value can serve as a guide in mathematical thinking. Within this broad idea, at least three currents can be distinguished as to how aesthetic value could, should, and did direct mathematical thought. Each connects to the same fundamental idea: mathematics that is aesthetically pleasing is in some sense ‘good’; aesthetic value points to other kinds of value.

Aesthetic value is a useful guide in seeking truth. A mathematician who faces a choice of a conjecture to make or of a proof technique to try should, *ceteris paribus*, first try the more aesthetically pleasing alternative. This position is much weaker than a simplistic equation of beauty and truth, although it parallels a notion of correctness that the art theorist Clive Bell (1881–1964) linked to aesthetic appreciation:

> ‘Before we feel an aesthetic emotion for a combination of forms, do we not perceive intellectually the rightness and necessity of the combination?’[1]

Aesthetic value points towards importance. Mathematicians should prefer to tackle problems or prove theorems that they consider beautiful, because these will turn out to be important.

Unconscious mathematical creativity involves aesthetic value. In the process of mathematical discovery, when the mathematician is trying to find a solution, aesthetic value plays a role in the unconscious generation of ideas. Naturally, trying to understand an unconscious process presents problems. The historian of mathematics Morris Kline (1908–92) said of the mental processes of the mathematical researcher:

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[1] Bell, *Art*, § 1.i.

> 'Just what goes on in his mind while he works on the problem we do not know, any more than we know exactly what thought processes inspired Keats to write fine poetry or why Rembrandt's hands and brain were able to turn out paintings that suggest great psychological depth.'[1]

And yet earlier in the same paragraph he affirmed that the mathematician must use 'imagination, insight, and creative ability'.[2] At least in outline, *something* can be said about the creative process, and, as will be discussed later, some have gone further and posited a deep role for aesthetic value.

POETICS

Several authors have seen parallels in the creative process in mathematics and literature, comparing mathematicians to poets.[3] In an oft-quoted phrase, the analyst Karl Weierstrass (1815–97) indicated that a kind of poetic creativity was essential to do mathematics well: 'it is true that a mathematician who is not a poet never will be a perfect mathematician'.[4]

[Unhappily, the passage from which this quotation is taken is otherwise repugnant, saying that a lack of imagination or intuition, 'a deficiency which is found in many highly intelligible people', afflicts 'especially those of the Semitic tribe'.[5]]

The quotation is sometimes ascribed to Sofia Kovalevskaya (1850–91), for she mentioned Weierstrass's sentiment in a letter to the writer A. S. Shabelskaya (Alexandra Stanislavovna Montvid?/Montvid-Monvizh?/Tolchinova?;[6] 1845–1921):

> 'it is a science which requires a great amount of imagination, and one of the leading mathematicians of our century[7] states the case quite correctly when he says that it is impossible to be a mathematician without being a poet in soul. [...] the poet has only to perceive that which others do not perceive, to look deeper than others look. And the mathematician must do the same thing.'[8]

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[1] Kline, *Mathematics in Western Culture*, p. 459.

[2] *Loc. cit.*

[3] Pycior, 'Mathematics and prose literature', p. 1641.

[4] Weierstrass, letter to Kovalevskaya, dated 27 Aug. 1883, repr. in Mittag-Leffler, 'Weierstrass et Sonja Kowalewsky', p. 191; trans. C.

[5] Weierstrass, letter to Kovalevskaya, dated 27 Aug. 1883, repr. in *ibid.*, p. 191; trans. C.

[6] Kuchar, 'Shabel'skaia', p. 287.

[7] The nineteenth century.

[8] Kovalevskaya, letter to Shabelskaya, dated 1890, repr. in Kovalevskaya, *Vospominaniya i pis'ma*, p. 314; trans. Kovalévsky, *Her Recollections of Childhood*, app. H, p. 316.

TACTICS

Gotthold Eisenstein (1823–52) wrote in an autobiography that what attracted him to mathematics, besides its content, was the mental processes it involved: the building up of clear and sound results into theories. For him, mathematical beauty could only be perceived by examining the relationship between individual theorems and seeing how they fit together in a theory.[1] And thus

> ‘there is such a thing as a mathematical *tact* or taste [Takt oder Geschmack] that enables one to see immediately whether an investigation will bear fruit, and to direct one’s thoughts and efforts accordingly.’[2]

Hermann Hankel (1839–73) gave as an example the kind of combinatorics practised by Carl Hindenberg (1741–1808), which had once been prominent but had been virtually forgotten.[3] Hankel thought that the cause of the fall of this school was that its members had succumbed to its internal elegance, forgotten about ends, turned the field inwards, and abandoned thought of applicability to other areas of mathematics.[4] He then pointed to a guard mathematicians had against such lapses, using the same terminology as Eisenstein:

> ‘It is, so to speak, a scientific *tact* [*Tact*], which must guide mathematicians in their investigations, and guard them from spending their forces on scientifically worthless problems and abstruse realms, a tact which is closely related to *esthetic* tact and which is the only thing in our science which cannot be taught or acquired, and is yet the indispensable endowment of every mathematician.’[5]

J. W. L. Glaisher* (1848–1928) echoed Hankel in his 1891 presidential address to the British Association for the Advancement of Science:

> ‘The mathematician requires *tact* and good taste at every step of his work, and he has to learn to trust to his own instinct to distinguish between what is really worthy of his efforts and what is not.’[6]

MONSTERS

‘ Logic sometimes breeds monsters. ’

— Henri Poincaré, ‘Mathematical Definitions and Education’, § 5.

The debate over pathological functions in the late nineteenth century was partly shaped by aesthetic feelings about

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[1] Eisenstein, ‘Eine Autobiographie’, pp. 156–7.

[2] Eisenstein, ‘Eine Autobiographie’, p. 157, emphasis added; trans. mod. Biermann, ‘Eisenstein, Ferdinand Gotthold Max’, p. 340.

[3] Hankel, *Die Entwickelung der Mathematik*, pp. 26–7.

[4] *Ibid*., p. 27.

[5] Hankel, *Die Entwickelung der Mathematik*, p. 28; trans. Moritz, *Memorabilia Mathematica*, § 622; emphases corrected.

* ▸ pp. 454–5

[6] Glaisher, ‘Presidential address’, p. 725, emphasis added.

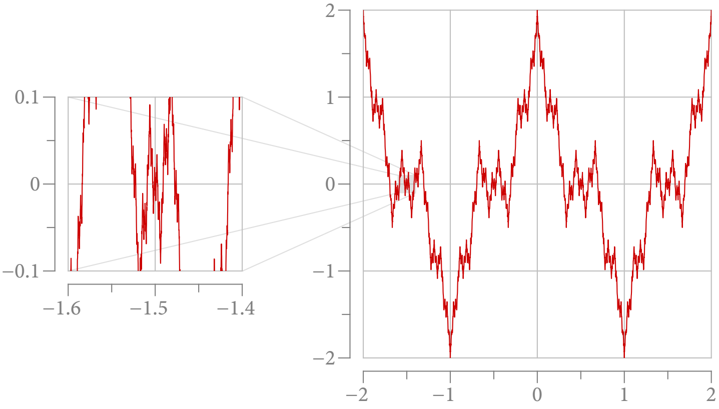

FIGURE 19.1
The Weierstrass function with $a = 3$ and $b = 0.5$ in the range $[-2, 2]$, with detail in the range $[-1.6, -1.4]$.

what a continuous function should be. Until around 1870, most textbooks on calculus stated and proved the theorem that any continuous function was differentiable except at isolated points, often using the intuitive idea that a continuous function must be piecewise monotonic,[1] despite it having been pointed out by 1861 by both Bernhard Riemann (1826–66) and Karl Weierstrass (1815–97) that continuity did not entail differentiability.[2] Hermann Hankel was the first to give a putative counterexample, although his reasoning was not entirely correct.[3] Phillipe Gilbert (1832–92) pointed out the problems in Hankel's arguments, and his tone suggests a certain emotional depth to his conviction that continuity implies differentiability: 'we put into relief some of the principal errors that they contain, so as to leave no doubt, we at least hope, about the inanity of the conclusions [l'inanité des conclusions]'.[4]

In 1872, Weierstrass presented to the Berlin Academy an example of a function that was continuous everywhere and differentiable nowhere, and his example was the first to withstand scrutiny. The so-called *Weierstrass function* is defined by

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n x\pi),$$

where a is a positive odd integer and b is positive and less than 1, is everywhere continuous (see Figure 19.1). It was shown to be nowhere differentiable when ab is greater than $1 + 3\pi/2$ by Weierstrass.[5] There were a number of improvements on the restriction of the product ab by various authors,[6] and its being nowhere differentiable when $ab \geqslant 1$ was proved by G. H. Hardy (1877–1947), who thought that the previous conditions were unnatural and were artifacts of the limitations of the proof methods used.[7]

The emergence of such pathological examples led to the disgust that Charles Hermite (1822–1901) famously expressed: 'I turn with dread and horror [effroi et horreur] from this lamentable plague [plaie lamentable] of continuous functions that have no derivatives.'[8]

[1] Hawkins, *Lebesgue's Theory of Integration*, pp. 43–4.

[2] *Ibid.*, p. 45.

[3] *Loc. cit.*

[4] Gilbert, 'Mémoire sur l'Existence de la Dérivée', p. v; trans. C.

[5] Weierstrass, 'Über continuirliche Functionen'.

[6] See Hardy, 'Weierstrass's non-differentiable function', § 1.2.

[7] *Ibid.*, Theorem 1.31 & p. 303.

[8] Hermite, letter to Stieltjes, dated 20 May 1893, repr. in Hermite & Stieltjes, *Correspondance*, vol. II, p. 318; trans. C.

Among the causes for the initial distaste shown for these functions, there may have been an underlying aesthetic influence. In 1921 the psychologist Oskar Lundholm (1891–1955) published the results of a study in which participants were asked 'to draw lines, each of which was to express the affective tone of an adjective given verbally'.[1] The results, as well as introspective statements made by the participants, indicate that a beautiful line has properties such as 'continuity, roundness of curves, lack of angles'.[2] Intuitively, therefore, continuity is bound up with beauty and roundness of curves, which, in the context of the graph of a function, suggests differentiability.

While it is perfectly legitimate to view the emergence of pathological examples as a failure of visual (and aesthetic?) intuition, one could also conclude that it is the rigorous definition of continuity that fails to encapsulate the intuitive idea of continuity.[3]

ERNST KUMMER

Ernst Kummer (1810–93) said that it was 'most lamentable [maxime dolendum]' that unique factorization into irreducibles does not extend from the rational integers $\mathbb{Z}$ to arbitrary rings obtained by adjoining complex numbers to $\mathbb{Z}$. This difference presents obstacles to proving certain results for these larger rings. For this reason, Kummer thought that the numbers constituting such rings were 'imperfect'.[4] Kummer therefore set out to *change* these rings, to restore unique factorization, so as to preserve the analogy with the rational integers.[5] Kummer seems to have felt that these rings did not have such a beautiful structure as the rational integers, and was thus moved to introduce the concept of 'ideal' numbers with the aim of restoring unique factorization.[6] For instance, in the ring $\mathbb{Z}[\sqrt{-5}]$, the number 6 factors into irreducibles as $2 \cdot 3$ and as $(1+\sqrt{-5})(1-\sqrt{-5})$. Kummer's idea was to adjoin new formal elements that would divide the various factors, so that they are no longer irreducible. Done in the appropriate way, one obtains a new ring containing $\mathbb{Z}[\sqrt{-5}]$ in which unique factorization holds. Kummer seems to have imagined that there should be an analogy between rings such as $\mathbb{Z}[\sqrt{-5}]$ and $\mathbb{Z}$ itself, to have found the absence of the analogy displeasing, and to have thought that there should be a route back to a more pleasing situation.

HENRI POINCARÉ

Henri Poincaré (1854–1912) complained, with evident distaste, of the emergence of the pathological functions:

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[1] Lundholm, 'The Affective Tone of Lines', p. 43.

[2] *Ibid.*, p. 60.

[3] Cf. Davis & Hersh, *Mathematical Experience*, p. 314.

[4] Kummer, 'Sur les Nombres Complexes qui sont Formés avec les ombres Entiers Réels et les Racines de l'Unité', p. 202.

[5] Boniface, 'A process of generalization', p. 491.

[6] Cf. Krull, 'The aesthetic viewpoint in mathematics', p. 51.

> 'Formerly, when a new function was invented, it was in view of some practical end. Today they are invented on purpose to show our ancestors' reasonings at fault, and we shall never get anything more than that out of them.'[1]

Poincaré seems to have had limited interest in a precisely rigorous definition and treatment of functions. He certainly saw its worth, for he described how the definition of continuity was distilled from an intuitive idea; however, he considered it important not to abandon the initial intuitive idea.[2] Elsewhere he wrote in favour of less rigorous arguments: his example was Felix Klein's (1849–1925) proof of the existence of a certain function on a Riemann surface using the idea of replacing the surface with a suitably chosen piece of metal and connecting two points on it to a battery: the resulting electrical current would define the required function. Poincaré said that Klein 'would probably believe he finds in it, if not a rigorous demonstration, at least a kind of moral certainty'.[3]

Poincaré distinguished two types of mind among mathematicians, which are sometimes termed 'geometers' and 'analysts'. These terms did not refer to the subjects upon which they worked, but rather their way of approaching mathematics. The 'geometers' proceeded by intuition, making 'sometimes precarious conquests';[4] the 'analysts' focus on logical rigour and proceeded carefully, 'leaving nothing to chance';[5] thus the 'geometers' were better called *intuitionalists* and the 'analysts' were better called *logicians*. (The term 'intuitionalist' has no connection with intuitionism in philosophy of mathematics.) Klein, Joseph Bertrand (1822–1900), Bernhard Riemann, Sophus Lie (1842–99), were intuitionists; Charles Méray (1835–1911), Charles Hermite, Karl Weierstrass, Sofia Kovalevskaya were logicians. The distinction was not found only among great mathematicians, but in students. Poincaré averred that a mathematician was not trained or educated into one of these ways of thinking; rather, they were born an intuitionist or a logician.[6] Yet he then retreated a little from this position:

> 'I […] insist on the place intuition should hold in the teaching of the mathematical sciences. Without it young minds could not make a beginning in the understanding of mathematics; they could not learn to love it.'[7]

For Poincaré, intuition was key to mathematical invention: 'to make arithmetic, as to make geometry, or to make any science, something else than pure logic is necessary. To designate this something else we have no word other than *intuition*.'[8] However, both types of mind were indispensable for the advance of mathematics: 'Logic […] is the instrument of demonstration; intuition is the instrument of invention.'[9] But more: *appreciation* of mathematics required both. Logic is required to check a proof, but examining each step of a proof with the closest attention did not give one a sense of 'that indefinable

[1] Poincaré, 'Mathematical Definitions and Education', § 5.

[2] Poincaré, 'Intuition and Logic in Mathematics', § vi; Poincaré, 'Mathematical Definitions and Education', § 6.

[3] Poincaré, 'Intuition and Logic in Mathematics', § i.

[4] *Loc. cit.*

[5] *Loc. cit.*

[6] *Ibid.*, § 1.

[7] *Ibid.*, § 4.

[8] Poincaré, 'Intuition and Logic in Mathematics', § iii; emphasis in original.

[9] *Ibid.*, § v.

something that constitutes the unity of the demonstration'; understanding the proof as a whole requires intuition.[1] Poincaré's analogy was that appreciation using logic was like gazing on the work of the mason, while appreciation using intuition was like gazing on that of the architect.[2] This analogy suggests that both forms of appreciation could be aesthetic, for the stonework cut by the mason and the edifice conceived by the architect could both be works of art. However, he elsewhere made another telling analogy: appreciating a proof was like observing a game of chess. Understanding the rules of the game sufficed only for witnessing that the game was a legal one; a deeper understanding was necessary in order to see why a player would make a given move and to perceive the game as an organized and harmonious whole.[3] This second analogy suggests that for Poincaré only the broader, more intuitive view was aesthetic.

Either analogy would dovetail with Poincaré's description of the process of mathematical creativity, in which aesthetic feeling was fundamental, as given in his 1908 essay 'Mathematical Discovery'. This account was based on an introspective study of his own thought-processes, but it agrees with psychological tests and interviews administered to him by Édouard Toulouse (1865–1947) in 1897.[4]

Poincaré held that the unconscious mind was responsible for sudden illumination that provided the solution to a problem one had previously worked on without success. After an initial unproductive effort, the unconscious continues to work on the problem during a period of rest, and the answer rose into the conscious mind later, either during later conscious work on the problem, or at some random moment. Poincaré gave several examples of this phenomenon from his own experience.[5] However, he said that he never solved problems when dreaming.[6]

Poincaré's description fits with the account given by William Rowan Hamilton (1805–65) of his discovery of the quaternions.* For years Hamilton had struggled to find a number system whose elements could be viewed as points in three-dimensional space, just as the complex numbers could be viewed as points in the plane. In Poincaré's terms, this time would constitute the initial periods of effort and incubation. The illumination of a four-dimensional space and the necessary algebraic rules came to Hamilton as he walked to the Royal Irish Academy.

Poincaré's account of the creative process is in parts similar to that given in 1891 by the scientist Hermann von Helmholtz (1821–94). Helmholtz said that he first had to make a long concentrated effort to examine all aspects of a problem. Ideas never arose in this phase, when he was tired or working at his desk. Only after recovering from fatigue would good ideas come, often when awakening, but also especially when walking.[7] Helmholtz's and Poincaré's accounts are clearly parallel, but there are also important differences. [There is no direct evidence that Poincaré knew of Helmholtz's account, although he did

[1] Poincaré, 'Mathematical Definitions and Education', § 6.

[2] *Loc. cit.*

[3] Poincaré, 'Intuition and Logic in Mathematics', § v; Poincaré, 'Mathematics and Logic', § ii.

[4] Toulouse, *Henri Poincaré*; Miller, 'Scientific Creativity'.

[5] Poincaré, 'Mathematical Discovery'.

[6] Toulouse, *Henri Poincaré*, p. 143.

* ▸ pp. 419–20

[7] Helmholtz, 'Erinnerungen', pp. 15–16.

refer to Helmholtz in some of his other essays]. Notably, Helmholtz also described other ways in which ideas arose: sometimes they crept into mind without being recognized until some later circumstance directed attention to their presence; sometimes they arrived without effort.[1] There is nothing in Poincaré's account about ideas *creeping* into mind, and, for Poincaré, prior effort was required. In any case, Helmholtz only described his perception of the emergence of ideas, without delving into the underlying mental process: his remarks were made during a speech at the celebration of his seventieth birthday, not in a philosophical essay.

Poincaré, however, went on to investigate the process that occurred between the initial effort and inspiration striking. The ideas that rose from the unconscious mind in those moments of illumination were 'generally useful and fruitful combinations, which appear to be the result of a preliminary sifting'.[2] That is, although myriad combinations of ideas were formed in the unconscious, only some were selected to pass into consciousness. Poincaré thought that those that do enter the conscious mind 'are those which, directly or indirectly, most deeply affect our sensibility',[3] those that satisfy 'the feeling of mathematical beauty, of the harmony of numbers and forms and of geometric elegance'.[4] He supported this assertion by pointing out that only the more intense sensory impressions can directly gain our attention, although lesser impressions may hold our attention through some secondary cause, such as our deliberate choice.

Now came the most delicate part of Poincaré's argument:

> 'What are the mathematical entities to which we attribute this character of beauty and elegance [...]? Those whose elements are harmoniously arranged so that the mind can, without effort, take in the whole without neglecting the details. This harmony is at once a satisfaction to our aesthetic requirements, and an assistance to the mind which it supports and guides. At the same time, by setting before our eyes a well-ordered whole, it gives us a presentiment of a mathematical law. [...] the only mathematical facts worthy of retaining our attention and capable of being useful are those which can make us acquainted with a mathematical law. Accordingly we arrive at the following conclusion. The useful combinations are precisely the most beautiful'.[5]

[In this argument, Poincaré stepped dangerously close to *circulus in demonstrando*. To be charitable, treat 'giving us a presentiment' and 'making us acquainted' as synonyms, and do not distinguish 'beauty' from 'beauty and elegance'. Define the following notation:

$\mathfrak{B}$ = beautiful mathematical ideas;
$\mathfrak{L}$ = mathematical ideas that give us a presentiment of a law;

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[1] Helmholtz, 'Erinnerungen', p. 15.

[2] Poincaré, 'Mathematical Discovery'.

[3] *Ibid.*

[4] *Ibid.*

[5] *Ibid.*

$\mathfrak{A}$ = mathematical ideas that are worthy of retaining attention;
$\mathfrak{C}$ = mathematical ideas that are capable of being useful;
$\mathfrak{U}$ = mathematical ideas that are useful.

The first three sentences of the above quotation say (P1) $\mathfrak{B} = \mathfrak{L}$; the fourth says (P2) $\mathfrak{A} \cap \mathfrak{C} = \mathfrak{L}$; the last two sentences draw the conclusion (C1) $\mathfrak{B} = \mathfrak{U}$. Now, (P1) and (P2) entail $\mathfrak{B} = \mathfrak{A} \cap \mathfrak{C}$, but to reach (C1) requires the hidden premiss (PX) $\mathfrak{A} \cap \mathfrak{C} = \mathfrak{U}$. Now, by definition $\mathfrak{U} \subseteq \mathfrak{C}$, so (PX) says that among the mathematical ideas that are *capable of* being useful, those that *are* useful are those that are 'worthy of retaining our attention'. This statement recalls Poincaré's earlier comments about our attention being engaged by things that appeal to our aesthetic sensibility. If this is what Poincaré meant by 'worthy of retaining our attention', then the argument seems to be circular. It is more likely that the phrase 'the only mathematical facts worthy of retaining our attention and capable of being useful' is imprecise, and should be read as 'the only useful mathematical facts'; with this reading, the apparent logical fallacy evaporates. This problem is not an artifact of translation; the relevant part of the original French reads: 'les seuls faits mathématiques dignes de retenir notre attention et susceptibles d'être utiles'.[1]]

Poincaré made a disclaimer about his conception of the role of the unconscious: it was 'only a hypothesis'.[2] However, he pointed out that occasionally false ideas arise out of the unconscious, and these, although discarded when scrutiny reveals their falsehood, are such that they would, if correct, appeal to the sense of mathematical elegance.

In addition to the role of aesthetic choice in the unconscious, Poincaré also saw a connection between aesthetic feeling and 'economy of thought'.* He held that this economy of thought, a concept that he drew from Ernst Mach's (1838–1916) philosophical work, had benefits for the development of new mathematics:

> 'the more we see this whole clearly and at a single glance, the better we shall perceive the analogies with other neighbouring objects, and consequently the better chance we shall have of guessing the possible generalizations.'[3]

That is, this economy of thought is both 'a source of beauty and a practical advantage'.[4] Mach himself held that economy of thought attained its zenith in mathematics:

> 'The greatest perfection of mental economy is attained in that science which has reached the highest formal development, and which is widely employed in physical inquiry, namely, in mathematics. Strange as it may sound, the power of mathematics rests upon its evasion of all unnecessary thought'.[5]

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[1] Poincaré, 'L'invention mathématique', p. 58.

[2] Poincaré, 'Mathematical Discovery'.

* ▸ pp. 477–8

[3] Poincaré, 'The Future of Mathematics'.

[4] Poincaré, *The Value of Science*, Preface.

[5] Mach, 'The Economical Nature of Physical Inquiry', p. 195.

For Poincaré, aesthetic feeling as a guide was so important in mathematics that ‘the man who has it not will never be a real discoverer’;[1] one who was incapable of it would never be counted a true mathematician: ‘all true mathematicians’[2] recognize it; it was a ‘special sensibility’ that ‘all mathematicians know, but of which laymen are so ignorant that they are often tempted to smile at it’.[3] However, he conceded that, at the highest level, the same was also true in other disciplines: ‘Only a privileged few are called to enjoy it fully, it is true, but is not this the case for all the noblest arts?’[4]

Finally, Poincaré drew a historico-political conclusion about the importance of mathematical beauty, a conclusion that today one must wince at:

> ‘If the Greeks triumphed over the barbarians and if Europe, heir of Greek thought, dominates the world, it is because the savages loved loud colors and the clamorous tones of the drum which occupied only their senses, while the Greeks loved the intellectual beauty which hides beneath sensuous beauty, and that this intellectual beauty it is which makes intelligence sure and strong.’[5]

AFTER POINCARÉ

E. W. Hobson. Poincaré’s view on the connection between aesthetic value as a guide to importance and ‘economy of thought’ was echoed, perhaps independently, by the mathematician E. W. Hobson (1856–1933), who said that the value mathematicians attached to elegance and conciseness was not merely aesthetic, although aesthetic instinct could serve as a guide. Rather, striving for these qualities often led to the discovery of new relationships and ‘to a synthesis in which unity takes the place of an earlier diversity, leading to a true economy of thought’.[6] [There is a faint echo of Jean-Pierre de Crousaz’s[*] (1663–1750) and Francis Hutcheson’s[†] (1694–1746) ‘uniformity in variety’ here, but no hint whether it was conscious or unconscious.]

Max Dehn. The mathematician Max Dehn (1878–1952) sounded a note of caution about aesthetic value as a guide. Generalization to concepts that were far from the starting intuitions, while potentially yielding beautiful results, could also lead to contradictions.[7] Dehn gave the example of naïve set theory. This field contains Georg Cantor’s (1845–1918) celebrated proof of the uncountability of the continuum, the so-called ‘diagonal argument’, often cited as a beautiful theorem.[8][‡] But the field also ultimately contains the fundamental contradiction known as Russell’s paradox. In naïve set theory, and in Gottlob Frege’s (1848–1925) axiomatization, one can posit a set

[1] Poincaré, ‘Mathematical Discovery’.

[2] *Ibid.*

[3] *Ibid.*

[4] Poincaré, ‘Analysis and Physics’, § I.

[5] Poincaré, *The Value of Science*, Preface.

[6] Hobson, *Mathematics*, p. 16.

[*] ▸ pp. 356–7

[†] ▸ pp. 356–7

[7] Dehn, ‘The Mentality of the Mathematician’, p. 23.

[8] See, for example, Hardy, *Apology*, § 13.

[‡] ▸ p. 848

comprising all sets that are not members of themselves. Symbolically, such a set R is defined by $R = \{S : S \notin S\}$. But then one has the contradiction $R \in R \Leftrightarrow R \notin R$, where the implication in both directions is immediate from the definition of R.[1]

Dehn also made the important point that, since the start of the nineteenth century, having some kind of guide as to the direction of research had become much more important for mathematicians. Before this time, mathematics (at least within the western tradition) was seen to have a unique subject-matter. Afterwards, with the emergence of non-euclidean geometries, it was realized that different but equally-valid systems could be devised. Some direction was required, for arbitrariness would likely result in ugliness and contradiction.[2]

Wolfgang Krull. In an inaugural lecture delivered in 1930, Wolfgang Krull* (1899–1971) described a tension between beauty and rigour which seems related to the distinction Poincaré made between intuition and logic. For Krull, beauty in mathematics might sometimes be achieved at the cost of complete rigour. He gave the example of Felix Klein's work, which had a 'particularly captivating charm',[3] due to the emphasis on geometric, visual reasoning, which was expressed through figures that 'illustrate the underlying mathematical relationships in an extremely simple and transparent way',[4] but which nevertheless contained flawed proofs. This tension did not mean that aesthetic value should be prized above rigour, but there was a question of balance between the two qualities:

> 'A mathematician who is concerned above all with the irrefutable certainty of his results will try to base his theorems on as few unproved assumptions as possible. Consequently he will only feel secure in geometry, for instance, when he has completely reduced that subject to arithmetic […]; he will even try to reduce the system of whole numbers to something even simpler, perhaps a system of logic. In short, he will devote himself to what people in mathematics nowadays call the study of foundations.
>
> The more aesthetically oriented mathematician will have less interest in the study of foundations, with its painstaking and often necessarily complicated and unattractive investigations. He will of course unfailingly fit his proofs to the rigor of his time, but he will not rack his brains about whether his theorems are proved in a way that will necessarily be considered absolutely flawless under all conditions for all eternity.'[5]

Krull also said that in doing mathematics he was motivated by creating harmonious order out of chaos: he would sense a hidden order behind the seeming disorder and this would drive him to search for the 'magic word' that would fit the pieces together.[6]

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[1] Russell, letter to Frege, dated 16 Jun. 1902, repr. in van Heijenoort, *From Frege to Gödel*, pp. 124–5.

[2] Dehn, 'The Mentality of the Mathematician', p. 24.

* ▸ pp. 492–4

[3] Krull, 'The aesthetic viewpoint in mathematics', p. 50.

[4] *Loc. cit.*

[5] *Loc. cit.*

[6] *Ibid.*, p. 52.

Raymond Wilder. In a 1944 examination of the nature of proof, Raymond Wilder (1896–1982) drew an important distinction between aesthetic value as a guide on the individual and on the community level. Individual intuition might be *usually* reliable, but could run into error. But he thought that the collective intuition of mathematicians, over the long term, could be relied upon. A conjecture, or a theorem with a proof of greater or lesser rigour, was accepted by the 'mathematical electorate' depending on whether it was a 'decidedly elegant and artistic piece of mathematical statement'.[1]

Marcus du Sautoy. The idea that some kind of aesthetic sense is necessary to be a successful mathematician is still present: Marcus du Sautoy (1965–) wrote in 2003, perhaps consciously echoing Poincaré, whom he cited elsewhere, that '[o]nly those with a special aesthetic sensibility are equipped to make mathematical discoveries'.[2]

JACQUES HADAMARD AND *THE PSYCHOLOGY OF INVENTION IN THE MATHEMATICAL FIELD*

'Few branches of mathematics were uninfluenced by the creative genius of Hadamard';[3] so concludes his entry in the *Dictionary of Scientific Biography*. Here the focus is not Jacques Hadamard's (1865–1963) creative genius, but his study of it in *An Essay on the Psychology of Invention in the Mathematical Field* (1945). This book, which was a conscious successor to Poincaré's essay 'Mathematical Discovery', was a study of the process and direction of discovery in mathematics, based upon Hadamard's introspection and upon correspondence with other mathematicians.

Hadamard held that aesthetic concerns played an important role mathematical discovery. He deliberately followed Poincaré's account of creativity in the first five sections of his book,[4] but his discussion was influenced[5] by four-stage model of creativity developed by the psychologist Graham Wallas (1858–1932) and described in his 1926 book *The Art of Thought*. Wallas identified four stages from the works of Helmholtz, Poincaré, and others. In Helmholtz, he discerned the first three stages: *preparation*, when the problem is closely and deliberately investigated; *incubation*, when the problem is not consciously considered; and *illumination*, when an idea emerges. From Poincaré, he added the fourth stage of *verification*, another period of deliberate effort in which the solution idea is verified.[6] Hadamard, though, split the fourth stage into three parts: first, the actual process of checking that inspiration has not deceived the mathematician; second, the process of 'precising', in which the idea is stated formally and exactly

[1] Wilder, 'The nature of mathematical proof', p. 318.

[2] du Sautoy, *The Music of the Primes*, p. 78.

[3] Mandelbrojt, 'Hadamard, Jacques', p. 5.

[4] See Hadamard, *Psychology of Invention*, p. 12.

[5] See, for example, *ibid.*, pp. 25, 28–39.

[6] Wallas, *The Art of Thought*, ch. IV, esp. pp. 79–81.

(which G. H. Hardy called 'a tiresome and subsidiary but essential process'[1]); finally, the process of continuing research.[2]

For Hadamard, the process of discovery fundamentally consisted in combining ideas. In the vast space of possible combinations of ideas, most lacked interest, and only a select few were potentially fruitful.[3] The conscious mind only perceived these few; the actual combination of ideas, to an extent random, was performed by the unconscious.[4] The few that were perceived by the conscious mind are sifted out by virtue of being 'those which, directly or indirectly, affect most profoundly our emotional sensibility'.[5] In sum, Hadamard held:

> 'that invention is choice
>
> that this choice is imperatively governed by the sense of scientific beauty.'[6]

Hadamard agreed with Poincaré that the selection of combinations of ideas is performed by the unconscious, although perhaps a different part of the unconscious from that which generates the combinations.[7] Introspection led Hadamard to be in complete agreement with Poincaré on this point:

> 'I feel exactly as he does [...] on account of my own auto-observation — because the ideas chosen by my unconscious are those which reach my consciousness, and I see that they are those which agree with my aesthetic sense.'[8]

Hadamard was quite emphatic that the only means of choosing which research to pursue is aesthetic, regardless of whether one assigns that name to it.

> 'The guide we must confide in is that sense of scientific beauty, that special esthetic sensibility [...].
>
> [...]
>
> Concerning the fruitfulness of the future result, about which, strictly speaking, we most often do not know anything in advance, that sense of beauty can inform us and I cannot see anything else allowing us to foresee. [...] Without knowing anything further, we *feel* that such a direction of investigation is worth following; we feel that the question *in itself* deserves interest, that its solution will be of some value for science, whether it permits further applications or not. Everybody is free to call or not to call that a feeling of beauty.'[9]

To support his view, Hadamard gave several examples from his own experience, and pointed to work of Élie Cartan (1869–1951) and Vito Volterra (1860–1940) that later found application in physics, and to Johann I Bernoulli's (1667–1748) brachistochrone problem,* which

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[1] Hardy, Review of *The Psychology of Invention*, p. 112.

[2] Hadamard, *Psychology of Invention*, ch. v.

[3] *Ibid.*, p. 29.

[4] *Ibid.*, pp. 29–30.

[5] *Ibid.*, p. 31.

[6] *Loc. cit.*

[7] *Ibid.*, p. 32.

[8] *Ibid.*, p. 39.

[9] *Ibid.*, p. 127, emphases in orig.

* ▸ pp. 331–3

led to the calculus of variations.[1] Another apposite example might be Hadamard's inequality: for an $n \times n$ complex matrix A,

$$|\det A|^2 \leqslant \prod_{i=1}^{n}\Big(\sum_{j=1}^{n}|a_{ij}|^2\Big).$$

This result was proved by Hadamard in 1893, and later found application in Fredholm theory.[2]

However, the aesthetic choice of the direction of research to pursue was made by the conscious mind, whereas the selection of combinations of ideas in inspiration was made by the unconscious mind.[3] Note, however, the difference between Hadamard and G. H. Hardy:* the latter would have mathematics pursued, guided by an aesthetic choice of research topics, regardless of any possible applications, while the former saw aesthetic value as a means of choosing fruitful topics to pursue. And fruitfulness, for Hadamard, included possible utility. Thus Hadamard did not present aesthetic value as a *justification* for mathematics, but merely as a guide. And he thought that one could not find practical applications by looking for them.[4] Thus, for Hadamard, the *only* way to obtain practically applicable mathematics was by following the aesthetic sense.

Hadamard quoted[5] Poincaré's remarks† on how real mathematicians had an aesthetic sense, which he thought was indispensable to the process of discovery.[6] Thus he implicitly joined Poincaré in judging that someone who lacked such an aesthetic sense for mathematics could not be a mathematician, but he did not explicitly emphasize this point.

In considering differences in mathematical minds, Hadamard said he had 'insuperable difficulty' in gaining a deep understanding of the theory of groups, while acknowledging that it contained 'most beautiful discoveries'.[7] It is regrettable that Hadamard did not expand upon this point. It might suggest that aesthetic appreciation of a mathematical theory, even for a mathematician of Hadamard's calibre, was *not* dependent on a profound grasp of it. On the other hand, it could equally suggest that Hadamard was able to aesthetically appreciate individual results without deep understanding of the theory as a whole.

AFTER HADAMARD

Albert Einstein. One of Hadamard's correspondents was Albert Einstein (1879–1955), whose letter was published as an appendix to the *Psychology of Invention*. Einstein wrote elsewhere that he had not strong enough mathematical intuition to distinguish fundamental and dispensable questions.[8] Ernst Straus (1922–83) wrote that Einstein repeatedly told him that he chose physics because mathematics

[1] Hadamard, *Psychology of Invention*, pp. 127–9.

[2] Maz'ya & Shaposhnikova, *Jacques Hadamard*, § 13.1.

[3] Hadamard, *Psychology of Invention*, pp. 130–1.

* ▸ pp. 507–9

[4] *Ibid.*, p. 124.

[5] *Ibid.*, p. 31.

† ▸ pp. 659–60

[6] *Loc. cit.*

[7] *Ibid.*, p. 115.

[8] Einstein, 'Autobiographical Notes', pp. 15, 17.

had too many beautiful problems that one could work on without finding the important ones.[1]

G. H. Hardy. G. H. Hardy reviewed the *Psychology of Invention* and agreed with much it had to say about the creative process, writing that 'a quite ordinary mathematician'[2] could recognize the process Poincaré and Hadamard described. Elsewhere, Hardy had written that conjecture formation came about by 'an effort of intuition'.[3] However, he was sceptical of the Poincaré–Hadamard view that the choice of ideas that rise into the conscious mind is due to an unconscious aesthetic selection, although he admitted that his dislike of mysticism might prejudice him.

M. L. Cartwright (1900–98) recalled that Hardy once said 'he preferred analytical arguments to geometrical ones because in analytical arguments one's pen guided one',[4] which fits with Hardy's own statement: 'I think almost entirely in words and formulae [...] I find thought almost impossible if my hands are cold and I cannot write in comfort'.[5] For Cartwright, what Hardy wrote here suggested that his work was mostly deliberate effort, owing 'comparatively little to incubation and sudden flashes'.[6] However, this inference seems too hasty, for the admittedly limited evidence suggests that the distinction between visual or non-visual mathematical thinking has no link with working through incubation and inspiration: Poincaré and Hadamard both experienced inspiration after incubation, but Poincaré thought in an auditory way and seemed not to make use of visual imagery at all,[7] while Hadamard thought in an entirely visual and non-linguistic way.[8] If Poincaré and Hadamard were correct about the aesthetic sifting in the unconscious, then this difference in their modes of thinking suggests that the action of the sieve was independent of conscious thought being visual or non-visual.

Helmut Hasse. The mathematician Helmut Hasse (1898–1979), who was noted for his work in algebra and number theory, thought beauty was often a guide to mathematical discovery, although his model of the process was very simple. He described how, confronted by an unsolved problem and no solution presenting itself, one could imagine what the solution would be if it were beautiful. Examples would confirm the imagined solution, which could then be established by a proof.[9] He saw a parallel between art, where the composition of a masterpiece should begin with an overall mental conception, and the development of a proof, which would begin only after seeing the truth of the result. And to see this truth, one should seek beauty.[10]

Marston Morse. In 1959, the mathematician Marston Morse (1892–1977) gave a lecture exploring parallels between mathematics and the arts. He spoke chronologically after Hadamard wrote the *Psychology of Invention*, but he mentioned only Poincaré's account of the

[1] Straus, 'Paul Erdős at 70', pp. 245–6.

[2] Hardy, Review of *The Psychology of Invention*, p. 112.

[3] Hardy, *Ramanujan*, p. 16.

[4] Cartwright, 'The Mathematical Mind', p. 43.

[5] Hardy, Review of *The Psychology of Invention*, p. 114.

[6] Cartwright, 'The Mathematical Mind', p. 43.

[7] Toulouse, *Henri Poincaré*, p. 166.

[8] Hadamard, *Psychology of Invention*, pp. 75 sqq.

[9] Hasse, *Mathematik als Wissenschaft, Kunst und Macht*, p. 12.

[10] *Loc. cit.*

experience of mathematical discovery.[1] His views certainly sat in the same tradition: mathematical discovery was grounded in some unconscious aesthetic selection. Morse emphasized that this process defied explanation:

> 'It is rather the result of mysterious powers which no one understands, and in which the unconscious recognition of beauty must play an important part. Out of an infinity of designs a mathematician chooses one pattern for beauty's sake, and pulls it down to earth, no one knows how.'[2]

For Morse, this puzzling aspect of discovery pointed to a 'psychological and spiritual'[3] affinity between mathematics and the arts.

J. E. Littlewood. J. E. Littlewood (1885–1977) gave in his collection of essays *A Mathematician's Miscellany* a brief account of mathematical creativity that largely paralleled the views of Poincaré and Hadamard. He included 'working out' among the phases of creativity,[4] which suggests an influence from Hadamard (who called it 'precising') rather than from Poincaré. That said, he assigned less importance to this phase, saying it was 'within the range of any competent practitioner, given the illumination'.[5]

Like Poincaré and Hadamard, Littlewood agreed that the creative process involves aesthetic appreciation, although the isolated comment he made about it suggests that he thought this appreciation was part of the period of concentrated effort:

> 'A mathematician's normal day contains hours of close concentration, and leaves him jaded in the evening. To appreciate something of high aesthetic quality needs close attention, easy to the unfatigued; but a strain for the fatigued mathematician.'[6]

However, Littlewood made an important distinction between the creative processes in science or mathematics and in music. In science or mathematics, the creative process led to a single crucial insight.

> 'In the case of a symphony, incubation would be a more continuous process and many separate illuminations would be called for. And then the final miracle of a great symphony is the welding into an organic whole.'[7]

Presumably, the same distinction would be present in the creation of literature; if so, possible parallels between the aesthetic appreciation of proof and of literature* might not be mirrored by closeness in creation.

[1] Morse, 'Mathematics and the Arts', p. 57.

[2] *Ibid.*, pp. 56–7.

[3] *Ibid.*, p. 56.

[4] Littlewood, *Miscellany*, p. 191.

[5] *Loc. cit.*

[6] *Ibid.*, p. 196.

[7] *Ibid.*, p. 192.

* ▸ pp. 634–50

SRINIVASA RAMANUJAN

> ‘ "He used to say that Ramanujan's mathematics was so beautiful it almost made him believe in God." ’
>
> — Gaurav Suri & Hartosh Singh Bal, *A Certain Ambiguity*, p. 38.

Paul Erdős (1913–96) recounted how G. H. Hardy (1877–1947) rated mathematicians by ‘pure talent’ on a scale of 0 to 100, giving himself a rating of 25, J. E. Littlewood (1885–1977) 30, David Hilbert[*] (1862–1943) 80, and Srinivasa Ramanujan (1887–1920) 100.[1] While Ramanujan's extraordinary brilliance in mathematics was evident during his early life in India, his devotion to the subject and concomitant neglect of other studies led to his failure to obtain a degree. He held a series of clerical posts while he continued to pursue mathematics.[2] He came to the notice and gained the support of P. V. Seshu Aiyar (1872–1935) and R. Ramachandra Rao (c. 1871–1936),[3] and in 1913 sent a now-famous letter listing some of his discoveries to Hardy. Hardy recognized Ramanujan's genius and, not without difficulty, brought him to Cambridge in 1914. His almost legendary mathematical successes, both alone and in collaboration with Hardy, were terminated only by his illness and untimely death.[4]

Ramanujan was a mathematician of the first class whose work seems to attract unvarying approbation for its beauty, but who lacked a systematic education in mathematics. Sometimes it took a trained eye to see the beauty in what he had found: Hardy noted that Ramanujan had ‘rediscovered an astonishing number of the most beautiful analytic identities’,[5] among which was the functional equation for the Riemann zeta function:[†]

$$\zeta(1-s) = 2(2\pi)^{-s}\cos\tfrac{1}{2}s\pi\,\Gamma(s)\zeta(s).$$

This formula appeared ‘in an almost unrecognisable notation’ in Ramanujan's notebooks.[6] Judgements of beauty like Hardy's, and Ramanujan's unconventional approach to discovery using induction and intuition in preference to rigorous proof, make it necessary to say something here about the role of beauty in his work.

Beautiful work

Ramanujan's works, both published and unpublished, seem to be almost universally acclaimed as beautiful. J. E. Littlewood, reviewing Ramanujan's collected works, wrote that ‘[t]he beauty and singularity of his results is entirely uncanny’,[7] though he noted their oddity. Hardy noted the beauty of many of the formulae Ramanujan had discovered before he came to Cambridge.[8] In his obituary of Ramanujan, he wrote that Ramanujan's work ‘has not the simplicity and the inevitableness of the very greatest work; it would be greater if it were less strange’, but that it had ‘profound and invincible originality’.[9]

Bruce C. Berndt (1939–), who analysed and published Ramanujan's notebooks, consistently and frequently described his math-

[*] ▸ pp. 462

[1] Berndt, *Ramanujan's Notebooks*, vol. I, p. 14.

[2] Seshu Aiyar & Ramachandra Rao, ‘Srinivasa Ramanujan’, pp. xii–xiii.

[3] *Ibid.*, pp. xiii–xiv.

[4] For biographical details, see, for example, Seshu Aiyar & Ramachandra Rao, ‘Srinivasa Ramanujan’; Hardy, *Ramanujan*, ch. I; Kanigel, *The Man Who Knew Infinity*.

[5] Hardy, *Ramanujan*, p. 14.

[†] ▸ p. 708

[6] *Loc. cit.*

[7] Littlewood, Review of *Collected papers*, p. 426.

[8] Hardy, *Ramanujan*, pp. 13–14.

[9] Hardy, ‘Srinivasa Ramanujan’, p. lviii.

ematics as beautiful;[1] the pattern continued when he and George E. Andrews (1938–) analysed and published the manuscripts that are romantically known as Ramanujan's 'lost notebook'.[2] Berndt once wrote that Ramanujan's work 'is normally characterized by beauty and elegance'.[3]

What of Ramanujan himself? The reconstruction of how he discovered certain results in his notebooks suggests that he may have been influenced by aesthetic concerns in choosing an order of presentation,[4] but the only instance of explicit aesthetic terminology that can be definitively attributed to Ramanujan occurs in a letter he wrote to Hardy a few months before he died:

> 'I discovered very interesting functions recently which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary ϑ-functions. […]
>
> If we consider a ϑ-function in the transformed Eulerian form e.g.
>
> $$\left.\begin{aligned} &1+\frac{q}{(1-q)^2}+\frac{q^4}{(1-q)^2(1-q^2)^2}\\ &\quad+\frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2}+\dots \end{aligned}\right\}\ \text{(A)}$$
>
> $$\left.\begin{aligned} &1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}\\ &\quad+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\dots \end{aligned}\right\}\ \text{(B)}$$
>
> and determine the nature of the singularities at the points $q = 1,\ q^2 = 1,\ q^3 = 1,\ q^4 = 1,\ q^5 = 1,\ \dots$ we know how beautifully the asymptotic form of the function can be expressed in a very neat and closed exponential form. For instance, when $q = e^{-t}$ and $t \to 0$,
>
> $$(\text{A}) = \sqrt{\frac{t}{2\pi}}\exp\left(\frac{\pi^2}{64}-\frac{t}{24}\right)+o(1),$$
>
> $$(\text{B}) = \frac{\exp\left(\frac{\pi^2}{15t}-\frac{t}{60}\right)}{\sqrt{\frac{5-\sqrt{5}}{2}}}+o(1),$$
>
> similar results at other singularities.'[5]

There is one other example of aesthetic terminology in Ramanujan's work, but it appears in a paper co-authored with Hardy, so it may not have been due to Ramanujan himself. Still, it is worth noting:

> 'The theory of partitions […] consists of an assemblage of formal identities — many of them, it need hardly be said, of an exceedingly ingenious and beautiful character.'[6]

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[1] Berndt, *Ramanujan's Notebooks*, *passim*.

[2] Andrews & Berndt, *Ramanujan's Lost Notebook*.

[3] Berndt, 'Ramanujan's Modular Equations', p. 320.

[4] Berndt, *Ramanujan's Notebooks*, vol. III, p. 282.

[5] Ramanujan, letter to Hardy, dated 12 Jan. 1920, repr. in Berndt, *Ramanujan: Letters*, p. 220; Andrews & Berndt, *Ramanujan's Lost Notebook*, vol. V, p. 312; note markers omitted.

[6] Hardy & Ramanujan, 'Asymptotic formulæ in combinatory analysis', p. 76.

So much of Ramanujan's work has been praised as beautiful or elegant by others that there is little point in giving further examples. But something should be said about his role in the study of the Rogers–Ramanujan identities.* These formulae, which Hardy found so beautiful, attracted Ramanujan's attention. They were first discovered and proved by L. J. Rogers (1862–1933) as corollaries of general results.[1] Ramanujan rediscovered them but had no proof. Rogers later supplied another proof, and in 1919 he and Ramanujan independently gave much simpler proofs using similar principles, which Hardy described as 'the simplest and most elegant proofs'.[2] However, Hardy did not consider any of these proofs to be truly '"simple" and "straightforward"' and said 'no doubt it would be unreasonable to expect a really easy proof'.[3]

[D. M. Bressoud (1950–) published in 1983 a proof which he characterized as 'easy'[4] and which Berndt called 'especially elegant and simple'.[5]]

Finally, it is worth noting that among the formulae in Ramanujan's lost notebook is a set of about forty whose beauty G. N. Watson† (1886–1965) thought comparable to that of the Rogers–Ramanujan identities.[6]

Discovery process

Ramanujan left no clues as to how he discovered many of his results.[7] According to Hardy, Ramanujan reached all of his results 'by a process of mingled argument, intuition, and induction, of which he was entirely unable to give any coherent account'.[8] Littlewood agreed: 'if a significant piece of reasoning occurred somewhere, and the total mixture of evidence and intuition gave him certainty, he looked no further'.[9]

But, thought Hardy, this approach did not mean that Ramanujan differed fundamentally from other mathematicians in the way he did mathematics.[10] The difference in ability was one of degree, not of kind. Hardy judged that Ramanujan had no contemporary rival in his own field.[11] But Hardy thought that there had been previous mathematicians with Ramanujan's virtuosity in series and formulae:

> 'It was his insight into algebraical formulæ, transformations of infinite series, and so forth, that was most amazing. On this side most certainly I have never met his equal, and I can compare him only with Euler or Jacobi.'[12]

But Ramanujan used induction from numerical examples far more than most mathematicians.[13] According to Hardy, he had a phenomenal memory: 'He could remember the idiosyncrasies of numbers in an almost uncanny way. It was Mr. Littlewood (I believe) who remarked that "every positive integer was one of his personal friends".'[14]

[The remark was indeed made by Littlewood, but Hardy only included the attribution after an exchange that Littlewood later recalled:

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* ▸ p. 7

[1] Rogers, 'Second Memoir', § 5.

[2] Rogers & Ramanujan, 'Proof of certain identities', Hardy, preface to.

[3] Hardy, *Ramanujan*, p. 91.

[4] Bressoud, 'An Easy Proof of the Rogers–Ramanujan Identities'.

[5] Berndt, *Ramanujan's Notebooks*, vol. III, p. 78.

† ▸ p. 5

[6] Andrews & Berndt, *Ramanujan's Lost Notebook*, vol. III, p. 218.

[7] See, for example, Berndt, *Ramanujan's Notebooks*, vol. II, pp. 5, 103.

[8] Hardy, 'Srinivasa Ramanujan', p. xxx.

[9] Littlewood, Review of *Collected papers*, p. 427.

[10] Hardy, 'Srinivasa Ramanujan', p. lvii; cf. Hardy, letter to Chandrasekhar, dated 19 Feb. 1936, quot. in Chandrasekhar, 'On Ramanujan', pp. 1364–5.

[11] Hardy, 'Srinivasa Ramanujan', p. xxxv.

[12] *Loc. cit.*

[13] *Loc. cit.*

[14] *Ibid.*, p. lvii.

> 'I read in the proof-sheets of Hardy on Ramanujan: "As someone said, each of the positive integers was one of his personal friends." My reaction was, "I wonder who said that; I wish I had." In the next proof-sheets I read (what now stands), "It was Littlewood who said..."
>
> (What had happened was that Hardy had received the remark in silence and with poker face, and I wrote it off as a dud. [...])'[1]

Presumably the parenthetical 'I believe' was Hardy's private joke in memory of this episode.]

In addition to these abilities, Hardy that thought Ramanujan had 'a power of generalisation, a feeling for form, and a capacity for rapid modification of his hypotheses, that were often really startling'.[2] The 'feeling for form' at least partly suggests an aesthetic instinct at work.

For instance, consider the following celebrated result of Hardy & Ramanujan:

THEOREM T. *'Suppose that*

$$\varphi_q(n) = \frac{\sqrt{q}}{2\pi\sqrt{2}}\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{\mathrm{e}^{C\lambda_n/q}}{\lambda_n}\right),$$

where $C = 2\pi/\sqrt{6}$ *and* $\lambda_n = \sqrt{n - 1/24}$, *for all positive integral values of* q*; that* p *is a positive integer less than and prime to* q*; that* $\omega_{p,q}$ *is a* $24q$*-th root of unity, defined when* p *is odd by the formula*

$$\omega_{p,q} = \left(\frac{-q}{p}\right)\exp\left[-\left\{\frac{1}{4}(2 - pq - p) + \frac{1}{12}\left(q - \frac{1}{q}\right)(2p - p' + p^2p')\right\}\pi\mathrm{i}\right],$$

and when q *is odd by the formula*

$$\omega_{p,q} = \left(\frac{-p}{q}\right)\exp\left[-\left\{\frac{1}{4}(q - 1) + \frac{1}{12}\left(q - \frac{1}{q}\right)(2p - p' + p^2p')\right\}\pi\mathrm{i}\right],$$

where (a/b) *is the symbol of Legendre and Jacobi, and* p' *is any positive integer such that* $1 + pp'$ *is divisible by* q*; that*

$$A_q(n) = \sum_{(p)} \omega_{p,q}\mathrm{e}^{-2np\pi\mathrm{i}/q};$$

and that α *is any positive constant, and* ν *the integral part of* $\alpha\sqrt{n}$*.*

Then

$$p(n) = \sum_{1}^{\nu} A_q\varphi_q + O(n^{-\frac{1}{4}}),$$

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[1] Littlewood, *Miscellany*, p. 61.

[2] Hardy, 'Srinivasa Ramanujan', p. xxxv.

so that $p(n)$ is, for all sufficiently large values of n, the integer nearest to

$$\sum_{1}^{\nu} A_q\varphi_q.\text{'}^{1} \qquad \text{(P)} \qquad \boxed{\text{T}}$$

[The front cover of the first edition of Hardy's *A Mathematician's Apology* was illustrated by a handwritten copy of the page containing the beginning of this theorem from Ramanujan's *Collected Papers*; presumably Hardy chose this page because Theorem T was a particular favourite of his.[2]]

Ramanujan had conjectured, before he came to Cambridge, that the first term of (P) was an approximation to $p(n)$. The next step was to consider an asymptotic series and make the summation limit ν a function of x; this argument involved function-theoretic methods that must have required Hardy's input. But Ramanujan was convinced that there was a way to define the function φ_q so that the error term would be small enough to give the result in the statement of Theorem T. There was essentially no numerical evidence from which to induct.[3] Without Ramanujan's intuitive insight and 'conviction that a closer asymptotic expression *must* exist',[4] the result would not have been conjectured.[5] That it *was* conjectured was ultimately due to what Hardy called Ramanujan's 'feeling for form', which seems to have been at least partly a kind of faith in the aesthetic rightness of mathematics — in this case, in the behaviour of partitions.

Errors and lacunae

While all mathematicians discover theorems using intuition and a feel for what is plausible, these guides are not reliable. Hardy felt this to be particularly true in the field of analytic number theory, 'where even Ramanujan's imagination led him very seriously astray'.[6] Hardy particularly pointed to errors Ramanujan had made in the distribution of primes.[7] Berndt, the editor of Ramanujan's notebooks, agreed that it was in number theory that Ramanujan made most errors. Sometimes his results were true but his reasoning was incorrect, often because he thought that approximations he had made were more accurate than they really were.[8]

Yet in general Ramanujan tended to be correct, if not rigorous. Even his notebooks, which were for personal use and not publication, contain very few serious errors.[9] And some of the errors are illuminating. For instance, define[10]

$$G(x) = \sum_{k=1}^{\infty} \frac{h_k x^{2k}}{(2k)^3}, \quad \text{where } h_k = \sum_{j=1}^{k} \frac{1}{2j-1}.$$

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[1] Hardy & Ramanujan, 'Asymptotic formulæ in combinatory analysis', Thm on pp. 84–5.

[2] Hardy, *Apology*, n. 1.

[3] Littlewood, Review of *Collected papers*, p. 427.

[4] Rice, 'Partnership, Partition, and Proof', p. 11, emphasis in orig.

[5] *Ibid.*, p. 11.

[6] Hardy, *Ramanujan*, p. 16.

[7] Hardy, 'Srinivasa Ramanujan', p. xxx.

[8] Berndt, *Ramanujan's Notebooks*, vol. IV, ch. 23, esp. pp. 51–2.

[9] Berndt, 'An Overview of Ramanujan's Notebooks', p. 146.

[10] Berndt, *Ramanujan's Notebooks*, vol. I, pp. 249, 255.

Ramanujan incorrectly claimed in his notebooks that[1]

$$G(1) = \frac{\pi}{4} \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+3)^3} - \frac{\pi}{3\sqrt{3}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3},$$

Nevertheless, despite the asserted equality being false, Berndt called it a 'beautiful formula'[2] in his editorial commentary. And in the 'lost notebook', there are false claims that editors, Andrews & Berndt, judge beautiful.[3] The apparent beauty found even in Ramanujan's incorrect claims suggests that his process of discovery was guided at least partly by aesthetic concerns.

FRANÇOIS LE LIONNAIS

François Le Lionnais (1901–84) was trained as a chemical engineer, but wrote on science, mathematics, chess, and literature, and was co-founder of the OuLiPo (*Ouvroir de Littérature Potentielle*; 'Potential Literature Workshop') literary movement. OuLiPo's aim was to explore new ways of writing under constraint, and its most impressive product might well be George Perec's (1936–82) novel *La disparition*, whose text contains no letters *e*. Le Lionnais's first manifesto for the movement declared that mathematics was a major inspiration, particularly the abstract structures found therein.[4]

Le Lionnais was the editor of a 1948 collection of essays on *Great Currents of Mathematical Thought* [*Les Grands Courants de la Pensée Mathématique*], enlarged in 1962 and published in English translation in 1971. He contributed an essay on beauty in mathematics,[5] in which he aimed to prepare the way for, but not to establish, a theory of aesthetics of mathematics.[6] It largely comprised a catalogue of examples and an outline of a typology of beauty in mathematics.

The first distinction in Le Lionnais's typology was that between the *facts* and the *methods* of mathematics. Facts comprised both results and objects; methods included proofs. Le Lionnais found beauty in both.[7]

For the second distinction, Le Lionnais looked to art. Works of art can be sorted under two headings: 'CLASSICISM, all elegant sobriety, and ROMANTICISM, which is full of striking effects'.[8] Le Lionnais went on to consider classical beauty and romantic beauty in mathematics, giving examples of each kind of beauty in both facts and methods. It is unclear whether Le Lionnais considered the products of mathematics to be works of art, or if he merely extended the definitions of classicism and romanticism to beauty in mathematics.

This division between classical and romantic beauty was, he held, only a convenience, not an unbending distinction. For example, he thought that the Koch curve (see Figure 19.2) had both romantic beauty and classical beauty, for it was wild in that it fails to have a

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[1] Berndt, *Ramanujan's Notebooks*, vol. I, p. 257.

[2] *Loc. cit.*

[3] Andrews & Berndt, *Ramanujan's Lost Notebook*, vol. III, pp. 193–4.

[4] Le Lionnais, 'La Lipo', pp. 21–2.

[5] Le Lionnais, 'La Beauté en Mathématiques', translated as Le Lionnais, 'Beauty in Mathematics', but sometimes very freely, and some quotations have been distorted by a twofold translation.

[6] Le Lionnais, 'La Beauté en Mathématiques', p. 438.

[7] *Loc. cit.*

[8] *Ibid.*, p. 438; trans. C.

derivative at infinitely many points (romantic) but had an essential unity in that any segment of it embodies the whole structure of the curve (classical).[1] The distinction was more proper in methods than in facts, for here the difference concerned the style of human handiwork and 'essentially reduces to the opposition between the will to equilibrium and the longing for giddiness'.[2] Le Lionnais here seems to have indicated that there was at least partial scope for choosing between methods, and that this choice could be driven by the aesthetic desire for classical or for romantic beauty.

Classical beauty

Le Lionnais wrote that a mathematical result is considered classically beautiful 'when it impresses us either by its austerity, by its control of variety, and even more so when it associates these two impressions in a harmoniously assembled whole'.[3] In the first part of this condition there is an echo of Bertrand Russell's (1872–1970) 'beauty cold and austere' from his essay 'The Study of Mathematics',[*] from which Le Lionnais elsewhere quoted other sentences.[4] But Le Lionnais's 'austerity' differed from Russell's: for Le Lionnais, austerity was a contributor to the perception of beauty; as for Russell, it marked a specific kind of beauty. The phrase 'control of variety' sounds vaguely Hutchesonian, but there is no definite indication that Le Lionnais knew of the work of Francis Hutcheson[†] (1694–1746). Read literally, Le Lionnais's statement that being impressed by a result's austerity or control of variety led to an ascription of beauty is difficult to believe: on its own, each quality seems insufficient for beauty. And an association of them in a harmonious whole, while more believable as entailing beauty, seems to reduce beauty to harmony: at best, a first step towards clarifying beauty.

An example of classical beauty is found in the composite body of results on the triangle,[5] and in particular in the existence of the nine-point circle, on which lie the midpoints of the three sides, the feet of the altitudes, and the points mid-way between the orthocentre and each vertex (see Figure 19.3).[6] Le Lionnais found the inverse relationship between the derivative and integral (the Fundamental Theorem of Calculus, though he did not name it) to be an instance of classical beauty.[7]

Mathematical objects that had classical beauty included magic squares, and Le Lionnais cited in particular Albrecht Dürer's (1471–1528) magic square from *Melencolia I*.[8][‡] Le Lionnais also located

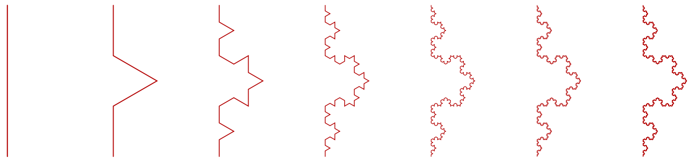

FIGURE 19.2
The first seven iterations of the generation of the Koch curve; the curve itself is the limit of this process.

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[1] Le Lionnais, 'La Beauté en Mathématiques', pp. 448–9.

[2] *Ibid.*, p. 449; trans. C.

[3] *Ibid.*, pp. 439–40; trans. C.

[*] ▸ p. 479

[4] *Ibid.*, pp. 437, 438.

[†] ▸ pp. 357–62

[5] *Ibid.*, p. 440.

[6] *Loc. cit.*

[7] *Ibid.*, p. 444.

[8] *Ibid.*, p. 440.

[‡] ▸ p. 254

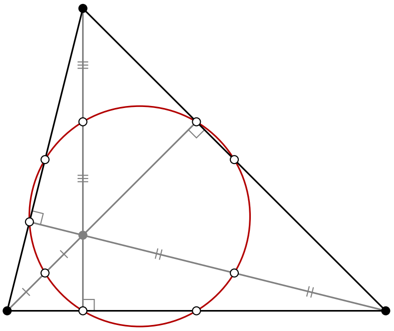

FIGURE 19.3
The nine-point circle: in any triangle, there is a circle on which lie the three midpoints of the sides, the feet of the three altitudes, and the midpoints of the three segments between the orthocentre and the vertices.

classical beauty in the cycloid,* recalling the epithet 'the Helen of geometry', and in the logarithmic spiral, which Jakob I Bernoulli (1655–1705) so admired.[1]†

For Le Lionnais, proof by induction was a method that possessed classical beauty, by virtue of its power and efficiency in leaping in a single bound over an infinite sequence of deductions. Concepts such as Riemann surfaces, which dissolve difficulties that otherwise arise, constituted methods of considerable classical beauty. Classical beauty was also found in methods of unification, such as showing that the circle, ellipse, hyperbola, and parabola all arise from the focus–directrix construction (see Figure 19.4), or appear as conic sections, or can transformed into one another by projective transformations, or are defined by a general quadratic equation in two variables.[2]

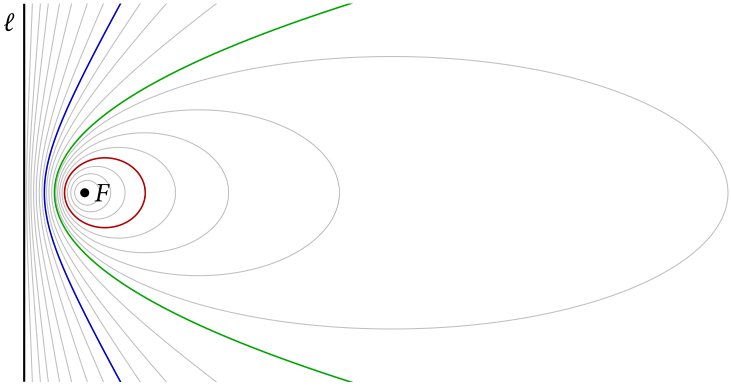

FIGURE 19.4
Any ellipse, parabola, or hyperbola arises as the set of points whose distances d_F from a point F (the *focus*) and d_ℓ from a line ℓ (the *directrix*) satisfy $d_F = ed_\ell$ for some e (the *eccentricity*). When $e < 1$, the points from an ellipse (shown in red for $e = .5$); when $e = 1$, a parabola (green); when $e > 1$, a hyperbola (blue for $e = 2$).

Romantic beauty

For Le Lionnais, the principle of romantic beauty was 'the worship of violent emotions, non-conformity, and strangeness'.[3]

In mathematical results, such romantic beauty appeared, for instance, in the solution of certain differential equations: such solutions could seem wild compared to the original equation. He quoted the

* ▸ pp. 296–7

[1] Le Lionnais, 'La Beauté en Mathématiques', pp. 440–1.

† ▸ pp. 329–30

[2] *Ibid.*, pp. 449–50.

[3] *Ibid.*, p. 444; trans. C.

physicist Silvanus Thompson's (1851–1916) classic text *Calculus Made Easy* on this point;[1] the original passage is as follows:

> 'The solution often seems as different from the original expression as a butterfly does from the caterpillar that it was. Who would have supposed that such an innocent thing as
>
> $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a^2 - x^2}$$
>
> could blossom out into
>
> $$y = \frac{1}{2a} \log_\varepsilon \frac{a+x}{a-x} + C \quad ?$$
>
> yet the latter is the *solution* of the former.'[2]

The volume of a ball of radius R in n-dimensional space and of the corresponding $n-1$ dimensional sphere (the ball's 'surface area') are given by

$$V_n(R) = \frac{\pi^{n/2}}{\Gamma(1+n/2)} R^n \quad \text{and} \quad A_n(R) = \frac{2\pi^{n/2}}{\Gamma(n/2)} R^{n-1}.$$

For fixed R, both $V_n(R)$ and $A_n(R)$ initially increase with n, but reach maxima and subsequently tend to 0 (see Figure 19.5). For Le Lionnais, this result for one of $V_n(R)$ and $A_n(R)$ was an instance of romantic beauty.[3] His account of this phenomenon was slightly confused: in the main text, he described how the 'measurement [mesure]' of the unit sphere varies with dimension. In the notes, he gave the sequence of values of the volume $V_n(1)$ for $n \geqslant 2$. But in the main text he claimed that the formula reaches its maximum at a non-integral value of n between 7 and 8, which is true only for $A_n(1)$. The formula for the volume $V_n(1)$ also reaches its maximum at a non-integral value of n, but it lies between 5 and 6.[4] Regardless of whether Le Lionnais confused the formulae, he was perturbed: the non-integer maximum was 'the most troubling' aspect of the behaviour.[5]

[An intuitive explanation for the behaviour of $V_n(R)$ and $A_n(R)$ is that when n is large relative to R, the inequality $\sum_{i=1}^{n} x_i^2 \leqslant R^2$ — which holds precisely when the point $(x_1, \dots, x_n)$ is a distance at most R from the origin — requires that most of the coördinates x_i are close to 0. Le Lionnais apparently did not see this as a situation where greater understanding lessens beauty.]

Among mathematical objects, Le Lionnais thought that the cycloid, in addition to classical beauty, also had romantic beauty. The latter kind of beauty arose from the cycloid being the solution of the tautochrone and brachistochrone problems,* which was contrary to intuition.[6]

Le Lionnais considered continuous functions with no derivatives to be another locus of romantic beauty.[7] Only forty-five years separated Le Lionnais's judgement here from the 'dread and horror' such functions aroused in Hermite.†

[1] Le Lionnais, 'La Beauté en Mathématiques', p. 446.

[2] Thompson, *Calculus Made Easy*, ch. xx, emphasis in orig.

[3] Le Lionnais, 'La Beauté en Mathématiques', p. 448.

[4] *Ibid.*, p. 448, n. (27) [p. 463].

[5] *Ibid.*, p. 448.

* ▸ pp. 309, 331–3

[6] *Ibid.*, p. 446.

[7] *Ibid.*, p. 448.

† ▸ p. 654

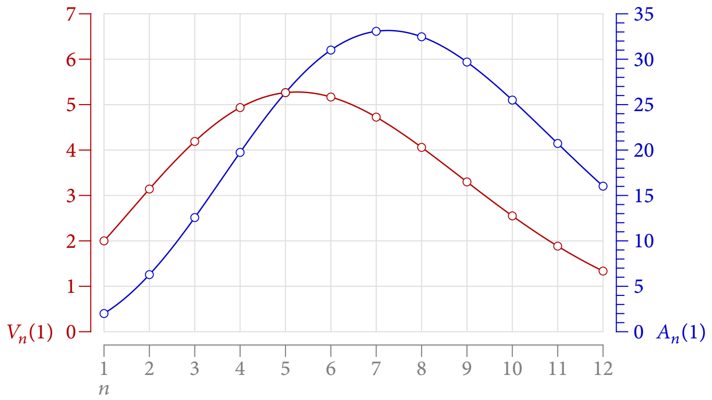

FIGURE 19.5
Graphs of $V_n(1)$ (red) and $A_n(1)$ (blue) for n between 1 and 12.

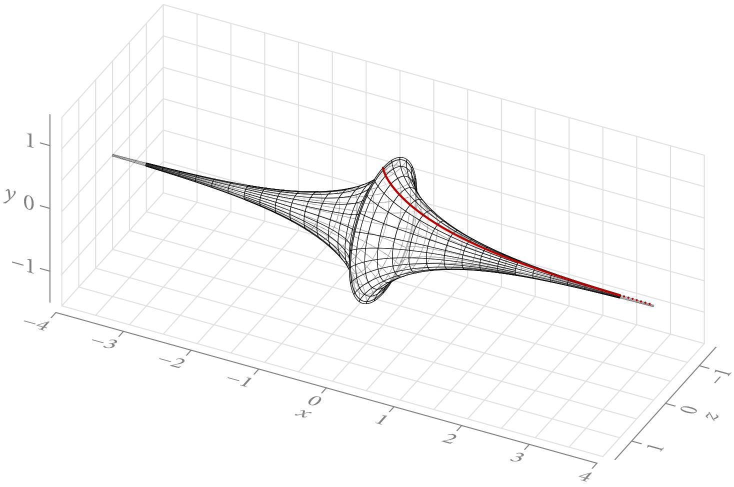

FIGURE 19.6
The tractroid, a pseudosphere formed by rotating the tractrix defined by

$$x = \log\frac{1+\sqrt{1-y^2}}{y} - \sqrt{1-y^2}$$

(shown for $x \geqslant 0$) about the x axis.

Another object that Le Lionnais thought had romantic beauty was the tractroid pseudosphere, for it was in some ways close to the sphere, having constant Gaussian curvature (which is negative, while the sphere's is positive), the same surface area as the sphere, and finite volume (equal to half of the sphere), but was in other respects wild, having infinite extent.[1]

The introduction of new concepts and definitions that broke the habits of intuition or of established practice, such as the developments of non-euclidean geometry,* could be a locus of romantic beauty.[2] Another locus consisted of facts that had no immediate explanation, such as the role of π — a number of geometric origin — in the following result due to Ernesto Cesàro (1859–1906) and rediscovered independently by J. J. Sylvester (1814–97): *The probability of two numbers randomly chosen from* $\{1, \ldots, n\}$ *being relatively prime approaches* $6/\pi^2$ *as n tends to infinity.*[3] Another example was Georg Cantor's† (1845–1918) treatment of infinity;[4] even non-commutative multiplication was, for Le Lionnais, a locus of romantic beauty.[5]

[1] Le Lionnais, 'La Beauté en Mathématiques', p. 462.

* ▸ pp. 427–34

[2] *Ibid.*, pp. 444, 446–7.

[3] Cesàro, 'Question 75'; Cesàro, 'Letter to Professor Sylvester'; Sylvester, 'On the number of fractions', p. 675; Sylvester, 'Sur les nombres de fractions', p. 86.

† ▸ pp. 456–7

[4] Le Lionnais, 'La Beauté en Mathématiques', p. 447.

[5] *Ibid.*, pp. 447–8.

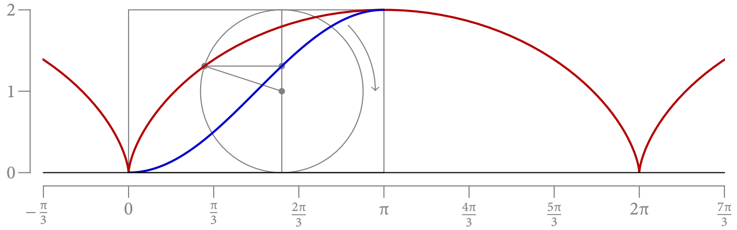

FIGURE 19.7
Roberval's argument for the area under a cycloid.

Le Lionnais thought that romanticism in methods of proof was often synonymous with difficulties.[1] Presumably Le Lionnais meant here that proofs having romantic beauty were often difficult, and not that difficult proofs often had romantic beauty. This interpretation would fit with his connection of romantic beauty to counterintuitive notions and approaches.

As an example of a source of romantic beauty in proofs, Le Lionnais pointed to the indirect approach of *reductio ad absurdum*. The traditional *reductio* proof of the irrationality of $\sqrt{2}$[*] was one of many elementary results whose proof methods were just as, if not more, pleasing than the actual results.[2] Being a mathematical platonist,[3] Le Lionnais found *reductio* proofs just as convincing as direct proofs, but thought that they did not yield illumination. *Reductio* produced 'a very characteristic state of dissatisfaction',[4] but Le Lionnais apparently did not think that this mental state detracted from their beauty.

The proscribed use of infinitesimals, such as in Gilles de Roberval's (1602–75) or Blaise Pascal's (1623–62) proofs that the area under a cycloid is three times that of its generating circle, was nevertheless delightful for Le Lionnais.[5] He did not dwell on this point, but one could imagine that he saw the romantic beauty as arising from the simplicity of the technically invalid proof, the knowledge that the result is actually correct and *can* be demonstrated rigorously, and perhaps a puckish delight in the use of forbidden tools. (Le Lionnais wrote before Abraham Robinson (1918–74) had re-established infinitesimals on a rigorous foundation using model theory.[6])

[Roberval's argument, illustrated in Figure 19.7, is as follows. The half-cycloid is generated as the circle rolls along a semicircumference. The rectangle enclosing the half-cycloid has area equal to twice the circle. The companion curve is generated by the projection of the cycloid generating point to the vertical line through the centre of the circle. This line bisects the enclosing rectangle, since there is a correspondence between vertical lines below and those above it. The area between the cycloid and the companion curve is equal to the area of the semicircle, since there is a correspondence between horizontal lines in these two areas. Combining these statements shows that the area under the half-cycloid is equal to 3/2 times that of the circle.[7]]

[1] Le Lionnais, 'La Beauté en Mathématiques', p. 454.

[*] ▸ p. 846

[2] *Ibid.*, p. 452.

[3] *Ibid.*, pp. 450, 452.

[4] *Ibid.*, p. 452; trans. C.

[5] *Ibid.*, pp. 452, 454.

[6] See Robinson, *Non-standard Analysis*.

[7] Struik, *A Source Book in Mathematics*, § IV.10.

Conclusions

Le Lionnais paid no attention to aesthetic value as a guide in mathematical practice. His most important contribution was in discerning romantic beauty in the kind of wildness that Hermite and Poincaré[*] and others had found distasteful. He seems to have been the first author to see in this wildness a separate species of beauty in mathematics, although some of his examples, such as proof by contradiction, had previously been cited as (generically) beautiful.

For Le Lionnais, mathematics as a totality was beautiful, and those moments when research revealed connections between different disciplines were themselves beautiful. Le Lionnais cited examples such as the Cartesian unification of algebra and geometry (and quoted the historian and philosopher Edgar Quinet (1803–75), who had also found it beautiful[†]); the connection of group theory to the solution of equations, following from Évariste Galois's (1811–32) ideas; and Klein's Erlangen programme of applying group theory to geometry.[1][‡] Despite Le Lionnais's championing of romantic beauty, this beauty in mathematics as a whole was unmistakably classical, and perhaps betrayed at least a non-exclusive preference for classical beauty.

Le Lionnais also made the point that while mathematics is objective, beauty in mathematics is subjective and so mutable. He seems to have been the first author to make this point. A result that at first elicited a feeling of beauty could, with experience, become banal. Le Lionnais gave three examples:

1) The function $y = \mathrm{e}^x$ is its own derivative: amazing, until one realized that it was unsurprising that $y = y'$ has a solution.
2) The equality $m \cdot n = \gcd(m,n) \cdot \operatorname{lcm}(m,n)$ (where $m, n \in \mathbb{N}$): surprising, until one realized that it follows from the definition of the greatest common divisor and least common multiple in terms of prime decompositions.[2]
3) Euler's equation,[§] in the form $\mathrm{i}^{\mathrm{i}} = \mathrm{e}^{-\pi/2}$ or $\mathrm{e}^{\pi\mathrm{i}} + 1 = 0$, was once 'generally considered THE MOST BEAUTIFUL FORMULA OF MATHEMATICS',[3] but had, according to Le Lionnais, come to be seen as obvious and dull.

In such cases, what Le Lionnais termed the *finality* ['finalité'] of the initial bare fact was broken and the underlying reasons show through. This finality, which Le Lionnais thought played a major role in aesthetic pleasure in the sciences, seems to have referred to an *ex nihilo* or apparently uncaused aspect of the fact.[4]

In closing, Le Lionnais briefly mentioned the concerns of utility and motivation:

> 'The beauty of mathematics does not, of course, guarantee its beauty or utility. But it grants to some the power to live incomparable hours, to others, the certainty that mathematics will continue to be cultivated for the greatest good of all and

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[*] ▸ pp. 654, 655–6

[†] ▸ pp. 391–2

[1] Le Lionnais, 'La Beauté en Mathématiques', pp. 456–7.

[‡] ▸ pp. 459–61

[2] *Ibid.*, p. 441.

[§] ▸ pp. 839–43

[3] *Ibid.*, p. 444; trans. C.

[4] *Ibid.*, p. 441.

> the greatest glory of the human adventure, by men who hope for no material profit for themselves.’[1]

HERMANN WEYL

The mathematician, physicist, and philosopher Hermann Weyl (1885–1955) has often been cited as having openly and humorously admitted that his work, both in mathematics and in physics, was guided more by beauty than truth. The actual origin of the celebrated quotation is a recollection that Freeman Dyson (1923–2020) included in his obituary of Weyl:

> ‘Characteristic of Weyl was an æsthetic sense which dominated his thinking on all subjects. He once said to me, half joking, “My work always tried to unite the true with the beautiful; but when I had to choose one or the other, I usually chose the beautiful”. This remark sums up his personality perfectly. It shows his profound faith in an ultimate harmony of Nature, in which the laws should inevitably express themselves in a mathematically beautiful form. It shows also his recognition of human frailty, and his humor, which always stopped him short of being pompous.’[2]

However, in a more serious vein, Weyl warned against unrestricted pursuit of beauty, just as Hankel had[*] before him: ‘Creative activity, if not supervised by reflection, is in danger of running away from all meaning, of losing contact and perspective.’[3]

In particular, choosing problems to tackle because of their interest and challenge, regardless of whether they were important to mathematics as a whole, rankled Weyl. This attitude is evident in the role he played in the Erdős–Selberg controversy over the elementary proof of the Prime Number Theorem,[†] which states that $\pi(n) \sim n/\log n$, where $\pi(n)$ is the number of primes less than n. There had long been a question over whether this result had an elementary proof which, unlike the first proofs by Jacques Hadamard and Charles Jean de la Vallée Poussin (1866–1964), did not rely on the concepts and machinery of complex analysis. Hardy, for instance, said that theorems such as this ‘lie deep’ and that an elementary proof would necessitate a re-evaluation of the depth of results in number theory.[4]

Atle Selberg (1917–2007) in 1948 proved that

$$\sum_{p<x} \log^2 p + \sum_{pq<x} \log p \log q = 2x \log x + O(x). \tag{S}$$

Paul Erdős heard of Selberg’s result and used it to prove[5] that *For any* $\varepsilon > 0$, *for sufficiently large* n *there is always a prime between* n *and*

[1] Le Lionnais, ‘La Beauté en Mathématiques’, p. 458; trans. C.

[2] Dyson, ‘Prof. Hermann Weyl’, p. 458.

[*] ▸ p. 653

[3] Weyl, ‘Axiomatic Versus Constructive Procedures’; cf. Weyl, ‘Levels of Infinity’.

[†] ▸ p. 407

[4] Hardy, ‘Goldbach’s Theorem’, pp. 5–6.

[5] Spencer & Graham, ‘The elementary proof of the prime number theorem’, p. 18.

$n(1 + \varepsilon)$; this result generalizes Chebyshev's theorem[*] that *There is always a prime between* n *and* $2n$. Using Erdős's result, Selberg could deduce the Prime Number Theorem. Erdős wanted to publish a joint paper about the deduction of the Prime Number Theorem from (S).[1] Perhaps as a consequence of hearing only the name of the more eminent Erdős attached to the news, Selberg published alone a paper containing a proof version that did not use Erdős's initial contribution, and which mentioned neither Erdős's role in the original elementary proof nor his part in the discovery that an elementary proof was possible.[2]

In an account of the controversy written in the 1970s for publication after those involved had died[3] (Erdős died in 1996, Selberg in 2007), Ernst Straus (1922–83) criticized Weyl for not having acted to conciliate the dispute. Straus reported that Weyl had strong aesthetic views about the work of both men. Weyl thus took the side of Selberg, whose work he rated very highly, actually standing to applaud Selberg's presentation of a result, against Erdős's, whose jumping from problem to problem he found repellent.[4]

According to Straus, Weyl had for this reason vetoed the renewal of Erdős's grant at the Institute for Advanced Study, and ensured that the *Annals of Mathematics* rejected Erdős's article and published Selberg's.[5]

GEORGE PÓLYA

Despite his many later contributions to the subject, George Pólya (1887–1985) showed no early special aptitude for mathematics. At the University of Budapest, he first studied law, then languages and literature, and only turned to mathematics when he was advised to study it to aid his understanding of philosophy.[6] It was to this late arrival in mathematics that Pólya attributed his lasting interest in teaching, heuristics, and problem-solving.[7]

In 1925, Pólya and Gábor Szegő (1895–1985) published *Aufgaben und Lehrsätze aus der Analysis* (translated as *Problems and Theorems in Analysis*), a collection of problems that took the novel approach of classification by solution method rather than subject.[8] This book was the precursor to Pólya's later works on solving mathematical problems and on mathematical discovery: *How To Solve It* (1945), *Mathematics and Plausible Reasoning* (1954), and *Mathematical Discovery* (1962, 1965).

Pólya believed that beauty should have a role in mathematics education. He wrote in 1969, when he was professor emeritus at Stanford University, that when high-school teachers came to the university for additional training, he gave them a set of twelve aphorisms, some his own, some quoted from others. Two of his own maxims declared:

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[*] ▸ p. 524

[1] Spencer & Graham, 'The elementary proof of the prime number theorem', pp. 21–2; Baas & Skau, 'The Lord of the Numbers, Atle Selberg', p. 644.

[2] Spencer & Graham, 'The elementary proof of the prime number theorem', pp. 21–2; Baas & Skau, 'The Lord of the Numbers, Atle Selberg', p. 643.

[3] Spencer & Graham, 'The elementary proof of the prime number theorem', p. 18.

[4] Straus, 'Note on the Controversy', p. 22.

[5] *Loc. cit.*

[6] Alexanderson & Lange, 'George Pólya', pp. 559–60.

[7] Pólya, quot. in Albers & Alexanderson, *Mathematical People*, p. 256.

[8] Alexanderson & Lange, 'George Pólya', p. 562.

> ‘5. Beauty in mathematics is seeing the truth without effort.
>
> 6. The beauty of a theorem (of a proof, or theory, or classroom presentation) is directly proportional to the number of ideas you can see in it and inversely proportional to the effort it costs you to see them.’[1]

The first of these statements seems to refer to beauty as an *experience*, the second to beauty as a *property*. But they are broadly compatible, and point to beauty (*qua* property) being related to qualities like depth and explanatoriness[*] and simplicity.

Pólya thought that teachers should cover some problems that have mathematical beauty, even if they are more difficult,[2] and followed this rule in his own writing.[3] And a person destined to become a mathematician ‘should enjoy and seek what seems to him simple or instructive or beautiful’.[4]

Pólya’s ideas about problem-solving and discovery allowed a role for aesthetic judgement, although its part was certainly not overmastering. For instance, he gave an account of Johann I Bernoulli’s solution to the brachistochrone problem,[5][†] and wrote that in diagrams such as Figures 9.14 and 9.15 on pages 332–3,

> ‘we may see intuitively the key idea of the solution. If we can see this idea clearly, without effort, anticipating what it implies, we may notice that there is *a real work of art* before us.’[6]

In tackling problems generally, vague aesthetic feelings can act as a guide in helping one judge if one is making progress towards a solution:

> ‘“It looks good to me,” or “It is not so good,” say the unsophisticated. More sophisticated people express themselves with some nuance: “This is a well-balanced plan,” or “No, something is still lacking and that spoils the harmony.”’[7]

Following such feelings frequently leads in the right direction. What we call inspiration is a strong and rapid emergence of such feelings.[8]

For Pólya, the kind of regularity that leads to conjecture formulation by analogy also had an aesthetic component. He noted that the centre of gravity of a homogeneous rod coincided with the centre of gravity of its two ends and divided the rod in a $1 : 1$ ratio, and that the centre of gravity of a homogeneous triangle coincided with the centre of gravity of its three vertices and divided the line connecting any vertex to the midpoint of the opposite side in a $2 : 1$ ratio. Given these facts, Pólya suggested that one should suspect that the centre of gravity of a homogeneous tetrahedron coincides with the centre of gravity of its four vertices and divides the line connecting any vertex to the centre of gravity of the opposite triangular face in a $3 : 1$ ratio. He then asserted:

> ‘It appears *extremely* unlikely that the conjectures suggested by these questions should be wrong, that such a beautiful

---

[1] Pólya, ‘Twelve Aphorisms’, repr. in [Steering Committee], ‘Teaching, Research, and the Faculty’, pp. 48–9.

[*] ▸ pp. 787–90

[2] Polya, *Mathematical Discovery*, p. 122.

[3] Polya, *Mathematics and Plausible Reasoning*, vol. I, p. vii.

[4] Polya, *How To Solve It*, pp. 205–6.

[5] Polya, *Mathematics and Plausible Reasoning*, [vol. I] § IX.4.

[†] ▸ p. 332

[6] *Ibid.*, [vol. I] § IX.4, p. 155, emphasis added.

[7] Polya, *How To Solve It*, pp. 183–4.

[8] *Ibid.*, p. 184.

> regularity should be spoiled. The feeling that harmonious simple order *cannot* be deceitful guides the discoverer both in the mathematical and in the other sciences, and is expressed by the Latin saying: *simplex sigillum veri* (simplicity is the seal of truth).'[1]

V. A. KRUTETSKII AND *THE PSYCHOLOGY OF MATHEMATICAL ABILITIES IN SCHOOLCHILDREN*

Vadim Andreyevich Krutetskii's (1917–91) book *The Psychology of Mathematical Abilities in Schoolchildren* (first published in Russian in 1968; English translation 1976) is focused on mathematics education, not research, but still pays attention to the role of aesthetic judgement in mathematical discovery. Krutetskii gave an overview of Poincaré's and Hadamard's accounts of mathematical creativity, including the roles they assigned to aesthetic considerations, and thought that Hadamard's was the more realistic.[2]

Krutetskii studied the general mathematical abilities of schoolchildren, but only the points where aesthetic considerations were involved are relevant here. Krutetskii characterized elegance as clarity, simplicity, and economy, and found that pupils who were capable in mathematics strove for the most elegant solution to a problem.[3] The more able the pupil, the more evident was this tendency.[4] Pupils whose ability was average or below showed no interest in the quality of their solutions, but capable pupils would usually be unsatisfied with a first solution and would consider whether there was one that was 'economical, rational, and "elegant"'.[5] According to Krutetskii, when they found or were shown such a solution, '[f]rom their emotional reaction it was entirely apparent that they experienced a distinctly aesthetic feeling':[6]

> '"A beautiful solution!" "This method, like a good chess combination, evokes a feeling of pleasure in me," the pupils said. And their whole demeanor testified to the aesthetic feeling they were experiencing: their eyes sparkled, they rubbed their hands in satisfaction and smiled, they invited one another to admire a keen train of thought or a particularly "elegant" solution.'[7]

And conversely they were openly annoyed or frustrated when the only solution they could find was '"crude", unwieldy, or complicated'.[8] Although Krutetskii did not emphasize this point, the pupils had an aesthetic sense that there *should be* an elegant solution.[9]

There was at least some recognition of the role for aesthetic judgement among school mathematics teachers: 48 % of a surveyed group of 56 agreed that '[s]triving for the economy of one's mental powers' was a component of mathematical ability, and some made statements indicated that aesthetic value, or specifically beauty, was involved in finding the simplest or most economical solution.[1]

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[1] Polya, *How To Solve It*, p. 45, emphases added.

[2] Krutetskii, *The Psychology of Mathematical Abilities*, pp. 44–6.

[3] *Ibid.*, p. 283.

[4] *Ibid.*, p. 284.

[5] *Ibid.*, p. 285.

[6] *Ibid.*, pp. 285–6.

[7] *Ibid.*, p. 347.

[8] *Ibid.*, p. 286.

[9] See *ibid.*, example on p. 286.

IMRE LAKATOS AND *PROOFS AND REFUTATIONS*

The best-known fallibilist account of mathematics is the book *Proofs and Refutations* by the philosopher of science and mathematics Imre Lakatos (1922–74). This work had its genesis in Lakatos's 1961 doctoral thesis at the University of Cambridge, parts of which were published in four issues of *The British Journal for the Philosophy of Science* in May 1963–February 1964. The published articles and other selections from the rest of his thesis were assembled into the book *Proofs and Refutations* after Lakatos's death.[2] Only the points where Lakatos's account involves aesthetic concerns are examined here.

The main part of *Proofs and Refutations* consists of two dialogues whose *dramatum personae* comprise a TEACHER and students ALPHA, BETA, GAMMA, …, MU, PI, RHO, SIGMA, OMEGA.[3] The first dialogue begins with the TEACHER presenting a proof of the following famous result:

EULER'S POLYHEDRON FORMULA. *For all polyhedra, $V - E + F = 2$, where V, E, and F are respectively the number of vertices, edges, and faces.* ☐

The TEACHER's proof involves removing a face and deforming the polyhedron (as if it were made of rubber) to become a planar graph (see Figure 19.8), which does not alter the sum $V + E - F$ since the unbounded area surrounding the graph substitutes for the removed face; triangulating the graph, which does not alter the sum $V + E - F$; and then using an induction on the number of triangles to prove that $V + E - F = 2$.[4] The students ALPHA, BETA, and GAMMA then air doubts about each of the main steps in the proof.[5]

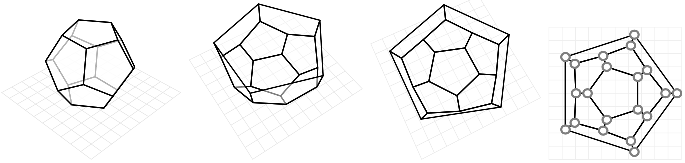

FIGURE 19.8
Deforming a regular dodecahedron to become a planar graph.

The TEACHER characterizes a proof as a thought-experiment that decomposes the original conjecture into subconjectures or lemmas.[6]

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[1] Krutetskii, *The Psychology of Mathematical Abilities*, p. 189.

[2] Larvor, *Lakatos*, p. 8.

[3] Lakatos, *Proofs and Refutations*, chs 1–2, *passim*.

[4] *Ibid.*, pp. 7–8.

[5] *Ibid.*, p. 8.

[6] *Ibid.*, p. 9.

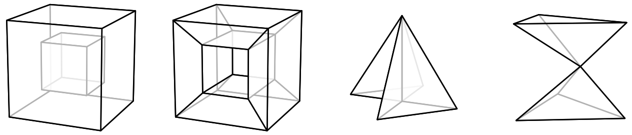

FIGURE 19.9
Some monster 'polyhedra' suggested by Lakatos. From left to right: a cube with a nested cubical void; a 'picture frame'; two tetrahedra joined at an edge; two tetrahedra joined at a vertex.

Some local counterexamples are advanced against certain steps of the proof, but the TEACHER is able to strengthen the proof to resist them.[1]

The discussion then turns to *global* counterexamples that refute the theorem. These are objects that are claimed to qualify as polyhedra but for which the theorem does not hold; Figure 19.9 shows some examples.[2] For Lakatos, there were various possible responses to finding global counterexamples to a conjecture: for instance, one could surrender and abandon the conjecture, one could modify the conjecture, or one could try to deny that the counterexamples were in fact within the scope of the conjecture.[3] This last response Lakatos called 'monster-barring' and the reasons he suggested for excluding the counterexamples were aesthetic:

> 'DELTA: [...] This pair of nested cubes is not a polyhedron at all. It is a *monster*, a pathological case, not a counterexample.'[4]

The character DELTA subsequently echoes Hermite's reaction to continuous non-differentiable functions:*

> 'DELTA: [...] I turn in disgust from your lamentable "polyhedra", for which Euler's beautiful theorem doesn't hold. I look for order and harmony in mathematics, but you only propagate anarchy and chaos. Our attitudes are irreconcilable.'[5]

Lakatos deliberately referred to Hermite,[6] and said that the disputes between refutationists and hermitian monster-barrers characterized the development of the modern theory of real functions in the late nineteenth and early twentieth centuries.[7]

DELTA's distaste for the counterexamples and his decrying them as monsters seem to be at least partly based on their not satisfying 'Euler's beautiful theorem'. That is, DELTA is motivated to exclude such counterexamples by a desire to restore the truth of the beautiful result. In general, DELTA objects to monsters because they 'never foster growth, either in the world of nature or in the world of thought', because they do not fit 'an harmonious and orderly pattern'.[8] Actually excluding these monsters entails refining the definition of 'polyhedron'.

Although DELTA also rejects as a monster the small stellated dodecahedron, one of Johannes Kepler's (1571–1630) star polyhedra,† his basis is less an aesthetic judgement and more an appeal to the physical concept of solidity.[9] On the other hand, the character SIGMA seems sympathetic to Kepler's aesthetic judgements of star polyhedra.[10]

Another character, BETA, dismisses the value-laden language of 'monsters' and says they should simply be called 'exceptions'.[11]

Later in the dialogue, the participants consider the 'method of proof and refutations'. When a global counterexample is discovered, a

[1] Lakatos, *Proofs and Refutations*, pp. 10–12.

[2] *Ibid.*, pp. 13 sqq.

[3] *Ibid.*, § 1.4(a–e).

[4] *Ibid.*, p. 14, emphasis in orig.

* ▸ p. 654

[5] *Ibid.*, p. 19.

[6] *Ibid.*, p. 19, n. 2.

[7] *Ibid.*, p. 19, n. 3 (cont. to p. 20).

[8] *Ibid.*, p. 21.

† ▸ pp. 289–90

[9] *Ibid.*, pp. 16–18.

[10] *Ibid.*, pp. 32–3.

[11] *Ibid.*, p. 24.

suitable lemma that is refuted by the counterexample should be added to the proof, and the conjecture replaced by one that incorporates this lemma as a condition.[1] But aesthetic judgement is required in choosing a *suitable* lemma, for one could exclude counterexamples like two tetrahedra joined at a vertex via a lemma like '[a]ll polyhedra have at least 17 edges':[2]

> 'LAMBDA: [...] The only safeguard against such *ad hoc* methods is to use *suitable* lemmas, i.e. lemmas in accordance with the spirit of the thought-experiment! Or would you drop the beauty of the proofs from mathematics and replace it by a silly formal game?'[3]

But Lakatos thought that seeking completely rigorous analyses of proofs, in the hope that the supply of counterexamples eventually dries up, is counter-productive on aesthetic grounds. Even if 'suitable' lemmas and modifications to the conjecture were used, they would cumulatively rob the conjecture and proof of their aesthetic value:

> 'ALPHA: Lambda's rules for "*rigorous proof-analysis*" deprive mathematics of its beauty, present us with the hairsplitting pedantry of long, clumsy theorems filling dull thick books, and will eventually land us in vicious infinity.'[4]

STANISŁAW ULAM

Stanisław Ulam (1909–84) considered in his memoirs some points previously discussed by Krull* and by Lakatos. He held that, in addition to its beauties, mathematics has a 'homely side':

> 'This homeliness has to do with having to be punctilious, of having to make sure of every step. In mathematics one cannot stop at drawing with a big, wide brush; all the details have to be filled in at some time.'[5]

Here Ulam echoed Krull's opposition between aesthetic value and rigour. But Ulam went further and said that there were results that were 'important, laborious and inelegant',[6] such as work on partial differential equations, which lacked beauty in 'form and style',[7] but which were still deep and had importance stemming from their applications in physics.

Ulam thought that in the decades after 1976, when he wrote, there would be developed an increased understanding of beauty 'even on a formal level',[8] though possibly by then the criteria for beauty would have changed and thus there would be 'super beauty in unanalyzable higher levels'.[9] Previous attempts to analyse aesthetic criteria of mathematics had been too narrow and had not taken into account

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[1] Lakatos, *Proofs and Refutations*, p. 50.

[2] *Loc. cit.*

[3] *Ibid.*, pp. 50–1, emphasis in orig.

[4] *Ibid.*, p. 54, emphasis in orig.

* ▸ p. 661

[5] Ulam, *Adventures of a Mathematician*, p. 274.

[6] *Ibid.*, p. 276.

[7] *Loc. cit.*

[8] *Loc. cit.*

[9] *Loc. cit.*

theories of the external world and the evolution of the human brain.[1] (Regrettably, Ulam did not make clear to which attempts he referred.)

ARMAND BOREL

For Armand Borel (1923–2003), mathematics could be counted as an art precisely because its growth was 'derived from, guided by, and judged according to aesthetic criteria'.[2] He seems to have belonged to the tradition that descended from Poincaré* via Hadamard† in which aesthetic choice plays a role in the unconscious part of mathematical discovery: he wrote that 'progress is achieved by intellectual means, many of which issue from the depths of the human mind, and for which aesthetic criteria are the final arbiters'.[3] But he offered no analysis of the process.

Borel also agreed that aesthetic judgement had a place in the selection of research problems. He gave the specific example of Heinz Hopf's (1894–1971) choice of problems in algebraic topology. They were partly motivated by rational factors, such as being accessible special cases of general problems that could not be tackled using existing techniques,[4] which was a feature of Hopf's work as early as his doctoral thesis.[5] But rational judgement was not the whole story:

> 'It is simply a part of the talent of a mathematician to be drawn to "good" problems, i.e., to problems which turn out to be significant later, even if it is not obvious at the time he takes them up. The mathematician is led to this partly by rational, scientific observations, partly by sheer curiosity, instinct, intuition, *purely aesthetic* considerations.'[6]

But in actually solving problems, aesthetic judgement took at best a subordinate place: any concern for elegance had to be pragmatically suppressed when one attempted a first proof or disproof of a conjecture.[7]

For Borel, what contributed to a judgement of beauty was not just 'purely' aesthetic factors such as elegance or simplicity or originality, but importance and applicability and consequences.[8] This view had the implication that the beauty of a piece of mathematics could change as a result of something external to it:

> 'I would naturally find a method of proof more beautiful if it found new and unexpected applications, although the method itself hadn't changed. It may have become more important, but in and of itself not more beautiful.'[9]

The context suggests that the word 'beautiful' in the second quoted sentence should be understood as indicating 'purely' aesthetic beauty, while in the first sentence it refers to possessing the compound beauty that involves factors such as importance or applicability. Borel seems to have been the first to have articulated explicitly this mutability of beauty according to external mathematical context.

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[1] Ulam, *Adventures of a Mathematician*, p. 276.

[2] Borel, 'Mathematics', p. 11.

* ▸ pp. 655–60

† ▸ pp. 662–4

[3] *Ibid.*, p. 12.

[4] *Ibid.*, p. 15.

[5] Frei & Stammbach, 'Heinz Hopf', pp. 994–5.

[6] Borel, 'Mathematics', p. 15, emphasis added.

[7] Borel, 'On the Place of Mathematics', p. 144.

[8] Borel, 'Mathematics', pp. 15–6.

[9] *Ibid.*, p. 16.

ROGER PENROSE

The mathematical physicist Roger Penrose (1931–) wrote on several occasions on the role of aesthetic judgement in mathematics and science.

Penrose saw aesthetic value in mathematics as having various loci, distinguishing the beauty of a theorem from the beauty of a proof, and observing that beauty is more often attributed to the latter.[1] Separate from both is elegance of presentation.[2] But he seems to have used the terms *elegance* and *beauty* interchangeably: when he wondered what could constitute elegance in mathematics, he suggested 'unexpected simplicity' as a contributing factor, but then mentioned 'the principle of *beauty* in unexpected simplicity'.[3] At the very least, he made no sharp distinction between the two forms of aesthetic value, only between the locus of content and the locus of presentation.

As regards creativity, Penrose's work was located in the Poincaré–Hadamard tradition, although in a 1974 essay on the subject, he remarked:

> 'There is also a more subtle rôle played by aesthetics [...], namely as a means for obtaining results. This is really an important aspect of the rôle of aesthetics which I do not think has been discussed very much.'[4]

The second quoted sentence suggests that, at the time, he was unaware of the work of Poincaré and of Hadamard.* Later, in his 1989 book *The Emperor's New Mind*, Penrose cited Poincaré's and Hadamard's accounts with approval, but admitted that on no occasion had an inspiration struck him suddenly and without warning, as in Poincaré's examples.[5] However, he sometimes had ideas when thinking of a problem in a 'low level' way while undertaking some other task. He recalled how a solution to a problem he had been thinking about once arose during a pause in conversation, but that when the conversation resumed, he forgot that the solution had occurred to him. Only later in the day did an inexplicable 'feeling of elation' cause him to search his memories for its cause, leading him to recall the solution that had earlier come to mind.[6] The context of this anecdote suggests that the feeling of elation was a consequence of the aesthetic value of the solution.

Penrose thought that '*aesthetic* criteria are enormously valuable in forming our judgements'[7] and that, although truth was a vital criterion in mathematics and science,

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[1] Penrose, 'The Rôle of Aesthetics', pp. 266–7.

[2] *Ibid.*, p. 267.

[3] *Ibid.*, p. 267, emphasis added.

[4] *Ibid.*, p. 267.

* ▸ pp. 655–60, 662–4

[5] Penrose, *The Emperor's New Mind*, pp. 541–2.

[6] *Ibid.*, pp. 543–4.

[7] *Ibid.*, p. 544, emphasis in orig.

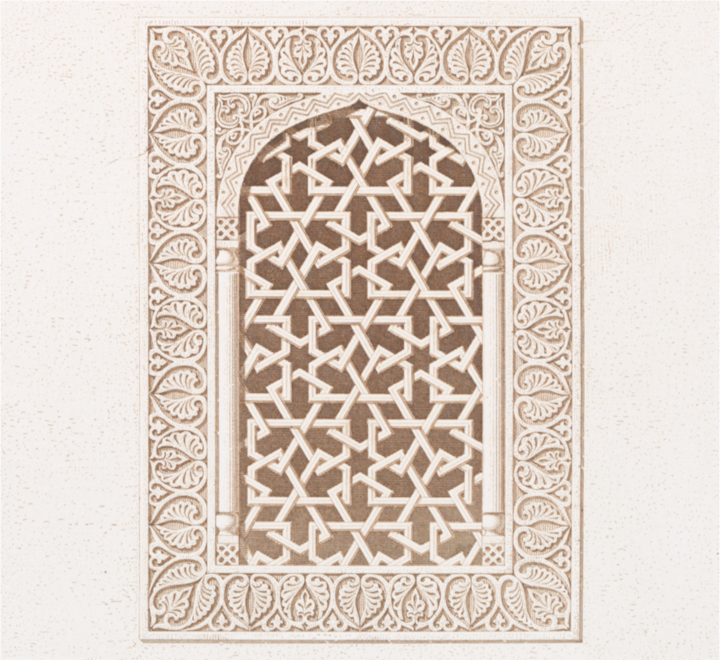

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FIGURE 19.10
Drawing by Émile Prisse d’Avennes of a window of the al-Nāṣir Muḥammad Mosque (Prisse d’Avennes, *L’Art Arabe*, vol. 1, pl. XLVI).

> ‘it seems to be impossible to separate one from the other when one considers the issues of inspiration and insight. [...] A beautiful idea has a much greater chance of being a correct idea than an ugly one.’[1]

That is, the aesthetic value of an inspiration seems to be connected closely to the sense that the inspiration, when it strikes, is valid.

Penrose also considered the relationship between visual and mathematical aesthetic value. A basic grid of squares is too monotonous to evoke any feeling of beauty, while a complex geometrical pattern has more aesthetic value. Penrose’s example was a drawing of a window of the fourteenth-century al-Nāṣir Muḥammad Mosque in Cairo (see Figure 19.10). The window has a geometric structure made up of interlocking trefoil knots, which align into straight lines crossing the arrangement, and the unexpected emergent pattern might contribute to its appeal.[2] But Penrose specifically wondered about the mathematical aspect of its greater aesthetic value compared to a plain grid: ‘I am not sure whether the appeal can be regarded as something just mathematical, or whether there is an essentially different kind of aesthetic appreciation also involved.’[3]

[This example has a pedigree among mathematicians and physicists: Penrose borrowed the image from Hermann Weyl’s[*] printed lectures *Symmetry* (1952);[4] Weyl in turn took the example from the third edition (1937) of Andreas Speiser’s (1885–1970) textbook *Die Theoerie der Gruppen* (first published 1923);[5] Speiser took it from its original appearance in Émile Prisse d’Avennes’s (1807–79) *L’Art*

[1] Penrose, *The Emperor’s New Mind*, p. 544.

[2] Penrose, ‘The Rôle of Aesthetics’, pp. 268–9.

[3] *Ibid.*, p. 269.

[*] ▸ pp. 679–80

[4] Weyl, *Symmetry*, Fig. 62 [p. 111].

[5] Speiser, *Die Theoerie der Gruppen*, Fig. 40 [p. 94].

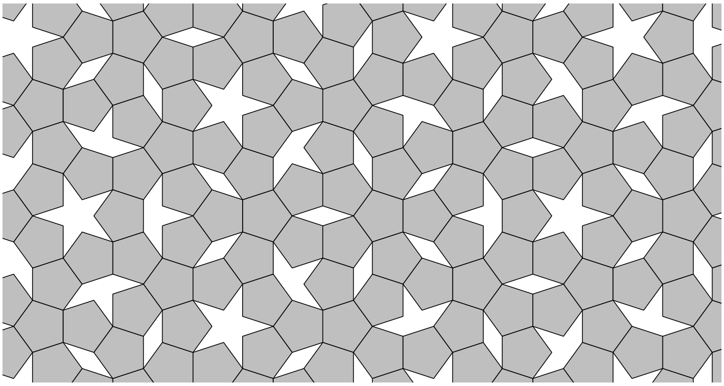

FIGURE 19.11
A portion of the aperiodic Penrose tiling whose tiles are pentagons, rhombi, pentagrams, and 'boats'.

Arabe.[1] It and other illustrations in Speiser's book fascinated the young Chen-Ning Yang* (1922–2025).]

Penrose emphasized that this combination of visual and mathematical aesthetic feeling was important in his own thinking. In particular, he was insistent that when he discovered the aperiodic tiling shown in Figure 19.11, it was the visual *and* mathematical aesthetic qualities of the pattern that gave him the inspiration and the certainty that that his four tiles could be augmented by matching rules so as to allow *only* this aperiodic tiling (and not, for instance, the periodic tiling of the plane using only the rhombic tiles):[2]

> 'Undoubtedly it was the aesthetic qualities of the first of these tiling patterns — not just its visual appearance, but also its intriguing mathematical properties — that had allowed the intuition to come to me [...] that its arrangement could be forced by appropriate matching rules'.[3]

For Penrose, the aesthetic qualities included the many overlapping regular decagons that appear, each surrounded by a ring of ten pentagons, and other, larger rings. Further, each line segment in the pattern lay on a line containing infinitely many segments.[4] There was also aesthetic value in the fact that the pattern almost repeated, but not quite.[5] The details of the matching rules are unimportant here, but they simply restrict which edges of tiles can be placed together in the tiling. They are not unlike those in Penrose's aperiodic kite-and-dart tiling (see Figure 19.12), which he derived from the earlier one.

[Others have recognized the beauty of the Penrose kite-and-dart tiles and tilings: for instance, Martin Gardner (1914–2010) said that the tiling had 'beauty and mystery';[6] Peter Renz (1937–) described the tiles as 'astonishing, fascinating, and beautiful'.[7]]

For Penrose, there was no essential difference between aesthetic value in mathematics and in art. It involved feeling for qualities such as beauty, simplicity, universality, and elegance. In mathematics and in the arts, the aesthetic judgements were very similar. The primary difference was that in mathematics, aesthetic value served both as a

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[1] Prisse d'Avennes, *L'Art Arabe*, vol. 1, pl. XLVI.

* ▸ pp. 916–18

[2] Penrose, *The Emperor's New Mind*, p. 545.

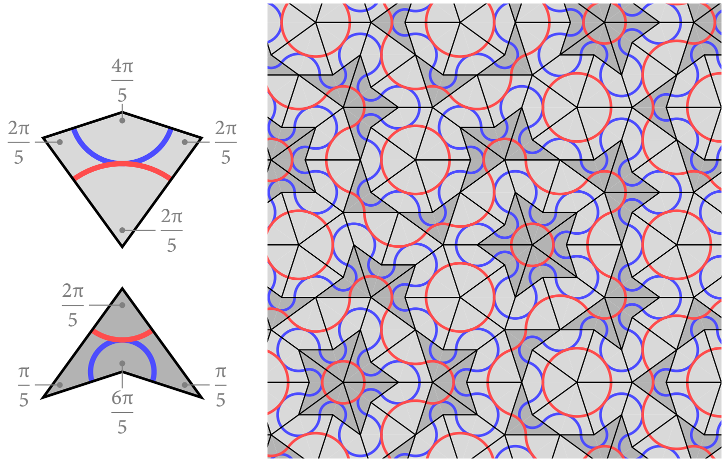
[3] *Loc. cit.*

[4] Penrose, 'Pentaplexity', p. 33; Penrose, 'The Rôle of Aesthetics', p. 270.

[5] Penrose, 'The Rôle of Aesthetics', p. 270.

[6] Gardner, *Penrose Tiles to Trapdoor Ciphers*, p. 7.

[7] Renz, 'Mathematical Proof', p. 84.



The ‘kite’ and ‘dart’ tiles.

A portion of the resulting tiling.

FIGURE 19.12 The kite-and-dart Penrose tiling. The tiles must be arranged so that the blue and red arcs combine into continuous wholly red and wholly blue curves.

means and as an end: as a means, it could guide one more easily to a solution, and as an end, it could motivate one to pursue mathematics.[1]

Penrose held that all evaluation in mathematics and science has, within the bounds set by logic and evidence, a strong aesthetic component.[2] He also thought, like Poincaré and Hadamard before him, that the unconscious mind did not throw out ideas at random, but that there were criteria for selection, which were largely aesthetic, but were influenced by conscious intent.[3] However, he suggested that when inspiration struck, the aesthetic component of conscious evaluation led to an idea being ‘quickly rejected and forgotten if it did not “ring true”’.[4] Such rejection might prevent unappealing ideas from reaching ‘any very appreciably permanent level of consciousness’.[5]

Although Penrose did not explore the consequences of this suggestion, it would have profound implications for accounts of creativity in the Poincaré–Hadamard tradition. If insights that are not aesthetically pleasing are rejected and forgotten, then the descriptions of mathematical creativity offered by this tradition stand on shaky introspective foundations. If only aesthetically pleasing insights are remembered, then introspection and memory would be biased in favour of such insights. There might have been many forgotten insights that were declined on aesthetic grounds, which would in turn weaken the evidence for any aesthetic sifting in the unconscious, for the sifting process might well have thrown up ugly ideas that could not later be recalled.

THE SILVER–METZGER STUDY

In 1989, Edward Silver (1948–) & Wendy Metzger published a study that in was in many ways a parallel for profes-

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[1] Penrose, quot. in Buckley & Peat, *A Question of Physics*, p. 50.

[2] Penrose, *The Emperor’s New Mind*, p. 545.

[3] *Ibid.*, p. 546.

[4] *Ibid.*, p. 545.

[5] *Ibid.*, p. 546.

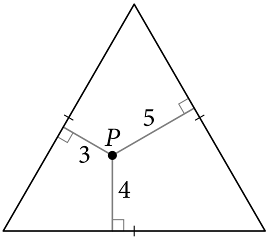

FIGURE 19.13
The 3–4–5 triangle problem: An equilateral triangle contains a point P a perpendicular distance 3, 4, 5 from the three sides. Show that the triangle has height 12.

sional mathematicians of Krutetskii's study of schoolchildren.* Five professors of mathematics, two doctoral students, and one master's student were interviewed and asked to think aloud while solving problems. The problems were drawn from algebra, number theory, and geometry. None was trivial and all lay within the scope of the general mathematical knowledge of the subjects.[1]

All of the subjects of the study were concerned with the aesthetic value of their actual and attempted solutions, and three of the professors were especially so.[2] Aesthetic value functioned both in evaluating solutions and as a guide to a solution.[3]

For instance, the subject 'Professor C' was presented with the '3–4–5 triangle problem' illustrated in Figure 19.13. An equilateral triangle contains a point P a perpendicular distance 3, 4, 5 from the three sides. The problem is to show that the triangle has height 12. Prof. C tried several different approaches, but never obtained a satisfactory solution because each time the equations became messy. He did however suspect that a clearer and simpler solution existed.[4] He was shown a solution involving drawing auxiliary lines from the vertices of the triangle to P (see Figure 19.14), thus creating three triangles whose bases are the sides of the original equilateral triangle and which have heights 3, 4, and 5; since their areas sum to the area of the original triangle, their heights must sum to the height of the original. Prof. C called this 'beautiful' and 'gorgeous' specifically because it avoided calculating the length of the side of the triangle.[5]

Prof. C also tackled the problem of proving that there are no prime numbers in the sequence $10001,\ 100010001,\ 1000100010001, \ldots$. He noticed that the first term factorized as $10001 = 137 \times 73$, which he found neat, though he said this would probably not be useful in solving the problem. Nevertheless he tried to use it as a basis for further investigation. However, this aesthetically-motivated approach was unsuccessful.[6] Silver & Metzger pointed out that there were two points to be noted here: Prof. C was willing to separate aesthetic and expected utility value; and that his pursuit of the aesthetically motivated pattern was flawed.[7]

Another participant, 'Professor D', presented with the problem of finding three points on a circle that are the vertices of the inscribed triangle of maximum area, surmised immediately that he could solve it using calculus, but wanted not to pursue that method because it would yield messy equations. He eventually found the solution —

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* ▸ pp. 682–3

[1] Silver & Metzger, 'Aesthetic Influences', pp. 63–4.

[2] *Ibid.*, p. 64.

[3] *Loc. cit.*

[4] *Ibid.*, p. 69.

[5] *Ibid.*, p. 65.

[6] *Ibid.*, pp. 66–7.

[7] *Ibid.*, p. 67.

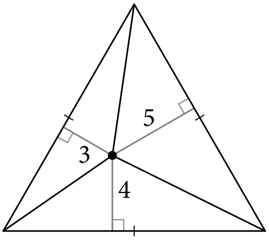

FIGURE 19.14
An elegant solution to the 3–4–5 triangle problem: draw lines from P to the vertices, yielding three triangles with heights 3, 4, 5. Since their areas must sum to that of the original triangle, their heights must sum to the height of the original.

three points that are $2\pi/3$ apart, forming the vertices of an equilateral triangle — using a mixture of symmetry arguments and calculus. But he was dissatisfied because the solution was not explanatory. He thought there should be an explanatory geometrical argument.[1] And he was similarly dissatisfied with the use of algebra rather than a 'neat little drawing' in another geometric problem.[2]

The rejection of potential solutions on aesthetic grounds was given an interesting characterization by one of the postgraduate students: 'You're *afraid* to do something because you feel it'll be so messy that you don't want to face it.'[3] That is, for this participant, it was less their aesthetic response than their *anticipated* aesthetic response that led to a preference for clean or elegant solutions over messy ones.

The strongest aesthetic response to a problem took the form of a complete rejection by a third participant, 'Professor S', who refused to work on the (not very well-defined) problem '[w]hat can you say about the product of any four consecutive integers?'[4] For her, there could be no beauty in the integers and their properties; they were merely a tool.[5] As Silver & Metzger pointed out, in Prof. S's reaction it is difficult to separate cleanly aesthetic value as a guide and aesthetic evaluation of results.[6]

The Silver–Metzger study was (and remains) important as the first systematic study focused on the place of aesthetic considerations in problem solving among professional mathematicians. (The part of Hadamard's study[*] that drew on inquiries addressed to other mathematicians[7] was not directly related to aesthetics.)

Possibly the single most important conclusion of the study was the observation that *when confronted with a more difficult problem, participants were less likely to express aesthetic feeling*. Silver & Metzger believed that this diminishing of aesthetic expression was due to the increased cognitive load concomitant with tackling a difficult problem. Conversely, when a mathematician knew that the problem was soluble *somehow*, there was free cognitive capacity to engage with aesthetic concerns. In support of this conclusion, they pointed to the observed greater incidence of aesthetic appreciation *after*, rather than *during*, problem-solving.[8] That said, Silver & Metzger did recognize that the requirement to think aloud while tackling the problems may have interfered with the expression of usually private aesthetic judgements.[9]

[1] Silver & Metzger, 'Aesthetic Influences', p. 66.

[2] *Loc. cit.*

[3] *Ibid.*, p. 68, emphasis added.

[4] *Ibid.*, p. 72.

[5] *Ibid.*, p. 68.

[6] *Loc. cit.*

[*] ▸ pp. 662–4

[7] Hadamard, *Psychology of Invention*, pp. 83 sqq.

[8] Silver & Metzger, 'Aesthetic Influences', p. 69.

[9] *Ibid.*, pp. 69–70.

PURSUIT OF MORE BEAUTIFUL PROOFS

> ‘ The overwhelming majority of research papers in mathematics is concerned not with proving, but with reproving ’
>
> — Gian-Carlo Rota, ‘Phenomenology of Mathematical Truth’, p. 116.

What motivation is there for re-proving a result? Perhaps the standards of rigour have changed, or perhaps the aim is to demonstrate the usefulness of some new technique or result that enabled the new proof. But another reason has been put forward: to replace existing proofs that lack aesthetic value.[1] When a conjecture is first formulated, the main aim is to prove or disprove it. Aesthetic value in the proof or counterexample that settles the conjecture would be a bonus, but is not a necessity. But re-proof that started from *a desire to re-prove* could be aesthetically motivated and could ultimately yield the best proof:

> ‘the first step is to establish the theorem, so that you can say, “I worked hard. I got the proof. It’s 20 pages. It’s ugly. It’s lots of calculations, but it’s correct and it’s complete and I’m proud of it.” If the result is interesting, then come the people who simplify it and put in extra ideas and make it more and more elegant and beautiful. And in the end you have, in some sense, the Book* proof.’[2]

[But later re-proofs of a theorem might not totally eclipse early ones: E. T. Bell (1883–1960) thought that the first attempts against a problem were more illuminating than later highly-polished and elegant approaches.[3]]

For instance, Erdős, Zhao Ke (Chao Ko; 1910–2002) & Richard Rado (1906–89) proved in 1961 the following fundamental result in extremal combinatorics:[4]

ERDŐS–KO–RADO THEOREM. *Suppose there are m distinct k-subsets of an n-set, where $k \leqslant n/2$, and that each two subsets have non-empty intersection. Then* $m \leqslant \binom{n-1}{k-1}$.

In 1972, Gyula O. H. Katona (1941–) gave a new proof, shorter than the original.[5] The following proof uses the same key ideas as Katona’s, though the exposition and notation are different:

Proof. Let $\mathcal{A}$ be a family of k-subsets as in the theorem. Choose a permutation σ of $\{1, \ldots, n\}$. Place the terms of σ around a circle and consider the sets of k elements that are consecutive in this arrangement. Suppose one such set C is a member of $\mathcal{A}$. Then there are $2k-2$ other sets of consecutive elements that intersect C. But these comprise $k-1$ pairs of disjoint sets, and so at most one of each pair of sets is a member of $\mathcal{A}$ (see Figure 19.15). So at most k of the sets of consecutive elements are in $\mathcal{A}$.

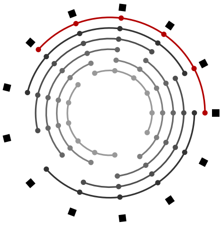

FIGURE 19.15
Illustration of how sets of k consecutive elements intersecting C (red) fall into $k-1$ disjoint pairs (grey).

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[1] Kline, *Mathematics in Western Culture*, p. 470.

[*] ▸ p. 828

[2] Ziegler, quot. in Klarreich, ‘In Search of God’s Perfect Proofs’.

[3] Bell, *The Queen of the Sciences*, p. 11.

[4] Erdős, Ko & Rado, ‘Intersection Theorems for Systems of Finite Sets’, Thm 1.

[5] Katona, ‘A Simple Proof of the Erdös-Chao Ko-Rado Theorem’.

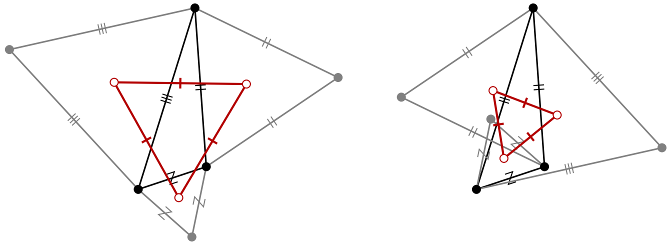

FIGURE 19.16 'Napoleon's theorem': for any triangle, the centres of the equilateral triangles erected on its sides, all outward (*left*) or all inward (*right*), form the vertices of an equilateral triangle. (Coxeter & Greitzer, *Geometry Revisited*, Thms 3.36–7)

Now count in two different ways the number of pairs (A, σ) where $A \in \mathcal{A}$ and A appears as a set of consecutive elements for σ. First, there are $|\mathcal{A}|$ choices for A and one can generate σ by choosing one of k permutations τ of A, one of $(n-k)!$ permutations υ of the elements not in A, and one of n cyclic shifts of the concatenation $\tau\upsilon$ from which to read off σ, giving $|\mathcal{A}|k!(n-k)!n$ possibilities. Second, there are $n!$ choices for σ and, by the first paragraph, at most k sets in A can appear as a set of consecutive elements for σ, giving at most $kn!$ possibilities.

Hence $|\mathcal{A}|k!(n-1)!n \leqslant kn!$ and so

$$|\mathcal{A}| = \frac{kn!}{k!(n-k)!n} = \binom{n-1}{k-1}. \qquad \square$$

The combinatorialists Noga Alon (1956–) & Michael Krivelevich (1966–) thought Katona's new proof beautiful;[1] Terence Tao (1975–) thought it elegant.[2] Martin Aigner (1942–2023) & Günter Ziegler (1963–) included it in each edition of *Proofs from THE BOOK*.[3][*]

And the elementary proof of the Prime Number Theorem by Selberg[†] was described by Ian Hacking (1936–2023) as 'a beautiful case of reducing the assumptions needed to prove a fundamental theorem'.[4] Its beauty may be something to do with the very unexpectedness of the existence of an elementary proof.

Philip Davis (1923–2018) expressed the desire for an elegant and simple proof that would make obvious the truth of 'Napoleon's theorem', which states *The centres of equilateral triangles erected on the sides of a triangle, all outward or all inward, form the vertices of an equilateral triangle*[5] (see Figure 19.16):

> 'there are proofs that can be given in four or five lines [...] but I was indifferent to them. [...] I wanted to append to the figure a few lines, so ingeniously placed that the whole matter would be exposed to the naked eye. I wanted to be able to say not ὅπερ ἔδει δεῖξαι (quod erat demonstrandum), as did the ancient Greek mathematicians, but simply, "Lo and behold! The matter is as plain as the nose on your face." This kind of simplicity and elegance has high appeal and ultimately great transparency, but its discovery may come harder.'[6]

These expressions of an aesthetic motivation in re-proving results are at least compatible with the conclusion of the Silver–Metzger

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[1] Alon & Krivelevich, 'Extremal and Probabilistic Combinatorics', p. 569.

[2] Tao & Vu, *Additive Combinatorics*, p. 280.

[3] Aigner & Ziegler, *Proofs from THE BOOK* (6th edn), pp. 214–15.

[*] ▸ p. 828

[†] ▸ pp. 679–80

[4] Hacking, *Why Is There Philosophy of Mathematics at All?*, p. 14.

[5] Coxeter & Greitzer, *Geometry Revisited*, Thms 3.36–7.

[6] Davis, *Mathematical Encounters of the Second Kind*, p. 17.

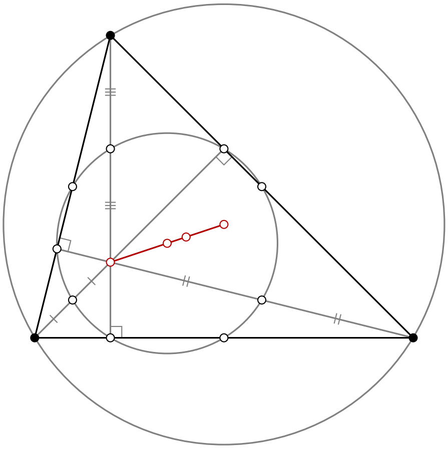


FIGURE 19.17
The Euler segment of a triangle. Four important points lie on this segment. From left to right, they are the orthocentre of the triangle (where the altitudes meet), the centre of the nine-point circle (see Figure 19.3 on page 674), the barycentre, and the centre of the circumcircle. The barycentre divides the segment in the ratio 2 : 1.

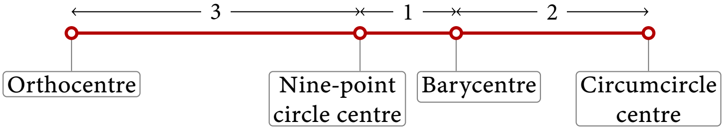
study,* that the cognitive load in tackling a difficult problem — in this case an initially unproved conjecture — tends to decrease aesthetic appreciation. Once the theorem has been proved, perhaps there is more capacity to evaluate aesthetically its proof and seek to improve it or replace it with an entirely new, more beautiful, demonstration.

* ▸ pp. 690–2

DOUGLAS HOFSTADTER

The cognitive scientist Douglas Hofstadter (1945–) thought that skilled mathematicians had a kind of instinct for potential proofs of a result. He draw a parallel: like an expert chess player, a mathematician had enough knowledge and experience to filter out paths that were unlikely to succeed.[1] This filtering is not unlike Poincaré's and Hadamard's posited role for unconscious aesthetic selection, but here Hofstadter did not explicitly postulate a role for aesthetic value.

In an account of his investigation of the role of the Euler and Nagel segments in the geometry of the triangle, Hofstadter described how he was directed by aesthetic judgements. He had found in Julian Lowell Coolidge's (1873–1954) book *A Treatise on the Circle and Sphere* (1916) a list of parallels between the Euler segment and the Nagel segment of a triangle. The *Euler segment* contains the orthocentre of the triangle (where the altitudes meet), the centre of the nine-point circle (see Figure 19.3 on page 674), the barycentre (the arithmetic mean point of the triangle), and the centre of the circumcircle (see Figure 19.17). The centre of the nine-point circle is the midpoint of this segment; the barycentre divides it in the ratio 2 : 1:



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[1] Hofstadter, *Gödel, Escher, Bach*, p. 286.

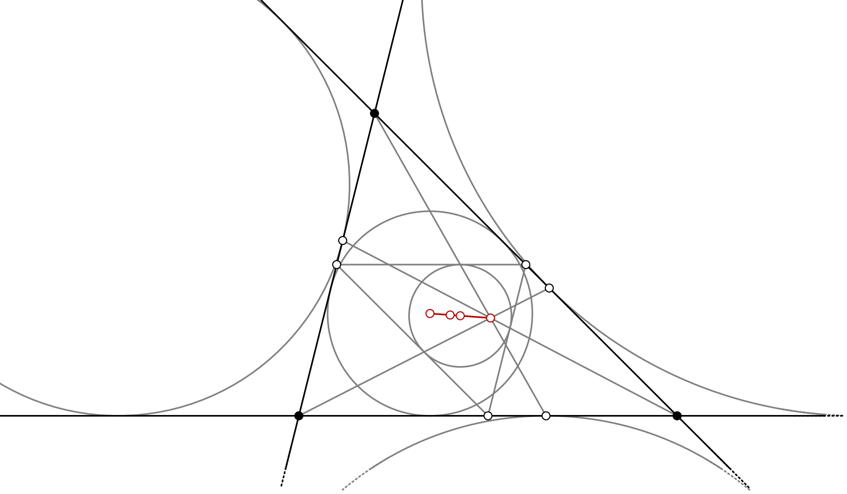
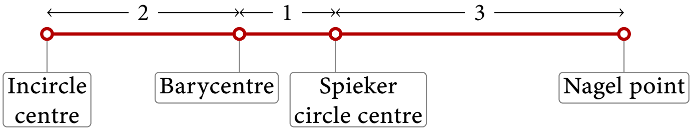


FIGURE 19.18
The Nagel segment of a triangle. Four important points lie on this segment. From left to right, they are centre of the incircle, the barycentre, the centre of the Spieker circle (the incircle of the midpoint triangle), and the Nagel point (where the lines joining vertices to tangent points of excircles meet).

The Nagel segment analogously contains the centre of the incircle, the barycentre, the centre of the Spieker circle (the incircle of the triangle whose vertices are the midpoints of the sides of the original triangle) and the Nagel point (where the lines connecting each vertex to the tangent point of the opposite excircle meet) (see Figure 19.18). The centre of the Spieker circle is the midpoint of the segment; again, the barycentre divides it in the ratio 2 : 1:



Coolidge also listed parallel properties of the nine-point and Spieker circles:[1]

Nine-point circle	Spieker circle
circumcircle of the midpoint triangle	incircle of the midpoint triangle
radius half that of the circumcircle	radius half that of the incircle
circumcircle of the triangle whose vertices are points midway from the orthocentre to the vertices	incircle of the triangle whose vertices are the points midway from the Nagel point to the vertices
passes through the points where the sides are cut by the lines linking the vertices to the orthocentre	tangent to the midpoint triangle where its sides are cut by the lines linking the vertices to the Nagel point

Hofstadter was impressed by what he called 'a beautiful analogy' and the '*analogical unity*'[2] of the segments, but thought that there was something unexplained:

> 'In mathematics, such a striking and intricate analogy can't just happen *by accident*! There's got to be a *reason* for it. But

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[1] Coolidge, *A Treatise on the Circle and Sphere*, p. 56.

[2] Hofstadter, 'From Euler to Ulam', p. 33, emphasis in orig.

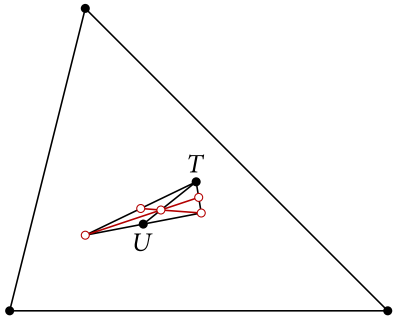

FIGURE 19.19
The Euler and Nagel segments viewed as the two of the three medians of a triangle.

> there was no discussion in Coolidge of why the parallels were so perfect and so systematic.'[1]

Witness Hofstadter's confidence that there had to be some deeper connection that explained the surface analogy. It was this type of faith about which he wrote

> 'I believe that precisely these kinds of undeniably emotional, somewhat irrational reactions to things are the true motivators of the quest for mathematical truth, which is based, after all, on a sense of beauty, something that one certainly cannot put one's finger on with any kind of cool objectivity.'[2]

After some initial investigation, Hofstadter realized that the division of the Euler and Nagel segments in the ratio $2 : 1$ by the barycentre meant that they could be viewed as two of the three medians (segments joining vertices to midpoints of opposite sides) of a triangle (see Figure 19.19).

> 'it seemed to me that since *two* of the medians were known segments having very important endpoints and midpoints (not to mention sublime properties), it almost *had* to be the case that this third median's endpoints and midpoint would also have absolutely fundamental properties.
>
> [...] My new triangle gave me a strong sense of symmetry and closure — the sense of bringing something beautiful but unfinished to its inevitable completion.'[3]

What remained was to discover the properties of the endpoints and midpoint of the new segment — properties that his aesthetic sense told him must exist. His investigations ultimately led him to a theorem that uses auxiliary triangles to connect, in a highly symmetrical way, the points on the Euler segment, the Nagel segment, and the new segment TU formed by the third median, which the barycentre divides in the ratio $2 : 1$. The *auxiliary triangle* of a point X is the triangle whose vertices are the midpoints of the segments connecting X to the vertices of the original triangle.[4] The theorem comprises nine statements; a single example is: *The point U is both the orthocentre of the auxiliary triangle of the Nagel point and the Nagel point of the auxiliary triangle of the orthocentre.*[5]

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[1] Hofstadter, 'From Euler to Ulam', p. 5, emphases in orig.

[2] *Ibid.*, p. 3.

[3] *Ibid.*, p. 9, emphasis in orig.

[4] *Ibid.*, p. 19.

[5] *Ibid.*, p. 15.

Moreover, while testing some of the cases of this theorem by computer before embarking on a proof, he found that one case seemed to be false. But his 'ever-strong belief in beauty and symmetry' led him to conclude that he had simply made an error in entering the geometric construction into the computer, which, it turned out, was in fact the case.[1]

Note that there is an important difference between Hofstadter's account and the Poincaré–Hadamard model. Hofstadter was focused on relating the *method* he used in his investigation, and he saw the role of aesthetic judgement primarily as a conscious guide. Unlike Poincaré and Hadamard, he said nothing about aesthetic choice playing any part in the unconscious generation of ideas.

M. C. ESCHER

An entirely different facet of aesthetic value as a guide to mathematical correctness is found in the work of the artist M. C. Escher (1898–1972), who is famous for his examination of the nature of space and the relationship between three-dimensional space and its depiction on a two-dimensional plane. There are hints of this exploration in some of his early works, in forms that include abstract tessellations, depictions of the regularity of human architecture set against the irregularity of the natural landscape, and pictures that show reflections in flat or spherical mirrors. But it was in 1936, after many years living and travelling in southern Europe, and especially after a second visit to the Alhambra* in May of that year, that he turned more definitively in this direction.[2]

One of Escher's fundamental themes was tessellations of the plane, which he called 'the richest source of inspiration that I have ever struck'.[3] He made a methodical investigation of regular divisions of the plane, using his own techniques,[4] which led to works such as *Day and night* (1938), *Sky and water I* and *II* (1938), and *Verbum* (1942). In a letter written in 1952, he acknowledged that his intentions were 'more reasoned, […] more conscious and cerebral'[5] than those of other artists, and in 1959[6] recognized his mathematical inclinations:

> 'Although I am absolutely innocent of training or knowledge in the exact sciences, I often seem to have more in common with mathematicians than with my fellow artists.'[7]

In his essay 'The Regular Division of the Plane', Escher used an extended metaphor to describe the relationship between his own investigations and those of mathematicians and crystallographers. He imagined the 'beautiful garden' of plane division as surrounded by a wall. He, unguided, clambered over the wall into an uncultivated area and thus initially had to cut his way through until he came upon the 'open gate of mathematics' from which 'well-trodden paths lead in every direction', enabling him to explore freely.[1]

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[1] Hofstadter, 'From Euler to Ulam', p. 18.

* ▸ pp. 228, 229–30

[2] Bool et al., *M. C. Escher*, pp. 50, 52–5.

[3] Escher, *The Graphic Work*, p. 7.

[4] Schattschneider, *M. C. Escher*, ch. 3; Schattschneider, 'The Mathematical Side of M. C. Escher', pp. 708–11.

[5] Escher, letter to Merema, dated 10 May 1952, quot. in Bool et al., *M. C. Escher*, p. 68.

[6] *Ibid.*, p. 55.

[7] Escher, *The Graphic Work*, p. 6.

His investigation of regular divisions of the plane led to an interest in creating patterns with a repeating motif that diminished to 'infinite smallness' as it approached a limit. He had succeeded in creating such patterns where the limit was a point, such as in his *Smaller and smaller* (1956), *Whirlpools* (1957), and *Path of Life I* and *II* (1958).[2] But he had been unable to create one where the limit was a surrounding circle.[3]

The geometer H. S. M. Coxeter (1907–2003) had seen Escher's works in Amsterdam during the 1954 International Congress of Mathematicians and in 1957 asked Escher's permission to use two of his drawings to illustrate a lecture to the Royal Society of Canada. Escher agreed, and as thanks Coxeter sent him the issue of the *Transactions of the Royal Society of Canada* that contained the printed version of his lecture 'Crystal Symmetry and Its Generalizations'. By his own account, Escher was 'shocked' to see in one of the figures in Coxeter's paper precisely the kind of circular limit he sought. The figure[4] depicted a Poincaré disc representation of a hyperbolic tessellation* by triangles.[5] He was greatly impressed by this tessellation, writing:

> 'A circular regular division of the plane, logically bordered on all sides by the infinitesimal, is something truly beautiful, almost as beautiful as the regular division of the surface of a sphere.'[6]

Escher set about trying to deduce how the hyperbolic tessellation had been created. He managed to deduce how to find the centres and radii of the circular arcs comprising the edges of the larger triangles, but hit an impass. He therefore wrote to Coxeter for a simple explanation of how to construct the tiling. Coxeter's reply was minimal insofar as it was understandable by a layman, and barely extended the knowledge Escher had already gleaned from his own empirical investigations of the diagram. Escher was acutely disappointed,[7] writing to his son that 'it seems to be very difficult for Coxeter to write intelligibly to a layman'.[8] Nevertheless, he continued his own study and was able to determine empirically an approximation sufficient to his needs. The outcome pleased him:

> 'My woodcut inspired by the Coxeter system is finished, and I consider it to be the finest of the "reductions" I have made. The circular limitation of infinitely small motifs, surrounding the whole thing logically and inevitably, is something I cannot take my eyes off. It approaches absolute beauty and purity.'[9]

In a later analysis of the geometry of Escher's *Circle Limit* series, Coxeter said that, although Escher had actually *misunderstood* the geometry of the construction, his aesthetic judgement had nevertheless guided him correctly. Escher thought that the white lines in his *Circle Limit III* ought to meet the circular boundary at a right angle.

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[1] Escher, 'The Regular Division of the Plane', p. 156.

[2] Escher, *The Graphic Work*, pp. 9–10, pls 19–21; Schattschneider, *M. C. Escher*, pp. 248–50.

[3] Escher, letter to Coxeter, dated Dec. 1958, quot. in Coxeter, 'The Non-Euclidean Symmetry of 'Circle Limit III'', p. 19.

[4] Coxeter, 'Crystal Symmetry and Its Generalizations', Fig. 7.

* ▸ p. 5

[5] Escher, letter to Coxeter, dated Dec. 1958, quot. in Coxeter, 'The Non-Euclidean Symmetry of 'Circle Limit III'', p. 19.

[6] Escher, letter, dated 9 Nov. 1958, quot. in Bool et al., *M. C. Escher*, p. 91.

[7] Schattschneider, 'The Mathematical Side of M. C. Escher', pp. 712–13.

[8] Escher, letter to his son G. A. Escher, dated 15 Feb. 1959, quot. in Bool et al., *M. C. Escher*, p. 92.

[9] Escher, letter to G. A. Escher, dated 7 Dec. 1958, quot. in *loc. cit.*

But he was incorrect, for these lines were not straight (in the hyperbolic sense): they were parts of equidistant curves.[1] Coxeter admired Escher for his decision to follow his intuition: ‘Escher’s integrity is revealed in the fact that he drew this angle correctly even though he apparently believed that it “ought” to be 90°.’[2]

The point is worth emphasizing: Escher did not do what his mistaken understanding of geometry told him to do: rather, he did what his own visual aesthetic judgement required of him and thus came to the mathematically correct solution.

[An only marginally less impressive example of Escher’s knack for intuiting mathematics is the creation of his *Print Gallery*, which shows a gallery, with a person looking at a picture, which shows a harbour, which includes a building, which contains the original gallery. To depict such a form, Escher, through trial and error and seeking what looked right to him, discovered a distortion of the plane that, on later analysis, was shown actually to approximate a fascinating logarithmic transformation.[3]]

THOMAS TYMOCZKO

The philosopher Thomas Tymoczko (1943–96) examined the role of aesthetic value in directing mathematics in a 1993 essay that was partly a response to Borel’s published lecture.[4][*] Tymoczko saw his essay as a first foray into drawing potentially useful parallels between mathematics and art.[5]

Tymoczko’s most important and novel contribution was an argument that criticism could shape aesthetic judgements of mathematics, just as it did in art. He pointed out that criticism was prior to philosophy of art: the former addressed particular works; the latter the general concepts.[6] (Tymoczko used different language.) A good critic — in the sense of a practitioner of art criticism — was a guide towards perceiving a work of art in a certain way; a great artwork provided multiple ways of seeing. Such a guide could enhance the appreciation of the object, or aid in articulating an unformulated opinion.[7]

But where were the critics of mathematics to be found? Tymoczko was perturbed by Borel’s seeming nonchalance about the inaccessibility of mathematical beauty to non-mathematicians, for such a lack of access implied that mathematical creators and their audience were identical, which would seem to be a block to any useful parallel between mathematics and art.[8] In art, a distance between the creators and the critics was required.[9]

Tymoczko agreed with Borel that one could and should hope for an educated audience, whether in art or in mathematics.[10] And he thus thought that to find critics, once should look to *performers*, rather than *creators*. His analogy was poetry: it was not too strong a

[1] Coxeter, ‘The Non-Euclidean Symmetry of ‘Circle Limit III’’, pp. 23–4.

[2] *Ibid.*, p. 24.

[3] de Smit & Lenstra, ‘The Mathematical Structure of Escher’s Print Gallery’.

[4] Tymoczko, ‘Value Judgments in Mathematics’, p. 67.

[*] ▸ pp. 686–7

[5] *Ibid.*, pp. 67, 76.

[6] *Ibid.*, p. 70.

[7] *Loc. cit.*

[8] *Ibid.*, p. 71.

[9] *Loc. cit.*

[10] *Loc. cit.*

requirement that an audience should be able to *recite* poetry, even if they could not compose it.[1]

He considered what this analogy would mean for proofs, for ‘if anything in mathematics is beautiful, proofs are’.[2] Proofs had two aspects: compositional, which is linked to metaphysical concerns, and performative, which is linked to epistemological concerns and must persuade the audience. The performative side comes to the fore when a mathematician presents the proof to themself or to others, whether working through it on a board or expounding it in a book. For Tymoczko, it was in this performative mode that the mathematician could engage in criticism.[3]

Thus the practice of mathematics, like art, had space for the critic. Tymoczko thought that criticism could serve as a check on the direction of mathematical progress, in particular as an alternative to natural science supplying such a check, such as John von Neumann had suggested.[4][*]

EDSGER DIJKSTRA

The computer scientist Edsger Dijkstra (1930–2002) assigned the highest importance to elegance and beauty in mathematics. He once expressed the value of the former in computer science using a deliberate hyperbole: ‘in the practice of computing, where we have so much latitude for making a mess of it, mathematical elegance is not a dispensable luxury, but a matter of life and death’.[5]

It is difficult to discern to what extent Dijkstra distinguished beauty and elegance. He variously wrote that he understood elegance as meaning ‘ingeniously simple and effective’[6] or ‘simple and surprisingly effective’.[7] But in his paper ‘A collection of beautiful proofs’, he wrote:

> ‘The purpose of this compilation is to collect the material that may enable us to come to grips with the main qualities that together constitute “mathematical elegance”’,[8]

which seems to indicate that he saw beauty either as an aspect of or as a necessary condition for elegance. He elsewhere wrote that the greatest virtues a computer program could have are elegance and beauty[9] and that

> ‘when we recognize the battle against chaos, mess, and unmastered complexity as one of computing science’s major callings, we must admit that “Beauty is our Business”’.[10]

But he thought that computer science should become a very formal branch of mathematics, one that would influence mathematics as a whole much more than physics ever had, and that ‘all this is very exciting and also very inviting because mathematical elegance will play such a central rôle'.[1]

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[1] Tymoczko, ‘Value Judgments in Mathematics’, p. 71.

[2] *Loc. cit.*

[3] *Ibid.*, p. 72.

[4] *Ibid.*, p. 70.

[*] ▸ p. 526

[5] Dijkstra, ‘My Hopes of Computing Science’, p. 442.

[6] Dijkstra, ‘Elegance and effective reasoning’, p. 1.

[7] Dijkstra, ‘On the nature of computer science’, p. 4.

[8] Dijkstra, ‘A collection of beautiful proofs’, p. 174.

[9] Dijkstra, ‘Some meditations on Advanced Programming’, p. 10.

[10] Dijkstra, ‘Some beautiful arguments using mathematical induction’, p. 1.

Dijkstra's desire for beauty or elegance was in part grounded in his experience of a phenomenon whereby some programmers found excitement in *not* fully understanding what they did,[2] which he compared to a lecture audience somehow feeling 'cheated' if a speaker delivers a perfectly clear and lucid talk.[3] But he thought that in large programs, which tended to be ugly, it was vital to seek elegance.[4] He therefore turned to what he termed the 'wider context of mathematical elegance' for insight from the older world of mathematics.[5] (Note that here he implicitly subsumed computer program elegance within mathematical elegance.)

In consulting mathematical colleagues, he found that they considered mathematical elegance to be too heavily based upon personal tastes to have any guiding value.[6] This conclusion would be completely at odds with the general trend which this chapter examines. But there was a twist: the responses he received led Dijkstra to the conclusion that there was *overwhelming* consensus among his colleagues concerning what was mathematically elegant.[7] And yet his mathematical colleagues' belief in the subjectivity of elegance led them to do clumsy things, which did not cause them problems because projects in mathematics were smaller than in programming.[8]

In a library of beautiful proofs he built up,[9] Dijkstra recognized several factors that contributed to elegance. Brevity was the most obvious property they all shared.[10] In particular, brevity tended to exclude repetition, which hid structure and entailed tedium.[11] But brevity alone was too crude a criterion for elegance: some proofs were short because their authors omitted large steps and left the work to the reader; some were short because they appealed to a too-vast library of previous mathematical knowledge ('an egg is being cracked with a sledgehammer').[12]

Dijkstra's concrete example was as follows: Consider a projectile fired with velocity v at an angle φ from the horizontal. Assuming flat ground, homogenous acceleration due to gravity g, and no air resistance, how far from the firing point does the projectile impact the ground? Most of the professional mathematicians he asked solved it using differential equations of motion, solving them, applying the initial conditions, and finally setting $y = 0$ in the trajectory equation. But two of them solved it in a more elegant way: they computed the time of flight from the time required for the vertical component of the velocity to change from $v \sin \varphi$ to the negative of this value, then multiplied this time by the constant horizontal velocity $v \cos \varphi$. In comparison, the theory of differential equations is a 'sledgehammer', which Dijkstra thought the mathematicians should have realized.[13]

[1] Dijkstra, 'On the nature of computer science', p. 4.

[2] Dijkstra, 'Essays on the nature and role of mathematical elegance (1–2)', p. 2.

[3] *Loc. cit.*

[4] *Ibid.*, p. 3.

[5] *Loc. cit.*

[6] *Loc. cit.*

[7] *Loc. cit.*

[8] *Loc. cit.*

[9] See Dijkstra, 'A collection of beautiful proofs'.

[10] Dijkstra, 'Essays on the nature and role of mathematical elegance (1–2)', p. 6.

[11] Dijkstra, 'Essays on the nature and role of mathematical elegance (3)', p. 1.

[12] Dijkstra, 'Essays on the nature and role of mathematical elegance (1–2)', pp. 6–7.

[13] *Ibid.*, pp. 7–8.

TIMOTHY GOWERS

Timothy Gowers (1963–) wrote that 'an appreciation of beauty is essential, or at least very useful indeed, for solving problems'.[1] But his was a pragmatic attitude: it was acceptable to reject a line of investigation as being too inelegant to work, but one could be wrong in making this decision. It was simply *efficient* to try elegant approaches first and turn to inelegant ones if they failed.[2]

Solving a problem or proving a theorem could involve using guesses and intuitions and heuristics. These moves might not survive later attempts to fill in the details and make them rigorous, but 'but it is a good rule of thumb that the more beautiful the guess, the more likely it is to survive'.[3] For Gowers, the initial non-rigorous approach was nicely illustrated by a result of Paul Erdős (1913–96) & Mark Kac (1914–84) on the distribution of the number of prime factors of natural numbers:[4]

THEOREM E. *For any real number z,*

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ m \leqslant n : \frac{\omega(m) - \log\log m}{\sqrt{\log\log m}} \leqslant z \right\} \right| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du \,.$$

Here, $\omega(n)$ denotes the number of distinct prime factors of n. Gowers found this result beautiful for at least five reasons:

1) The shape of the normal distribution (see Figure 19.20) was aesthetically satisfying.
2) The result was simple, the normal distribution being possibly the simplest non-trivial distribution. (Gowers here said 'unexpectedly simple', but the word 'unexpectedly' seems to function mainly as an intensifier and it was the simplicity that was important.)
3) It was unexpected: the primes are defined deterministically and behave irregularly, and yet hidden within them is the normal distribution, which usually describes random phenomena.
4) The result stood beyond experience and experiment, for no feasible amount of computer time could calculate an actual distribution that would closely approximate the normal distribution.
5) The proof was very satisfying, for *it took the form of a heuristic argument that could be made concrete.*[5]

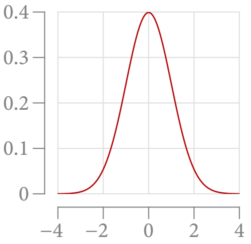

FIGURE 19.20
The standard normal distribution ($\mu = 0, \sigma^2 = 1$).

Heuristic proof. a) The Hardy–Ramanujan theorem, which says that *If φ is a function such that $\varphi(n)$ that tends to infinity as n tends to infinity, then*

$$|\omega(n) - \log\log n| < \varphi(n)\sqrt{\log\log n}$$

for almost all n.[6] Informally, when n is large, most numbers near n have roughly $\log\log n$ prime factors.

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[1] Gowers, 'The Importance of Mathematics', p. 22.

[2] *Ibid.*, pp. 22–3.

[3] *Ibid.*, p. 23.

[4] Erdös & Kac, 'The Gaussian Law of Errors'.

[5] Gowers, 'The Importance of Mathematics', pp. 19–20.

[6] See Hardy & Ramanujan, 'The Normal Number of Prime Factors', Thm B.

b) Thus most prime factors of numbers near n are small, since a significant number of large prime factors would multiply to a larger number than n.

c) If m is close to n, then for small primes p, events 'm is divisible by p' are roughly independent. By the 'Brun sieve' technique, if one assumes they are exactly independent, the conclusions drawn will be approximately correct.

d) A large number of exactly independent events yields a normal distribution. $\boxed{\text{E}}$

This pleasing heuristic argument actually works when formalized.[1]

For Gowers, the strategy of seeking a proof by trying aesthetically pleasing ideas first was born of historical experience showing that 'natural, elegant, interesting statements almost always involve other natural, elegant, interesting statements'.[2] But making good use of this strategy required one to have a good idea of what *was* 'natural, elegant, interesting'. Thus it would be useful to have a theory that explained these and other informal judgements in mathematics.[3]

Gowers suggested that one could approach such a theory by considering what makes mathematical statements and proofs memorable,[4] in particular because there seemed to be a correlation between being memorable and being 'natural, elegant, interesting'.[5] A mathematician did not memorize a proof by rote, but stored only those key ideas necessary to generate the proof by applying general principles (and some proofs relied wholly upon such general principles, so that it was hardly necessary to remember anything).[6] Gowers suggested that the notion of *width* could clarify how memorable a proof was. The width of a proof was, loosely, the amount one had to hold in one's memory to work through a proof, though this measure was complicated by how the proof was divided and arranged into lemmata or other steps that were recalled and used later.[7]

Gowers stopped short of associating the width of a proof with its aesthetic value, but some readers made an explicit conjecture: Gila Hanna (1934–) & John Mason (1944–) wondered whether '[a]n elegant proof [...] could well be one of very low width that relies upon a single key idea of broader mathematical relevance'.[8]

FOUNDATIONS OF SET THEORY

The logician Kurt Gödel (1906–78) is most famous for his startling incompleteness theorems, which effectively showed that the quest to ground all of mathematics on a provably consistent and complete system of axioms was in vain. [The first incompleteness theorem states that in any consistent formal system that is strong enough to encompass a certain portion of elementary arithmetic, there are statements that are not provable or disprovable in that system. The second incompleteness theorem says that in a formal system that is strong enough to encompass a certain portion of elementary arithmetic, the consistency of the system is not provable in that system.]

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[1] Gowers, 'The Importance of Mathematics', p. 20.

[2] Gowers, 'Mathematics, Memory, and Mental Arithmetic', p. 37.

[3] *Loc. cit.*

[4] *Ibid.*, p. 39.

[5] *Ibid.*, p. 37.

[6] *Ibid.*, pp. 44–8.

[7] *Ibid.*, pp. 55–7.

[8] Hanna & Mason, 'Key Ideas and Memorability in Proof', p. 14.

In his philosophical work, Gödel defended mathematical platonism.[1] He wrote in 1963 that what he termed mathematical intuition allowed perception of the objects of set theory, and this perception meant that certain axioms 'force themselves upon us'.[2] But the axioms of set theory leave 'gaps': consider the Continuum Hypothesis, which asserts that *There is no set with cardinality strictly greater than that of the set of natural numbers and strictly less than that of the continuum.* Put another way, it asserts the cardinality of the continuum — or the set of real numbers — which equals that of the power set of the set of natural numbers, is the smallest uncountable cardinal: more concisely, $2^{\aleph_0} = \aleph_1$. Gödel proved in 1939 that if the Zermelo–Fraenkel axioms of set theory together with the axiom of choice are consistent, then the Continuum Hypothesis is consistent with these axioms. Paul Cohen (1934–2007) proved in 1963 that the negation of the Continuum Hypothesis is also consistent with these axioms. Thus the Continuum Hypothesis is independent of the usual axioms of set theory.

This independence means that the truth of the Continuum Hypothesis cannot be judged in a mathematically conventional way. One must add one or more new axioms, so that the new extended collection of axioms implies the truth or falsity of the Continuum Hypothesis.[3] Unless one accepts set-theoretic relativism, there is a decision to be made on which axioms it is correct to add. One could appeal to intuition of the objects of set theory, but Hao Wang (1921–95) warned, in an essay written in 1969–70,[4] that even if successive extensions to the axioms converged to the intuition, they might not yield a unique model in which the Continuum Hypothesis was true or false.[5] Furthermore, the very intuition of a set was subject to development.[6]

Paul Benacerraf (1931–2025) & Hilary Putnam (1926–2016), in the introduction to a collection of essays in which Wang's article was reprinted, suggested that there was an aesthetic element to the choice of axioms.[7]

The mathematical logicians Juliette Kennedy (1955–) & Jouko Väänänen (1950–) pointed out that the foundations of set theory constituted one of certain fundamental areas of mathematics where truth and beauty could blur together. At least, it could become difficult to distinguish aesthetic and alethic judgements.[8] They suggested that the acceptability of additional axioms, such as those necessary for making the Continuum Hypothesis true or false, was ultimately judged on aesthetic grounds:

> 'the quandary set theorists now find themselves in — having to decide the truth of independent statements — is an *aesthetic* quandary'.[9]

[1] Gödel, 'Some basic theorems on the foundations of mathematics and their implications', esp. 322–3.

[2] Gödel, 'What is Cantor's continuum problem?', pp. 483–4.

[3] Kennedy & Väänänen, 'Aesthetics and the Dream of Objectivity', p. 87.

[4] Wang, *From Mathematics to Philosophy*, p. xiii.

[5] Wang, 'The concept of set', p. 558.

[6] *Ibid.*, p. 553.

[7] Benacerraf & Putnam, 'Introduction', p. 32.

[8] Kennedy & Väänänen, 'Aesthetics and the Dream of Objectivity', p. 85.

[9] *Ibid.*, pp. 88–9, emphasis in orig.

According to the proof theorist Gaisi Takeuti (1926–2017), who studied under Gödel at Princeton, Gödel himself thought that the Continuum Hypothesis was false on aesthetic grounds. Takeuti recalled once asking Gödel if he judged axioms valid by considering their consequences. Gödel replied:

> 'Not necessarily. However, the case of the continuum hypothesis is similar to the situation in which one feels very odd when just one note is out of tune in a beautiful and simple melody. That is why I believe it is false.'[1]

Thus, wrote Takeuti, Gödel hoped for a new axiom that would entail the negation of the Continuum Hypothesis.[2]

Takeuti also reported that Gödel thought, but was not certain, that $2^{\aleph_0} = \aleph_2$.[3] This cardinality for the continuum came about via what Kennedy & Väänänen termed a 'beautiful turn', where there was a balance between many real numbers existing and a collapsing of cardinals together.[4] But, in this case, there should be some natural subset of the real numbers with cardinality α_1, and from such a set one could construct non-measurable sets of real numbers, which are considered 'pathological'.[5]

There has been a general practice among mathematicians of accepting principles such as the axiom of choice, even though it could lead to what Kennedy & Väänänen called 'unnatural, even "ugly" objects'.[6] But choosing a more limited foundation that excluded such ugly consequences came with the price of a different kind of ugliness, in the diminished mathematics that could be built on such a foundation.[7] Thus mathematicians are often caught between the Scylla of one form of ugliness and the Charybdis of another.[8]

At minimum, in set theory, permitting the definition of certain objects with beautiful properties also required allowing some ugliness to enter into the mathematical universe. The alternative was to restrict oneself to constructive mathematics, excluding most of the ugly or pathological things; the price here was that 'mathematics becomes so complicated that it can hardly be called beautiful'.[9] The classical mathematician's acceptance or the constructivist's denial of actual infinity were perhaps both aesthetic judgements.[10]

Thus, suggested Kennedy & Väänänen, at least in the context of certain set theoretical questions, judgement of truth may ultimately be a certain special kind of judgement of aesthetic value.[11]

PAUL ZEITZ

'[T]he great mathematician and teacher of problem solving'[12] was how Pólya* was described by one of his disciples, the mathematician and educator Paul Zeitz (1958–). But Zeitz's approach to problem-solving placed a greater emphasis on the role of aesthetic judgements than Pólya's. Zeitz thought that 'problem solving, at its best, is a passionate, aesthetic endeavor'[1] and that the 'experience is an esthetic one'.[2]

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[1] Gödel, quot. in Takeuti, *Memoirs of a Proof Theorist*, p. 45.

[2] *Loc. cit.*

[3] *Ibid.*, p. 46.

[4] Kennedy & Väänänen, 'Aesthetics and the Dream of Objectivity', p. 89.

[5] *Loc. cit.*

[6] *Loc. cit.*

[7] *Ibid.*, p. 90.

[8] *Loc. cit.*

[9] *Loc. cit.*

[10] *Loc. cit.*

[11] *Ibid.*, p. 93.

[12] Zeitz, *The Art and Craft of Problem Solving*, p. 13.

* ▸ pp. 680–2

While aesthetic pleasure is experienced in seeking and finding a solution to a mathematical problem, aesthetic value can be a guide, and a guide that should be cultivated. As a rule, any beautiful idea, when encountered, should be admired, appropriated, and applied enthusiastically, limitlessly, and remorselessly.[3] Ideally, a single application of such an idea would yield a solution.[4]

Such ideas included symmetry arguments, for they gave 'free' information: it would be enough to understand one part of a problem in order to understand all the symmetrical parts.[5] More generally, one should seek harmony and beauty, which lacked the formal definiteness of symmetry:

> 'Look for harmony, and beauty, whenever you investigate a problem. If you can do something that makes things more harmonious or more beautiful, *even if you have no idea how to define these two terms*, then you are often on the right track.'[6]

Any change that enhances simplicity, symmetry, or beauty (which, thought Zeitz, were often synonymous) was usually, but not always, to be preferred.[7] An exception that Zeitz noted was in determining the relative order of 1998/1999 and 1999/2000. This problem was most easily solved by defining a function for such values and then transforming it:

$$f(x) = \frac{x}{x+1}, \quad \text{or, equivalently,} \quad f(x) = \frac{1}{1+\frac{1}{x}}; \tag{Z}$$

from the modified expression, it is clear that $f(x)$ is increasing in x and so $1998/1999 < 1999/2000$. But Zeitz thought that the changed expression in (Z) was uglier than the original, which meant that the solution did not fit his usual aesthetic guidelines.[8]

CAVEATS

Although this chapter has described how many prominent mathematicians have pointed to aesthetic value as a guide, a degree of caution is required. In Leone Burton's (1936–2007) study of how mathematicians work,[*] only one of the seventy interviewed mathematicians made a claim of being guided by aesthetics. The others who discussed beauty in mathematics located it in the products, not the process, of mathematics.[9]

And at least one first-rate mathematician, the 2005 Abel laureate Peter Lax (1926–2025), denied that beauty could be a useful guide, precisely because it was subjective:

[1] Zeitz, *The Art and Craft of Problem Solving*, p. 10.

[2] *Ibid.*, p. 20.

[3] *Ibid.*, p. 16.

[4] *Ibid.*, p. 82.

[5] *Ibid.*, p. 59.

[6] *Ibid.*, p. 60, emphasis in orig.

[7] *Ibid.*, p. 151.

[8] *Loc. cit.*

[*] ▸ pp. 544–5

[9] Burton, *Mathematicians as Enquirers*, p. 63.

> ‘LAX: There’s an aesthetic quality, yes, but if you try to pin that down, you are just begging the question. What is beautiful is purely subjective. Saying something is beautiful may be no different from saying that you have a feeling that something is important.
>
> [...]
>
> LAX: Beauty is in the eye of the beholder. *It’s a poor guide, aesthetics is.* You have to feel that what you are doing is beautiful but, after all, someone used to classical art regards modern art as horrible and ugly.’[1]

RIEMANN HYPOTHESIS

The *Riemann zeta function* is defined as the analytic continuation to the whole complex plane of the function defined for complex numbers s with $\Re s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots . \tag{Z}$$

Its important role in number theory is suggested by the *Euler product formula*, which connects it to the primes: for $\Re s > 1$,

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right).$$

If s is a negative even number, then $\zeta(s) = 0$; the negative even numbers are the *trivial zeroes* of ζ; all other zeroes are *non-trivial*. The known non-trivial zeroes of ζ lie on the so-called *critical line*: that is, the line $\Re s = 1/2$ in the complex plane; the *Riemann hypothesis* asserts that *all* the non-trival zeroes lie on the critical line.

There is computational evidence that the Riemann hypothesis is correct.[2] Number theorists have a motivation for wanting it to be true, for *if* it holds, it would be a powerful tool and much could be deduced with its aid.[3] But some mathematicians seem to desire its truth or to think it true for aesthetic reasons. Atle Selberg (1917–2007) was one: he said that ‘it would really be a *blot* on mathematics if a zero turned up off the critical line’,[4] and elsewhere that ‘[i]f anything at all in our universe is correct, it has to be the Riemann Hypothesis, if for no other reasons, so for purely esthetical reasons’.[5] Enrico Bombieri (1940–) was even more emphatic:

> ‘If the Riemann hypothesis turns out to be false, there will be huge oscillations in the distribution of primes. In an orchestra, that would be like one loud instrument that drowns out the others — an aesthetically distasteful situation. As a follower

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[1] Albers, Alexanderson & Reid, *More Mathematical People*, pp. 155–6, emphasis added.

[2] Borwein, Choi et al., *The Riemann Hypothesis*, ch. 4.

[3] *Ibid.*, ch. 7.

[4] Albers & Alexanderson, *Fascinating Mathematical People*, p. 271, emphasis added.

[5] Selberg, quot. in Baas & Skau, ‘The Lord of the Numbers, Atle Selberg’, p. 618.

Riemann hypothesis

> of William of Occam, I reject that conclusion, and so I accept the truth of the Riemann hypothesis.'[1]

And the number theorist Steve Gonek (1951–) seemed partly to echo Bombieri:

> 'If there *are* lots of zeros off the line — and there might be — the whole picture is just horrible, horrible, very ugly. It's an Occam's razor sort of thing, you either have absolutely beautiful behavior of prime numbers, they behave just like you want them to behave, or else it's *really* bad.'[2]

Marcus du Sautoy (1965–) wrote that: '[i]f the Riemann Hypothesis turns out to be false it will shatter our sense of how beautifully we believe the primes are laid out', but pointed out that aesthetic desires make no difference to whether a beautiful potential theorem actually holds.[3] Aleksandar Ivić (1949–2020) limited himself to declaring a *desire* for the truth of the Riemann hypothesis based on its beauty and elegance and power, and made it clear that he did not have any belief in its truth or falsity.[4]

Don Zagier (1951–) was exasperated by the attitude of believing in the Riemann hypothesis on aesthetic grounds.[5] After computer calculations showed that the first 3×10^8 non-trivial zeroes are on the critical line, he declared himself just as convinced of the truth of the Riemann hypothesis as Bombieri, but was at pains to deny that his conviction was founded on aesthetic or metaphysical concerns: rather he believed it true 'not a priori because it is too beautiful and is such an elegant thing or because of the existence of God, but because of this evidence'.[6]

And yet, underlying Zagier's belief on empirical grounds was a kind of faith — some might call it aesthetic — that the mathematical universe is not so badly designed as to have the first 3×10^8 non-trivial zeroes of the Riemann zeta function *on* the critical line and even just one non-trivial zero *off* the critical line. There are, after all, infinitely many non-trivial zeroes, and despite the many results showing that a certain proportion of the zeroes must be on the critical line, the first 3×10^8 is not a representative sample of infinity. Even adding in the various heuristic arguments for the truth of the Riemann hypothesis still left open the possibility that mathematical reality was not as 'good' as one hopes. Ultimately, Zagier and others reject the idea that mathematics is playing a trick, appealing to a credo like Einstein's 'Subtle is the Lord, but malicious he is not'.[7] One had to have some confidence in what Philip Davis (1923–2018) & Reuben Hersh (1927–2020) called the 'orderliness or "rationality"' of numbers.[8]

[1] Bombieri, 'Prime Territory', p. 36.

[2] Gonek, quot. in Sabbagh, *Dr. Riemann's Zeros*, p. 112.

[3] du Sautoy, 'Exploring the mathematical library of Babel', p. 22.

[4] Ivic, quot. in Sabbagh, *Dr. Riemann's Zeros*, p. 229.

[5] du Sautoy, *The Music of the Primes*, p. 214.

[6] Zagier, quot. in *ibid.*, p. 219; context suggests in conversation with du Sautoy.

[7] Quot. in Pais, *Subtle is the Lord*, p. 113.

[8] Davis & Hersh, *Mathematical Experience*, p. 367.

BERNOULLI NUMBER DEFINITIONS

What are now called the Bernoulli numbers were discovered empirically by Jakob I Bernoulli (1655–1705) by considering expressions for sums of the m-th powers of the first n natural numbers.[1] In modern notation, he noticed that

$$\begin{aligned}1^m + 2^m + \ldots + \ell^m &= \frac{1}{m+1}\left[B_0\ell^{m+1} + \binom{m+1}{1}B_1\ell^m + \binom{m+1}{2}B_2\ell^{m-1} + \ldots + \binom{m+1}{m}B_m\ell\right]\\ &= \frac{1}{m+1}\sum_{n=0}^{m}\binom{m+1}{n}B_n\ell^{m+1-n}\end{aligned}$$

for certain constants B_n. The numbers B_n are now known as the *Bernoulli numbers*, and first few terms of the sequence $(B_n)_{n\geqslant 0}$ are as follows:[2]

n	0	1	2	3	4	5	6	7	8	9	10	11	12
B_n	1	$\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$

If, however, one considers the sum of m-th powers up to but not including ℓ^m, one obtains the similar expression

$$1^m + 2^m + \ldots + (\ell-1)^m = \frac{1}{m+1}\sum_{n=0}^{m}\binom{m+1}{n}B_n^-\ell^{m+1-n}.$$

The sequence $(B_n^-)_{n\geqslant 0}$ is equal to the sequence $(B_n)_{n\geqslant 0}$ except that the first terms differ: $B_1 = \frac{1}{2}$ and $B_1^- = -\frac{1}{2}$.

Most modern texts consider the numbers B_n^- to be *the* Bernoulli numbers and simply denote them by B_n. In the early twenty-first century, a debate arose about preferring the original numbers B_n. The debate began with an open letter from Peter Luschny to Donald Knuth (1938–), one of the authors of the popular discrete mathematics textbook *Concrete Mathematics*, urging him to 'correct' the book's definition of Bernoulli numbers to have $B_1 = \frac{1}{2}$.[3] Knuth's initial response was sceptical, leading to Luschny issuing the 'The Bernoulli Manifesto', which presented various arguments for adopting the $B_1 = \frac{1}{2}$ definition. The arguments in the 'Manifesto' generally concerned the myriad relationships between the Bernoulli numbers and other mathematical concepts, and were largely aesthetic in nature. In the 'Manifesto', aesthetic value was a guide of a kind different from that discussed earlier in this chapter. *Correctness* was not a concern; any result stated using B_n can be stated with appropriate modifications using B_n^-. The question was rather of choosing the definition that ensured beauty in the theory that depended on it.

Bernoulli number definitions

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[1] Bernoulli, *The Art of Conjecturing*, pp. 215–6.

[2] OEIS, A164555, A027642.

[3] Luschny, 'Open Letter to Donald E. Knuth'.

One argument concerned the generating functions of the two sequences $(B_n)_{n\geqslant 0}$ and $(B_n^-)_{n\geqslant 0}$, which are

$$\frac{z}{1-\mathrm{e}^{-z}} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n; \tag{G1}$$

$$\frac{z}{\mathrm{e}^{z}-1} = \sum_{n=0}^{\infty} \frac{B_n^-}{n!} z^n. \tag{G2}$$

Knuth pointed out[1] that the generating function (G2) satisfies the identity

$$\frac{z}{\mathrm{e}^{z}-1} - \frac{z}{\mathrm{e}^{z}+1} = \frac{2z}{\mathrm{e}^{2z}-1},$$

which had many applications; Luschny responded that (G1) satisfies what he called 'an identity of no less beauty':[2]

$$\frac{z}{1-\mathrm{e}^{-z}} - \frac{z}{1+\mathrm{e}^{-z}} = \frac{2z}{\mathrm{e}^{z}-\mathrm{e}^{-z}}.$$

Certain other arguments concerned the connections between the Riemann zeta function and the Bernoulli numbers. If one expands the series (Z) on page 708 using the Euler–Maclaurin formula, one finds that

$$\begin{aligned}\zeta(s) &= \sum_{k=0}^{K} \frac{B_k}{k!} s^{\overline{k-1}} + R(s,K)\\ &= \frac{B_0}{0!} s^{\overline{-1}} + \frac{B_1}{1!} s^{\overline{0}} + \frac{B_2}{2!} s^{\overline{1}} + \ldots + \frac{B_K}{K!} s^{\overline{K-1}} + R(s,K)\end{aligned}$$

where $R(s,K)$ is a remainder term and the overlined superscript denotes the rising factorial power: $s^{\overline{m}} = \Gamma(s+m)/\Gamma(s)$. (When m is a positive integer, $s^{\overline{m}} = s(s+1)\cdots(s+m-1)$.) Here, thought Luschny, 'everything blends in beautifully, as long as we select $B_1 = 1/2$'. In contrast, if one prefers B_1^-, one must treat the $k=1$ term specially.[3]

Another reason for preferring $B_1 = 1/2$ is that in this case the Bernoulli numbers are related to the Riemann zeta function by a simple formula. Define $B(s) = -s\zeta(1-s)$ for $s \neq 0$, with $B(0) = 1$, so that $B(0) = \lim_{s\to 0} B(s)$. Then $B_n = B(n)$ for all $n \in \mathbb{N} \cup \{0\}$.[4] It is of course possible to obtain a function B^- such that $B_n^- = B^-(n)$ for all $n \in \mathbb{N} \cup \{0\}$: one could take $B^-(s) = -s\zeta(1-s)\cos(\pi s)$, but both the function definition and arguably the graph of the function would be less pleasing (see Figure 19.21).

Luschny advanced several other arguments for preferring $(B_n)_{n\geqslant 0}$ over $(B_n^-)_{n\geqslant 0}$. Ultimately the choice of which are 'the' Bernoulli numbers is at least partly an aesthetic one. One could work around any disadvantage that arose from the choice, but preferring $B_1 = 1/2$ yielded more beautiful interactions with other mathematical concepts.

[1] Knuth, email to Luschny, dated 23 Jul. 2004, repr. in Luschny, 'Open Letter to Donald E. Knuth'.

[2] Luschny, 'The Bernoulli Manifesto'.

[3] *Ibid.*

[4] *Ibid.*

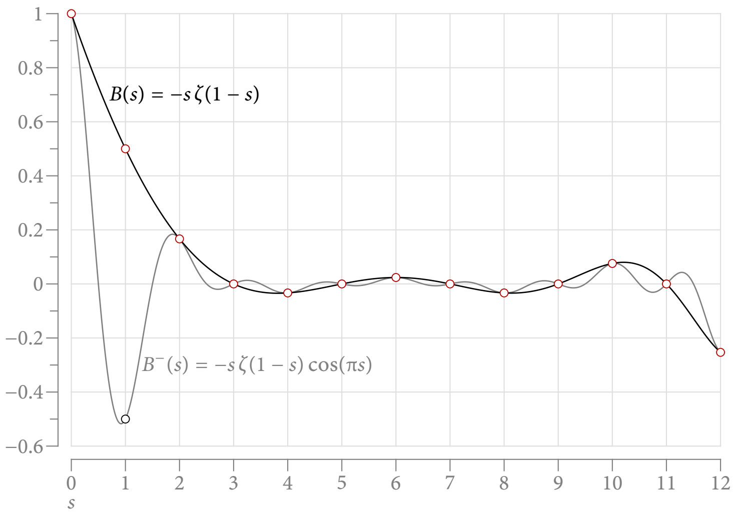

FIGURE 19.21
Plots of the functions

$$B(s) = -s\zeta(1-s),$$
$$B^{-}(s) = -s\zeta(1-s) \cdot \cos(\pi s)$$

(extended to $B(0) = B^{-}(0) = 1$ by taking the limits as s tends to 0), with the Bernoulli numbers B_n shown in red and B_1^{-} in black.

Knuth was convinced by Luschny's arguments, although he seems to have given greater weight to the historical arguments than to the aesthetic ones, and in 2022 revised *Concrete Mathematics* to use $(B_n)_{n\geqslant 0}$ as *the* Bernoulli numbers.

τ OVER π

A perhaps similar definitional question concerns the role of π. Rob Palais (1958–) thought that the 'beautiful idea' of a dimensionless measure of angles had been 'sabotaged' by the historical accident that the customary version of the circle constant, π, was defined to be the quotient of the circumference by the diameter.[1] The definition of π means that the angle $\pi/2$ is a quarter-turn and $\pi/3$ is a one-sixth turn, and a whole turn is 2π. Thus the number 2 turns up alongside π in such diverse contexts as Stirling's approximation and the Gaussian normal distribution in mathematics:[2]

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^{n}; \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-x^2/2}\,\mathrm{d}x = 1.$$

Palais argued that it would be better if the circle constant were the quotient of the circumference by the radius, and used a special 'three-legged' variant π for $2\pi = 6.283\,185\,307\,1+$ to illustrate how formulae would look with this convention:[3]

$$\mathrm{e}^{\mathrm{i}\pi} = 1, \qquad n! \sim \sqrt{\pi n}\left(\frac{n}{\mathrm{e}}\right)^{n}, \qquad \hbar = \frac{h}{\pi}.$$

This notation failed to gain traction, doubtless in part because of the absence of a suitable vocalization for π. [All prior publications that discussed Palais’s symbol simply superimposed two glyphs π rather than create a new glyph; the ugliness of the result might also be a consideration.]

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[1] Palais, 'π Is Wrong!', p. 7.

[2] *Loc. cit.*

[3] *Ibid.*, p. 8.

The physicist Michael Hartl (1973–) thought that reading Palais’s article had ‘definitely opened his eyes’ to the problems with π,[1] and on 28 June 2010 he released ‘The Tau Manifesto’, arguing for the adoption of τ, denoting 2π, the ratio of the diameter of a circle to its radius, as the conventional circle constant. He thought that it was a conceptual improvement, since an angle τ/n would represent $1/n$ of a full turn.[2]

Hartl addressed in particular the new form that Euler’s equation would take, precisely because, as he pointed out, it is sometimes thought of as the most beautiful of equations.[3]* Using τ, it becomes

$$e^{i\tau} = 1. \tag{ET}$$

In words, the equation (ET) is the result *The complex exponential of the circle constant is* 1. And, furthermore, since multiplication by $e^{i\vartheta}$ corresponds to rotation by ϑ in the complex plane, the geometric expression of the new form of Euler’s equation is *A rotation by one full turn is* 1. Perhaps more significantly, $e^{i\tau/2} = -1$: a half-turn corresponds to multiplication by -1. Palais seems to have found this form superior to the original.[4] David Richeson at least agreed that there was no aesthetic diminution: ‘we can replace Euler’s lovely identity $e^{i\pi} = -1$ with its *equally beautiful* τ counterpart: $e^{i\tau} = 1$’.[5] In response to those who complain that (ET) only involved four numbers, not the ‘five most important numbers’ 0, 1, i, e, π that appear in Euler’s equation, Hartl pointed out that $e^{i\tau} = 1 + 0$ or, by putting $\vartheta = \tau$ in Euler’s formula, $e^{i\tau} = 1 + i0$.[6]

◆

[1] Hartl, ‘The Tau Manifesto’, § 5.1.1.

[2] *Ibid.*, § 2.1.

[3] *Ibid.*, § 2.3.

* ▸ pp. 839–43

[4] Palais, ‘π Is Wrong!’, p. 7.

[5] Richeson, *Tales of Impossibility*, p. 300, emphasis added.

[6] Hartl, ‘The Tau Manifesto’, § 2.3.1.

Beauty Contested

20

Deconstruction of aesthetic value in mathematics

‘ *Tortoise:* […] I’ve never had the time to analyze Beauty.
It’s a Capitalized Essence; and I never seem to have the time
for Capitalized Essences. ’

— Douglas R. Hofstadter, *Gödel, Escher, Bach*, p. 29.

‘ Interpretation itself must be evaluated ’

— Susan Sontag, ‘Against interpretation’, § 3.

In 1964, the historian of mathematics Morris Kline (1908–92), considering whether mathematics could be thought of as an artistic pursuit, wrote the following (which immediately follows the passage quoted on page xxix):

> ‘No doubt many people feel that the inclusion of mathematics among the arts is unwarranted. The strongest objection is that mathematics has no emotional import. Of course this argument discounts the feelings of dislike and revulsion which mathematics induces in some people.’[1]

‘[M]athematics has no emotional import’: a claim any reasonable person will doubt on watching Andrew Wiles (1953–) recalling the moment when he saw the solution to the problem that had been found in his original proof of Fermat’s last theorem. A transcript barely does justice to the emotion:

> ‘I was trying… convincing myself that it didn’t work, just seeing exactly what the problem was, when suddenly, totally unexpectedly, I had this incredible revelation. [*Pause. Voice begins to break.*] I… I realized that what was holding me up was exactly what would resolve that problem I’d had with my Iwasawa theory attempt three years earlier. It was… [*Long pause.*] It was the most… the most important moment of my working life. [*Long pause.*] It was so indescribably beautiful. It was so simple and so elegant.’[2]

And yet, even if the emotion is acknowledged, both mathematicians and philosophers have argued over whether the experience behind the emotion is genuinely an aesthetic one. Against the background of the aesthetic justification for mathematics there emerged a theme

[1] Kline, *Mathematics in Western Culture*, p. 466.

[2] Andrew Wiles, in Singh, ‘Fermat’s Last Theorem (Documentary)’, 41:00 sqq.; cf. Singh, *Fermat’s Last Theorem*, ch. 7.

of re-interpretation or even counter-interpretation, whereby some scholars argued that what were claimed to be aesthetic responses to mathematics were actually only quasi-aesthetic or not aesthetic at all.

Chapter 11 described how, in the first half of the nineteenth century, there was less philosophical interest in mathematical beauty than in the eighteenth.[*] Victor Cousin's[†] (1791–1867) mid-century remarks stand in isolation and reflect the views of a philosopher much influenced by an earlier age. But the lack of interest did not, in general, lead to active repudiation of mathematical beauty. Even Friedrich Theodor von Vischer[‡] (1807–87), though critical of arguments supporting beauty in mathematics, did not directly confront the notion itself. It was at the end of the nineteenth century and the beginning of the twentieth century that philosophy started to pay new attention to beauty in mathematics, and there quickly emerged disagreement over whether it was *genuine* beauty.

MATHEMATICAL BEAUTY AND THE QUESTION OF SENSE-PERCEPTION

> ‘consider what kind of knowledge
> and what sort of a world we should be left with
> if we removed from them the experience of
> the so-called secondary and tertiary qualities;
> if we were left only with the comfortless company of
> the dreary formulæ of mathematics, the sciences and philosophy.’
>
> — Philip Leon, ‘Æsthetic Knowledge’, p. 199.

Christian Hermann Weiße

An isolated exception to the nineteenth-century philosophical passivity with regard to mathematical beauty was the theologian and philosopher Christian Hermann Weiße (1801–66). In his 1830 book *System der Aesthetik*, he held that only ‘concrete natural entities [concrete Naturwesen]’ could be beautiful. Although he interpreted this class widely, so as to include souls, he maintained that it excluded ‘abstract, logical, or metaphysical conceptual definitions’ such as ‘mathematical definitions, figures, and shapes’. These things, not being concrete, could not be beautiful.[1] As Weiße put it bluntly, ‘[t]he mathematical, owing to its abstract deadness, is incapable of beauty’.[2]

Although he was aware that the concept of beauty had previously been thought to be applicable to mathematical entities, and to geometric constructions that show symmetry or harmony, he thought that this position was absurd. If beauty was attributed to logical or mathematical entities, they would ‘fall under the category of non-things [Undinge] or spectres [Gespenster]’: because they were eternal and necessary ‘form-determinations [Formbestimmungen]’ of being, they

[*] ▸ pp. 387–8
[†] ▸ pp. 385–6
[‡] ▸ p. 388

[1] Weiße, *System der Aesthetik*, vol. 1, p. 105; trans. C.

[2] *Ibid.*, vol. 1, p. 107, n. ∗ cont. from p. 106; trans. C.

participated in all particular beings — and also in the beautiful — but they did not themselves have a separate existence.[1]

George Santayana

George Santayana (1863–1952) was a philosopher, a poet, and a public figure whose autobiography and only novel were best-sellers.[2] He took his doctorate at Harvard and taught there from 1889,[3] and based his book *The Sense of Beauty* (1896) on his lectures on aesthetics. He apparently did not think highly of the book, telling Arthur Danto (1924–2013) late in life that it was a 'wretched potboiler', written only so that he could remain at Harvard. Nevertheless, *The Sense of Beauty* had a lasting influence and was read far more than any of his numerous other works.[4]

For Santayana, the existence of each value was grounded on human appreciation of it.[5] Thus beauty, in particular, did not exist apart from experience. But what made aesthetic experience distinctive? Unlike many previous thinkers, Santayana looked neither to disinterest, nor to universality. Aesthetic pleasure was not disinterested, because it could be preliminary to the acquisitive desire. Rather, aesthetic pleasure was *immediate*: it was not based on calculation or the anticipation of some other future pleasure.[6] And aesthetic pleasure was not universal: 'It is unmeaning to say that what is beautiful to one man *ought* to be beautiful to another.'[7] This fallacy came about because 'we think those beauties *are in the object*'.[8] But this was a *reductio ad absurdum*, since beauty was a value that cannot exist separately from its perception: 'A beauty not perceived is a pleasure not felt, and a contradiction.'[9]

Santayana was thus led to his own definition: 'beauty [...] is value positive, intrinsic, and objectified'.[10] It was a pleasure of the perceptor, but was regarded as a quality of a thing.[11] Succinctly, '[i]t is pleasure objectified.'[12]

Having reached this definition, Santayana argued in the rest of *The Sense of Beauty* that there was a wide range of qualities of objects that could cause an aesthetic response.[13] But while there could be *pleasure* in mathematics, it was not *aesthetic* pleasure. Rather, it was evoked by the activity of relating things, and the same pleasure came from other forms of manipulating abstract relationships, such as in puzzles. Santayana maintained that the pleasure did not come from the objects, and so was not aesthetic.[14] He went so far as to deny that mathematicians made judgements of beauty of their subject:

> 'We think of more or less interesting problems or calculations, but it never occurs to the mathematician to establish a hierarchy of forms according to their beauty. Only by a metaphor could he say that $(a+b)^2 = a^2 + 2ab + b^2$ was a more beautiful formula than $2 + 2 = 4$.'[15]

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[1] Weiße, *System der Aesthetik*, vol. 1, p. 105; trans. C.

[2] Saatkamp & Coleman, 'George Santayana', Introduction.

[3] *Ibid.*, § 1.

[4] Danto, intro. to Santayana, *The Sense of Beauty*, pp. xv–xvi.

[5] *Ibid.*, § 2.

[6] *Ibid.*, § 8.

[7] *Ibid.*, § 9, emphasis in orig.

[8] *Ibid.*, § 10, emphasis in orig.

[9] *Ibid.*, § 10.

[10] *Ibid.*, § 11.

[11] *Ibid.*, § 10.

[12] *Ibid.*, § 11.

[13] Guyer, *Hist. Modern Aesthetics*, vol. 2, p. 248.

[14] Santayana, *The Sense of Beauty*, § 49.

[15] *Loc. cit.*

Mathematical beauty and sense-perception

The artefacts of mathematics were, for Santayana, too abstract. Aesthetic terms could only be applied literally to more definite and objective things.[1]

But Santayana's discussion of the perception of beauty in music came frustratingly close to fitting the perception of beauty in mathematics. He thought that music was the presentation of abstract relations via the senses, and the pleasure in these relations was the pleasure in perceiving the presentation of a form that could have beauty.[2] Why could not one simply replace *music* here with *mathematics*? Santayana supplied no answer. Presumably, at least at this early point in his career, Santayana simply did not know that mathematicians made aesthetic judgements of their work, and thus felt it unnecessary to delve further into the question.

Jonas Cohn

In the latter part of the nineteenth century in Germany, a philosophical movement developed that was oriented anew towards Kant's philosophy. Jonas Cohn (1869–1947) was a second-generation member of the 'southwestern school' of these neo-Kantians, and his *General Aesthetics* [*Allgemeine Ästhetik*] (1901) affords the best idea of this school's approach to aesthetics.[3] Although not he did not directly argue against Santayana, he wrote in this book a defence of beauty in mathematics that can be read as opposing Santayana's view.

The first chapter of Cohn's book opens with the assertion that an aesthetic term like *beautiful* could only be properly applied to a direct or immediate experience [unmittelbares Erlebnis]. A rose *qua* individual flower might be beautiful, the rose *qua* species — which was an abstract thing — was not. An act was only beautiful as it immediately appeared; if one looked into the motives behind it, the judgement became ethical, and to apply the term *beautiful* was to conflate it with *good*.[4]

Cohn recognized that objections could be raised to his starting-point. For instance, while it was meaningless to say that the *concept* of a triangle is beautiful (and he thought that no-one would make such a claim), he allowed that a mathematician could call a formula or a proof beautiful.[5] But, thought Cohn, examining the experience of mathematical beauty showed that it did not contradict his initial assertion, but rather confirmed it. A mathematical formula was not beautiful merely because it was true and fruitful, but also because someone with enough knowledge could, just by looking at the formula, immediately see many of its consequences.[6] What might be first conceived of as the beauty of the *concept* could be traced back to the immediate experience, and it was this experience that justified the application of the aesthetic term to the abstract thing.[7] For Cohn, the role of this immediate experience in the aesthetic appreciation of ab-

[1] Santayana, *The Sense of Beauty*, § 49.

[2] *Loc. cit.*

[3] Guyer, *Hist. Modern Aesthetics*, vol. 2, p. 346.

[4] Cohn, *Allgemeine Ästhetik*, p. 18.

[5] *Ibid.*, pp. 18–19.

[6] *Ibid.*, p. 19.

[7] *Ibid.*, pp. 19–20.

stract things served to refute Weiße's argument against mathematical beauty.[1]*

Mathematical beauty and sense-perception

Cohn also argued that the role of knowledge or expertise in making perceptible certain beauties did not invalidate his starting-point, for immediate experience was unconsciously informed by mental processes that were not part of sense perception. The basic process of sorting a mass of visual inputs into mental apprehensions of separate solid things was informed by experience accumulated over a lifetime. The automatic integration of acquired experience into sense-input was a *part* of the immediate experience to which aesthetic judgement applied.[2] (Although Cohn only discussed this objection in the context of beauty in art, it immediately follows his discussion of mathematical beauty and clearly applies to it.)

For Cohn, a judgement of mathematical beauty was a special kind of value-judgement. He considered aesthetic judgements to attribute a value that was 'purely intensive',[3] meaning intrinsic or non-instrumental or disinterested.[4] Epistemic or ethical judgements were different: they were related respectively to truths or to goals beyond the thing that was judged.[5] While many things could be subjected to both aesthetic and non-aesthetic judgement, there were certain places where the two types of evaluation were united. Cohn gave the examples of 'moral beauty or the beautiful solution of a mathematical problem',[6] but it seems likely that he meant mathematical beauty generally, because, as discussed, he had argued that the immediate experience that gave validity to aesthetic judgements of a piece of mathematics included the perceptions of its consequences. Thus the non-aesthetic judgements of truth were intermeshed with the aesthetic judgements.

DeWitt Parker

DeWitt H. Parker's (1885–1949) book *The Principles of Æsthetics* (1920), based on his lectures at the University of Michigan, was for some time a widely-used introduction to the subject.[7]

Parker did not identify art with expression, as Benedetto Croce (1866–1952) had before him. He thought that all art was free expression, but not all free expression was art, and the most important example of non-artistic free expression was science, within which he seems to have included mathematics:[8]

> 'In science [...] there is much free expression; but beauty not yet. No abstract expression such as Euclid's *Elements*, Newton's *Principia*, or Peano's *Formulaire*, no matter how rigorous and complete, is a work of art. We admire the mathematician's formula for its simplicity and adequacy; we take delight in its clarity and scope, in the ease with which it enables the mind to master a thousand more special truths, but we do not find it beautiful.'[9]

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[1] Cohn, *Allgemeine Ästhetik*, p. 20, n. 1.

* ▸ pp. 715–16

[2] *Ibid.*, pp. 20–2.

[3] *Ibid.*, p. 23.

[4] Guyer, *Hist. Modern Aesthetics*, vol. 2, p. 347.

[5] Cohn, *Allgemeine Ästhetik*, pp. 23 sqq.; see also Guyer, *Hist. Modern Aesthetics*, vol. 2, p. 347.

[6] Cohn, *Allgemeine Ästhetik*, p. 225; trans. C.

[7] Ducasse, Frankena & Blanshard, 'DeWitt Henry Parker'.

[8] Parker, *The Principles of Æsthetics*, p. 19.

[9] *Ibid.*, pp. 19–20.

Parker saw this differing attitude to science, including mathematics, and to art as a consequence of science being 'intentionally objective': its products were meant to exclude any emotional response.[1]

But Parker used beauty in nature to illustrate his argument that 'In themselves, things are never beautiful. [...] But beauty may belong to our *perceptions* of things.'[2] Perhaps curiously, he seems not to have considered that beauty might reside, not in the products of mathematics, but in the perception of them.

David Wight Prall

David Wight Prall (1886–1940) came to philosophy via a roundabout route. For his bachelor's degree, he studied mathematics, chemistry, and English and German literature; for his master's, German and rhetoric. Only after a period as an instructor in English did he focus on the philosophical study of value. His two major contributions to philosophical aesthetics were his books *Æsthetic Judgement* (1929) and *Æsthetic Analysis* (1938). In the former work, which was well-received when it appeared,[3] the notion of mathematical beauty was closely connected to Prall's explanation of differing aesthetic judgements.

Aesthetic judgement, wrote Prall, 'follows and records' direct aesthetic experience, the feeling of the beauty of some object.[4] Aesthetic judgement was not the same as criticism: the former made an explicit record of the experience and at least initially made no comparative judgements; the latter attempted to justify the judgement, to classify the object, and to relate something of the experience.[5]

Beauty, however apprehended, was a quality that supervenes upon the changing and moving reality that the subject inhabited and perceived. Just as redness was a quality that occurred in a *transaction* involving the photoreceptors of the eye, so too was beauty a quality that occurred in a complicated and little-understood transaction involving light or sound transmission, nerve activity in sense-organs, brain activity starting from perceptual processing, muscle movements, and coordinating responses.[6] Every particular beauty was an eternal and unique thing that might be embodied and experienced as a quality of particular object at a particular time and place, or might never be embodied.[7] Thus a beautiful vision could only occur *once*. While there might be a vision of precisely the same form, it would be a different *event*, and so the second beauty, as a quality, would occur in a different transaction from the first.[8]

Aesthetic judgements were peculiar because of how disagreements are treated. A disagreement on a non-aesthetic judgement could be resolved by measurement and comparison. A disagreement on an aesthetic judgement cannot be so resolved, because two people could contemplate the same object and still not agree on whether it was beautiful,[9] even though beauty was an objective quality of the object,

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[1] Parker, *The Principles of Æsthetics*, p. 20.

[2] *Ibid.*, p. 23, emphasis added.

[3] Ross, intro. to Prall, *Æsthetic Judgement*, p. xx.

[4] *Ibid.*, p. 5.

[5] *Loc. cit.*

[6] *Ibid.*, pp. 7–8.

[7] *Ibid.*, pp. 8–9.

[8] *Ibid.*, p. 9.

[9] *Ibid.*, pp. 15–17.

Mathematical beauty and sense-perception

albeit a *tertiary* quality distinct from the qualities involved in simple perception.[1]

Since beauty was a quality that was present in certain transactions, when persons disagree aesthetically, they must therefore have participated in *different transactions*.[2] That is, in apprehending a quality such as beauty, the transaction depended on the apprehending process as well as on the object. Thus, while the object remained the same, the apprehending process could vary from person to person depending on nature or training.[3] For Prall, this variation explained why some persons could not apprehend beauty in an object that others found beautiful:

> 'As a blind man misses colors, an æsthetically deficient man misses beauty. As a lazy man misses the exhilaration of physical activity or an unskilled worker the feeling of skillfulness in doing his work, so a man who has not exercised his perceptive powers to discriminate fine differences, or a man who has not learned a given technique, is simply incapable, in the presence of objects or events of certain sorts, of responding in the particular and complex dynamic response that is one part of the event upon which beauty supervenes before him as an objective quality upon, and of, the object or event apprehended.'[4]

Even a person who has been endowed with the ability and who has acquired through practice and training the necessary acumen to perceive beauty in an object on one occasion may fail to see it at another time. Prall disagreed with theorists who would argue that in such circumstances the aesthetic object has changed. Rather, the viewer — or, more precisely, the transaction that is partly dependent on the mental and bodily functioning of the viewer — has changed, whether because of the viewer's mood, comfort, fatigue, health, or otherwise.[5]

Prall considered the beauty experienced by mathematicians in the context of his explanation of varying aesthetic appreciation. [He specifically mentioned 'æsthetic pleasure in [...] mathematical objects', but it seems that he intended the word 'objects' to indicate things experienced aesthetically by mathematicians; that is to say, any loci of mathematical beauty.[6]] He accepted that beauty in mathematics was compatible with his theory, and that the requirement of specialized knowledge would make this beauty inaccessible to many and render it an atypical example of aesthetic experience. Prall thought that when the mathematician was actively working on mathematics, most of their attention was focused on how to proceed rather on the point they had reached. But when they halted, having found a proof or completed an entire theory, they could look upon what they had made and feel aesthetic pleasure:[7]

> 'as soon as he stops his mathematical operations and notices their form, he is a contemplating subject with precisely the sort

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[1] Prall, *Æsthetic Judgement*, p. 22.

[2] *Ibid.*, pp. 17–18.

[3] *Ibid.*, pp. 18–19.

[4] *Ibid.*, pp. 22–3.

[5] *Ibid.*, pp. 23–6.

[6] *Ibid.*, p. 27.

[7] *Loc. cit.*

> of absorbing object of attention that is typical of all æsthetic experience.'[1]

But Prall seems to have been slightly ambivalent about the experience of mathematical beauty. What was presented to the senses was only mathematical symbolism, and so the processes accompanying the 'color and sound and richness' of other sensory content were not activated:

> 'If it is the purest and most elevated æsthetic pleasure, it is at any rate also the thinnest and most meagre. It is frugal if not ascetic.'[2]

Prall seems not to have considered the role in aesthetic experience of the *meaning* represented by the mathematical symbolism. In contrast, he argued that beauty in literature largely concerned the meaning that was communicated *through* the symbolic medium of words and sentences.[3] In literature, there could be beauty in the medium, on the level of words and sentences and paragraphs, which was 'not an altogether different sort of beauty from that called economy or elegance in mathematical symbolism and its structures', although in literature it was 'far more subtle and complex and personal and even arbitrary'.[4]

Prall revisited what he saw as the meagre aesthetic experience of mathematics when he discussed what he considered to be two errors that descended from Plato's elevation of the objects of mathematics, metaphysically and aesthetically, above the sensible world.[5]* The first error was

> 'that the highest æsthetic appreciation is the mathematician's, an error easy to make plausible by using the term purity instead of emptiness or thinness, in the same way in which ignorance is so often called innocence, and so lauded as virtue when it should be merely tolerated, or actually condemned'.[6]

In light of this later statement, the previously-quoted phrase 'If it is the purest and most elevated æsthetic pleasure' should perhaps be read as 'If it is *called* [...]'. In any case, for Prall, while there was beauty in mathematics, it was (relatively) empty, frugal, meagre, thin.

The second error Prall identified was in seeing the simple geometric objects like the circle as the most beautiful. Prall maintained that, since complicated geometrical objects have a higher degree of intellectual content, Plato and his descendants should have seen complicated objects as more beautiful and the circle as among the least beautiful.[7]

Louis Arnaud Reid

Louis Arnaud Reid's (1895–1986) book *A Study in Aesthetics* (1931) was the dissertation for which he received his doctorate from the University of Edinburgh. He began the book with a defence

[1] Prall, *Æsthetic Judgement*, p. 27.

[2] *Loc. cit.*

[3] *Ibid.*, pp. 293–5.

[4] *Ibid.*, p. 295.

[5] *Ibid.*, pp. 119–20.

* ▸ pp. 71–4

[6] *Ibid.*, p. 120.

[7] *Ibid.*, pp. 121–2.

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of aesthetic inquiry.[1] Even if beauty was relative to us, and was an individual thing rather than a general concept, there were 'certain *kinds* of objects which many people agree in calling beautiful, and certain *kinds* of mental experience called aesthetic', which indicates that there was *something* common to all the individual objects and experiences.[2]

Reid's starting-point for his investigation proper was a very provisional consideration of aesthetic experience.[3] He thought that it was clear at the outset that *aesthetic* experience is not merely perceptual experience, for the experience of watching a drama or listening to a symphony was more than just perceiving: some additional quality was involved.[4] The experience of poetry raised another problem, because what was perceived were the words on the page, but the aesthetic experience involved what was imagined as a result of reading those words.[5] So the fundamental issue was 'whether aesthetic experience is always (at least) either perceptual experience, or *imaginary* perceptual experience, or both'.[6]

Many aesthetic objects involved a perceptual or imaginary component: the various plastic arts, dance, music, literature. But, asked Reid, could there be aesthetic experience of things that had no perceptual aspect, direct or imaginary, such as a moral action, a moral character, a mathematical proof, a scientific or philosophical system? In particular, 'May not a mathematical proof be beautiful?'[7]

Reid started by admitting that the question was difficult to answer. He admitted that aesthetic terms were often misused — he seems to have been unwilling to allow even metaphorical usage — but he rejected the solution of asserting that the use of aesthetic terms in such contexts was what he bluntly termed a 'mistake':[8]

> 'this method of outright denial could hardly be applied here, since the claim that mathematics (for example) possesses beauty is frequently made by persons of discrimination, judgment, and taste, who are also familiar with those aesthetic experiences which do contain perceptual elements, such as experiences of any of the arts.'[9]

As illustration, Reid specifically compared Russell speaking of 'the rapture of pure mathematics' as an escape from the mundane world[*] with Arthur Schopenhauer's (1788–1860) writing on aesthetic experience.[10][†] He mentioned one other person explicitly: 'The Prince Consort used to say that a mathematical proof affected him in the same way as a movement in a piece of music.'[11] Reid supplied no citation (and the present author has been unable to find a source) for this claim, but presumably he thought that Albert of Saxe-Coburg and Gotha had 'discrimination, judgment, and taste'.

A possible solution was that in the process of understanding a mathematical proof, a moral character, or other non-perceptual and non-imaginative object, one pictured them. The symbols that made up a proof — whether marks seen on a paper or words heard in a

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[1] Reid, *A Study in Aesthetics*, pp. 13–28.

[2] *Ibid.*, p. 19.

[3] *Ibid.*, pp. 31–2.

[4] *Ibid.*, pp. 31–2, 41–2.

[5] *Ibid.*, pp. 32–3.

[6] *Ibid.*, p. 33, emphasis added.

[7] *Ibid.*, p. 34.

[8] *Ibid.*, pp. 34–5.

[9] *Ibid.*, p. 35.

[*] ▸ p. 479

[10] *Ibid.*, p. 36.

[†] ▸ p. 473

[11] *Loc. cit.*

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lecture-hall — might not themselves be aesthetically satisfying, but they might express a picture than was imagined and experienced aesthetically. In this case, the aesthetic experience of mathematical proof was encompassed by imagined perceptual experience.[1] In his later book *Meaning in the Arts*, Reid said explicitly that he could 'see' simple proofs.[2] But, at least at the time of writing *A Study in Aesthetics*, he only claimed to have experienced 'a certain limited amount of satisfaction in such things'.[3]

But what of aesthetic experience of things that remain completely unpictured, and in particular of unvisualized mathematics? Reid confessed that he did not know. At this preliminary stage, his investigation had not yet characterized the essence of aesthetic experience, so to deny that such experiences were aesthetic would be to come close to circular reasoning,[4] and he thought that it was clear that he had 'no positive right' to make such a denial.[5] He made the pragmatic decision to focus *A Study in Aesthetics* on those aesthetic experiences that had a direct or imaginative perceptual aspect, for, as he saw it, even if non-perceptual aesthetic experiences existed, they constituted only a minuscule minority of all aesthetic experiences.[6]

Reid defended a view that 'beauty is perfect expressiveness'.[7] He accepted that properties such as serenity, proportion, harmony, balance could be, and perhaps always were, present in beautiful things, but he did not think that it was useful to define beauty in such terms.[8] Aesthetic experience and mere perception differed in the expressiveness of the object:

> 'in aesthetic, as distinct from other kinds of contemplation, the object is so regarded that the very arrangement of the perceptual data as we apprehend them seems in itself to embody some valuable meaning, something the apprehension of which moves, interests, excites us.'[9]

Since aesthetic contemplation consisted in this kind of meaningful apprehension of objects, there could be aesthetic experience of nature,[10] which G. W. F. Hegel (1770–1831) had excluded from consideration a century before.[*] And, although Reid did not pursue his investigation in this direction, 'some valuable meaning [...] which moves, interests, excites us' could easily be interpreted in a way compatible with beauty in mathematics.

[Reid also criticized Santayana's view of beauty as 'pleasure objectified',[†] less because of the *objectification* (which, for Santayana, entailed excluding mathematical concepts from *aesthetic* pleasure) than because of what he saw as the superficiality of reducing aesthetic experience to pleasure.[11]]

Samuel Alexander

Samuel Alexander's (1859–1938) early work was influenced by idealism, but he moved towards a naturalistic position that culminated in his 1916–18 Gifford lectures at the University of Glasgow, which were published as his *magnum opus* *Space, Time, and Deity* (1920). In this work, he set out a metaphysical theory in which the world was seen as a hierarchy in which space-time formed the fundamental level, and where the successive levels of matter, life, mind, values, and deity were each emergent from the lower levels.[1] It was a consequence of his emergentist position that '[t]he highest so-called values, truth, beauty, and goodness, are all of them human inventions, and the valuable is what satisfies certain human instincts or impulses'.[2]

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[1] Reid, *A Study in Aesthetics*, p. 35.

[2] Reid, *Meaning in the Arts*, p. 212.

[3] Reid, *A Study in Aesthetics*, pp. 35–6.

[4] *Loc. cit.*

[5] *Ibid.*, p. 36.

[6] *Loc. cit.*

[7] *Ibid.*, p. 31, n. 1.

[8] *Loc. cit.*

[9] *Ibid.*, p. 39.

[10] See Guyer, *Hist. Modern Aesthetics*, vol. 3, p. 179.

[*] ▸ p. 388

[†] ▸ pp. 716–17

[11] Reid, *A Study in Aesthetics*, p. 94.

Alexander's last book and main work on aesthetics was *Beauty and Other Forms of Value*, which was published in (1933) but which incorporated passages from various earlier works. He started from the position that all values were attached by the mind to a perceived object.[3] For Alexander, the creation of art was characterized by *homo additus naturae*: the addition of man to nature. Art was made by the artist mixing meaning with the material.[4] [The thought expressed in the Latin phrase seems to have originated with Francis Bacon (1561–1626): 'Ars sive additus rebus Homo'.[5] The altered version used by Alexander was in circulation by the late nineteenth century.[6]] An artwork was beautiful if it was the object of the same creative impulse in the one who perceived it, although the impulse was no longer practical but contemplative.[7] The appreciation of natural beauty likewise included an element contributed by the spectator. It was 'creation in the less exacting form of *selection*'.[8] Beauty was seen in nature not because there *was* beauty there, 'but because we select from nature and combine, as the artist does'.[9]

Alexander considered the possibility of beauty in mathematics and science in *Space, Time, and Deity*:

> 'In the same way, just as beauty is one part of the good and to pursue it is a virtue, so goodness and truth are species of the beautiful, or they have their aesthetic side. Some parts of mathematics have been described as poetry and certain methods in science are, to indicate an exceptional excellence, justly called beautiful; and good actions may have beauty or grace or sublimity, or a life may be a true poem. The aesthetic feeling in these cases in distinguishable from the mere "logical" sentiment for truth or the moral sentiment of approval.'[10]

These words also appear in *Beauty and Other Forms of Value*.[11] In both books, they are followed by slightly different descriptions of how the role of the spectator served to distinguish judgements of truth and beauty in the case of mathematics and science. Of the two descriptions, the latter is the more lucid:

> 'What is true or good is treated much as we treat a piece of natural beauty, such as a beautiful face or the beautiful figure of a man, where as it happens no selection or supplementation

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[1] Alexander, *Space, Time, and Deity*, *passim*; for a summary, see Thomas, 'Samuel Alexander', § 3.2.

[2] Alexander, 'Morality as an Art', p. 143.

[3] Alexander, *Beauty and Other Forms of Value*, p. 7.

[4] *Ibid.*, pp. 19 sqq.

[5] Bacon, 'Descriptio Globi Intellectualis', p. 731.

[6] See, for example, Blanc, *Grammaire des Arts du Dessin*, p. 18.

[7] Alexander, *Beauty and Other Forms of Value*, p. 22.

[8] *Ibid.*, p. 30, emphasis added.

[9] Alexander, *Beauty and Other Forms of Value*, p. 30; cf. Alexander, *Space, Time, and Deity*, vol. II, pp. 288–90.

[10] Alexander, *Space, Time, and Deity*, vol. II, pp. 298–9, emphasis added.

[11] With trivial differences; see Alexander, *Beauty and Other Forms of Value*, p. 273.

> from the spectator is necessary, though the spectator still needs to recognise it with his aesthetical appreciation or approval.'[1]

That is, the judgement of truth was similar to the judgement of natural beauty, but the selective contribution of the spectator only applied to the latter, not the former.

But these passages require careful scrutiny, especially in light of what Alexander wrote about beauty in mathematics and science earlier in *Beauty and Other Forms of Value*:

> 'Scientific work, especially mathematical work which is purely conceptual, may indeed possess the *appearance* of beauty, because of the inner coherence which it shares with fine art, or may resemble a piece of architecture. But *it is called beautiful rather by analogy than intrinsically*.'[2]

These sentences indicate that, for Alexander, what was called beauty in science and mathematics was only a semblance, so that the aesthetic term was not being used in the same way as when it was applied to a work of art. In both *Space, Time, and Deity* and *Beauty and Other Forms of Value*, Alexander stopped short of admitting that there was beauty in mathematics and science: he reported only that '[s]ome parts of mathematics have been *described* as poetry and certain methods in science are [...] justly *called* beautiful'.[3] For Alexander, applying the term *beauty* to mathematics or science might have indicated excellence, and perhaps even an *aesthetic* excellence, but this excellence seems to have been distinct from beauty as found in art or nature.

Edgar F. Carritt

Edgar F. Carritt (1876–1964) wrote numerous books on aesthetics, ethics, and political philosophy. He edited a widely-used anthology of writing on aesthetics from ancient Greece to his own time, and published two introductory texts on the subject.[4] In the first of these two books, *What is Beauty?* (1932), he declared that when mathematicians used the term *beautiful* to describe a solution that is 'ingenious or brief', they were applying the word in a loose sense that reflection showed to be improper.[5] There was a recognizable difference between 'what is strictly beautiful and what is merely pleasant, like a drink, or instructive, like a medical case, or ingenious, like the solution of a cross-word puzzle'.[6] A mathematical text or a medical case or a drink *could* be beautiful, but this aesthetic interest should be distinguished from the intellectual or medical or hydrational interest.[7] What this seems to imply is that Carritt thought that any beauty (in a non-loose sense) of a mathematical text would be *visual*, being evoked by the symbolism and physical features of the text, without any connection to the mathematical content it conveyed.

[1] Alexander, *Beauty and Other Forms of Value*, vol. II, p. 273–4.

[2] *Ibid.*, pp. 51–2, emphases added.

[3] Alexander, *Space, Time, and Deity*, vol. II, p. 298; Alexander, *Beauty and Other Forms of Value*, 273, emphases added.

[4] Guyer, *Hist. Modern Aesthetics*, vol. 3, pp. 158–9.

[5] Carritt, *What is Beauty?*, p. 7.

[6] *Ibid.*, p. 8.

[7] *Loc. cit.*

John Dewey

In his *Art as Experience* (1934), John Dewey (1859–1952) maintained that the word *beauty*, which he hardly mentioned in the first third of the book, was primarily a term indicative of a specific emotion that indicated 'the capacity of the object to arouse admiration that approaches worship'.[1] He thought that it was not a useful term to use in aesthetics, for, insofar as it denoted the aesthetic quality of a particular experience, it was better to examine the experience directly.[2]

There was another, more limited, use of *beauty*, when it described a contrast with other types of aesthetic quality, positive or negative. Dewey thought that analysis of the use in practice of *beauty* in this sense indicated one of two meanings: either i) 'the striking presence of decorative quality, of immediate charm for sense', or ii) 'the marked presence of relations of fitness and reciprocal adaptation among the members of the whole'.[3] Dewey continued: 'Demonstrations in mathematics [...] are thus said to be beautiful'.[4] This seems to point to Dewey having believed that meaning 20 lay behind a description of a proof as beautiful. Dewey did not expand further on this characterization, instead turning to the implications of the failure of theorists to unite meanings 20 and 20.

Monroe C. Beardsley

In his 1958 book *Aesthetics: Problems in the Philosophy of Criticism*, Monroe C. Beardsley (1915–85) relegated to a note the observation that, in tackling the problems of aesthetics, '[i]t is helpful to ask in what sense a mathematical system, or a game of chess, is an aesthetic object as well as a new truth.'[5] The closest he came to offering an answer was when he listed four criteria, which he thought would be uncontroversial, that characterized what made *aesthetic experience* distinct:

1) In an aesthetic experience, attention was fixed on 'heterogeneous but interrelated components of a phenomenally objective field', such as a visual or auditory pattern or characters and events in literature.
2) The experience was intense and motivates the viewer to devote their attention.
3) The experience had unity, or coherence.
4) The experience was complete, with its elements being resolved or balanced by other elements.[6]

These criteria could be found separately in other, non-aesthetic, experiences. For example, watching a competitive sport could fulfil the latter three criteria, but usually lacked a dominant pattern on which the attention was fixed. But sometimes it *had* such a pattern and so

[1] Dewey, *Art as Experience*, p. 129.

[2] *Ibid.*, p. 130.

[3] *Loc. cit.*

[4] *Loc. cit.*

[5] Beardsley, *Aesthetics*, p. 397.

[6] *Ibid.*, pp. 527–8.

could then also be an aesthetic experience.[1] Beardsley then turned to science and mathematics:

> 'Carrying through a triumphant scientific investigation or the solution of a mathematical problem may have the clear dramatic pattern and consummatory conclusion of an aesthetic experience, but it is not itself aesthetic experience unless the movement of thought is tied closely to sensuous presentations, or at least a phenomenally objective field of perceptual objects.'[2]

Here, Beardsley allowed aesthetic experience of mathematics, but only when the mental process closely follows sensory perception or 'a phenomenally objective field of perceptual objects'. It is not clear what he considered to distinguish mathematical objects and their relations from characters and events in literature, which did satisfy the first of his criteria, except possibly that the literary persons and happenings can be imagined. If so, Beardsley might have admitted visual thinking about mathematics as an aesthetic experience.[*]

J. N. Findlay

J. N. Findlay's (1903–87) is most remembered for his work on metaphysics and logic, and he made only one contribution devoted to aesthetics,[3] a lecture he delivered to the British Society of Aesthetics in 1966. He opened by acknowledging that he was not a practitioner in aesthetics, and admitted that did not closely follow the field. But he seems to have felt informed enough to rubbish contemporaneous practitioners as 'fantastically misguided' and believed that what he had to say would have some value.[4]

Findlay placed a high value on aesthetic experience, which he thought was 'a type of experience uniquely marked out, extraordinary in its delight'.[5] He thought that beauty did not represent a particular feature of certain objects or a particular mental attitude. It was rather part of being conscious of any object, though much more intensely present in some objects and the experience of them.[6] Beauty, he thought, might not be definable, but it could be 'fairly well pinned down'.[7]

Findlay thought that Franz Brentano's (1838–1917) tripartite division of mental phenomena was valuable for aesthetics, for it separated the presentation of the object being contemplated from the decision on the acceptance of its involvement in reality and the decision on the attitude towards it.[8] [In Brentano's scheme, the three parts were: presentation, or mere conception of a object; judgement, or the acceptance or denial of the reality of the object; and emotion, or the feelings towards the object.[9]] When one took an aesthetic attitude, one was interested in the presentation itself, not in its correspondence to reality.[10]

[1] Beardsley, *Aesthetics*, p. 530.

[2] *Loc. cit.*

[*] ▸ pp. 780–2

[3] Lackey, 'John Niemeyer Findlay', § 7.

[4] Findlay, 'The Perspicuous and the Poignant', p. 3.

[5] *Ibid.*, p. 4.

[6] *Ibid.*, p. 5.

[7] *Loc. cit.*

[8] *Ibid.*, pp. 8–9.

[9] Findlay, 'The Perspicuous and the Poignant', p. 8; see also Huemer, 'Franz Brentano', § 3.1.

[10] Findlay, 'The Perspicuous and the Poignant', p. 9.

He maintained that an aesthetic response could arise '*whatever* the object'.[1] He explicitly rejected the restriction of the aesthetic attitude to objects of sense-perception. He thought that this limitation, accepted by Hegel and Croce among others, conflicted with practice.[2] Usage of words and concepts pointed to the vast complexity of the phenomenology of the aesthetic.[3] There was contemplative pleasure in exploring the structure of a sense-object, and there was also contemplative pleasure in exploring the structure *independently* of the sense-object.[4] In the latter case, 'it is both natural to call such pleasure "aesthetic" and also natural to say that it is aesthetic *in exactly the same sense as* sensory structure'.[5] The appreciation of a well-written story, a well-constructed joke, a well-argued philosophical discourse could evoke aesthetic pleasure,[6] and '[i]t is not in virtue of a mere metaphor that we are said to take aesthetic pleasure in a mathematical proof or a scientific theory or a philosophical argument.'[7] Sense-objects were *important* in aesthetic appreciation, for there was a greater pleasure in seeing beautiful colours than in having them conveyed in poetry, but the difference was only in degree, and there was similarly greater pleasure in reading a logical argument set out in well-chosen notation and diagrams than in reading an argument written in poor notation.[8]

MATHEMATICS AND ART THEORY I

Clive Bell

The art critic Clive Bell (1881–1964) thought that the foundation of any theory of aesthetics must be 'the personal experience of a peculiar emotion'.[9] Bell thought that, although the term *beauty* was applied widely, the genuine aesthetic emotion arose only in narrow contexts. It was a different, non-aesthetic, emotion that lay behind applying *beautiful* to a natural thing such as a flower; similarly, it was desire that lay behind applying the term to a person.[10] For Bell, the genuine aesthetic emotion was caused by art, and he described its mental effect as being indistinguishable from that produced by mathematics:

> 'Art transports us from the world of man's activity to a world of aesthetic exaltation. For a moment we are shut off from human interests; our anticipations and memories are arrested; we are lifted above the stream of life. The pure mathematician rapt in his studies knows a state of mind which I take to be similar, if not identical. He feels an emotion for his speculations which arises from no perceived relation between them and the lives of men, but springs, inhuman or super-human, from the heart of an abstract science. I wonder, sometimes, whether the ap-

[1] Findlay, 'The Perspicuous and the Poignant', p. 9, emphasis in orig.

[2] *Ibid.*, p. 10.

[3] *Ibid.*, p. 7.

[4] *Ibid.*, p. 10.

[5] *Ibid.*, p. 10, emphasis in orig.

[6] *Ibid.*, pp. 10–11.

[7] *Ibid.*, p. 10.

[8] *Ibid.*, p. 11.

[9] Bell, *Art*, § I.i.

[10] *Loc. cit.*

> preciators of art and of mathematical solutions are not even more closely allied.'[1]

The transportation by art or mathematics 'to a world of aesthetic exaltation' was likely a conscious echo of Bertrand Russell's (1872–1970) early view of mathematics as a refuge.* Bell, as a member of the Bloomsbury Group, doubtless knew Russell's 1907 essay 'The Study of Mathematics', which had been published in *The New Quarterly*, edited by Desmond MacCarthy (1877–1952), another member.

Roger Fry

The painter and critic Roger Fry (1866–1934) thought that there were two uses of the word *beauty* in art: first, when an artwork had 'sensuous charm', and only the perceptual side of imagination was involved; second, when an artwork, perhaps ugly in the first sense, still produced an aesthetic approval in a 'supersensual' way, involving an intense emotional response.[2] He noted in the introductory essay to his 1926 anthology *Transformations*, that while one could divide with some sharpness aesthetic responses — that is, responses to art — from those to ordinary life, it was more difficult to distinguish them from the response to doing pure mathematics.[3] The distinction he suggested was concerned less with the emotional response, and more with the motivation of the object:

> 'in the case of works of art the whole end and purpose is found in the exact quality of the emotional state, whereas in the case of mathematics the purpose is the constatation of the universal validity of the relations without regard to the quality of the emotion accompanying apprehension.'[4]

R. G. Collingwood

The philosopher and archaeologist R. G. Collingwood (1889–1943) grew up in an artistic household and encountered philosophy early.[5] It is thus perhaps fitting that he is especially remembered for his work on aesthetics and the philosophy of art. In his philosophical writings, he put forward two distinct aesthetic theories. He adumbrated the first theory in a chapter of his 1924 book *Speculum Mentis* and explained it at greater length the following year in *Outlines of a Philosophy of Art*. When the latter work went out of print, he chose, since he had changed his mind 'on some things', to write an entirely new book rather than to prepare a revised edition.[6] In fact, the theory he set out in the new book, *The Principles of Art* (1938), posited a very different relationship between beauty and art.

Art, for the early Collingwood, was imagination.[7] Aesthetic contemplation is imaginative contemplation; without imagination, a thing could not be contemplated as an artwork.[8] Art aimed at beauty,[9]

[1] Bell, *Art*, § 1.i.

* ▸ pp. 479–85

[2] Fry, 'An Essay in Æsthetics', p. 20.

[3] Fry, 'Some Questions on Aesthetics', p. 6.

[4] *Loc. cit.*

[5] Guyer, *Hist. Modern Aesthetics*, vol. 3, p. 190.

[6] Collingwood, *The Principles of Art*, p. v.

[7] Collingwood, *Outlines of a Philosophy of Art*, pp. 11–12.

[8] *Ibid.*, pp. 25–6.

[9] *Ibid.*, pp. 7, 19, 86.

but while beauty involved structure, organization, unity, coherence, harmony, symmetry, congruity, it could not be truly defined other than as what was imagined: ‘That which distinguishes beauty from these is that it is not any unity, but an imaginative unity.’[1] Beauty was not a property of a perceived object or a concept. One was conscious of it simply as an emotion: ‘an emotional colouring which transfuses the entire experience of the imagined object’.[2]

While this concept of beauty left space for beauty in mathematics, Collingwood did not discuss it explicitly in either *Speculum Mentis* or *Outlines of a Philosophy of Art*, though he was aware that mathematics could be hedonic, and in the two books he made passing references to joy in mathematical discovery and pleasure in mathematical truth.[3]

In *Outlines of a Philosophy of Art*, Collingwood maintained that to apply terms like *sublime* or *idyllic* or *lyrical* to an artwork was to say something about it *qua* work of art. But to call it a *seascape* or *villanelle* or *fugue* was to assign ‘a predicate with no aesthetic significance whatever’.[4] But when he came to write *The Principles of Art* more than a decade later, he took an almost opposite position. Beauty no longer had any connection with art;[5] the word *aesthetic* indicated specifically a concern with art;[6] and ‘aesthetic theory is the theory not of beauty but of art’.[7] Terms like *beauty* and *beautiful*, however they were applied, only described excellence or admiration:

> ‘The words “beauty”, “beautiful”, as actually used, have no aesthetic implication. We speak of a beautiful painting or statue, but this only means an admirable or excellent one. Certainly the total phrase “a beautiful statue” conveys an implication of aesthetic excellence, but the aesthetic part of this implication is conveyed not by the word “beautiful” but by the word “statue”.’[8]

The difference here between his earlier and later views is located in which word carries aesthetic significance: for the earlier Collingwood, in a phrase like ‘sublime seascape’, it would be only the first word; for the later Collingwood, in ‘beautiful statue’, it would be only the second.

For the later Collingwood, the term *beautiful* could be applied to a variety of things to indicate that they were excellent or well-made or desirable.[9] There was an associated emotional response, but it was not an *aesthetic* — that is to say, not an *art-relevant* — response.[10] In this context, Collingwood made his single explicit mention of mathematical beauty:

> ‘The word “beautiful” is used in such a case no otherwise than as it is used in such phrases as “a beautiful demonstration” in mathematics or “a beautiful stroke” at billiards. These phrases do not express an aesthetic attitude in the person who uses them towards the stroke or the demonstration; they express an attitude of admiration for a thing well done, irrespective

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[1] Collingwood, *Speculum Mentis*, p. 66; see also Collingwood, *Outlines of a Philosophy of Art*, p. 21.

[2] Collingwood, *Outlines of a Philosophy of Art*, pp. 21, 27.

[3] Collingwood, *Speculum Mentis*, p. 40; Collingwood, *Outlines of a Philosophy of Art*, p. 27.

[4] Collingwood, *Outlines of a Philosophy of Art*, p. 31.

[5] Collingwood, *The Principles of Art*, pp. 36–41.

[6] *Ibid.*, p. vi.

[7] *Ibid.*, p. 41.

[8] *Ibid.*, pp. 38–9.

[9] *Ibid.*, p. 39.

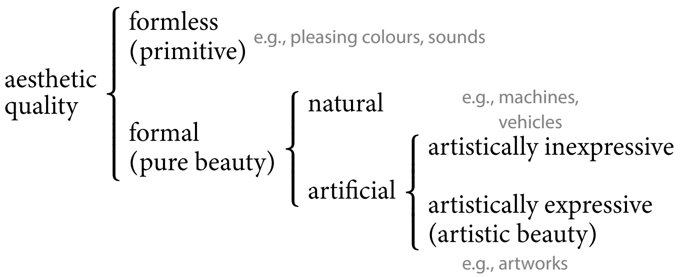
[10] *Loc. cit.*

> of whether that thing is an aesthetic activity, an intellectual activity, or a bodily activity.'[1]

Thus the phrase 'beautiful proof' would denote a proof that was thought admirable or excellent, and presumably Collingwood would have admitted that the term *beautiful* can be applied to other mathematical artefacts to indicate similar admiration or excellence. But, for Collingwood, the response to a beautiful work of mathematics would not be same as that to beautiful work of art, not because the term *beautiful* took different meanings in the two contexts, but because mathematics was an intellectual rather than an aesthetic activity.

Theodore Meyer Greene

For Theodore Meyer Greene (1897–1969), 'aesthetic quality' was a part of anything that could cause 'an aesthetically re-creative response in a sensitive person'.[2] He divided aesthetic qualities into *formal* and *formless*; the latter included colours and tones that could produce an aesthetic response aside from any formal organization or pattern. Formal qualities split into natural and artificial, and the latter into those that were artistically expressive or inexpressive:[3]



He chose to limit *beautiful* to formal aesthetic quality, so that to use the term was to say something about one's response to the aesthetic character of a formal organization or pattern.[4] He thought that this limitation was consistent with the usual philosophical usage and still permitted the word to be applied in diverse contexts. *Any* formal organization that was 'intrinsically satisfying' could be called beautiful; thus, in particular, there was 'formal beauty in mathematics (where it is usually entitled "elegance")'.[5] Thus, for Greene, 'elegance' in mathematics served as a term of art for beauty in that specific context, and did not denote a distinct aesthetic category.

Greene seems to have thought that the notion of beauty in mathematics was uncontroversial, or required at most the very simplest defence: 'The ascription of aesthetic quality to mathematical and other types of excellence is quite justifiable, since notable achievements in *any* medium may exhibit an aesthetic character.'[6] His straightforward acceptance of beauty in mathematics perhaps made it natural for him to use the trained mathematician's judgement of 'elegance' to illustrate his observation that, in any field, expertise was required to make competent aesthetic judgements.

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[1] Collingwood, *The Principles of Art*, p. 39.

[2] Greene, *The Arts and the Art of Criticism*, p. 5.

[3] *Ibid.*, pp. 5–6.

[4] *Ibid.*, p. 7.

[5] *Loc. cit.*

[6] *Ibid.*, p. 124, emphasis added.

The only caveat upon which Greene insisted was that the wide range of applicability of formal beauty did not annul the distinction of 'artistic and *mere* aesthetic quality';[1] this was the difference he had previously described between being artistically *expressive* and *inexpressive*. He did not make it quite clear whether he thought that mathematics could have artistic as well as aesthetic quality. He emphasized the impersonal and objective ideal of scientific and mathematical exposition,[2] but admitted that in making this argument he had ignored 'elegance' in proof, 'to which mathematicians […] attach importance and which the philosopher of art might profitably investigate'.[3] This admission, and particularly its last clause, seem to indicate that Greene did not rule out the possibility of finding artistic quality in mathematics.

D. W. Gotshalk

D. W. Gotshalk's (1901–73) book *Art and the Social Order* (1947) is, as far as the author has been able to determine, the earliest work on philosophical aesthetics or the philosophy of art that engages, however briefly, with G. H. Hardy's (1877–1947) *A Mathematician's Apology*.* Gotshalk distinguished fine art as having the aim of '[t]he construction of objects of aesthetic experience'.[4] The focus on the 'intrinsic perceptual appeal' of its products differentiated fine art from the sciences, which aimed to build conceptual structures that contain knowledge either of something beyond themselves (such as physics building theories of matter and force) or of the consequences of certain premisses (such as mathematics and logic exploring the implications of axioms).[5]

Gotshalk accepted that the conceptual structures of science could have 'great beauty or elegance'.[6] He continued:

> 'those who confuse fine art with the creation of beauty may contend that the sciences are indistinguishable from fine art or that some single science, such as mathematics, is.'[7]

At this point, Gotshalk cited Hardy's *Apology* without comment. If he meant to suggest that Hardy was one of 'those' to whom he referred, the insinuation is unfair. Hardy showed no such confusion in the *Apology*, nor did he argue for the indistinguishability of mathematics and fine art; at most, he saw parallels between them. But, whatever Gotshalk's view of the *Apology*, it is plain that he accepted that mathematics could be beautiful.

Karl Aschenbrenner

The philosopher Karl Aschenbrenner (1911–88) mentioned in passing the aesthetics of mathematics in his 1985 book *The*

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[1] Greene, *The Arts and the Art of Criticism*, pp. 124–5, emphasis added.

[2] *Ibid.*, p. 260.

[3] *Loc. cit.*

* ▸ pp. 498–523

[4] Gotshalk, *Art and the Social Order*, p. 31.

[5] *Loc. cit.*

[6] *Loc. cit.*

[7] *Loc. cit.*

Concept of Coherence in Art. From the brief comments that he made it is difficult to extract a coherent view. On the one hand, he admitted that it was sometimes held that 'pure objects of thought, such as numbers and theories about them' were objects of aesthetic interest.[1] But he tempered this by writing:

> 'the aesthetic attributes these have are conceivable only in relation to a perceptible symbolism, a phenomenal or imaginal vehicle to present the thought.'[2]

What this seems to say is that the aesthetic attributes were assigned to mathematical statements (theorems, theories, proofs), and not to their objects. This conclusion is reinforced by his later characterization of aesthetic data as 'wholes whose character is more than or other than can be anticipated from the parts'.[3] Mathematical quantities did not satisfy this condition; they were 'conditioned by their components, not by their contexts'.[4] That is, a mathematical quantity — or, more, generally, a mathematical *object* — was never more than the sum of its parts, but a mathematical *statement* could be, and thus could be aesthetic data. However, the aesthetic value of mathematical statements was firmly subordinated to their epistemic value:

> 'aesthetic appeal [...] is severely limited since considerations of its truth override all aesthetic considerations. It can scarcely sustain even the mathematician's aesthetic interest if he knows that the thought it conveys is logically false or confused.'[5]

It is natural to wonder if the phrase '*even* the mathematician's aesthetic interest'[6] indicates a certain disdain for aesthetic value in mathematics on Aschenbrenner's part.

CONSTRUCTIONAL MATHEMATICAL BEAUTY

The mathematician and philosopher L. E. J. Brouwer (1881–1966) gave prime place to mathematical beauty in a 1948 account of how consciousness arose and engaged with the world. For Brouwer, consciousness was dependent on a 'move of time' that creates a distinction between the sensations of present and past.[7]

Causal attention was the differentiation and organization of experienced sensation, which led to causal thinking — the ability to predict how the experienced world will unfold — and causal acting — the ability to influence the experienced world with the aim of achieving certain desired outcomes.[8] Science was an 'economical and efficient' way of cataloguing causal predictions.[9] While mathematics was a basis of science, it did not depend on the causal attitude to the experienced world, but was rather the bare subject that was left

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[1] Aschenbrenner, *The Concept of Coherence in Art*, p. 9.

[2] *Loc. cit.*

[3] *Ibid.*, p. 12.

[4] *Loc. cit.*

[5] *Ibid.*, p. 9.

[6] Emphasis added.

[7] Brouwer, 'Consciousness, Philosophy, and Mathematics', p. 1235 ⟦p. 480⟧.

[8] *Ibid.*, pp. 1235–6 ⟦p. 480–1⟧.

[9] *Ibid.*, p. 1237 ⟦p. 482⟧.

> ‘creating new mathematical entities in the shape of *predeterminately or more or less freely proceeding infinite sequences* of mathematical entities previously acquired’.[1]

In this emergence of consciousness, Brouwer asked where values such as beauty and truth could be found.[2] There was a beauty in the ‘self-revelation of consciousness’ and in works of art.[3] But there was also a *constructional beauty*, when things were *playfully* built,[4] in the sense of the action not being determined by some desired and foreseen change in the world.[5] This constructional beauty allowed the free creation of things that had ‘power, balance, harmony’, and was found in its purest form in mathematics:

> ‘the fullest constructional beauty is the *introspective beauty of mathematics*, where instead of elements of playful causal acting, the basic intuition of mathematics is left to free unfolding. This unfolding is not bound to the exterior world, and thereby to finiteness and responsibility; consequently its introspective harmonies can attain any degree of richness and clearness.’.[6]

Thus, for Brouwer, while the beauty of mathematics was of a kind distinct from the kind of beauty found in art, it was nevertheless a genuine beauty, and moreover was the *ne plus ultra* of that special kind of beauty.

QUASI-AESTHETIC APPRAISALS

The philosopher of science Horace Romano ‘Rom’ Harré (1927–2019) published in 1958 an essay arguing that the application of aesthetic terms in mathematics and the sciences did not reflect genuinely aesthetic appraisals, but only *quasi*-aesthetic appraisals: that is, they did not lie within the usual range of aesthetic judgements.[7] He seems to have maintained this view, for it appears in abbreviated form in the second edition (1983) of his book *An Introduction to the Logic of the Sciences*.[8]

Harré’s first degree was in chemical engineering and mathematics[9] and he mainly considered judgements of mathematics; he thought that quasi-aesthetic appraisals occurred much more frequently in mathematics than in science.[10] He claimed that, in mathematics, aesthetic terms were usually applied to proofs and not to formulae. But only a limited subset of the aesthetic vocabulary was applied: one could see proofs described as ‘“elegant”, “pretty”, “neat”, and sometimes “beautiful”’ and, on the negative side, as ‘“clumsy”, “ugly”, “uninspired”, “dull”’, but not as ‘“charming”, “delightful”, “lovely” or “handsome”’.[11] This restricted vocabulary alone, for Harré, made a *prima facie* case for distinguishing judgements of mathematics as quasi-aesthetic.[12] There was a parallel between applying these terms in judging mathematics and in making genuinely aesthetic appraisals, but this parallel did not justify thinking that that their use in mathematics was aesthetic.[1]

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[1] Brouwer, ‘Consciousness, Philosophy, and Mathematics’, p. 1237 ⟦p. 482⟧, emphasis in orig.

[2] *Ibid.*, p. 1238 ⟦p. 483⟧.

[3] *Ibid.*, p. 1238 ⟦p. 483⟧.

[4] *Ibid.*, pp. 1238–9 ⟦p. 483–4⟧.

[5] *Ibid.*, p. 1236 ⟦p. 481⟧.

[6] *Ibid.*, p. 1239 ⟦p. 484⟧, emphasis in orig.

[7] Harré, ‘Quasi-Aesthetic Appraisals’, p. 133.

[8] Harré, *Logic of the Sciences*, pp. 146–8.

[9] Christensen & Brock, ‘Obituary: Rom Harré’, p. 153.

[10] Harré, ‘Quasi-Aesthetic Appraisals’, p. 133.

[11] *Ibid.*, pp. 134–5.

[12] *Ibid.*, p. 135.

Harré claimed that the term used most frequently for aesthetic appraisals in mathematics was *elegant*.[2] A proof could be called elegant if it was short, or if the proof-steps were uncomplicated, or if it used some ingenious device.[3] (For Harré, a proof device might be *neat*, *ingenious*, or *pretty*, but never *elegant*: it was not elegant in itself, but the cause that elegance was in a larger proof.[4]) In short, judgements of elegance depended on 'quite *formal* features of proofs'.[5]

But Harré's view that *elegant* was the most-used aesthetic term in mathematics suggested to him another factor. The proper scope of application of aesthetic terms varied: *beautiful* could be applied to a statue or a dress or a mountain; *elegant* only seemed applicable to the first two. The term *elegant* suggested some deliberate intent to create a pleasing effect: a person might be beautiful by the gift of nature but elegant by dint of effort.[6] In short, *elegant* signified 'both that the aesthetically pleasing character of the subject of the judgement is contrived (not there by accident) and that it is achieved in some simple, neat and economical way'.[7] That is, there was an 'overtone of artifice' and an 'overtone of simplicity'.

These connotations, for Harré, explained why *elegant* appeared often in mathematics in the phrase 'simple and elegant' but not, so far as Harré knew, in the phrase 'elegant and simple'.[8] While *elegant* carried with it a suggestion of simplicity, the juxtaposition 'simple and elegant' involved an aesthetic appraisal that the term *simple* alone did not, and made a statement about the *way* in which an object achieved a pleasing effect.[9] Applied to a proof, *simple* on its own pointed to the shortness and lack of conceptual complexity of a proof; 'simple and elegant' pointed to simplicity in this sense and said that this simplicity was not accidental.[10] Since *elegant* had this supplementary sense, the formulation was always 'simple and elegant'.[11] The use of *elegant* also indicated that the proof achieved its goal in some specially satisfactory way. Harré claimed that this satisfaction was only mentioned in contrast to some less satisfactory proof that was known or suspected to exist.[12]

But, said Harré, this satisfaction was *secondary*. Discovering a proof brought the satisfaction of mathematical achievement; reading and understanding such a proof brought a similar satisfaction of mathematical achievement. These were the *primary* satisfactions of mathematics.[13] And here, for Harré, lay the distinction between aesthetic and quasi-aesthetic judgements. A genuinely aesthetic appraisal was *first-order*: it was the immediate response to a work of art and determined its success or failure. The first-order response to a mathematical proof was a judgement of correctness, of whether the proof succeeded or failed; the judgement of elegance was *second-order*. It signified more than just mathematical success.[14] Hence there was a

[1] Harré, 'Quasi-Aesthetic Appraisals', p. 136.

[2] *Ibid.*, p. 133.

[3] Harré, *Logic of the Sciences*, p. 147.

[4] Harré, 'Quasi-Aesthetic Appraisals', p. 136.

[5] *Ibid.*, pp. 136–7, emphasis in orig.

[6] *Ibid.*, pp. 132–3.

[7] *Ibid.*, p. 133.

[8] *Loc. cit.*

[9] *Ibid.*, p. 134.

[10] *Ibid.*, p. 135.

[11] *Loc. cit.*

[12] *Ibid.*, p. 136.

[13] *Loc. cit.*

[14] *Ibid.*, p. 137.

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kind of contradiction in speaking of an 'inelegant work of art' that did not exist in speaking of an 'inelegant proof': an artefact surely fails as a work of art if it is inelegant, but a proof can be correct but inelegant.[1]

◆

In a response, Christine Nicholson argued that there was a genuinely aesthetic element in what Harré called 'quasi-aesthetic appraisals'.[2] She pointed out that the distinction of first-order and second-order satisfactions may not be as clear-cut as Harré had believed. A sculpture may be beautiful, but it was at least conceivable that on completing it the sculptor gained a non-aesthetic satisfaction in the achievement,[3] perhaps in having succeeding in creating a technically difficult piece or having achieved an accurate likeness.

Nicholson also pointed out that objects such as buildings, furniture, and pottery have both functional and aesthetic value without the latter value being less important.[4] In his reply, Harré argued that these examples did not contradict his thesis, but rather showed that the distinction of aesthetic and quasi-aesthetic appraisals actually applied more widely than he had originally thought.[5]

Nicholson noted that there could be different kinds of aesthetic value in the same artwork, lying in the result and in the method. In a Bach fugue, there was an almost mathematical structure that gave 'an aesthetic delight quite separable — although not, in normal listening, always separated — from that of the sound of the whole fugue'.[6] That is, there was the satisfaction produced by the fugue — the result — and there was the satisfaction of appraising how the result was produced.

Perhaps curiously, Nicholson said that in mathematics, unlike in art, 'results cannot be aesthetically appraised'.[7] This statement went far beyond Harré's position, and in any case stood in opposition to a vast history of such appraisals. Taken at face value, Nicholson's assertion would imply that all these appraisals would be at best quasi-aesthetic. In any case, as Nicholson said, the impossibility of aesthetically appraising results did not imply the impossibility of aesthetically appraising methods.

Nicholson also argued that, even if an extensive survey supported Harré's assertion that 'elegant and simple' was used but 'simple and elegant' was not, the natural fluidity and deliberate reinvention of idiom meant that nothing could be inferred from it.[8]

[1] Harré, 'Quasi-Aesthetic Appraisals', p. 137.

[2] Nicholson, 'Harré on Quasi-Aesthetic Appraisals', p. 155.

[3] *Loc. cit.*

[4] *Ibid.*, p. 156.

[5] Harré, 'In Reply to Mrs. Nicholson', p. 157.

[6] Nicholson, 'Harré on Quasi-Aesthetic Appraisals', p. 156.

[7] *Loc. cit.*

[8] *Ibid.*, p. 157.

MATHEMATICS AND ART THEORY II

Harold Osborne

Harold Osborne (1905–87), philosopher, founding member of the British Society of Aesthetics, and founder of the *British Journal of Aesthetics*,[1] may have once been sceptical that there could be beauty in mathematics, but on several later occasions considered intellectual beauty, in mathematics and science.

Osborne's earlier scepticism is suggested by a passage in his 1953 introductory text *Theory of Beauty*. In considering the psychology of aesthetic experience, he pointed out that many people of taste had denied that the appreciation of beauty produced a specific emotional state, while a few have insisted upon it.[2] He suggested that the few who asserted such a state had erred in not recognizing that the emotion they described was the 'special sort of pleasure or emotion' that often accompanied 'the unimpeded and intense activation of any skilled faculty', but felt at a higher degree.[3] The aesthetic faculty could evoke this emotion, but it was not unique in doing so. The emotional response

> 'may be experienced by Mr. Clive Bell in the contemplation of beauty, by Russell in the process of mathematical discovery, by a skilled craftsman in the exercise of his craft, by Goossens in perfect execution on the oboe or by an athlete in smooth bodily coordination.'[4]

The comparison made here between Bell's contemplation of beauty and Russell's mathematical discovery suggests that Osborne thought that Russell had fallen prey to the same error when he wrote about mathematical beauty.*

Whatever his earlier views, by 1964 Osborne had come to believe that the notion of intellectual beauty had been neglected, with few authors giving it more than a token mention or granting it weight equal to sensory beauty. He suggested that it was curious that mathematicians and physicists paid more attention to it than philosophers.[5]

Science sought rational understanding of the universe; art sought direct aesthetic satisfaction. Osborne thought that both aimed at order of some kind, but the two varieties of order were so different that it could seem pointless to attempt to unify them conceptually.[6] For Osborne, science involved a kind of 'classificatory perception' in which data were abstracted from sensory input,[7] and this kind of perception was 'repugnant to aesthetic experience'.[8] If an object did not provoke continued contemplation for its own sake — 'an aesthetic attitude' — then either the viewer lost interest entirely or else their attention shifted to analysis, rationalization, dissection.[9] Objects that could support aesthetic contemplation, without such a shift occurring, were the 'proper things of aesthetic awareness [...]. The most prominent of these things are [...] those that are called *artworks*'.[10] Scientific theories could also create this kind of response, and the appeal might be like that of masterpieces of art.[1] However, aesthetic objects were 'unique and individual in experience'.[2] This caveat points to a careful distinction that Osborne seems to have made: intellectual beauty attached to the *statements* of scientific laws or mathematical theorems, not to what they describe. That is, the intellectual beauty was not in the phenomena of nature or mathematics (although Osborne made no ontological commitment about mathematics) but is rather in the laws or theorems that unified, described, and predicted these phenomena.[3]

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[1] Diffey, 'Harold Osborne', esp. p. 304.

[2] Osborne, *Theory of Beauty*, p. 68.

[3] *Loc. cit.*

[4] *Loc. cit.*

* ▸ p. 479

[5] Osborne, 'Notes on the Aesthetics of Chess', p. 160.

[6] Osborne, 'Concepts of Order', p. 293.

[7] *Ibid.*, pp. 291–2.

[8] *Ibid.*, p. 292.

[9] *Loc. cit.*

[10] *Ibid.*, p. 292, emphasis in orig.

The first duty of science was to the truth, and theories that did not agree with observation had to be discarded.[4] Nevertheless, Osborne concluded that a theory could be beautiful but false, quoting in support Erwin Schrödinger's (1887–1961) description of Lamarckian evolution as 'beautiful, elating, encouraging and invigorating'.[5] Osborne also quoted the physicist Dennis Sciama (1926–99), without giving a source, as saying that Fred Hoyle's (1915–2001) steady-state cosmology was very beautiful but in conflict with the observational evidence.[6] Sciama certainly expressed the sentiment of the unsourced quotation: after he had abandoned it in favour of the Big Bang theory,[7] he wrote that '[t]he steady-state theory has a sweep and beauty that for some unaccountable reason the architect of the universe appears to have overlooked'.[8] The test against reality, for Osborne, marked the motivational division between pure mathematics and the sciences: pure mathematicians were willing to pursue theories of great beauty without concern for connection to observation or application, while scientists were not.[9]

Osborne agreed with Kline* that mathematics could be classed as an art, and in particular that 'some proofs convey that immediate sense of absolute rightness which enters into the aesthetic appreciation of works of art'.[10] But he thought that it should be possible to discern the qualities that contributed to mathematical beauty.[11] He drew on Aristotle and contemporary scientists to identify such properties: order, commensurability, precision, harmonious coherence among ideas, conceptual lucidity, elegance, clarity, economy, significance, depth, simplicity, comprehensiveness, insight.[12] He called Hardy's *Apology* '[t]he best modern account of mathematical beauty',[13] though he noted that it was not a rigorous account and that some of the qualities Hardy discussed were difficult to pin down.[14] Some qualities that contributed to mathematical beauty are present in *all* mathematics, such as order, precision, clarity, and harmonious coherence among ideas, but theorems, theories, and proofs could exhibit them in greater or lesser degrees.[15]

Gideon Engler

Gideon Engler's (1936–) essay 'Aesthetics in science and in art', published in 1990, considered whether there was a genuinely aesthetic basis for mathematicians' and scientists' feeling of beauty, or whether it was only a sense of excitement and satisfaction in achievement in their work.[1]

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[1] Osborne, 'Concepts of Order', pp. 292–3.

[2] *Ibid.*, p. 293.

[3] *Loc. cit.*

[4] Osborne, 'Interpretation in Science and in Art', p. 12; cf. Osborne, 'Mathematical beauty and physical science', p. 292.

[5] Schrödinger, *Mind and Matter*, p. 107.

[6] Osborne, 'Interpretation in Science and in Art', p. 292.

[7] Rees, 'Dennis Sciama', p. 366.

[8] Sciama, Review of *The Measure of the Universe*, p. 293.

[9] Osborne, 'Interpretation in Science and in Art', p. 12.

* ▸ p. 714

[10] Osborne, 'Mathematical beauty and physical science', p. 292.

[11] *Ibid.*, p. 293.

[12] *Loc. cit.*

[13] *Ibid.*, p. 294.

[14] *Ibid.*, pp. 294–5.

[15] *Ibid.*, p. 296.

His essay, however, is something of a puzzle. He interpreted both Aschenbrenner[*] and Osborne as having held that aesthetics was inapplicable to science. In Osborne's writing, wrote Engler, 'the scientific mode of thinking, discursive thinking, is basically contrasted with the "proper things" required for aesthetic appreciation'.[2] He did admit that Aschenbrenner's and Osborne's reactions, as he saw them, were nuanced, and in Osborne's case, contradictory. He saw this aspect of their views as a demonstration of the difficulty of fitting science into any traditional conception of aesthetic appreciation while having to take account of expressions of aesthetic value by scientists.[3] (The accounts of Aschenbrenner and Osborne offered in this chapter do not lead to any such contradiction: both scholars admitted that theorems and theories could be appreciated aesthetically, although Aschenbrenner did not think this appreciation extended to the objects or phenomena of mathematics or science.) Thus, though Engler argued in favour of the reality of aesthetic experience in mathematics and science, his essay was partly shaped as a response to his particular interpretation of Aschenbrenner and Osborne.

Engler independently made the same point as Harré had about the limited aesthetic vocabulary used in mathematics and the sciences.[4][†] In particular, he thought that *pretty* and *neat* usually played the subsidiary role of describing a device that yielded an elegant proof.[5] But Engler also thought that the application of aesthetic terms to modern art and in particular to constructivist art was mostly limited to the same vocabulary as was used for mathematics and science;[6] this lessens the force of Harré's argument.

Engler thought that unity would be key in the aesthetics of both science and art, but saw important differences in the perception of unity. Both in art and in mathematics and science, the whole was more than the parts. But in mathematics and science the whole was perceived only through the parts: mathematically-expressed concepts and relations.[7] This view led Engler to reject Bell's[‡] notion of 'significant form'[8] as being too bound to wholes.[9] It also led him to conclude that aesthetic concepts in mathematics and science had to be those that could be represented mathematically, such as the classical concepts of symmetry, simplicity, order, coherence, unity, elegance, and harmony.[10]

Engler interpreted symmetry in the sense of invariance under transformation rather than in the broader sense of the Greek *symmetria* [συμμετρία]. Simplicity was a vaguer term but could refer to the number of elements or the complexity of the construction. Order, coherence, and unity followed from the regularity and structure. Elegance could refer to the shortness of a proof or the use of an unforeseen idea, but was distinct from simplicity in that the proof had to 'achieve its conclusion in a particularly satisfactory way'. Harmony

[1] Engler, 'Aesthetics in science and in art', p. 25.

[*] ▸ pp. 732–3

[2] *Ibid.*, p. 26.

[3] *Ibid.*, pp. 26–7.

[4] *Ibid.*, pp. 32–3.

[†] ▸ p. 734

[5] *Ibid.*, p. 33.

[6] *Loc. cit.*

[7] *Ibid.*, p. 27.

[‡] ▸ pp. 728–9

[8] See Bell, *Art*, ch. I.I.

[9] Engler, 'Aesthetics in science and in art', p. 27.

[10] *Ibid.*, p. 28.

concerned pleasing relations between parts.[1] With the exception of symmetry, it is not clear that these concepts had, as Engler thought, a 'representation in mathematical forms and relations',[2] and he did not go into detail on this point.

Returning to the fundamental question, Engler thought that mathematicians and scientists felt pleasure in both 'perceptual-intellectual and sensuous-emotional modes'[3] and that these were hard to differentiate. The former was non-aesthetic: it was 'a deep satisfaction […] from the explanatory power achieved and its value of truth';[4] the latter is aesthetic: 'it is profound pleasure […] from the deep-rooted conviction that his accomplishments were guided by aesthetic concepts'.[5] The distinction, Engler conceded, was ambiguous, but was a natural consequence of the general problem of dividing the aesthetic and non-aesthetic. But Engler's characterization of the aesthetic mode is problematic. Engler thought that pleasure in this mode comes from a 'deep-rooted conviction', not directly from experience or the response to an experience, which would seem to undermine the reason for calling it 'aesthetic'. Further, this position seems to have stepped close to reducing the aesthetic feeling to a non-aesthetic 'conviction'.

Marcus Giaquinto

In a 2016 essay, the philosopher Marcus Giaquinto considered whether the application of aesthetic terms to proofs might just be 'loose talk', in the sense that '"beautiful" abounds as a general feel-good word', whose use can signify a non-aesthetic evaluation.[6] In the context of what Giaquinto termed the 'paradigm cases' of the appreciation of beauty — of a piece of art or music — an aesthetic judgement depended on the sense experience. A judgement of a proof was not dependent thus: *prima facie*, this difference could count against the use of aesthetic terms signifying a genuinely aesthetic judgement. But outside these paradigm cases, examples such as poetry or prose could be aesthetically appreciated, and their appreciation depended much less on sense-experience. In particular, a novel could be judged aesthetically from a translation, despite the sense-experience of seeing the words on the page or hearing them read aloud being entirely different from the corresponding experience one would obtain in the original language.[7] If one admits that aesthetic judgements could depend on *comprehension* rather than on perception or imagination, the abstractness of a proof ceases to be an obstacle to it being the object of such judgement.[8]

Thus Giaquinto disposed of one argument against there being genuine aesthetic judgements of mathematics. But he seems to have thought that the real test would be whether one could point to features of a proof that contribute to its aesthetic value, which he did via an examination of Steve Fisk's (1946–2010) proof of the 'art gallery theorem', in which he identified contributing properties such as clarity, brevity, imagination.[1] [His arguments on the genuineness of aesthetic judgements in mathematics was a prelude to a study of the relationship between beauty and explanatoriness, which is examined in the next chapter.*]

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[1] Engler, 'Aesthetics in science and in art', pp. 28–31.

[2] *Ibid.*, p. 28.

[3] *Ibid.*, p. 31.

[4] *Loc. cit.*

[5] *Loc. cit.*

[6] Giaquinto, 'Mathematical Proofs', pp. 57–8.

[7] *Ibid.*, pp. 58–9.

[8] *Ibid.*, p. 59.

OVER-DISTANCE AND UNDER-DISTANCE

Edward Bullough (1880–1934) published in 1912 an essay arguing that what he termed *Distance* could clarify some of the key problems of philosophical aesthetics. Distance, for Bullough, denoted a psychological position relative to some perceived object. He illustrated the concept by inviting the reader to imagine themself on a fogbound ship at sea: such a situation would be inconvenient, for one's travel would likely be disrupted by delay, and would be stressful and possibly dangerous, for there would be an increased risk of collision and disaster. However, one could also enjoy the experience of the fog and its effects of distorting visual and aural perception. The difference in perspectives came from the intercession of Distance. The disruption and anxiety directly and immediately affected one's personal needs and cares; the pleasurable contemplation was possible when a mental Distance separated the phenomenon from the self.[2]

Bullough thought that Distance served to distinguish the agreeable from the beautiful: '*the agreeable is a non-distanced pleasure*',[3] for the agreeable satisfied immediately personal needs or cares, such as water slaking a thirst. The beautiful (construed broadly as aesthetic pleasure) was distanced; it satisfied no such needs. Thus Bullough's Distance was a relative of the disinterestedness that descended from Shaftesbury† (1671–1713): 'It is Distance which makes the aesthetic object "an end in itself."'[4] Distance was also one of the criteria that served to separate aesthetic values from practical, scientific, or social ones.[5]

But Distance admitted degrees. People differed in their ability to maintain a greater or lesser Distance; objects differed in the Distance they evoked; an individual could differ in their ability, depending on the object.[6] This gradation entailed two ways in which Distance could prevent an aesthetic experience from occurring: *over*-distancing or *under*-distancing.[7] The former was when the object was too far for the perceiver to engage aesthetically; the latter was when the object struck too close to the viewer's personal needs. Overly pure classical music might be for many people over-distanced; light or catchy tunes might be so under-distanced that they passed from art to mere amusement.[8]

Jerry King (1935–2020) suggested in his 1992 book *The Art of Mathematics* that Bullough's concept of Distance explained the contrasting attitudes to mathematics of mathematicians on the one hand

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[1] Giaquinto, 'Mathematical Proofs', pp. 61–3.

* ▸ pp. 804–8

[2] Bullough, '"Psychical Distance" as a Factor in Art', pp. 89–90.

[3] *Ibid.*, p. 108, emphasis in orig.

† ▸ pp. 353–6

[4] *Ibid.*, p. 117.

[5] *Ibid.*, p. 118.

[6] *Ibid.*, p. 94.

[7] *Loc. cit.*

[8] *Ibid.*, p. 98.

and humanists, scientists, and engineers on the other. For humanists, mathematics failed to produce an aesthetic experience because of over-distancing. For scientists and engineers, mathematics was under-distanced: it served the immediate practical needs of the scientist in formulating or working with a theory or the engineer in designing a structure.[1]

King drew the conclusion that the two cultures divide between the sciences and the humanities was reflecting the under-distancing and over-distancing of mathematics as a potential object of appreciation. Hence mathematics could serve as a bridge between the two cultures, if its teaching could be reformed so as to widen the zone within which people could maintain an appropriate Distance.[2]

[Although King did not explore this point, the application of the Distance concept seems to be linked to an association of mathematical beauty to lack of utility, or at least to a lack of interest in utility. An awareness of practical applications would tend to lessen the Distance between the contemplator and the mathematics, risking under-distancing.]

REINTERPRETATION I: BEAUTY AS ENLIGHTENMENT

Gian-Carlo Rota

The mathematician Gian-Carlo Rota (1932–99), in a paper presented to a 1996 symposium on 'Proof and Progress in Mathematics', made an attempt to analyse the use of the term *beauty* in mathematics, and his analysis led him to reject any genuinely aesthetic component.

Rota gave five examples of loci of beauty in mathematics:[3] i) theorems, ii) proofs, iii) theories, iv) definitions, and v) 'brilliant proof steps' in otherwise unexceptional proofs. (He seems to have subsumed axiom systems under definitions.[4]) He emphasized that these loci were often independent: for instance, a beautiful theorem might lack a beautiful proof. As examples, Rota said that the Prime Number Theorem[*] and the theorem that there are exactly five regular convex polyhedra were thought to be beautiful, but lacked beautiful proofs.[5]

He also took pains to maintain a distinction between the intrinsic beauty of mathematics and the form of its exposition.[6] 'Mathematical elegance has to do with the presentation of mathematics, and only tangentially does it relate to content.'[7] Thus a beautiful theorem, proof, or theory could be presented either elegantly or inelegantly; he gave the example of Hermann Weyl's[†] (1885–1955) proof of the equidistribution theorem, but did not cite examples of elegant or inelegant presentations.[1] One could make an effort towards achieving elegance, he thought, but one could not strive for beauty.[2]

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[1] King, *The Art of Mathematics*, pp. 207–8.

[2] *Ibid.*, pp. 267–8.

[3] Rota, 'Phenomenology of Mathematical Beauty', pp. 171–2.

[4] *Ibid.*, p. 173.

[*] ▸ p. 407

[5] *Ibid.*, p. 172.

[6] *Ibid.*, pp. 173–5.

[7] *Ibid.*, p. 178.

[†] ▸ pp. 679–80

Rota disagreed with G. H. Hardy's (1877–1947) assessment that surprise (or, to use Hardy's term, unexpectedness[3]) was an element on which the beauty of a proof depended: while the perception of beauty in mathematics was often accompanied by a feeling of surprise, there were very surprising results that had never been thought beautiful.[4] He suggested two examples of results that were surprising but not beautiful: Morley's trisector theorem, which he emphasized possessed neither a beautiful statement nor a beautiful proof, and the existence of non-equivalent differential structures on spheres of high dimension. (Others, however, have found these results beautiful.[*]) Rota added that many theorems seemed surprising when first published, but were never considered beautiful.[5] Rota's argument, however, was based on a flawed understanding of Hardy, who listed three 'purely aesthetic qualities'[6] possessed by beautiful theorems and proofs: unexpectedness, inevitability, and economy.[†] Thus, for Hardy, a beautiful theorem and its proof generally featured unexpectedness, but he did not claim that unexpectedness alone was *sufficient* for beauty. Rota's examples thus did not truly confront Hardy's position (which would rather be weakened by an example of a theorem or proof which was beautiful but which did not feature unexpectedness).

Rota innovated in the weight he placed on beauty in definitions, which no earlier scholar had considered. He believed that it was a special feature of twentieth-century mathematics that theories had emerged in which the definitions were more beautiful than the theorems.[7] His primary example was category theory, which was 'rich in beautiful and insightful definitions and poor in elegant proofs'. (Given Rota's distinction between beautiful content and elegant presentation, he presumably made a linguistic slip here and meant 'beautiful proofs'.) The basic notions of the field were intrinsically beautiful, but the theorems were 'clumsy and dull'.[8] And there were other examples of beauty in definitions: he called 'quite beautiful' Alonzo Church's (1903–95) axiomatization of the propositional calculus and H. S. M. Coxeter's (1907–2003) version of the axioms of euclidean geometry.[9]

For Rota, whatever beauty was, it was of the same import as truth. Mathematical truth and mathematical beauty were both objective properties. Neither was arbitrarily settled. But neither was absolute: mathematical truth was measured by the age's standard of rigour; mathematical beauty by its standard of taste.[10]

Rota thought that a proof that was considered beautiful was usually the definitive one, and a lack of beauty in a proof was often associated with a lack of definitiveness and a concomitant search for better proofs.[11] And beauty was often associated with brevity of statement or proof.[12]

Rota argued that all of the effort that went into understanding a piece of mathematics, including the work required to assimilate the

[1] Rota, 'Phenomenology of Mathematical Beauty', p. 178.

[2] *Ibid.*, p. 177.

[3] Hardy, *Apology*, § 18.

[4] Rota, 'Phenomenology of Mathematical Beauty', p. 172.

[*] ▸ pp. 832–3

[5] *Loc. cit.*

[6] Hardy, *Apology*, § 18.

[†] ▸ pp. 502

[7] Rota, 'Phenomenology of Mathematical Beauty', p. 172.

[8] *Ibid.*, p. 173.

[9] *Loc. cit.*

[10] *Ibid.*, pp. 175–6.

[11] *Ibid.*, p. 178.

[12] *Ibid.*, p. 179.

necessary background, vanished from consciousness when one perceived its beauty, and was replaced by an impression of instantaneous realization through a flash of insight: a 'light-bulb moment'.[1] For Rota, the reality of mathematical beauty was revealed by considering this mistaken impression. When a mathematician had made the effort to understand a piece of mathematics, and had succeeded in formally checking it, but had not attained this state of insight, they had the feeling of something being missing. The step-by-step logical checking had been completed without yielding *enlightenment*: there was no sense of the role that this piece of mathematics played in the larger context.[2]

And it was as enlightenment that Rota interpreted beauty. A beautiful theorem or proof was one that enlightened the reader:

> 'We acknowledge a theorem's beauty when we see how the theorem "fits" in its place, how it sheds light around itself [...]. We say that a proof is beautiful when such a proof finally gives away the secret of the theorem, when it leads us to perceive the inevitability of the statement that is being proved.'[3]

He argued that, because enlightenment admitted degrees, mathematicians, accustomed to absolutes and to certainty, hid the concept behind the term 'mathematical beauty', which he rather curiously suggested did not admit degrees.[4] (The respondents to questionnaires that ask for results to be scored for beauty would presumably beg to differ.[*])

Rota's thesis, however, was undermined by some of his other statements. He rejected a particular proof as 'far too sophisticated to be beautiful',[5] and yet it is conceivable that a sophisticated proof could still yield enlightenment. (Note that it matters not whether the proof in question — Coxeter's proof of Morley's trisector theorem[6] — was enlightening or beautiful or neither or both; the point is the *non sequitur* in Rota's *reason* for thinking it not beautiful.) Furthermore, the enlightenment interpretation does not explain how 'many occurrences of mathematical beauty eventually fade or fall into triviality as mathematics progresses':[7] it is unclear how a proof, for instance, can be made less enlightening by further mathematical developments. Rota's use of 'triviality' here also hinted at some role for importance or difficulty in mathematical beauty.

Even more puzzling is his statement that a proof might only be seen as beautiful in contrast to other clumsier or longer proofs.[8] It is difficult to imagine a proof that grants one enlightenment only if one has read another proof that does not.

Fundamentally, Rota seems to have mistaken beauty for a property that contributed to beauty.[9] If one judges as beautiful some highly symmetrical object, it does not follow that one is erroneously applying the term *beautiful* to describe the property of being symmetrical.[10] Similarly, from judging as beautiful some enlightening pieces of mathematics, one cannot infer that the term *beautiful* actually describes

[1] Rota, 'Phenomenology of Mathematical Beauty', p. 179.

[2] *Ibid.*, pp. 180–1.

[3] *Ibid.*, p. 182.

[4] *Ibid.*, p. 180.

[*] ▸ pp. 834–8

[5] *Ibid.*, p. 172.

[6] See Coxeter, *Introduction to Geometry*, pp. 24–5.

[7] Rota, 'Phenomenology of Mathematical Beauty', p. 175.

[8] *Ibid.*, p. 179.

[9] Montano, *Explaining Beauty in Mathematics*, p. 13.

[10] *Ibid.*, p. 14.

the property of being enlightening. Even if every piece of mathematics judged as beautiful were enlightening, this fact would only suggest that being enlightening is a property necessary to evoke a judgement of beauty.

[The idea that there could be pleasure in enlightenment has been applied also to other fields. Francis Hutcheson[*] (1694–1746) gave the example of how in history it is dull to read a bare enumeration of events, but enjoyable to find an explanation of diverse events as consequences of hitherto unknown causes.[1]]

Ulianov Montano

As part of a broader study of mathematical beauty,[†] Ulianov Montano (1973–) rejected the notion that aesthetic judgements in mathematics were non-literal.[2] He advanced various arguments against this notion and in particular against Rota's beauty-as-enlightenment thesis.[‡] But one of his primary arguments contra Rota was curious. He wrote that *reductio ad absurdum* had been considered one of the most beautiful methods of proof since Euclid, but that a proof by *reductio* was not enlightening.[3] The first part of this statement is historiographically doubtful: even looking just at the mid-twentieth century, one can find explicit disagreement on the aesthetic value of such proofs.[§] The second part of Montano's assertion, that a proof by *reductio* is not enlightening, seems to have been inherited from François Le Lionnais[¶] (1901–84). Le Lionnais neither defended nor developed this point and it served no further role in his discussion. It seems not to withstand scrutiny. For instance, the proof of the irrationality of $\sqrt{2}$ by appeal to the Fundamental Theorem of Arithmetic[♠] is a *reductio ad absurdum* that is enlightening, moreso than more elementary proofs.

Montano also criticized Rota's position on the grounds that it did not lend itself to a unified approach to beauty, ugliness, or elegance.[4] This disunity was partly a matter of definition: elegance, for Rota, concerned the presentation of mathematics, and 'only tangentially' its content.[5] But as regards beauty and ugliness, the criticism is valid. If beauty is equated with enlightenment, then a lack of beauty is equated with a lack of enlightenment. Rota claimed that '[m]athematicians seldom use the word "ugly"', preferring '"clumsy", "awkward", "obscure", "redundant", [...] "technical", "auxiliary", and "pointless"'.[6] The terms substituted for *ugly* all seem to indicate a lack of enlightenment, which was equivalent — in Rota's view — to a lack of beauty. But ugliness surely means something *more* than a mere lack of beauty: it is in some sense opposed to beauty. But there seems to be nothing opposed to enlightenment with which ugliness could be equated.

[*] ▸ pp. 357–62

[1] Hutcheson, *Inquiry Concerning Beauty*, Art. VI.vi.

[†] ▸ pp. 795–802

[2] Montano, 'Beauty in Mathematics', ch. 2, *passim*.

[‡] ▸ pp. 742–5

[3] Montano, *Explaining Beauty in Mathematics*, p. 12.

[§] ▸ p. 851

[¶] ▸ p. 677

[♠] ▸ p. 846

[4] *Ibid.*, p. 14.

[5] Rota, 'Phenomenology of Mathematical Beauty', p. 178.

[6] *Loc. cit.*

Carlo Cellucci

The philosopher Carlo Cellucci (1940–) thought that the reinterpretation of mathematical beauty as enlightenment was problematic, because the term *enlightenment* suggested an instantaneous moment of realization in which one grasped a proof. For Cellucci, focusing on such a single act of mental apprehension seemed to contradict Rota's warning about thinking of the 'light-bulb moment'.[1] The distinction that Cellucci seems to have made was perhaps subtle: Rota thought that the experience of mathematical beauty is only *remembered* as a light-bulb moment, with the preceding effort forgotten, while for Cellucci the enlightenment actually *was* a light-bulb moment. Thus it was a mistake to reinterpret mathematical beauty as enlightenment. Rather, 'when mathematicians say that a piece of mathematics is beautiful, they mean to say that it gives understanding'.[2]

Cellucci did not explore mathematical understanding in depth, but said that it meant 'recognition of the fitness of the parts to each other and to the whole'.[3] With this meaning, understanding would be an aesthetic property, since there is a long-standing tradition associating fit and beauty.[4]*

SCEPTICISM, THE AESTHETIC, AND THE EPISTEMIC

The philosopher Cain S. Todd (1976–), in his 2008 essay 'Unmasking the truth beneath the beauty', argued that it was more plausible that putative aesthetic properties claimed to be found in mathematics and the sciences were in fact epistemic properties.[5] This conclusion, he thought, should lead one to adopt a sceptical position with regard to genuinely aesthetic properties in mathematics and the sciences. He did not argue that apparently aesthetic judgements could be reduced to epistemic judgements,[6] just that the burden of proof fell on those who maintained that the putative aesthetic properties were genuinely aesthetic, and he held that no aesthetic theory of mathematics or the sciences had convincingly argued how to distinguish aesthetic properties from epistemic ones.[7] Fundamentally, the question turned on what was actually meant by 'aesthetic' and 'epistemic'[8] and on the observation that the practice of mathematics or science and the practice of art differed: mathematics and science had as a (not necessarily *the*) primary purpose the attainment of something other than the aesthetic.[9]

Todd's 2008 essay mainly concerned the sciences, but he summarized the application of his argument to mathematics in another essay in 2018.[10] The account given here concentrates on mathematics.

[1] Cellucci, 'Mathematical Beauty, Understanding, and Discovery', p. 344.

[2] *Loc. cit.*

[3] Cellucci, 'Mathematical Beauty, Understanding, and Discovery', p. 344; cf. Cellucci, *Rethinking Knowledge*, pp. 324 sqq.

[4] Cellucci, 'Mathematical Beauty, Understanding, and Discovery', p. 344.

* ▸ pp. 816–17

[5] Todd, 'Unmasking the truth beneath the beauty', p. 63.

[6] *Ibid.*, pp. 63, 75.

[7] *Ibid.*, pp. 62–3.

[8] *Ibid.*, p. 62.

[9] *Ibid.*, p. 73.

[10] Todd, 'Fitting Feelings and Elegant Proofs', pp. 212–16.

Todd pointed out that there were two basic positions that one could hold concerning the aesthetic and epistemic evaluations of proofs or other mathematical statements. The *conjunctive* view affirmed that there was some connection between these evaluations; the *disjunctive* view insisted that there was none.[1]

The conjunctive view was compatible with thinking of aesthetic judgements as some aspect of epistemological judgements. One could argue, say, that a proof was more likely to be seen as beautiful if it was more enlightening or explanatory. But such an argument did not explain how aesthetic value could be a guide to truth, and it did not explain how the pleasure in such a proof came to be perceived as aesthetic.[2] As Todd noted, a defence of this position should offer some account of the relationship between aesthetic and epistemic criteria. Since there seemed to be no such account, one was entitled to be suspicious that there was in fact a conflation of aesthetic and epistemic criteria.[3] Within the conjunctive position, one could also argue for an equivalence between aesthetic and epistemic criteria, as Peter Kivy (1934–2017) did in the sciences:

> 'there never is a contest between beauty and truth in theoretical science, understood as an attempt to represent nature. It cannot represent nature beautifully, in the fullest sense, without representing it truthfully.'[4]

Todd objected that the representational goals and criteria of success are different in art and in science: art aims for aesthetic value for its own sake; science aims at an accurate representation of nature.[5]

It is plausible to apply Kivy's argument to mathematics, if one views mathematics as striving to represent a platonic heaven. In this case, Todd's objection applies *mutatis mutandis*: art aims for aesthetic value for its own sake; mathematics aims at an accurate representation. However, the beauty of nature is at least partly accessible to the unmediated senses; the beauty of the platonic mathematical reality is accessible *only* through mathematical representation.

The disjunctive viewing was that aesthetic and epistemological value were independent. Todd's opposition to this position was based on the observation that it allowed for the possibility that a theorem or proof, or a scientific theory, was true but not beautiful or *beautiful but not true*. At minimum, it would be possible to appreciate aesthetically a theorem or proof, or a scientific theory, *regardless* of whether it was true. But this conclusion seemed to clash with the truth-functional natures of theorems, proofs, and theories.[6] In general, there seems to be much scepticism among scientists regarding the idea that a theory could be beautiful but not true,[7] although there are exceptions. Lamarckian evolution and the steady-state cosmology have already been mentioned.* A subtler example might be general relativity, which was still considered beautiful long after it was known to be *incomplete*.[8]

This clash between aesthetic and epistemic value is perhaps a point where Todd's argument, which focused primarily on the sci-

[1] Todd, 'Unmasking the truth beneath the beauty', p. 64.

[2] Cf. *ibid.*, p. 65.

[3] *Loc. cit.*

[4] Kivy, 'Science and aesthetic appreciation', p. 193.

[5] Todd, 'Unmasking the truth beneath the beauty', p. 66.

[6] *Ibid.*, p. 67.

[7] McAllister, *Beauty & Revolution*, ch. 6.

* ▸ p. 738

[8] See, for example, Nahin, *Dr. Euler's Fabulous Formula*, p. xxxi.

ences, does not apply directly to mathematics. In mathematics, rigour is an important factor in whether a proof is judged true. A proof can be judged non-rigorous and thus incorrect (in the sense of not establishing the truth of the theorem), but still be considered beautiful, especially if the theorem is already known to be true. Wolfgang Krull (1899–1971) observed that there can even be a tension between beauty and rigour.[*] But even a non-rigorous proof must offer an argument that increases a reader's belief that the theorem is correct, or conveys some intuitive idea of why the result is correct. That is, even a non-rigorous proof has some truth-functional role.

Todd noted that if someone used an aesthetic term like *beautiful* to describe a painting, it was conceivable that they were actually expressing a non-aesthetic interest, such as desire to own it. Such a reductive use of the word would be revealed by inquiring into the reasons that supported its use.[1] Todd pointed out that the arguments mathematicians made to support their aesthetic judgements tended to cite features that are also criteria of whether proofs are valid or adequate.[2] Properties such as simplicity, symmetry, elegance,[3] economy, insightfulness, significance, breadth, depth, seriousness, rigour, efficiency,[4] unexpectedness, inevitability, (the last two being inferred from Todd's citation of Hardy's *Apology*)[5] were either truth-functional or expressed other 'epistemic virtues' like comprehensibility or enlightenment.[6] (Unexpectedness — which, admittedly, Todd did not list explicitly — might seem to be an oddity among putatively epistemic criteria, but could be considered as contributing to memorability, which might count as an epistemic value.) At minimum, each criterion could be interpreted as aesthetic or non-aesthetic, evaluative or non-evaluative.[7] Todd also thought that these criteria for beauty led to the scope of aesthetic terms in science and mathematics being more clearly delimited than in art.[8] Further, there were important differences between the use of such criteria in mathematics and in art: in mathematics, none of them seemed to be used to justify negative aesthetic judgements, unlike in art. 'Could a clumsy proof or ugly theorem be so *in virtue of* its symmetry or simplicity?'[9]

Todd saw the partial coincidence of defences put forward for aesthetic and epistemic evaluations of proofs, agreed with Harré's observation about the limited aesthetic vocabulary used in mathematics and science,[10][†] noted the problems inherent in both the conjunctive and disjunctive positions, and perceived a lack of discord in aesthetic claims in mathematics.[11][‡] He therefore wondered how one could be sure that what was *called* aesthetic value in mathematics actually *was* based upon aesthetic pleasure as opposed to intellectual pleasure in the correctness of a proof or the foreseen career benefits of such success. More generally, how could one distinguish aesthetic and non-aesthetic uses of a given term?[12]

In writing his 2018 essay, Todd was apparently unaware of, or at least did not cite, Rota's 'The Phenomenology of Mathematical

[*] ▸ p. 661

[1] Todd, 'Unmasking the truth beneath the beauty', p. 74.

[2] *Ibid.*, p. 71.

[3] *Ibid.*, p. 73.

[4] Todd, 'Fitting Feelings and Elegant Proofs', p. 214.

[5] Todd, 'Unmasking the truth beneath the beauty', p. 71.

[6] *Ibid.*, p. 73.

[7] Todd, 'Fitting Feelings and Elegant Proofs', p. 215.

[8] Todd, 'Unmasking the truth beneath the beauty', p. 71.

[9] Todd, 'Fitting Feelings and Elegant Proofs', p. 215, emphasis in orig.

[10] Todd, 'Unmasking the truth beneath the beauty', p. 71.

[†] ▸ p. 734

[11] *Ibid.*, pp. 71–2.

[‡] ▸ pp. 829–30

[12] *Ibid.*, p. 71.

Beauty'.[1]* His independently-formulated scepticism was clearly compatible with Rota's beauty-as-enlightenment thesis. Todd, however, did not deny that the mathematics labelled 'beautiful' could produce genuinely affective experiences,[2] and in later work he offered an explanation of such experiences in terms of aesthetic-epistemic feelings, which will be examined in the next chapter.†

REINTERPRETATION II: BEAUTY AS METAPHOR

Nick Zangwill

The 2001 book *The Metaphysics of Beauty* by Nick Zangwill (1957–) led in 2009 to one of the few instances of a direct debate between two philosophers on the aesthetics of mathematics: John Barker argued against Zangwill's position, and Zangwill made a response.

In his book, Zangwill acknowledged that there was pleasure to be had in contemplating a proof or theory or chess move: an *intellectual* pleasure, certainly. But, he asked, was it *aesthetic* pleasure? Was it the response to *aesthetic* properties?[3] The use of aesthetic terms in evaluating a proof was evidence that an aesthetic judgement was taking place, but it was not decisive, for such terms might be used metaphorically, 'like calling a company takeover "elegant," a hand of cards "beautiful" or a mood "ugly"'.[4] Such metaphorical uses indicated admiration or dislike for a thing, but did not signify admiration or dislike for its aesthetic properties.

Zangwill argued that the application of aesthetic terms to proofs was just such a metaphorical use. First, he held that aesthetic properties depended on sensory properties.[5] Barker argued that a more precise formulation of this point was the following:

> 'SENSORY/AESTHETIC EXPLANATION. For any object X with an aesthetic property A, there are sensory properties that play a role in explaining why X has property A, although non-sensory properties may also play such a role.'[6]

Zangwill did not mention this reformulation in his reply to Barker, so presumably he found it unobjectionable.[7] This dependence on sensory properties seemed to imply that mathematical objects could not have aesthetic properties, although, as Barker noted, this conclusion might depend on precisely what was meant by an explanatory role.[8] If mathematical objects did not have aesthetic properties, the application of aesthetic terms to them had to be metaphorical. However, Zangwill did not make positive arguments for the Sensory/Aesthetic Explanation thesis (in his original formulation); instead, he defended

[1] Todd, 'Unmasking the truth beneath the beauty', References.

* ▸ pp. 742–5

[2] Todd, 'Fitting Feelings and Elegant Proofs', p. 217.

† ▸ pp. 823–4

[3] Zangwill, *The Metaphysics of Beauty*, p. 140.

[4] *Ibid.*, p. 141.

[5] *Ibid.*, p. 127.

[6] Barker, 'Mathematical Beauty', p. 63.

[7] Zangwill, 'Reply to John Barker'.

[8] Barker, 'Mathematical Beauty', p. 63.

it against certain counterexamples that might be proposed. Thus his accepting it as the default position seems to have been based on it holding for works of art and other objects that were subjects of aesthetic appreciation.

Another argument Zangwill advanced for his thesis that aesthetic terms were used metaphorically when applied to proofs, scientific theories, and chess moves was that each of these abstract entities had a purpose. The purpose of a proof was to demonstrate rigorously the truth of a result on the basis of the truth of earlier results. The purpose of a theory was to explain the phenomena (or at least to model and predict them). The purpose of a chess move was to advance the aim of winning the game. Admiration for proofs, theories, and chess moves depended solely on their effectiveness in attaining their purpose. A proof could not be admired if it was invalid and incorrect proofs could not be beautiful. A scientific theory that did not account for much of the experimental data could not be beautiful. A chess move could not be beautiful if it lost the game.[1]

Zangwill also denied that a proof could have *dependent beauty*. A dependent aesthetic property was one that arose from the *way* in which a thing expressed its purpose. It was possible for a thing to have a dependent aesthetic property but to fail in its purpose: a building that was meant to be sturdy could aesthetically express that sturdiness but actually be weak. A proof could not be dependently beautiful, for, in the case of proofs, the aesthetic expression of purpose and the actual fulfilment of that purpose could not separate. That is, a proof aesthetically expressed the purpose of demonstrating a result if and only if it actually fulfilled the purpose of demonstrating the result. Thus the pleasure one felt in a 'beautiful' proof was not genuine aesthetic appreciation, but just an admiration of how effectively it fulfilled its purpose.[2]

Berys Gaut

Berys Gaut (1958–) briefly argued against Zangwill in his book *Art, Emotion and Ethics* (2007). He pointed to the longstanding tradition in which mathematicians and physicists held theories and proofs to be beautiful, in which the aesthetic objects were apprehended via contemplation rather than sense-perception. Unlike a metaphor asserting that 'a man is a swine', such statements by mathematicians and physicists were not *prima facie* false when taken literally.[3]

Further, Gaut also disagreed with Zangwill's argument that the expression of a purpose and the fulfilment of a purpose were inseparable in the case of abstract qualities. A theory could express the purpose of yielding the truth and yet fail to do so.[4] Similarly (though this was not one of Gaut's examples), a proof might express the purpose of establishing a theorem and yet be invalid.

[1] Zangwill, *The Metaphysics of Beauty*, p. 141.

[2] *Ibid.*, pp. 141–3.

[3] Gaut, *Art, Emotion and Ethics*, p. 126.

[4] *Loc. cit.*

John Barker

Barker rejected Zangwill's conclusions. He pointed out Zangwill's lack of positive arguments for the position he reformulated as the Sensory/Aesthetic Explanation thesis and argued that it could only be accepted as a default position upon an evidence base from which mathematical objects were preëmptively excluded.[1]

Barker also disagreed with Zangwill's position that using aesthetic terms to describe proofs was too closely tied to the evaluation of the purpose of a proof to indicate genuinely aesthetic judgements, and that the application of aesthetic terms rather expressed an admiration for the effectiveness of the proof. Barker held that 'effectiveness is an all-or-nothing affair when it comes to proofs':[2] a proof either demonstrated the result or it does not. It was thus not possible for one proof of a result to be more effective at fulfilling its purpose than another proof of that result. Hence it followed from Zangwill's position that all proofs of a given result were aesthetically identical.[3]

Yet some proofs of a result were elegant and others were not. An elegant proof did not make the theorem more true, so it could not fulfil its purpose more effectively than an inelegant proof. Thus, Barker concluded, when a proof was judged elegant, it was *not* being judged on its effectiveness.[4]

For Barker, an elementary proof was one that could be understood without advanced mathematical knowledge. Therefore elementary proofs tended to use methods that were not fruitful in the sense of being patterns that were generally conducive to success: the methods and strategies were basic and less interesting. But Barker held that such proofs could be elegant *because* they are elementary, not *despite* their being elementary.[5] Similarly, simplicity was seen as contributing to elegance, but did not seem to be conducive to success.[6]

[Barker's definition of 'elementary' was perhaps slightly different from the one used by most mathematicians, for whom an elementary proof does not make use of mathematical concepts or objects from outside the field of the theorem. For example, an elementary proof of a result in number theory might use very difficult and advanced arguments, but would only use the integers, not the real or complex numbers. The methods of elementary proofs can be very complicated and possibly very interesting, but they are perhaps *specialized*, and so Barker's point about them not being *generally* conducive to success may still hold.]

Barker thought that it could be argued that explanatoriness was in some sense success-conducive, and that explanatoriness was not an aesthetic property.[*] But he pointed out that a proof can be explanatory and inelegant, or elegant and non-explanatory. There was often a trade-off, for an elegant proof might be short and clever and thus mystifying.[7] Barker gave as an example the Second Recursion Theorem (also called Kleene's Recursion Theorem), although his explanatory footnote suggests that he was thinking of Hartley Rogers's (1926–2015)

[1] Barker, 'Mathematical Beauty', pp. 63–4.

[2] *Ibid.*, p. 65.

[3] *Loc. cit.*

[4] *Loc. cit.*

[5] *Loc. cit.*

[6] *Ibid.*, p. 66.

[*] ▸ pp. 787–90

[7] *Ibid.*, pp. 66–7.

simpler version.[1] The short proof of which Barker was presumably thinking relies on defining a *deus ex machina* auxiliary function that quickly establishes the theorem. [The present author agrees that the proof is non-explanatory but, perhaps due to not being a logician, hesitates to call it elegant. The notion of *deus ex machina* has been explored more generally in connection with the aesthetics of proof.[*]]

Elegance thus could not be identified with the effectiveness of the proof or with features that were conducive to that effectiveness. But Barker agreed that elegance was not wholly separate from effectiveness: it would be strange to call an unsuccessful proof elegant.[2] He questioned Zangwill's rejection of dependent elegance in proof, noting that the inference from 'a proof lacking success cannot be elegant' to 'what is called elegance in proofs is not a dependent aesthetic property' required another principle, something like:

> 'NON-INSTRUMENTALITY. When a dependent aesthetic property *P* is connected to a function *F*, objects must be capable of having property *P* while completely failing to fulfill the function *F*.'[3]

As Barker pointed out, Zangwill did not give an argument for this principle, just examples such as the building that falsely appeared sturdy. Barker strengthened his argument by pointing out that the manifest properties of a building were not tied to its actual sturdiness. The aesthetic judgement of sturdiness depended on the building's manifest properties, so the divide between the aesthetic judgement of sturdiness and the actual weakness did not lie between the aesthetic judgement and the manifest properties, but between the manifest properties and the actual weakness. In a proof, the correctness was manifest. A proof could not appear correct while being completely devoid of properties that were conducive to correctness. If a proof appeared correct but was in fact invalid, then it usually contained much that was correct and only a few subtle errors. Consider, for instance, Alfred Bray Kempe's (1849–1922) famous 1879 invalid proof of the four-colour theorem.[4] Its flaw is that two transformations of a four-colouring, each of which can be performed separately, may not be possible *simultaneously*; nevertheless, enough can be salvaged from Kempe's proof to establish that *five* colours suffice to colour any planar graph.[5][†]

Thus, since the possible division between manifest properties and purpose in Zangwill's examples did not apply to proofs, Barker concluded that Zangwill had not excluded dependent elegance. Barker did accept that there was some separation between aesthetic properties and purpose. A proof could not have aesthetic properties simply by being correct. It could be correct but not elegant; it could not be elegant but (manifestly) not correct.[6]

Barker proceeded to note some similarities between judgements of mathematical beauty and elegance and what were ordinarily accepted as aesthetic judgements. First, judgements of elegance were

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[1] Rogers, *Theory of Recursive Functions*, Thm 11.2.1.

[*] ▸ pp. 637–48

[2] Barker, 'Mathematical Beauty', p. 67.

[3] *Loc. cit.*

[4] Kempe, 'On the Geographical Problem of the Four Colours'.

[5] Heawood, 'Map-Colour Theorem'.

[†] ▸ pp. 602–3

[6] Barker, 'Mathematical Beauty', pp. 68–9.

Reinterpretation II: Beauty as metaphor

universal and subjective in the Kantian sense.[*] The judgement attributed an aesthetic property to a proof rather than simply reporting a state of mind. One expected that others would broadly agree. Such judgements also seemed to be based on subjectively felt responses to the proof: there certainly seemed to be a closely associated response of pleasure when one contemplated an elegant proof, and it was difficult to see what they *could* be based on, if not on a felt response.[1] And such judgements were not inferential: one would know it when one saw it. But knowledge could bear upon these judgements by influencing how one perceived a thing: a dendrologist saw a tree as an elm where another person saw it only as a tree; someone with relevant experience would perceive aesthetic features that another person would miss. But the knowledge and experience altered the perception rather than supporting inference from it.[2] Barker noted in this regard that *understanding* a proof, as opposed to a line-by-line verification, required non-conceptual knowledge. The proof consisted of certain written statements and diagrams; the understanding was 'something else', an insight that was gained in a non-inferential way by following and thinking about the proof. But this 'something else' could not be communicated in language and diagrams; at most, one could try to explain facets of the proof to someone else in the perhaps vain hope that they could achieve a similar insight. Understanding was thus paralleled by elegance: the judgement was an insight, not an inference, and although the judgement could be communicated in language, the recognition of it could not be.[3]

Barker listed certain features of proofs that tended to contribute to elegance. The first was simplicity, which he admitted was hard to define, being another 'know it when one sees it' quality, but which included, for instance, avoidance of division into many cases. Another contributing feature was economy: 'a large payoff for a small investment: e.g., a simple but well-placed move creates a large effect in terms of advancing the proof'.[4] A third was directness, in the sense of not straying from the natural territory of the proof.[5]

Barker's features echoed but differed from the Hardian triad.[†] Barker's 'directness' and Hardy's 'inevitability' seem closely related. Hardy's 'economy' was about 'avoiding complications of detail'[6] rather than the profitability of the moves; instead, Barker's 'simplicity' might overlap with Hardy's 'economy'.

In any case, Barker's aim was to note that such features, seen in a work of art, might (in some instances) be conducive to a judgement of beauty in that work, and that this parallel supported his thesis that a judgement of elegance in proof was also aesthetic.

Zangwill's response to Barker

In reply to Barker, Zangwill declared that his confidence in his position had been shaken by Barker's arguments, but gave another example to replace the apparently sturdy building: a suspension bridge I) was elegant *as a bridge* (that is, as a solution for transport across a body of water, not as a structure with a particular form); II) appeared flimsy; III) was strong. The elegance of the bridge was a dependent aesthetic property, being connected with its purpose, but unconnected with its manifest properties. For Zangwill, this example undermined the objection that Barker had based on his distinction of manifest properties.[1]

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[*] ▸ p. 376

[1] Barker, 'Mathematical Beauty', pp. 69–70.

[2] *Ibid.*, p. 71.

[3] *Ibid.*, p. 72.

[4] *Ibid.*, p. 73.

[5] *Ibid.*, pp. 72–3.

[†] ▸ pp. 502

[6] Hardy, *Apology*, § 18.

Further, Zangwill noted that Barker had placed emphasis on proofs that were correct but not elegant, but thought that the real work in his own argument was on the *lack* of proofs that were invalid but elegant:

> 'this is puzzling and difficult to explain for the believer in aesthetic mathematical elegance. If the elegance of mathematical proofs is an aesthetic property of them, then why cannot there be elegant but ineffective proofs? The best explanation, surely, is that "elegance" does not denote an aesthetic property as it does normally, and it is being used metaphorically when applied to mathematical proofs.'[2]

MATHEMATICS AND ART THEORY III

In a 2011 paper, the philosopher of art Rob van Gerwen (1957–) argued against the expression 'mathematical beauty' denoting a genuine aesthetic experience, by formulating comparisons with other forms of beauty and with art.

Comparisons

Van Gerwen pointed out that various traditions of aesthetics have agreed that recognition of beauty involves an awareness of the 'proper logic' of an event or thing.[3] This broadly kantian notion was explained as no external purpose applying to a thing that determined or caused its beauty. The experience of natural beauty was a sudden perception of such proper logic, but there was no inference to nature's proper logic (which was the purview of biology), only an instant awareness of the beauty.[4] Artistic beauty similarly depended on the perception of an artwork that fitted together in a manner similar to natural beauty.[5] Facial beauty involved the awareness of the 'personal logic that shows in it', and it was facial beauty, if anything, that exemplified the meaning of 'beauty'.[6]

[Van Gerwen's consideration of natural, artistic, and facial beauty is problematic in various way that are not directly relevant to the present discussion, for the purposes of which one can simply accept

[1] Zangwill, 'Reply to John Barker', pp. 76–7.

[2] *Ibid.*, p. 77.

[3] van Gerwen, 'Mathematical Beauty and Perceptual Presence', p. 251.

[4] *Ibid.*, pp. 252–3.

[5] *Ibid.*, p. 253.

[6] *Ibid.*, p. 255.

his positions. As a single example, he made the curious claim that natural beauty cannot be conveyed by a photograph or film because it must be experienced by all of one's senses.[1] But contemplation of a photograph of a beautiful landscape can engender a feeling of beauty. Maintaining that it is not an experience of *natural* beauty surely requires some argument to place the response into some other category of beauty.]

These archetypes of the phenomenology of beauty led van Gerwen to conclude that all recognition of beauty required an actual or imagined perceptual experience.[2] Beauty depended on the perception of a thing, and, though it was 'answerable' to something in the object, it was not simply built upon its properties.[3] Indeed, he held that our attention was only directed to non-aesthetic properties *after* we had perceived the aesthetic properties.[4] The mental event of perception had to involve an element of surprise, for the apprehension is guided by the object of perception, not by the mind or senses.[5]

But van Gerwen thought that there was no perceptual aspect in recognizing what was called 'mathematical beauty'. What there was instead was the understanding of the mathematical argument. Thus the phenomenology of 'mathematical beauty' was different from the phenomenologies of natural, artistic, or facial beauty, which were fundamentally based on perception and which were also open to those lacking specialist training.[6] (Van Gerwen specifically noted that the accessibility of mathematical beauty only to mathematicians was not, on its own, problematic.[7]) This difference meant that one must conclude that 'mathematical beauty' did not denote an actual aesthetic value like natural or artistic beauty but was a term that was applied loosely to express *awe* about the simplicity of proof,[8] about the intricate and consistent structure of mathematics,[9] or about a neat calculation or theorem.[10]

Van Gerwen considered potential rejoinders:

> 'Mathematicians might respond by arguing that apparently, not all cases of beauty require perception: mathematical beauty conforms to another aspect of beauty, but it does so in a more strictly intellectual way [...]. *That response, though, threatens to beg the question.*'[11]

But van Gerwen had already begged the question by considering as archetypes natural and artistic and facial beauty — concepts of beauty that required perceptual experience. If one started from a wider conception of the experience of beauty, including forms of intellectual and moral beauty, one would not conclude that sensory perception was unneeded for such experience.

Van Gerwen also considered how the awareness of 'proper logic' applied to mathematics. One could defend mathematical beauty on the ground that such awareness arose in a purely intellectual way.[12] Van Gerwen's counter-argument seems to have been that with natural or artistic or facial beauty, the beauty was perceived first and only

[1] van Gerwen, 'Mathematical Beauty and Perceptual Presence', p. 253.

[2] *Ibid.*, p. 250.

[3] *Ibid.*, p. 256.

[4] *Loc. cit.*

[5] *Ibid.*, p. 257.

[6] *Ibid.*, p. 259.

[7] *Ibid.*, p. 249.

[8] *Ibid.*, p. 251.

[9] *Ibid.*, p. 262.

[10] *Ibid.*, p. 263.

[11] *Ibid.*, pp. 250–1, emphasis added.

[12] *Ibid.*, p. 259.

afterwards was attention directed to the proper logic. With mathematical beauty, this order was reversed.[1] There are two problems with this argument. The first is that the 'proper logic' does not refer to a formal logic in the setting of the natural or artistic or facial beauty, so it does not follow that the 'proper logic' refers to the actual logic of (for instance) a proof in mathematics. It could refer to a rather less well-defined set of impressions of the proof, such as whether it explains a result, how it fits into a broader theory. The second is that the order is perhaps in fact *not* reversed in mathematics. One could recognize the beauty of a proof before one has been convinced that the details of the proof are correct or (referring to the previous objection) before one has considered what it is about the proof and its setting that could contribute to its beauty.

For van Gerwen, conceptual art, which presented only the idea of an event or object, was allowed to be beautiful.[2] He considered the possibility that conceptual art was perceptual in a derivative way through the viewer's imagination of its reality. In both literature and conceptual art, imagination might function as what van Gerwen called 'offline perception'.[3] But he pointed out that applying the term 'beauty' to imagined perception was a novelty and had not been elaborated.[4] Even conceding that this application was valid, Van Gerwen wrote:

> 'I fail to see how "mathematical beauty" would require such an imaginative enactment, not even if we would construe the offline perceptual experience as of geometrical figures — the figures could not surprise one perceptually.'[5]

This sentence is probably the single weakest part of van Gerwen's argument. An 'imaginative enactment' could be the mental apprehension of the meaning of a theorem or proof beyond the basic syntactic form. And *why* should not the figures surprise one? Morley's trisector theorem, presented as a diagram,* seems to be acknowledged as surprising. Perhaps it — or the imagination of the figure of a general instance of the theorem — does not 'surprise one perceptually', but this possibility seems to lead to a circularity: it is not perceptual surprise, for there is no perception; there is no perception, for there is no perceptual surprise.

Ineffability

Van Gerwen considered the following statement of Paul Erdős (1913–96) (which he quoted in a modified form):

> 'It's like asking why Beethoven's Ninth Symphony is beautiful. If you don't see why, someone can't tell you. I *know* numbers are beautiful. If they aren't beautiful, nothing is.'[6]

The phrase 'someone can't tell you' led van Gerwen to consider what he termed 'ineffability'.[7] He pointed out, correctly, that ineffability

[1] van Gerwen, 'Mathematical Beauty and Perceptual Presence', p. 263.

[2] *Ibid.*, p. 254.

[3] *Loc. cit.*

[4] *Ibid.*, p. 255.

[5] *Loc. cit.*

* ▸ p. 3

[6] Erdős, quot. in Hoffman, 'The Man Who Loves Only Numbers', p. 68, emphasis in orig.; see also Hoffman, *The Man Who Loved Only Numbers*, p. 44.

[7] van Gerwen, 'Mathematical Beauty and Perceptual Presence', pp. 257 sqq.

denoted not a property but rather a state of mind, and claimed that this state was 'the absence of categorial clarity'.[1] This state was not a sufficient condition for beauty. It might be a necessary one, although van Gerwen maintained that the ineffability of Beethoven's Ninth was different from that of numbers: Beethoven's music guided one through an aesthetic appreciation; numbers did not.

Erdős was perhaps guilty of a certain oversimplification in referring to 'numbers'. The original appearance of the quotation was in a popular mathematics article, and perhaps he spoke with this audience in mind (assuming that he was quoted accurately). It is likely that he was referring to the system of numbers as a whole, or the whole of number theory, or even using 'numbers' to refer to mathematics generally. Van Gerwen wrote that because numbers' main role was in calculation, '[f]ollowing a calculation equals doing mathematics'.[2] Taken literally, this was another oversimplification. But one could interpret it liberally as 'following a proof equals doing mathematics'. Again, the literal 'equals' is problematic, since it would exclude conjecture formation, the process of discovery, analogous reasoning, …, but certainly following a proof is one *aspect* of doing mathematics, and certainly one that is claimed to lead to an experience of beauty. With this interpretation, van Gerwen's claim was that following a proof was 'analogous rather to performing the Ninth than listening to it'.[3] But the same analogy with performance seems to apply to reading Shakespeare's *Hamlet*: when one reads a drama, one mentally performs it and imagines the characters interacting. Thus if van Gerwen's argument counts against mathematical beauty, it seems also to have implications for beauty in literature.

Van Gerwen also asked why, per Erdős, we had to give up because 'someone can't tell you' why mathematics was beautiful. After all, if someone did not experience beauty in listening to Beethoven's Ninth, one could ask them to listen again and perhaps try other performances, while pointing out certain features and drawing comparisons.[4] But van Gerwen seems to not to have considered that a similar process could happen in mathematics: if someone did not appreciate the beauty of a proof, one could point out important features, draw comparisons with other proofs of the result, and show how it drew on the resources of the broader theory. Neither in music nor in mathematics would this be 'telling' why a thing is beautiful: all one would do is help towards a more precise perception of the thing, and hope that the experience of beauty occurs. As van Gerwen noted, beauty was something one had to see for oneself.[5]

Difference with art

Van Gerwen also pointed out a difference between mathematics and art: the properties of a work of art reflected the artist's intentions and the observer could see that the work realized those intentions, whereas the reader of a proof did not see those realized intentions, for, if manifest, they would conflict with the 'feel of inevitability that (some) mathematical proofs or theorems have'.[1] Van Gerwen seems to have assumed here that the mathematician who discovered a proof played a role analogous to a painter or composer. Yet there is another possibility: the role of a mathematician was like that of an explorer who opened up new natural beauties. Such an analogy is perhaps more appealing to a mathematical platonist, but it does not require any actual ontological commitment, for the parallel is epistemological.

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[1] van Gerwen, 'Mathematical Beauty and Perceptual Presence', p. 257.

[2] *Ibid.*, p. 258.

[3] *Loc. cit.*

[4] *Loc. cit.*

[5] *Ibid.*, pp. 249–51.

Van Gerwen did not expand upon how a mathematician's individual style would be problematic for inevitability. Furthermore, it is not clear that the 'feel of inevitability' is absent in artworks. Take van Gerwen's example of Beethoven's Ninth: is there not some kind of perceived inevitability in how the music unfolds? And is it not inevitable that Oedipus will be driven to his fate in Sophocles' *Oedipus Rex*? It is also possible that the intention of the artist is not reflected — or is only partially reflected — in the work. In poetry and literature, for instance, unexpected meanings can creep in: '*there is no true sense of a text*. No authority of the author. Whatever he *wanted to say*, he wrote what he wrote.'[2]

REINTERPRETATION III: BEAUTY AS SATISFACTION

The philosopher, physicist, and mathematician Henri Poincaré (1854–1912) once wrote that 'the sentiment of mathematical elegance is *nothing but* the satisfaction due to some conformity between the solution we wish to discover and the necessities of our mind'.[3] However, Poincaré's extensive account of the role of aesthetic feeling in mathematical discovery* suggests that he did not here intend to reduce all aesthetic experience in mathematics to some non-aesthetic feeling.

A January 2010 conference talk by Robert Langlands (1936–) offers a better example of an eminent mathematician doubting whether what was called *beauty* in mathematics was *actually* beauty as understood more generally.[4] Langlands was already a noted mathematician who had originated what became known as the *Langlands programme*, which aims to establish deep connections between algebraic number theory and the representation theory of Lie groups,[5] and who became the 2018 Abel laureate. At an interview shortly before the award ceremony, he reiterated the uncertainty about beauty he had expressed in his earlier talk.[6]

Though Langland's talk putatively addressed the question 'Is there beauty in mathematical theories?', he claimed that terms such as

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[1] van Gerwen, 'Mathematical Beauty and Perceptual Presence', pp. 258, 260.

[2] Valéry, 'Au Sujet du *Cimetière Marin*', trans. C.; see also Eco, 'Postscript', esp. § 'The Title and the Meaning'.

[3] Poincaré, 'The Future of Mathematics', emphasis added.

* ▸ pp. 655–60

[4] Langlands, 'Is there beauty in mathematical theories?', § 1.

[5] Gelbart, 'An Elementary Introduction to the Langlands Program'.

[6] Dundas & Skau, 'Interview with Abel Laureate Robert P. Langlands'.

beauty and *aesthetics* were too difficult for him, and hoped rather in his lecture simply to show his audience — mainly comprising philosophers — *how* the terms were applied to mathematics.[1] Part of his motivation was that he thought that while mathematicians often referred in earnest to the beauty of their subject, they did not reflect on their use of the term.[2] For example, Langlands related the view that mathematics was a divine creation, and that human mathematicians merely discovered its beauties. (This was perhaps — at least in part — a rhetorical expression of mathematical platonism.) Langlands neither endorsed this view nor offered an alternative, though he admitted that the many defects in human mathematics pointed to its mundane origin.[3]

Langlands saw an 'attraction' in mathematics, and accepted that some might call it *beauty* and might even compare it to beauty in plastic arts like architecture. But the two species of beauty were different.[4] They, and other types of beauty — personal and natural — seemed unlike the beauty of mathematics. In mathematics, Langlands was at best ambivalent:

> 'Beauty is not quite the right word. It is satisfying — it is intellectually satisfying — that things fit together, but beauty? I say it's a pleasure when things fit together. [...] I avoided the word beauty because I don't know what it means to say that a mathematical theorem is beautiful. It is elegant, it is great, it is surprising — that I can understand, but beauty?!'[5]

So there was *pleasure* to be found in mathematics, but not necessarily beauty. Elegance, for Langlands, seems to have been a less slippery concept, perhaps not even an aesthetic one. Despite his ambivalence, he sometimes let slip an unequivocal aesthetic assertion, such as his judgement that the theory of the division of elliptic integrals was 'unsurpassed in its intrinsic beauty and in its intellectual influence'.[6]

In any case, Langlands himself had concluded that *greatness*, *majesty*, and *endurance* were more important for mathematics than beauty *per se*. In the term *endurance* one perhaps hears an echo of Hardy's trumpeting of the permanence of mathematical ideas.[*] *Greatness* and *majesty* seem to have referred to the eminence of the ideas, but beyond his declaration that 'greatness is unthinkable without that quality that practitioners refer to as beauty',[7] Langlands did not explore what factors, social or intrinsic, might make a piece of mathematics great or majestic.

Sometimes an abstract mathematical theory could be closely connected to visual beauty, such as the study of cyclotomic polynomials in the last chapter of Gauss's *Disquisitiones Arithmeticae*, which led to the characterization of the regular polygons constructible using straightedge and compass.[8][†] In contrast, while there might be beauty in the abstract symmetries that make up a Galois group, it was not a visual beauty.[9]

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[1] Langlands, 'Is there beauty in mathematical theories?', § 1.

[2] *Loc. cit.*

[3] *Ibid.*, § 2.

[4] Dundas & Skau, 'Interview with Abel Laureate Robert P. Langlands', p. 19.

[5] *Loc. cit.*

[6] Langlands, 'Is there beauty in mathematical theories?', § 2.

[*] ▸ p. 508

[7] *Ibid.*, § 1.

[8] *Ibid.*, § A.3.

[†] ▸ p. 395

[9] *Ibid.*, § A.4.

Langlands even asked whether the highly abstract results about numbers revealed by Galois groups, so far removed from vision, could be considered beautiful. His question may have been rhetorical, and he avoided offering a direct answer about their beauty or value, beyond a cynical and presumably self-deprecating remark about their professional value for professional mathematicians.[1]

REINTERPRETATION IV: BEAUTY AS PLEASURE

Michael Harris (1954–), himself a mathematician, suggested that when mathematicians spoke of 'beauty', they really meant pleasure — a peculiarly mathematical pleasure.[2] Harris supported this view using a sociological study of the ethos of mathematicians by Bernard Zarca (1941–). In Zarca's survey, 88 % of mathematicians rated the pleasure of doing mathematics as 'extremely' or 'very important',[3] and more mathematicians chose to value beauty and pleasure equally than valued beauty above pleasure or valued pleasure about beauty.[4] Zarca interpreted this result as indicating a close link between beauty and pleasure.[5]

Harris suggested that mathematicians prefer to talk of beauty because it was 'poor form' to confess to being motivated by pleasure,[6] a mindset that Harris thought went back to Plato's *Philebus* and the ranking of pleasure last among human 'possessions'.[*] Aesthetics was the solution through which one could reconcile the pleasure with what Harris, in conscious reference to Russell,[†] called the '"lofty habit of mind"'.[7]

BRAINS AND BEAUTY

The application of neuroscience to aesthetics, and indeed to any field of human wit or endeavour, is based upon the idea that all individual human behaviour has a neural counterpart, in the sense that any action, thought, or emotional response corresponds to some activity in the brain, and that studying such activity should elucidate the faculties that are in play.[8]

There have been numerous studies examining the neural activity correlated with the experience of beauty or with aesthetic judgement. The human brain has specific modules dealing with senses like vision or touch. But there is no similar aesthetic or art module;[9] rather, aesthetic experience seems to involve a combination of sensory, emotional, and cognitive systems.[10] In particular, empirical investigation has shown that aesthetic experience can be informed by knowledge.

[1] Langlands, 'Is there beauty in mathematical theories?', § A.4.

[2] Harris, *Mathematics without Apologies*, ch. 3.

[3] Zarca, 'The Professional Ethos of Mathematicians', Appendix, Tbl. 2.

[4] *Ibid.*, Appendix, Tbl. 1.

[5] *Ibid.*, p. 172.

[6] Harris, *Mathematics without Apologies*, ch. 10.

[*] ▸ pp. 77–8

[†] ▸ p. 484

[7] *Loc. cit.*

[8] Chatterjee, *The Aesthetic Brain*, p. xi.

[9] *Ibid.*, p. 183.

[10] *Ibid.*, pp. 183–4.

For example, a 2009 study demonstrated that test subjects who had expertise in architecture had more neural activity in the hippocampus than non-experts when they looked at pictures of buildings than when they looked at pictures of faces, indicating that their memories of buildings had been activated. But when they viewed buildings, they also had greater activity than non-experts in the volumes of the brain that involve pleasure and reward: their expertise was involved in the pleasure.[1]

Here, the focus is on research related to the scrutiny of brain activity during the aesthetic appreciation or evaluation of mathematics. The neuroscientist Semir Zeki (1940–) was involved in a series of studies that examined the brain activity correlated with the experience of beauty using functional magnetic resonance imaging (fMRI), and which led up to a 2014 study examining the correlates of *mathematical* beauty. In 2004, Hideaki Kawabata & Zeki published such a study of visual beauty. Volunteers first rated pictures of paintings on a numerical scale. These rating were used to select examples of beautiful, indifferent, and ugly pictures. The volunteers then evaluated these examples while their brain activity was monitored using fMRI.[2] Analysis of the results showed that the experience of visual beauty was correlated with increased activity in the orbito-frontal cortex (OFC).[3] Ugliness, however, was not associated with activity in any specific volume; rather, there was a relative decrease in activity in the OFC and greater activity in the motor cortex.[4]

In 2011, Tomohiro Ishizu & Zeki published an analogous study on the experience of visual and of musical beauty. The experience of both kinds of beauty was correlated with increased activity in volume A1 of the medial orbito-frontal cortex (mOFC). [Perhaps curiously, there was no common volume showing activity correlated with ugliness in both vision and in music.[5]] This led Ishizu & Zeki to suggest a neurobiological definition of beauty, quite unlike analyses of previous thinkers who had attempted to identify some common characteristic in things judged beautiful:

> ‘We propose that all works that appear beautiful to a subject have a single brain-based characteristic, which is that they have as a correlate of experiencing them a change in strength of activity within the mOFC and, more specifically, within field A1 in it.’[6]

As Ishizu & Zeki pointed out, such a characterization of beauty, dependent wholly on the subject and not on the object, would lend support to the subjectivity of aesthetic judgements.[7] However, such judgements were, at least in principle, *quantifiable*: the correlation of the intensity of beauty experienced with activity in the mOFC meant that for any individual subject, the degree of beauty they perceived in a thing could, again in principle, be measured independently of their ability to express their experience.[8] Ishizu & Zeki’s proposal did not offer an explanation of what beauty was in any traditional

[1] Kirk, Skov et al., ‘Brain correlates of aesthetic expertise’, esp. § 4.

[2] Kawabata & Zeki, ‘Neural Correlates of Beauty’, pp. 1699–1700.

[3] *Ibid.*, p. 1702.

[4] *Loc. cit.*

[5] Ishizu & Zeki, ‘Toward A Brain-Based Theory of Beauty’, p. 9.

[6] *Ibid.*, p. 8.

[7] *Loc. cit.*

[8] *Ibid.*, pp. 8–9.

philosophical sense, for such questions were not the concern of neuroscience,[1] but rather redefined the problem of explanation in terms of observable and measurable brain activity.

In 2014, Zeki, the neuroscientist John Romaya, the theoretical physicist Dionigi Benincasa and the mathematician Michael Atiyah (1929–2019) published a study examining the neural activity that occurred when a subject experienced mathematical beauty.[2] The experimental procedure was similar to the earlier studies.[3] Fifteen test subjects, all mathematicians at postgraduate or postdoctoral level,[4] were given to study in their own time a list of sixty formulae and asked to rate each on a scale from −5 (ugly) to +5 (beautiful). A few weeks later, each subject re-rated each formula on an abridged scale of ugly–neutral–beautiful while their brains were scanned using fMRI. The earlier ratings were used to gather the equations into three groups of twenty low-, medium-, and high-rated equations.[5] During the scan, each subject viewed the equations in a pseudo-random order in which there were no consecutive equations in the same grouping. Some days after scanning, each subject was asked to report their understanding of each formula on a scale of 0 (no understanding) to 3 (profound understanding).[6]

There was a significant but imperfect correlation between beauty and understanding. This imperfection allowed for separation of the effects of beauty and of understanding.[7] The resulting analysis showed that brain activity that was related to the experience of beauty, after accounting for the effects of understanding, was confined to field A1 of the mOFC.[8] Previous studies had shown that this same volume was active during the experience of visual, musical, or moral beauty,[9] while neighbouring but distinct volumes were active during the experience of enjoying food or erotic pleasure.[10]

Notice, however, that the stimuli in this study were only *equations*. It is conceivable that different neural activity arises in experiencing the beauty of proofs or theorems or theories or objects.

[Zeki, Romaya, Benincasa & Atiyah had initially intended their study to extend to twelve non-mathematicians, but very few of these participants reported their understanding of any formula to be higher than 1 (on the scale of 0 to 3), and the majority denied experiencing any emotional response to beautiful equations.[11] Zeki et al. suggested that these participants were aesthetically evaluating the formulae as visual forms, independently of their semantic content. This hypothesis was confirmed by activity in the visual volumes of the brain.[12]]

[1] Cf. Zeki, 'Neural Correlates of the Experience of Beauty', 10:15 sqq.

[2] Zeki, Romaya et al., 'The experience of mathematical beauty'.

[3] *Ibid.*, pp. 1–2.

[4] *Ibid.*, p. 1.

[5] *Ibid.*, pp. 2, 3.

[6] *Ibid.*, p. 2.

[7] *Ibid.*, p. 6.

[8] *Loc. cit.*

[9] *Ibid.*, p. 8.

[10] *Ibid.*, p. 10.

[11] *Ibid.*, p. 6.

[12] *Ibid.*, p. 10.

REINTERPRETATION V: BEAUTY AS SUCCESS

The philosopher Axel Gelfert (1975–) pointed out in a 2017 essay that the very fact that the term *mathematical beauty* was applied to such different artefacts of mathematics (theorems, proofs, definitions, theories, objects, ...) might indicate that it did not label a single unified concept.[1] Without delving deeply into the problem, he suggested that, while some of the objects of mathematics might exhibit 'an intrinsic aesthetic appeal', the application of terms like *beauty* and *beautiful* indicated non-aesthetic qualities. Applied to a proof, they described its '*performative success*' in demonstrating the result; applied to a theory, they described qualities like explanatoriness, depth, economy, coherence, and seriousness.[2]

And the philosopher Juliet Floyd (1960–) argued in the same year that the pleasures experienced in doing mathematics arose from the cognitive achievement involved in solving a problem or understanding a proof or object. They were, simply, unnecessary for *doing* mathematics.[3] Whatever feelings obtained were appendages, 'experiences on the margins of something stronger and more rigid'.[4] The phenomena to which aesthetic terms were applied, such as a solution to a long-standing problem or a theorem that had far-reaching and unexpected consequences, were not properly aesthetic phenomena at all. They could be explained and communicated without the use of aesthetic terms, although moments of 'aha!' insight occurred often. Following a slight hyperbole of Rota,[5] Floyd said that, even if initially seen as beautiful, all mathematics ultimately came to be seen as trivial.

TAKING MATHEMATICAL BEAUTY LITERALLY

Adam Rieger held that the burden of proof was on those who denied that the use of aesthetic vocabulary in mathematics indicated genuine aesthetic experiences.[6] Opposed to such denials stood the testimonies of mathematicians and the suggestive results of Zeki et al.'s study;[7][*] most importantly, the introspection of someone trained in mathematics seemed to be utterly convincing on this point, difficult though it might be to use this experience to persuade others.[8]

Rieger found unconvincing Harré's point about the limited aesthetic vocabulary used in the sciences.[†] Even if the restricted vocabulary reflected a restricted range of putatively aesthetic experience in mathematics, it did not follow that the experiences were not in fact aesthetic.[9]

Rieger also made the point that the search for better proofs could be at least partly aesthetic. If the first proof secured the theorem, there must be some motivation for seeking another. Greater explanatory power might be one, but Rieger[1] pointed out that Hutcheson had said that beauty could impel the creation of new proofs 'even after they have sufficient Knowledge and Certainty in all these Truths from distinct independent Demonstrations'[2] and that Morris Kline had said that the quest for new proofs could be driven by the lack of aesthetic appeal in existing proofs.[3]

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[1] Gelfert, 'The Unreasonable Attractiveness of Mathematics to Artists and Scientists', p. 67.

[2] *Ibid.*, p. 67, emphasis in orig.

[3] Floyd, 'The Fluidity of Simplicity', p. 161.

[4] *Ibid.*, p. 162.

[5] Rota, 'Phenomenology of Mathematical Truth', p. 118.

[6] Rieger, 'The Beautiful Art of Mathematics', p. 240.

[7] *Ibid.*, pp. 240–1.

[*] ▸ pp. 760–2

[8] *Ibid.*, p. 241.

[†] ▸ p. 734

[9] *Loc. cit.*

Regarding Zangwill's position that aesthetic qualities were connected to sensory perception, Rieger suggested that there was a kind of vicious circle at work: the historical focus of aesthetics on the visual arts and music had led to an assumption that sensory perception must define the limits of aesthetic experience. But had mathematical beauty and other forms of intellectual beauty been (or stayed) part of the traditional remit of aesthetics, the sensory-dependence thesis would not have become so entrenched.[4] The fact that Zangwill's own position led him hesitantly[5] to the conclusion that a literary work had aesthetic properties only in its words, not in its contents (the characters, setting, story, and so forth), was, for Rieger, close to a *reductio ad absurdum* for the supposed sense-dependence of aesthetic qualities.[6]

Rieger also noted that Hardy's platonism illuminated an important distinction: 'In mathematics, the main aesthetic value lies with the thing represented, not the representation.'[7] Of course, an individual presentation of a piece of mathematics might be described in aesthetic terms, but aesthetic value was mainly seen in theorems or proofs. Here mathematics differed from literature, painting, and sculpture, where the aesthetic value was seen in the representation. However, Rieger did not think that there was a sharp contrast:

> 'Arguably the most valued paintings have beautiful subjects, as well as being themselves beautiful representations; part of what the artist is commended for is having successfully conveyed a beautiful part of reality.'[8]

[One could note in contrast that some of the supreme works of literature depict events, characters, or settings that are certainly not beautiful: Dante's *Inferno*, say, or much of tragic literature.]

Against Zangwill's position that a proof could not have dependent beauty because the aesthetic expression of purpose and the actual fulfilment of purpose could not separate, Rieger pointed out that Zangwill did not consider the possibility that *some* dependent beauty might depend on success in fulfilling the purpose. Thus, he concluded, Zangwill's argument seemed to require extra premisses, all questionable:

1) *The only possible mathematical beauty was the dependent beauty of a proof in fulfilling the purpose of proving the theorem.* Proofs certainly had purposes, but it was not clear that theorems or objects or other loci of mathematical beauty did. Also, there seemed

[1] Rieger, 'The Beautiful Art of Mathematics', p. 241.

[2] Hutcheson, *Inquiry Concerning Beauty*, Art. III.v.

[3] Kline, *Mathematics in Western Culture*, p. 470.

[4] Rieger, 'The Beautiful Art of Mathematics', pp. 241–2.

[5] Zangwill, *The Metaphysics of Beauty*, p. 137, n. 11.

[6] Rieger, 'The Beautiful Art of Mathematics', p. 242.

[7] *Ibid.*, p. 247.

[8] *Ibid.*, p. 248; typo corrected.

to be more in the supposed beauty of a proof than merely its effectiveness in proving the theorem, for otherwise any two valid proofs would have the same beauty.

2) *All genuine cases of dependent beauty arose from the expression of a purpose and not from actually fulfilling that purpose.* But a painting could have dependent beauty because it accurately represented its subject.
3) *In mathematics, the expression of purpose could not come apart from fulfilling it.* But a proof could be strictly invalid but still be beautiful because of ideas it contains; an example might be Euler's proof that $\pi^2/6 = \sum_{n=1}^{\infty} 1/n^2$.[1][*]

Rieger thought that there had to be some connection between truth and beauty in mathematics: he thought it 'at least *roughly* right that truth (or validity in the case of a proof) is a necessary condition for beauty',[2] although a non-rigorous proof could still be beautiful. Rieger also suggested, without giving an example, that there could be a flawed theorem that was untrue but possessed the same kind of flawed beauty as a cracked vase.[3]

Turning to the question of whether mathematics was an art, Rieger thought that there was a wittgensteinian 'family resemblance' between mathematics and representational painting and literature. Opposed to this was the idea that truth was too important in mathematics. But, for Rieger, this objection was not decisive, for a painting was not disqualified from being art by representational accuracy.[4] Another comparison Rieger might have made is with photography, which offers a more exact representation than painting. And certain genres of non-fiction literature — history, say — aim at accurate representation, and, regardless of how well they succeed, a Herodotus or a Gibbon stands with the great artists. [Photography and literature are perhaps closer parallels to mathematics given their reproducibility.[5]]

INTRINSIC MATHEMATICAL BEAUTY

Ewa Bigaj's 2020 doctoral thesis was 'Three Essays on Aesthetic Experience'. In the first part, entitled 'No Mathematics Without Beauty', she emphasized the importance of beauty to mathematicians. As a thought experiment, Bigaj imagined that beauty was objective and could be evaluated according to a set of rules, and that, with access to such rules, one could make precise and accurate aesthetic judgements. Then she asked whether one would trade aesthetic *experience* — the mental states that come about through perceiving beauty or other aesthetic values — for a book of such rules of aesthetic judgement. She imagined that Poincaré[†] or a similar mathematician would recoil, for losing aesthetic experience would mean losing the capacity for mathematical understanding and creation.[1] The one exception might be an altruistic mathematician who sacrificed their own abilities so as to be able to make blind and pleasureless aesthetic judgements that would aid the understanding of others.[2]

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[1] Rieger, 'The Beautiful Art of Mathematics', pp. 242–3.

[*] ▸ pp. 339–40

[2] *Ibid.*, p. 243, emphasis in orig.

[3] *Ibid.*, p. 243.

[4] *Ibid.*, pp. 246–7.

[5] Benjamin, 'The Work of Art'.

[†] ▸ pp. 655–60

Bigaj then considered how mathematics was appreciated aesthetically, setting it against the background of Kant and other eighteenth-century thinkers on philosophical aesthetics. She thought that the diagrams used in the well-known rearrangement proof of the pythagorean theorem (Figure 20.1) could be the subject of visual aesthetic appreciation which was independent of their mathematical meaning. She compared them on this level to an abstract Mondrian-esque image (see Figure 20.2), and suggested that contemplation of the parts of both images led to the Kantian free play of the imagination and understanding.[3]* But when one realized that the equality of the grey areas in the left and right configurations in Figure 20.1 establishes the pythagorean theorem, the aesthetic pleasure is deepened.[4]

[The rearrangement proof is conjectured to be of ancient origin, perhaps a pre-Euclidean proof available to the early pythagoreans.[5] The earliest written appearance that the present author has been able to trace is an essentially equivalent proof in Henry Boad's *Artium Principia* (1733); Boad attributed it only to 'a *learned Friend*'.[6]]

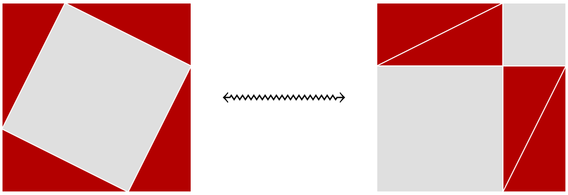

FIGURE 20.1
Rearrangement proof of the pythagorean theorem.

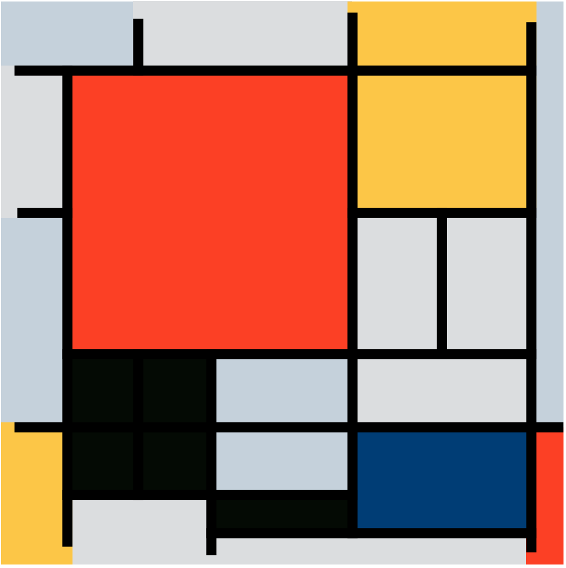

FIGURE 20.2
Structure of Piet Mondrian's (1872–1944) *Composition with large red plane, yellow, black, gray and blue* (1921).

Public domain

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[1] Bigaj, 'Three Essays', pp. 16–17.

[2] *Ibid.*, p. 17.

[3] *Ibid.*, pp. 23–4.

* ▸ p. 376

[4] *Ibid.*, p. 26.

[5] Heath, notes to Euclid, *Elements*, Prop. I.47 [vol. 1, p. 354].

[6] Boad, *Artium Principia*, pp. 56–7, emphasis in orig.

For Bigaj, this parallel in visual appreciation pointed to a structural similarity: 'visual proofs as well as art can lead to a visual play which is further enhanced by noticing the representational role of the visual properties'.[1] A visual proof served as a compressed proof: a blueprint or mnemonic from which a more formal uncompressed proof could be constructed.[2] While the unaugmented visual proof could be an object of aesthetic contemplation, Bigaj held that aesthetic appreciation of mathematics involved both compressed and uncompressed proofs and moving between the two perspectives, which was analogous to shifting attention between detail and overall structure in a painting.[3] The aesthetic experience consisted in the contemplation involved in this shifting; whether a proof was beautiful depends on whether it engaged attention in this way.[4]

Thus while the rearrangement proof was beautiful, the proof using similarity shown in Figure 20.3 was more beautiful: it proceeds by projecting the right angle of a right-angled triangle onto the hypotenuse, so dividing the original triangle into two smaller triangles, each similar to the original. Each of the three triangles is viewed as being constructed on its own hypotenuse. Since the areas of similar triangles are to each other as the squares on their hypotenuses, the pythagorean theorem follows.[5]

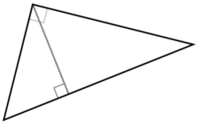

FIGURE 20.3
Proof of the pythagorean theorem.

This proof depends on knowledge of similarity and of its consequences, but Bigaj thought that, for someone who has this knowledge, the visual proof was at the limit of simplicity.[6] It was in some sense a borderline case of a visual proof, for *without* knowledge of similarity, the diagram would seem trivial.[7] Certainly there seems to be little visual aesthetic value in the bare diagram; it does not engage free play like Figure 20.1.

Bigaj considered the possibility that mathematical beauty might not be *disinterested*, which philosophers like Shaftesbury,[*] Hutcheson,[†] and Kant[‡] thought was a feature of beauty; rather, it might be *egotistical*. One could argue I) that aesthetic pleasure in mathematics was coincident with gaining epistemic goods such as knowledge or understanding; II) therefore, such aesthetic pleasure was pleasure at the epistemic gain; III) therefore, the pleasure is egotistical.[8] To support the first inference, one could consider the analogy of the owner of a painting who may genuinely believe that the pleasure they take in it is aesthetic, despite the pleasure increasing when the painting increases in value.[9]

But Bigaj pointed out weaknesses in this argument. One gained some knowledge in viewing a beautiful painting: at minimum, one learned what it was a painting *of*.[10] It was absurd to think this increased knowledge compromised the disinterestedness of the aesthetic appreciation. Monetary value could be detached from the painting, unlike the knowledge of what the painting *depicted* or the knowledge of what the proof *proved*. One would not consider it worthwhile to

[1] Bigaj, 'Three Essays', p. 27.

[2] *Loc. cit.*

[3] *Ibid.*, p. 31.

[4] *Ibid.*, pp. 32–3.

[5] Raman-Sundström, 'The Notion of Fit', pp. 274–5; see Euclid, *Elements*, Prop. VI.31.

[6] Bigaj, 'Three Essays', pp. 45–6.

[7] *Ibid.*, p. 47.

[*] ▸ pp. 353–6

[†] ▸ pp. 357–62

[‡] ▸ pp. 374–82

[8] *Ibid.*, p. 61.

[9] *Ibid.*, p. 62.

[10] *Loc. cit.*

contemplate aesthetically the painting or the proof without this knowledge.[1]

Further, Bigaj thought that an incorrect proof could be beautiful, though she offered no support and gave no examples. She thought that '[s]uch proofs fail to provide understanding while still providing a pleasurable experience'.[2] It was arguable that if one read a fallacious proof but did not notice the error, then one might gain an incorrect impression of understanding: again, an example might be Kempe's flawed proof of the four-colour theorem.[*] Bigaj maintained that when the flaw was noticed, the aesthetic pleasure could be diminished but not extinguished: the epistemic value might have gone, but some aesthetic value remained.[3]

MATHEMATICAL BEAUTY AS ARCHETYPICAL

Mathematicians seem to have admitted readily that beauty in mathematics was somehow *different* from other kinds of beauty. But, as Timothy Gowers (1963–) noted in 2002, beauty in music was also different from beauty in a painting, a poem, or a person.[4] So it was perhaps natural that some philosophers did not *argue for* mathematical beauty or more generally for the aesthetic experience of mathematics, but simply took it as a given and explicitly let it inform their analyses of aesthetic experience. This final section briefly surveys some such philosophers.

Jerome Stolnitz

In his 1960 introductory book on aesthetics, Jerome Stolnitz (1925–) defined the aesthetic attitude as 'disinterested and sympathetic attention to and contemplation of any object of *awareness* whatever, for its own sake alone'.[5] In casting the net of aesthetics widely, he followed and cited[6] Curt J. Ducasse (1881–1969), who maintained in his 1929 book *Philosophy of Art* that '[*a*]*ny entity whatever may be viewed as aesthetic object*'.[7] Ducasse emphasized that this statement applied to abstract entities, 'even the completely impersonal entities of mathematics and logic'.[8]

Stolnitz deliberately chose the word *awareness* instead of *perception* for his definition of the aesthetic attitude, so as to include objects of aesthetic attention that were not perceived through the senses. His example was a mathematician pausing in the act of problem-solving to contemplate the logical structure; the resulting experience was aesthetic, as shown by the use of aesthetic terms such as *elegant* to express it.[9]

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[1] Bigaj, 'Three Essays', p. 63.

[2] *Ibid.*, p. 64.

[*] ▸ pp. 602–3

[3] *Ibid.*, pp. 64–5.

[4] Gowers, *Mathematics*, p. 51.

[5] Stolnitz, *Aesthetics and Philosophy of Art Criticism*, p. 35, emphasis added.

[6] See *ibid.*, esp. ch. 7.

[7] Ducasse, *The Philosophy of Art*, p. 223, emphasis in orig.

[8] *Ibid.*, p. 224.

[9] Stolnitz, *Aesthetics and Philosophy of Art Criticism*, 41–2; cf. p. 183.

Kendall L. Walton

Kendall L. Walton (1939–) initially studied music and only later turned towards philosophical aesthetics, and a substantial number of his essays bear particularly on music.

In 1970, in one of his earliest published papers, he drew an analogy between the experiences of musical and mathematical beauty. At its most basic, the sonata form comprised three large-scale sections: exposition, development, and a recapitulation.[1] In the second section, there was created and intensified a structural dissonance relative to the original key from the exposition; the tension was resolved with a return to the home key in the recapitulation. Certain choices of new key in the development may seem 'alien and brashly adventurous' and lead to 'a distinct feeling of unease, tension, uncertainty, as the time for the recapitulation approaches'. A successful return to the original key in the last section would, in hindsight, give the whole 'a sense of correctness and perfection'.[2] Walton thought that mathematics could have a similar effect:

> 'Our impression of it is likely, I think, to be very much like our impression of a "beautiful" or "elegant" proof in mathematics. (Indeed the composer's task in this example is not unlike that of producing such a proof.)'[3]

[Douglas Hofstadter (1945–) later made a similar comparison between the tension built up and resolved, locally or globally, in a piece of music, in the structure of a dialogue, and in a mathematical proof,[4] and stated, without further discussion, that '[t]he mathematician's sense of tension is intimately related to his sense of beauty'.[5]]

In a later essay asking 'What is Abstract About the Art of Music?' (1988), Walton explored three ways in which '(absolute)' music might differ from more obviously representational arts like much of painting and sculpture and most literature. The first line of investigation, which is the one relevant here, examined whether music 'lacks meaning or semantic content' in comparison to representational art.[6]

Walton noted that a novel described certain fictional events and characters and, furthermore, could in some sense be said to be *about* love, or life, or ambition. This latter observation suggested where one could look for the value in representational art. But if abstract music like Bach's had no subject-matter, why did the listener feel a physical thrill?[7] It had been suggested that the music expressed something — was *about* something — that included human emotion,[8] although the terminology that was used suggested that it could only express a limited emotional range: music might be called *joyful* or *tense* or *anguished*, but never *embarrassed* or *jealous*.[9]

Walton quoted from the critic Eduard Hanslick's (1825–1904) influential book *Of Musical Beauty* [*Vom Musikalisch-Schönen*] (1854) the position that when one listened to music purely, without importing external ideas, what was expressed was nothing but musical ideas, and that a musical idea was an end in itself, not a representation of something else.[1] Walton thought that this attitude was supported by the comparison to intellectual beauty:

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[1] Caplin, *Classical Form*, pp. 195–6.

[2] Walton, 'Categories of Art', p. 351.

[3] *Loc. cit.*

[4] Hofstadter, *Gödel, Escher, Bach*, pp. 129–30, 227.

[5] *Ibid.*, p. 227.

[6] Walton, 'What is Abstract About the Art of Music?', p. 352.

[7] *Ibid.*, p. 353.

[8] *Loc. cit.*

[9] *Ibid.*, pp. 353–4.

> ‘That there is something to such purist attitudes is suggested by this comparison: Chess moves and chess games can be marvelous, beautiful, elegant. So can proofs in logic and mathematics; so can scientific theories. But it seems strained, to say the least, to attribute these aesthetic qualities to “expressiveness,” and downright implausible to suppose that they consist in the expression of human emotions. A chess game or a proof is not beautiful because it expresses joy or anguish or determination or resignation, nor because it is in some sense “about” love or ambition or the human condition, let alone because it represents, say “fate knocking on the door.”’[2]

Walton here called upon the aesthetic value of mathematics in support of a point concerning aesthetic value in music, which indicated that he saw the former as an acceptable basis on which to argue. He then continued:

> ‘If this is so there would seem to be a kind of aesthetic value which is not to be explained in terms of subject matter, one which might be important in music.’[3]

The antecedent ‘this is so’ presumably referred to the claim that the chess game or proof was beautiful in spite of the absence of expression or ‘about’-ness. The consequent assertion of the existence of aesthetic value not explained in terms of subject-matter seems problematic: the chess game and the proof *are* the subject-matter. They are the analogue of Hanslick’s musical ideas; their communication on the chessboard or blackboard, or via notation, is analogous to the musical sounds.

Roger Scruton

Roger Scruton (1944–2020), who made a number of passing references to beauty in mathematics in his books,[4] used the aesthetic appreciation of mathematics to support arguments he made in *The Aesthetics of Architecture* (1979). He noted that philosophers had often thought that *anything* can be considered aesthetically, including mathematical proofs, and thus one should be able to so consider buildings.[5] But, he pointed out, it did not follow that in appreciating a building *as a building*, one was appreciating it aesthetically.[6] One could be contemplating instead its structural soundness or its efficiency in serving some purpose. To back up this point, he wrote:

> ‘When we regard a proof from the aesthetic point of view we do not consider it only as mathematics, and we can fully

[1] See Hanslick, *Vom Musikalisch-Schönen*, pp. 8, 59.

[2] Walton, ‘What is Abstract About the Art of Music?’, p. 354.

[3] *Loc. cit.*

[4] See, for example, Scruton, *Beauty*, pp. 1, 20.

[5] Scruton, *The Aesthetics of Architecture*, p. 16.

[6] *Loc. cit.*

> understand its mathematical validity without being concerned with its aesthetic power.'[1]

Scruton perhaps fell into clumsy phrasing here. Leaving aside possible visual aesthetic value in diagrams, a proof appreciated aesthetically was surely still appreciated solely *as mathematics*. Rather, what Scruton seems to have meant is that a proof appreciated aesthetically was not being appreciated *only* because it proved the theorem, but for other reasons.

Berys Gaut

In his 2007 book *Art, Emotion and Ethics*, Berys Gaut (1958–) acknowledged that trying to describe what it meant to experience a thing aesthetically could be frustrating. If one ventured too far towards non-aesthetic concerns, one struck the Scylla of counterexample; if one stayed within aesthetic concerns, one drifted into the Charybdis of circularity.[2]

Gaut first considered what he termed 'the *narrow* sense of the aesthetic', which limited aesthetic judgement to beauty and ugliness or to particular species of each.[3] He pointed out that one could not think of such judgements solely in terms of pleasure in sensory perception, for there was beauty in things not perceived through the senses; his first example was mathematical proof.[4]

When it came to 'the *wide* sense' of the aesthetic, which encompassed concepts such as being unified, integrated, balanced, dynamic, exciting, vivid,[5] Gaut noted that these terms were used to evaluate artworks, and so he considered the hypothesis that the aesthetic value of an artwork *W* was simply the value of *W qua* artwork. From this hypothesis, one could define the aesthetic properties of any thing *W* to be those evaluative properties that gave *W* value *qua* artwork, and one could say that adopting an aesthetic attitude towards *W* meant considering it *qua* artwork.[6]

But Gaut saw that this definition ran into a problem, namely how to treat things like natural objects and mathematical theorems, which had aesthetic value but which were not works of art.[7] Precisely because of this difficulty, the definition had to be refined. Gaut thus thought that aesthetic properties were *primarily* those of artworks as previously defined. An aesthetic property of a natural object, and presumably of a mathematical theorem (though Gaut did not say so explicitly), was any of its properties that could be an aesthetic property of an artwork; thus aesthetic concepts applied primarily to artworks and had a derivative application elsewhere.[8]

◆

[1] Scruton, *The Aesthetics of Architecture*, pp. 16–17.

[2] Gaut, *Art, Emotion and Ethics*, p. 26.

[3] *Ibid.*, pp. 26–7, emphasis in orig.

[4] *Ibid.*, p. 27.

[5] *Ibid.*, p. 27, emphasis in orig.

[6] *Ibid.*, pp. 34–5.

[7] *Ibid.*, p. 35.

[8] *Loc. cit.*

21

The Smile of Reason

Aesthetic experience and epistemic judgement in mathematics

‘ The smile of reason may seem to betray
a certain incomprehension of the deeper human emotions ’
— Kenneth Clark, *Civilisation*, p. 171.

‘ Naturalness, fruitfulness, conceptual power, and elegance
are other qualities valued in a piece of mathematics.
Are these merely transitory aesthetic judgements,
or is there something “objective” to them? ’
— David Corfield, *Towards a Philosophy of Real Mathematics*, p. 34.

The psychiatrist and author Anthony Storr (1920–2001) spoke in 1976 about the possibility of building bridges across the divide between the ‘two cultures’, which his friend and former tutor C. P. Snow (1905–80) had discussed in a famous — or perhaps notorious — lecture.[1] Storr pointed out that scientists and artists had aesthetic appreciation in common and cited G. H. Hardy’s (1877–1947) ‘brief masterpiece, *A Mathematician’s Apology*’ in support.[2] Storr continued:

> ‘In considering works of art there has been a tendency to think of aesthetic appreciation — that is, the appreciation of form and order — as intellectual, whilst regarding the appreciation of content as emotional. This is, I believe, *a false dichotomy*. The thrill we get from appreciating the perfect shape of a Mozart quartet, and the thrill we get from formulating or recognizing an explanatory hypothesis in science, are not dissimilar.’[3]

Storr here suggested common ground between the aesthetic response to music and the epistemic experience in science.

There seems to be such a commonality in the aesthetic and epistemic aspects of mathematics. More broadly, there is an established tradition of seeking a rational grounding for aesthetic judgements of mathematics, and this tradition is the focus of the present chapter. One facet of this search has been to find non-aesthetic properties that contribute to aesthetic value, such as the Hardian triad of unexpectedness, inevitability, and economy in theorems and proofs.* Subsequent discourse has drawn on a wider range of concepts, some

[1] Snow, *The Two Cultures*.

[2] Storr, ‘Bridging the Two Cultures’, p. 73.

[3] *Ibid.*, p. 73, emphasis added.

* ▸ pp. 502

of which are introduced in the first five sections of the chapter: explanation in mathematics, visual proof, the experience of illumination or revelation, epistemic feeling, and processing fluency.

◆

Two special journal issues were foci for many facets of the recent discourse on these and related concepts. The first was the January 2016 issue of the *Journal of Humanistic Mathematics* on *The Nature and Experience of Mathematical Beauty*, edited by Manya Raman-Sundström, Lars-Daniel Öhman & Nathalie Sinclair (1970–), which grew out of a seminar held at Umeå University; the second was the June 2018 issue of *Philosophia Mathematica* on *Aesthetics in Mathematics*, edited by Angela Breitenbach & Davide Rizza. The contributions to these issues are treated where appropriate.

EXPLANATION

Roughly, scientific knowledge can be distinguished into two kinds: descriptive knowledge (knowledge *that*) and explanatory knowledge (knowledge *why*). Science sometimes provides only a description of the phenomena; sometimes it provides an explanation of those phenomena. The distinction was made by Aristotle (384–322 BCE), who thought that a demonstration was explanatory when the reasoning mirrored the causal relationship of the phenomena.[1] Philosophy of science has paid much attention to scientific explanation, with the watershed that marks the beginning of the modern study being the seminal 1948 essay 'Studies in the Logic of Explanation' by Carl G. Hempel (1905–97) & Paul Oppenheim (1885–1977). This study has led to the emergence of and debate over various models of scientific explanation.[2]

In mathematics itself, Proclus (410/12–485 CE) noted that *reductio ad absurdum* proof were non-causal, and pointed to certain other proofs in Euclid as also being non-causal; such proofs were, by Aristotelian standards, non-explanatory.[3] The psychological motivation for explanatory proofs was discussed by Antoine Arnauld (1612–94) & Pierre Nicole (1625–95) in their '*Port-Royal Logic*' (first ed. 1662). While Arnauld & Nicole lauded geometers' attention to *convincing* readers of their reasoning, they lamented the comparative lack of attention to *enlightening* readers, and in particular the readiness to resort to non-explanatory *reductio ad absurdum* proofs when alternative proofs were possible. Explanatory proofs should be preferred: without explanation, the mind remained unsatisfied.[4]

Mathematical explanation — in the specific sense of explanation *internal to* mathematics, rather than mathematical explanation in natural science — seems to differ fundamentally from scientific ex-

[1] Aristotle, *Posterior Analytics*, BEK I.13.78a21–b33.

[2] Salmon, 'Four Decades of Scientific Explanation'.

[3] Proclus, *Commentary on Euclid*, FRI 202, 206–7.

[4] Arnauld & Nicole, *Logic or the Art of Thinking*, ch. IV.9, 'Mistakes I, III'.

planation.[1] The latter is bound up with causality. Take a famous example due to Sylvain Bromberger (1924–2018): given the elevation of the Sun in the sky, one could deduce the length of the shadow of a flagpole from its height, or its height from the length of its shadow. But the length of the shadow seemed to be *caused by* and thus *explained by* the height, and not vice versa.[2] Causation played no such role in mathematical explanation, which instead depended, at least in part, on logical consequence: axioms entailed theorems, and not vice versa; presumably explanation should have the same direction.[3]

Mathematicians have certainly expressed desire for explanation in various forms.[4] For instance, in a lecture given late in his life, Hermann Weyl (1885–1955) used a memorable expression to evoke the difference between proving and explaining:

> 'We are not content to get *convicted*, as it were, rather than *convinced* of a mathematical truth by a long chain of formal inferences and calculations leading us blindfolded from link to link.'[5]

The difference lay in seeing not only the goal, but the context and the relationship of the ideas to mathematics generally.[6] In his 2007 book *How Mathematicians Think*, William Byers (1943–) seemed to make explanatoriness vital for proofs to be praiseworthy: he characterized a '"good" proof' as 'one that brings out clearly the reason why the result is valid'.[7] And the mathematician Joseph Auslander (1930–) was even more emphatic: 'Almost by definition, a proof is supposed to *explain* the result.'[8]

Whatever a mathematical explanation is, it seems to grant understanding, or insight, or enlightenment, though it has been debated whether examining these mental states actually helps clarify the notion of mathematical explanation.[9]

Explanatory and non-explanatory proofs

In the main, philosophical scrutiny of mathematical explanation has focused on explanation in proof. A proof, by definition, must show that the result holds. But a proof can go further than this base requirement: the proof can show, not just *that* the result holds, but *why* it holds: it can be *explanatory*. To prepare the way for what follows, it makes sense to begin with some examples of explanatory and non-explanatory proofs.

Mutilated chessboard

In 1991, Alan Robinson (1930–2016) used the celebrated *mutilated chessboard problem* to illustrate the distinction between a 'proof by search' and a 'proof by explanation'. The problem, first stated by the philosopher Max Black (1909–88) in his 1946 book *Critical Thinking*, is as follows:

[1] Mancosu, Poggiolesi & Pincock, 'Mathematical Explanation', § 2.1.

[2] Salmon, 'Four Decades of Scientific Explanation', p. 47.

[3] Cf. Lange, 'Aspects of Mathematical Explanation', pp. 486–7.

[4] Hafner & Mancosu, 'The Varieties of Mathematical Explanation', pp. 218–21.

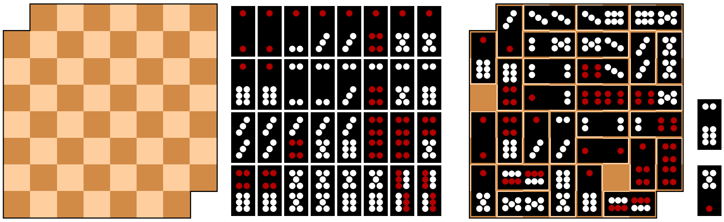
[5] Weyl, 'Axiomatic Versus Constructive Procedures', p. 195, emphasis added.

[6] *Ibid.*, pp. 195–6.

[7] Byers, *How Mathematicians Think*, p. 337.

[8] Auslander, 'On the Roles of Proof in Mathematics', p. 66, emphasis added.

[9] Lange, 'Explanation, Existence and Natural Properties in Mathematics', p. 436, n. 2; cf. Lange, 'Depth and explanation in mathematics', pp. 196–7.



A chessboard with two opposite corners removed. The question is whether it is possible to cover the board with 31 dominoes.

Placing 30 dominos leaves two squares of the same colour uncovered.

FIGURE 21.1
The mutilated chessboard problem.

> ‘An ordinary chessboard has had two squares — one at each end of a diagonal — removed. There is on hand a supply of 31 dominoes, each of which is large enough to cover exactly two adjacent squares of the board. Is it possible to lay the dominoes on the mutilated chessboard in such a manner as to cover it completely?’[1]

(See Figure 21.1 (left).)

Robinson explained that one could prove that the answer is ‘No’ using a brute-force computer search.[2] But the ‘simple, clear, elegant and convincing’[3] solution was to note that the two removed squares have the same colour, so that, having placed 30 dominoes, each of which must cover one black and one white square, the remaining uncovered squares were of the same colour and so were not adjacent (see Figure 21.1 (right)). This was the difference:

> ‘The parity argument [...] proves the nonexistence of a covering not just by designing a search which fails to find one, but *by revealing why no such search could ever be successful*. What this example illustrates is the difference between a proof which merely certifies that there is no counterexample to some proposition and thereby establishes its truth, and a proof which not only demonstrates the nonexistence of a counterexample but also supplies some kind of reason why not.’[4]

Desargues’s theorem

Two proofs of one direction of Desargues’s theorem perhaps offer a more genuinely mathematical example, which was used by the philosopher Marc Lange (1963–),[5] to illustrate the contrast between explanatory and non-explanatory proof.

DESARGUES’S THEOREM. *Let ABC and $A'B'C'$ be triangles in the plane. Then AA', BB', and CC' meet at a point if and only if the intersections of AB and $A'B'$, of BC and $B'C'$, and of CA and $C'A'$ are collinear.* (See Figure 21.2.)

The proofs here are only of the ‘only if’ part of the theorem. The first proof uses the signed version on Menelaus’ theorem:

[1] Black, *Critical Thinking*, p. 142.

[2] Robinson, ‘Formal and Informal Proofs’, p. 272.

[3] *Ibid.*, p. 271.

[4] *Ibid.*, p. 274, emphasis in orig.

[5] Lange, ‘Explanation, Existence and Natural Properties in Mathematics’, § 2.

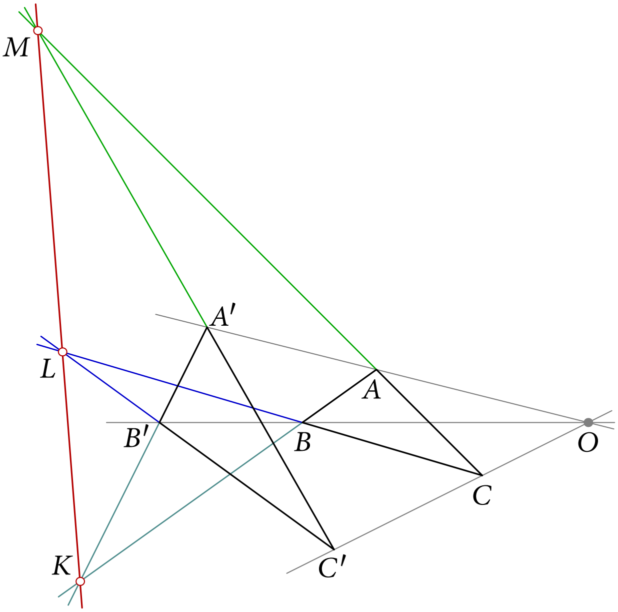


FIGURE 21.2
An instance of Desargues's theorem.

MENELAUS' THEOREM (Signed version). *Let PQR be a triangle and let X, Y, and Z lie respectively on PQ, QR, and RP. Then X, Y, and Z are collinear if and only if*

$$\frac{PX}{XQ} \cdot \frac{QY}{YR} \cdot \frac{RZ}{ZP} = -1. \qquad \square$$

In this result, terms such as PX are *signed* lengths, so that $PX = -XP$, and PX/XQ is positive if and only if X lies between P and Q. There is a weaker version of Menelaus' theorem that uses unsigned lengths.*

* ▸ p. 640

First proof. Apply the signed version of Menelaus' theorem thee times: to $\triangle OBC$ and the points B', L, and C'; to $\triangle OAB$ and A', K, B'; to $\triangle OCA$ and C', M, A'. These applications give:

$$\frac{OB'}{B'B} \cdot \frac{BL}{LC} \cdot \frac{CC'}{C'O} = -1;$$
$$\frac{OA'}{A'A} \cdot \frac{AK}{KB} \cdot \frac{BB'}{B'O} = -1;$$
$$\frac{OC'}{C'C} \cdot \frac{CM}{MA} \cdot \frac{AA'}{A'O} = -1.$$

Multiply the left sides together and the right sides together and simplify via $OB'/B'O = -1$ and analogous relationships. The result is:

$$\frac{BL}{LC} \cdot \frac{AK}{KB} \cdot \frac{CM}{MA} = -1.$$

By the signed version of Menelaus' theorem applied to $\triangle ABC$, the points K, L and M are collinear. ☐

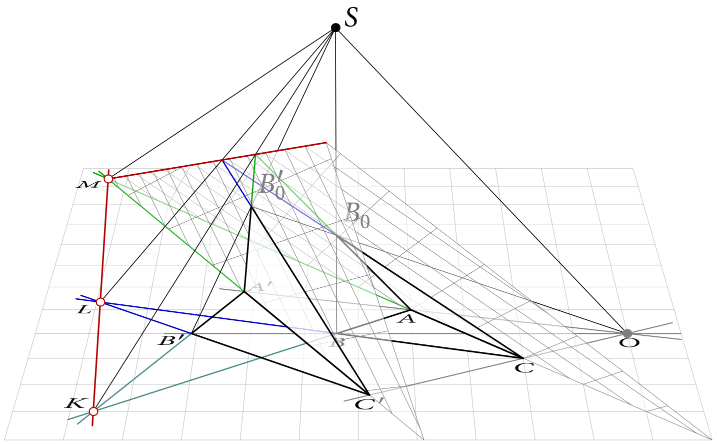


FIGURE 21.3
Three-dimensional proof of Desargues's theorem using projection

The second proof, again of the 'only if' part, proceeds by extending into three dimensions and using projection from a point outside the plane.

Second proof. Choose a point S not in the plane containing $\triangle ABC$ and $\triangle A'B'C'$. Choose collinear points B_0 and B_0' not in the original plane that project from S to B and B' respectively. The intersection of the planes containing $\triangle AB_0C$ and $\triangle AB_0'C$ is a straight line ℓ on which lie the intersections of AB_0 and $A'B_0'$, of B_0C and $B_0'C'$, and of CA and $C'A'$. Since lines project to lines, the intersections of AB and $A'B'$, of BC and $B'C'$, and of CA and $C'A'$ lie on the projection of ℓ from S back to the original plane. ☐

Zvezdelina Stankova (1969–) pointed out that the first proof was simple, being little more than a mechanical application of the signed version of Menelaus' theorem. But she said 'it doesn't give us a clue *really why* Desargues' Theorem works'.[1] But Lange said that the three-dimensional proof explained the theorem, because the two planes *united* the three pairs of corresponding sides of the triangles.[2] The key was that in three dimensions, a line on a plane need not be defined by two points, but by another plane.[3]

Some relevant philosophical analyses

The modern philosophical scrutiny of mathematical explanation began with a 1978 essay by Mark Steiner (1942–2020). He took it as given that mathematicians did indeed distinguish explanatory proofs from those that 'merely demonstrate'.[4]

He began by considering whether there was an identification of explanation with generality or abstraction. A general setting, he thought, seemed to give greater insight into mathematical results, such as how complex analysis illuminated the behaviour of the prime numbers.[5]

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[1] Stankova, 'Geometric Puzzles and Constructions', p. 175, emphasis in orig.

[2] Lange, 'Explanation, Existence and Natural Properties in Mathematics', p. 450.

[3] *Loc. cit.*

[4] Steiner, 'Mathematical explanation', p. 135.

[5] *Ibid.*, pp. 135–6.

But the putative identity of explanation with generality or abstraction failed when confronted with example. There is a standard inductive proof that $\sum_{i=1}^{n} i = n(n+1)/2$ for any $n \in \mathbb{N}$: first, for $n = 1$, one has $\sum_{i=1}^{1} = 1 = 1(1+1)/2$; then, assuming that the result is true for all $n < k$, one has

$$\sum_{i=1}^{k} i = \left(\sum_{i=1}^{k-1} i\right) + k = \frac{(k-1)\big((k-1)+1\big)}{2} + k$$
$$= \frac{(k-1)k + 2k}{2} = \frac{k(k+1)}{2}.$$

Steiner thought that this proof was simple, but that 'a more illuminating proof'[1] was given by pairing each term of the sequence with the corresponding term of the reverse sequence:

$$\begin{array}{ccccccc} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow \\ n & n-1 & n-2 & \dots & 3 & 2 & 1 \end{array}$$

Each of the n pairs of terms sums to $n+1$. The sum of the series and its reverse is thus $n(n+1)$; by dividing by two, one recovers the sum of the original series. [Mathematical folklore of doubtful origin has long ascribed the idea of this proof to the young Carl Friedrich Gauss* (1777–1855).]

* ▸ p. 401

Steiner pointed out that this second proof was more abstract, since it quantified over *series*, not just numbers. But he thought the visual configuration of the proof, not its abstractness, was the reason for its greater explanatoriness. He considered that the visual proof shown in Figure 21.4 (which is perhaps the canonical example of a 'proof without words', which will be discussed later in the chapter), was 'even more explanatory'.[2][3]

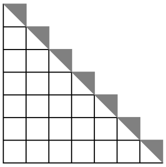

FIGURE 21.4
A visual proof of
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

[How does this visual proof 'work'?[4] The philosopher of science James Robert Brown (1949–) noted that mathematicians and philosophers were divided into two roughly equal groups. One group held that the diagram *encoded* an induction, and was thus a legitimate proof. The other group thought that the diagram was a heuristic device that *suggested* an induction, and that it was this suggested induction that was a legitimate proof.[5] Brown himself maintained that there was no induction present, arguing that '*Some "pictures"* [...] *are windows to Plato's heaven*',[6] or, less poetically but perhaps more accurately, that 'As telescopes help the unaided eye, so some diagrams are instruments [...] which help the unaided mind's eye.'[7]]

The visualizability of a proof may be linked to explanation, but Steiner thought it too subjective to offer a satisfactory account of mathematical explanation.[8] He instead proposed a criterion that would distinguish explanatory proofs:

> 'an explanatory proof makes reference to a characterizing property of an entity or structure mentioned in the theorem, such

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[1] Steiner, 'Mathematical explanation', p. 136.

[2] *Ibid.*, p. 137.

[3] *Ibid.*, pp. 136–7.

[4] This paragraph is based on the author's essay 'Visual thinking and simplicity of proof'.

[5] Brown, *Philosophy of Mathematics*, pp. 40–1.

[6] *Ibid.*, p. 40, emphasis in orig.

[7] *Ibid.*, p. 40.

[8] Steiner, 'Mathematical explanation', p. 143.

> that from the proof it is evident that the result depends on the property.'[1]

A 'characterizing property' was a property that uniquely distinguished an entity or structure within a family of such entities or structures.[2]

* * *

Michael D. Resnik (1938–) & David Kushner pointed to analyses of why-questions showing that there were variant questions subsumed under 'Why is it the case that P?'. Such a question implicitly asked 'Why is it the case that P and not some other member of a class of alternatives X?', and answers were only satisfactory relative to such a particular question. If the question 'Why did Adam eat the apple?' was answered with 'Because he was hungry', the answer was satisfactory if the particular question was 'Why did Adam eat the apple instead of handing it back to Eve?', but not if it was 'Why did Adam eat the apple instead of the pear?'.[3] There was thus no such thing as a single absolute explanation, but rather multiple context-dependent explanations. Resnik & Kushner suggested that the intuition that there were explanatory proofs arose from certain proofs revealing so little about the objects of the theorem that many such particular why-questions were left unanswered. Perhaps each individual why-question could be satisfactorily answered in context, but their conflation and subsumption under a catch-all 'Why?' led to the incorrect idea that there was some objective distinction between proofs that failed to explain and proofs that supply an explanation simpliciter.[4]

* * *

The philosopher of science Marc Lange (1963–) offered an argument that proofs by induction could not be explanatory. His starting point was that an explanation (or at least an explanatory proof) of a result $\mathfrak{P}$ was a deduction of $\mathfrak{P}$ from various other mathematical truths $\mathfrak{Q}_i$ that collectively helped explain why $\mathfrak{P}$ holds. He further assumed that explanation could not be circular. An inductive proof of $\mathfrak{P}$ normally comprised a proof of $\mathfrak{P}(1)$ and a proof that $\mathfrak{P}(k)$ implied $\mathfrak{P}(k+1)$ for all $k \in \mathbb{N}$. But it was in principle possible, and logically sound, to modify such an inductive proof to comprise a proof of $\mathfrak{P}(5)$, a proof that $\mathfrak{P}(k)$ implied $\mathfrak{P}(k+1)$ for all $k \in \mathbb{N}$, and a proof that $\mathfrak{P}(k+1)$ implied $\mathfrak{P}(k)$ for all $k \in \mathbb{N}$. Then in the first proof, it seemed that $\mathfrak{P}(1)$ was a component of the explanation of $\mathfrak{P}(5)$, while in the second proof, it seemed that $\mathfrak{P}(5)$ was a component of the explanation of $\mathfrak{P}(1)$. This outcome clashed with the assumption that explanation could not be circular.[5]

Contra Lange, several mathematicians and philosophers defended explanation in inductive proofs.[6] John T. Baldwin (1944–), for instance, pointed out that Lange's argument was not about induction *per se*, but showed that any specific instance could not be part of the explanation of a general result. (More formally, in Baldwin's view, Lange's argument showed that for any domain $\mathcal{A}$, element $a \in \mathcal{A}$, and proposition $P(■)$, the statement $P(a)$ could not be part of the explanation of $\mathcal{A} \models \forall x.P(x)$.) This conclusion ran counter to common mathematical techniques of beginning an argument by choosing a normal form or representative. For Baldwin, such arguments were often explanatory, and showing that one could reduce to the normal form was part of the explanation.[1]

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[1] Steiner, 'Mathematical explanation', p. 143.

[2] *Loc. cit.*

[3] van Fraassen, *The Scientific Image*, p. 127.

[4] Resnik & Kushner, 'Explanation, Independence and Realism in Mathematics', pp. 153–4.

[5] Lange, 'Why proofs by mathematical induction are generally not explanatory', pp. 205–10.

[6] Mancosu, Poggiolesi & Pincock, 'Mathematical Explanation', § 2.

VISUAL PROOF

> "to see" is often "to understand."
>
> — Roger B. Nelsen, *Proofs Without Words*, p. v.

Visual proofs have oft been considered a locus of beauty. The writer, expositor, and recreational mathematician Martin Gardner (1914–2010) thought that a dull proof often had 'a geometric analogue so simple and *beautiful* that the truth of a theorem is almost seen at a single glance'.[2] Gardner here did not use the term *geometric* as a strict limitation on the form of the proof; pre-mathematical spatial perception might be all that was required for what he called 'Look-See Proofs'.[3]

As a historical example, there is a visual proof of the convergence of the geometric series $\sum_{i=1}^{\infty} r^i$ for $r \in (0,1)$, which goes back at least to the second edition (1914) of Horatio Scott Carslaw's (1870–1954) textbook *Plane Trigonometry*,[4] though Carslaw did not claim to have originated the proof. It comprises Figure 21.5, a sight of which is apt to convince the viewer that the geometric series converges to the height above the base of the intersection of the lines with gradients 1 and r. The teacher of mathematics W. J. Dobbs (1868–1940) called it 'very beautiful and illuminating'.[5]

The *calisson problem* concerns tilings of a regular hexagon of integral side length using rhombi, each composed of two equilateral triangles of unit side length. [The name *calisson* refers to a type of French confectionary with the same rhombic shape.] In such a tiling (see Figure 21.6 (left)), each rhombus has one of three orientations.

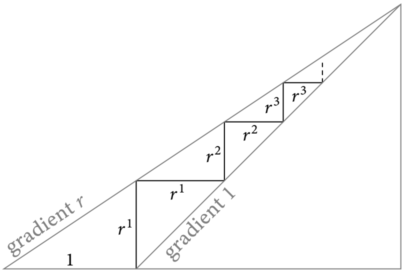

FIGURE 21.5
Visual proof that a geometric series with common ratio $r \in (0,1)$ converges (See Carslaw, *Plane Trigonometry*, Fig. 82 on p. 243).

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[1] Baldwin, 'Foundations of Mathematics', p. 79.

[2] Gardner, *Knotted Doughnuts and Other Mathematical Entertainments*, p. 192, emphasis added.

[3] *Ibid.*, p. 192.

[4] Dobbs, 'The Introduction to Infinite Series', p. 245. The author has been unable to check the 1st ed. (1909) of Carslaw, *Plane Trigonometry*.

[5] Dobbs, 'The Introduction to Infinite Series', p. 245.

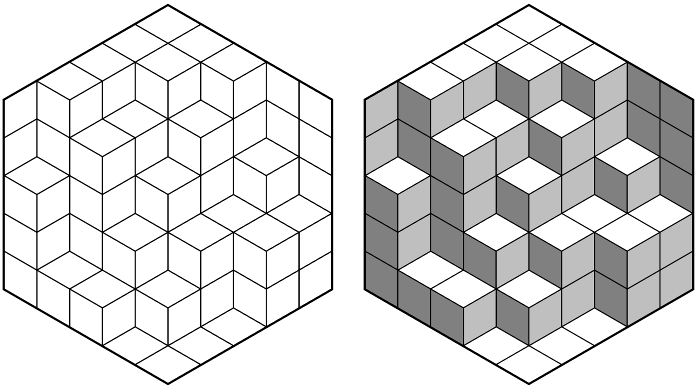

FIGURE 21.6

A proof without words of the calisson problem. Shading the rhombi according to their orientation makes it clear that a tiling of the regular hexagon with rhombi is an isometric projection of cubes stacked within a cubical space. Thus there must be an equal number of rhombi of each orientation. (David & Tomei, 'The Problem of the Calissons')

And in any such tiling, there are the same number of rhombi of each orientation. To see this, it suffices to shade each rhombus according to its orientation, which shows that the tiling of the hexagon with rhombi is an isometric projection of stacked cubes. Since these stacked cubes project exactly onto the walls of a cubical space, there must be the same number with each orientation.[1]

A *proof without words* is a visual proof of an extreme degree: a diagram, perhaps augmented by a few symbols, that alone serves to establish the theorem. Aristotle's explanation of the pythagorean association of odd numbers with 'limit' being grounded on pebble arithmetic[*] in effect supplies a proof without words that a sum of consecutive odd numbers, starting at 1, is a square number.[2]

A more advanced proof without words establishes the combinatorial identity $\sum_{i=1}^{n-1} i = \binom{n}{2}$ by the bijection shown in Figure 21.7.[3] The viewer is supposed to grasp the idea that each pair of the n elements in the bottom row (of which there are $\binom{n}{2}$) uniquely determines one of the elements in the top $n-1$ rows (of which there are $\sum_{i=1}^{n-1} i$), and vice versa, the particular number n not being noticed. There has, however, been debate over whether it is truly understandable.[4]

◆

Mathematics Magazine and *The College Mathematics Journal* had popular regular 'proofs without words' features, which were collected and published in two eponymous books edited by Roger Nelsen (1942–).[5] In the introduction to the first collection (1993), Nelson wrote: 'Of course, "proofs without words" are not really proofs.'[6] He cited a 1991 report by Theodore Eisenberg (1942–) & Tommy Dreyfus (1946–), both professors of mathematics education, that colleagues had not considered 'proofs without words' to be proofs at all, but rather to be mnemonics that might aid in producing a proof.[7] Nelsen seemed to accept this:

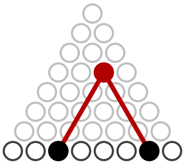

FIGURE 21.7

A bijective visual proof that $\sum_{i=1}^{n-1} i = \binom{n}{2}$.

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[1] David & Tomei, 'The Problem of the Calissons'.

[*] ▸ pp. 46–7

[2] Cf. Nelsen, *Proofs Without Words*, p. 71.

[3] Larson, 'A Discrete Look at $1 + 2 + \cdots + n$', § 5.

[4] `https://mathoverflow.net/a/8847/49388`

[5] Nelsen, *Proofs Without Words*; Nelsen, *Proofs Without Words II*.

[6] Nelsen, *Proofs Without Words*, p. vi.

[7] Eisenberg & Dreyfus, 'On the Reluctance to Visualize in Mathematics', p. 31.

> '[proofs without words] are pictures or diagrams that help the observer see *why* a particular statement may be true, and also to see how one might begin to go about proving it true.'[1]

But in the sequel collection (2000), Nelsen quoted as an epigraph[2] a rather contrary statement by the mathematician Andrew Gleason (1921–2008), who recalled that he often said to his students:

> 'proofs really aren't there to convince you that something is true — they're there to show you *why* it is true.'[3]

(Gleason said this in an interview published in 1990, so he was certainly not responding to either Nelsen or Eisenberg & Dreyfus.) Nelsen had also become more circumspect about the status of proofs without words: 'Of course, *some argue* that [proofs without words] are not really "proofs"'.[4]

ILLUMINATION

The historian of science Horace Judson (1931–2011) described how the experience of discovery — 'what ought to be meant [...] by the tarnished phrase "the moment of truth"'[5] — was one of 'exhilaration and tranquillity. It is luminously clear. It is beautiful.'[6] Although the specific context of Judson's discussion was mathematical discovery, his description was about scientific discovery generally,[7] and appeared in a book which was based on conversations with many different scientists.[8]

According to Roger Penrose (1931–), Michael Atiyah (1929–2019) remarked that elegance in mathematics was close to being synonymous with simplicity. Penrose thought that while the two were indeed related, there was more to be said: he thought that elegance was connected specifically with *unexpected* simplicity. Complexity and perhaps toil are foreseen, and yet there comes a revelation that the situation is much simpler and easier than anticipated.[9]

The very phenomena of unexpectedness and illumination in mathematics are dependent upon *implicational opacity*. This term names the inability to see immediately all the consequences of certain mathematical statements, and more generally to see how concepts in mathematics are linked to each other.[10] Unexpectedness, and possibly illumination, in mathematics comes about when the opacity is penetrated and a hitherto unknown logical or conceptual connection becomes visible.[11] The philosopher Jody Azzouni (1954–) suggested that the psychological effect of implicational opacity was to motivate the practice of mathematics:[12] without it, the consequences, connections, and context of any part of the subject of mathematics could be encompassed at a glance. He thought that implicational capacity had 'a major role in aesthetic reactions to proofs.'[13]

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[1] Nelsen, *Proofs Without Words*, p. vi, emphasis in orig.

[2] Nelsen, *Proofs Without Words II*, p. ix.

[3] Gleason, quot. in Albers, Alexanderson & Reid, *More Mathematical People*, p. 86, emphasis added.

[4] Nelsen, *Proofs Without Words II*, p. x, emphasis added.

[5] Judson, *The Search for Solutions*, p. 2.

[6] *Loc. cit.*

[7] *Loc. cit.*

[8] *Ibid.*, p. 206.

[9] Penrose, 'The Rôle of Aesthetics', p. 268.

[10] Azzouni, 'Applying Mathematics', pp. 222–3.

[11] *Ibid.*, p. 211.

[12] *Ibid.*, p. 223.

[13] *Loc. cit.*

Timothy Gowers (1963–) tried to explain to a lay audience mathematicians' use of aesthetic terminology — *elegant*, *beautiful*, *witty* — in evaluating proofs by suggesting a parallel with aesthetic judgements of music:

> 'we may be entranced when a piece moves in an unexpected harmonic direction that later comes to seem wonderfully appropriate, or when an orchestral texture appears to be more than the sum of its parts in a way that we do not fully understand. Mathematical proofs can provide a similar pleasure with sudden revelations, unexpected yet natural ideas, and intriguing hints that there is more to be discovered.'[1]

In both mathematics and music, the unforeseen turn led to a surprising revelation of correctness or at least of fit, and pointed to hidden depths.

David Ruelle (1935–) closed his 2007 study of the psychology of mathematicians with a short chapter on the beauty of mathematics and its role in motivation.[2] He suggested that

> '*the beauty of mathematics lies in uncovering the hidden simplicity and complexity that coexist in the rigid logical framework that the subject imposes*.'[3]

The interleaving of simplicity and complexity were not special to the beauty of mathematics, but were found in art and in other forms of beauty. But the beauty of mathematics — its intertwined simplicity and complexity — was, thought Ruelle, not a product of human craft; it was natural and pre-existing and yet fulfilled our desire for simplicity and complexity.[4] He ended by emphasizing that it was in doing mathematical research that one truly saw its beauty, when the tangle of emergent complications yielded to a fundamental simplicity:

> 'In those moments a piece of a colossal logical structure is illuminated, and some of the meaning hidden in the nature of things is finally revealed.'[5]

Francis Su explicitly contrasted the patient effort needed to access mathematics with the kind of sudden illumination wherein 'the beauty of mathematics can also reveal itself to a math explorer in a blinding instant'.[6] This revelation he termed the *'aha!' moment*. But a degree of care is needed with the 'aha!' As will be seen, by this expression some authors have meant the instant and the experience of sudden illumination, while some have meant the mental state of understanding and of judging an explanation as adequate.

[1] Gowers, *Mathematics*, p. 51.

[2] Ruelle, *The Mathematician's Brain*, pp. 127–8.

[3] *Ibid.*, p. 129, emphasis in orig.

[4] *Ibid.*, p. 129.

[5] *Ibid.*, p. 130.

[6] Su, *Mathematics for Human Flourishing*, ch. 5.

EPISTEMIC FEELINGS

The philosopher Gilbert Ryle (1900–76) pointed out that the word *feeling* was used in manifold ways. Among these were a *perceptual* use (feeling the shape of a thing); a related metaphorical *mock-perceptual* use (feeling the noose around one's neck); a *localized sensation* use (feeling an itch); a *general condition* use (feeling thirsty or uneasy); and a *something-is-the-case* use (feeling that an argument is flawed).[1] William P. Alston (1921–2009) noted another distinction: 'feeling sorry for *x*' suggested a set of dispositions — to feel sad when contemplating the troubles of *x* and if possible to help — rather than the states of consciousness suggested by Ryle's examples.[2]

The feeling that an argument is flawed is an example of an *epistemic feeling*: it is a feeling *about* the state or process of knowledge. There are a variety of epistemic feelings, including the *feeling of curiosity* and the *feeling of competence*.[3] The *feeling of familiarity* might be judged as having appeared early in the history of philosophy: perhaps in the *Meno* Plato was trying to portray such a feeling in the servant boy who recognized as valid Socrates' duplication of the square.[4][*] Such epistemic feelings may serve a meta-cognitive function of allowing the subject to act in spite of mental uncertainty.[5]

The most relevant epistemic feelings here are the following:

Feeling of knowing. The feeling that one has available in memory a certain datum despite attempts to retrieve that datum being unsuccessful.[6] As William James (1842–1910) pointed out in an early discussion of the phenomenon, if incorrect candidates for the datum were proposed by others, they were immediately rejected as incorrect;[7] if the correct datum was proposed, it was immediately recognized.

Feeling of understanding. A positive feeling of confidence and satisfaction that an explanation of a phenomenon is adequate, accurate, and, correct.[8] This feeling is sometimes called the 'aha!' feeling.[9]

But, although good explanations tend to evoke a feeling of understanding, this feeling is neither a necessary nor a sufficient condition for an explanation to be good. History is replete with scientific explanations that doubtless produced this sense and that were later shown to be inadequate or simply wrong.[10] On the other hand, a phenomenon may only admit (correct) explanations that are too large or too complicated or too alien to cause this feeling. Quantum mechanics, for instance, is conceptually so far divorced from everyday human life that it fails to produce a feeling of understanding in many, if not most: the Nobel laureate Murray Gell-Mann[†] (1929–2019) called it 'that mysterious, confusing discipline, which none of us really understands but which we know how to use.'[11]

Feeling of certainty or rightness. The feeling on completion of a reasoning or decision process that its execution was successful.[12]

[1] Ryle, 'Feelings', pp. 193–6.

[2] Alston, 'Feelings', p. 6.

[3] Arango-Muñoz & Michaelian, 'Epistemic Feelings, Epistemic Emotions', p. 102.

[4] de Sousa, 'Epistemic Feelings', p. 189.

[*] ▸ p. 74

[5] Arango-Muñoz & Michaelian, 'Epistemic Feelings, Epistemic Emotions', p. 103.

[6] de Sousa, 'Epistemic Feelings', pp. 192, 196; Arango-Muñoz & Michaelian, 'Epistemic Feelings, Epistemic Emotions', pp. 99–100.

[7] James, *Principles of Psychology*, p. 243.

[8] Arango-Muñoz & Michaelian, 'Epistemic Feelings, Epistemic Emotions', p. 101; Trout, 'Scientific Explanation', pp. 213–14.

[9] Arango-Muñoz & Michaelian, 'Epistemic Feelings, Epistemic Emotions', p. 101.

[10] Trout, 'Scientific Explanation', pp. 212–13.

[†] ▸ pp. 955–7

[11] Gell-Mann, 'Questions for the Future', pp. 169–70.

[12] Arango-Muñoz & Michaelian, 'Epistemic Feelings, Epistemic Emotions', p. 100.

> Such a feeling arrests (at least temporarily) further inquiry and allows action based on the conclusion whose correctness is now assumed.[1]
>
> This feeling is neither a necessary nor a sufficient condition for a reasoning process to be correct. An invalid mathematical demonstration may produce this feeling; a valid one may fail to produce it.

PROCESSING FLUENCY

Rolf Reber (1959–), Norbert Schwarz (1953–) & Piotr Winkielman considered in a 2004 paper whether it was possible to identify cognitive processes that grounded aesthetic experience. They proposed the hypothesis that such experience arose from the degree of *fluency* with which the perception of an object is processed: greater fluency evoked a more positive aesthetic response.[2] *Processing fluency* applied to different aspects of the cognitive process of perception.[3] *Perceptual fluency* concerned the low-level processes that dealt primarily with the sense-input of the perceived object. It could be influenced by the repetition or duration of stimuli, the contrast in those stimuli, and priming via preliminary stimuli such as seeing an accurate outline before being shown a distorted complete image.[4] *Conceptual fluency* concerned the high-level processes that dealt primarily with how the perceived object related to pre-existing knowledge structures. It could be influenced by conceptual priming or predictability on a semantic level. The two types of processes interacted, especially when the interpretation of inadequate or insufficient sense-data had to be aided by previous experience.[5]

In an earlier essay, Winkielman, Schwarz, Tedra A. Fazendeiro & Reber distinguished *objective fluency* and *subjective fluency*. Both terms referred to a process that was rapid, undemanding, and accurate, but the former term denoted a process that did not necessarily rise to a conscious level, while the latter denoted a conscious experience. The two did not necessarily go together: objective fluency might apply to the mental processes of a person utilizing an oft-practised skill while they were consciously focusing on something else.[6]

It seemed that some metacognitive feedback mechanism granted the subject access to the degree of processing fluency.[7] Processing fluency was hedonically marked: it led to an experience of pleasure and was subjectively judged positively. A person making an aesthetic — or any kind of evaluative — judgement drew on subjective experience, and so this positive hedonic experience contributed to positive aesthetic judgements.[8] But the use of fluency in judgements was affected by subjects attempting to attribute the fluency to an appropriate

[1] de Sousa, 'Epistemic Feelings', p. 192.

[2] Reber, Schwarz & Winkielman, 'Processing Fluency', p. 365.

[3] *Ibid.*, p. 366.

[4] Winkielman et al., 'Hedonic Marking', pp. 193–4; Reber, Schwarz & Winkielman, 'Processing Fluency', p. 367.

[5] Winkielman et al., 'Hedonic Marking', p. 194.

[6] *Ibid.*, p. 193.

[7] *Loc. cit.*

[8] Reber, Schwarz & Winkielman, 'Processing Fluency', pp. 365–6; Winkielman et al., 'Hedonic Marking', p. 195.

source, and indirectly by its effect upon cognitive operations and the way in which information was represented mentally.[1]

High fluency might generate a positive response because it was associated with something positive about the world or about the subject's cognitive state.[2] It might be associated with the successful recognition of a stimulus or the ease with which it was understood and interpreted in context, or with its familiarity and thus (*ceteris paribus*) harmlessness.[3] It might indicate that there was successful progress in cognition that was underway.[4]

Properties that had historically been held to be conducive to beauty, such as symmetry, contributed to processing fluency.[5] More generally, objects carrying less information — in particular those with symmetry — were easier to process than those without.[6] This ease of processing fitted with a well-known human preference for symmetry[7] and with characterizations of beauty such as Francis Hutcheson's (1694–1746) 'uniformity admist variety'.[*]

Reber, Schwarz & Winkielman defined beauty as 'a pleasurable subjective experience that is directed toward an object and not mediated by intervening reasoning' and treat as synonymous the terms *beauty* and *aesthetic pleasure*.[8] This definition of beauty was novel and expansive: it seems to collapse together different positive aesthetic values such as beauty and elegance and neatness, and it treats beauty as a conscious mental state of reaction towards an object, rather than any quality that caused that reaction.

This stance fitted with fluency being more than a function of properties of the object. Fluency could be affected by variables external to the object, such as repetition, duration, or priming.[9] Education and experience in painting, sculpture, music, or poetry would presumably increase the fluency with which such artworks, and especially complex or abstract or otherwise unusual artworks, could be processed. Thus the fluency hypothesis accorded with the different preferences of novices and experts.[10] It was a consequence of the fluency hypothesis that beauty was not a property of the object, but lay, if not in the eye of the beholder, then in a function of the perceptual processes of the beholder. But these processes and the subjective experiences to which they gave rise were a consequence of properties of the object, of the beholder's perceptual and cognitive history, and ultimately of the structure, function, and evolution of the human brain.[11]

Furthermore, the *effect* of fluency was moderated by factors such as expectation: an experience of fluency in a surprising situation made a strong positive impression.[12] In particular, there was a particularly strong aesthetic pleasure when processing was expected to be difficult — that is, *un*fluent — and yet fluency unexpectedly occurred.[13] [This response to unforeseen fluency is compatible with characterizations of beauty and elegance as 'unexpected simplicity'.[†]]

As regards the relationship in the sciences between beauty and truth, Reber, Schwarz & Winkielman quoted the remark that Freeman

[1] Oppenheimer, 'The secret life of fluency', pp. 239–40.

[2] Reber, Schwarz & Winkielman, 'Processing Fluency', p. 366; Winkielman et al., 'Hedonic Marking', p. 197.

[3] Reber, Schwarz & Winkielman, 'Processing Fluency', p. 366; Winkielman et al., 'Hedonic Marking', p. 195.

[4] Winkielman et al., 'Hedonic Marking', pp. 196–7.

[5] Reber, Schwarz & Winkielman, 'Processing Fluency', pp. 365, 369; Winkielman et al., 'Hedonic Marking', p. 196.

[6] Reber, Schwarz & Winkielman, 'Processing Fluency', pp. 368–9.

[7] See, for example, Eisenman, 'Complexity-simplicity I'; Rhodes et al., 'Facial symmetry'; Osborne, 'Symmetry as an aesthetic factor'.

[*] ▸ pp. 357–62

[8] Reber, Schwarz & Winkielman, 'Processing Fluency', p. 365.

[9] Winkielman et al., 'Hedonic Marking', p. 191.

[10] Reber, Schwarz & Winkielman, 'Processing Fluency', p. 374.

[11] *Ibid.*, p. 378.

[12] *Ibid.*, pp. 366, 372.

[13] *Ibid.*, p. 373.

[†] ▸ pp. 782, 814

Dyson (1923–2020) remembered that Hermann Weyl (1885–1955) humorously made about favouring the beautiful over the true,[*] and the third-hand account via Dyson and Subrahmanyan Chandrasekhar (1910–95) of him actually preferring beauty in the gauge theory of gravitation,[†] and considered these accounts to be illustrations of scientists at least accepting beauty as an intuitive route to truth.[1] The link, they suggested, could be supplied by fluency serving as a non-analytic source of judgements of both beauty and truth.[2]

Sascha Topolinski & Reber offered an account of 'aha!' experiences in terms of processing fluency. They identified four features of such experiences as described in the literature:

suddenness: the insight was immediate, not formed over a discernible period;

easy: the insight, once gained, was understood without difficulty;

enjoyable: the experience of insight was pleasurable;

a feeling of rightness: the insight was confidently held to be true.[3]

Topolinski & Reber argued that the enjoyment of the moment and the subsequent ease and the feeling of rightness could be explained by a sudden increase in processing fluency.[4] In this case, one would also expect a positive aesthetic response to the 'aha!' experience via the greater fluency.

BEAUTY AND EXPLANATORINESS

The testimonies of working mathematicians suggest that the quality of explanatoriness, or related qualities of insight or understanding, are somehow connected with aesthetic value. But it is important to note at the outset that although the idea that explanatory proofs are beautiful is *compatible* with Gian-Carlo Rota's (1932–99) reinterpretation of mathematical beauty as enlightenment,[‡] it does not *necessitate* such reinterpretation. Furthermore, the philosopher John Barker argued that a proof can be explanatory and inelegant or non-explanatory and elegant.[§]

Alfred North Whitehead. In the third lecture in the series *Modes of Thought*, delivered in 1937–8, the philosopher Alfred North Whitehead (1861–1947) addressed the notion of *understanding*. Understanding was something more than mere factual knowledge. It came about when our perception of a thing was so clear that it required no epistemological foundation beyond itself: 'self-evidence is understanding'.[5] Proof, therefore, whether in the sense of a mathematical demonstration or other argument, was 'a feeble second-rate procedure'.[6] While a proof might *convince*, it only granted *understanding* if it produced self-evidence and thus annulled its own purpose.[7] As far

[*] ▸ p. 679

[†] ▸ p. 895

[1] Reber, Schwarz & Winkielman, 'Processing Fluency', p. 377.

[2] *Loc. cit.*

[3] Topolinski & Reber, 'Gaining Insight Into the "Aha" Experience', p. 402.

[4] *Ibid.*, p. 404.

[‡] ▸ pp. 742–5

[§] ▸ pp. 751–2

[5] Whitehead, *Modes of Thought*, p. 47.

[6] *Ibid.*, p. 48.

[7] *Loc. cit.*

as understanding was concerned, a proof was only a tool to extend self-evidence.[1]

Like understanding, 'aesthetic experience is another mode of the enjoyment of self-evidence':[2] a thing was judged beautiful because it was self-evidently beautiful. Whitehead thought that philosophy should attend to mathematicians' judgements of some proofs as being more beautiful than others, and more broadly to an analogy between aesthetic and logical experience.[3] The distinction between the two was in their degree of abstraction: logic was focused upon high abstraction, aesthetics on the concrete.[4] In logic, understanding was the enjoyment of the combination of more primitive ideas into a unity; in aesthetic experience, '[t]he whole precedes the details'.[5]

Whitehead did not develop the analogy further, but his posited connection between understanding and aesthetic experience foreshadowed later discourse on explanation and mathematical beauty.

Ludwig Wittgenstein. Ludwig Wittgenstein (1889–1951) indicated, in a parenthetical remark embedded in a discussion of mathematicians' conception of the continuum and related problems, a wholly negative view of any connection between elegance in proof and understanding:

> '(The only point there can be to elegance [Eleganz] in a mathematical proof is to reveal certain analogies in a particularly striking manner, when that is what is wanted; otherwise it is a product of stupidity and its only effect is to obscure what ought to be clear and manifest. The stupid pursuit of elegance is a principal cause of the mathematicians' failure to understand their own operations; or perhaps the lack of understanding and the pursuit of elegance have a common origin.)'[6]

[Wittgenstein's isolated remark appears in a portion he did not revise of the 1932–3 typescript upon which Part II of the posthumously-published *Philosophical Grammar* was based.[7]]

W. W. Sawyer. In his 1943 popular mathematics book *Mathematician's Delight*, the mathematician and educator W. W. Sawyer (1911–2008) connected the beauty found in complex numbers with how it illuminated the ordinary numbers the 'feeling of having been taken behind the scenes'[8] and seeing the natural reasons behind hitherto mysterious results. For complex numbers, 'results frequently take forms which one can *see*, and remember'.[9] Sawyer only briefly sketched the importance of the complex numbers in pure and applied mathematics, but an example of this 'behind the scenes' phenomenon might be Richard Dedekind's (1831–1916) proof of Fermat's two-square theorem.[*] This proof shows that the expressibility of any prime number of the form $4k + 1$ as a sum of two squares is a consequence of the notion of primality in the ring of Gaussian integers:[10]

[1] Whitehead, *Modes of Thought*, p. 50.

[2] *Ibid.*, p. 60.

[3] *Loc. cit.*

[4] *Ibid.*, pp. 60–1.

[5] *Ibid.*, p. 61.

[6] Wittgenstein, *Philosophical Grammar*, p. 462.

[7] *Ibid.*, p. 487.

[8] Sawyer, *Mathematician's Delight*, p. 232.

[9] *Loc. cit.*

[*] ▸ p. 7

[10] Dedekind, 'Supplemente', p. 444.

Proof. Suppose p is a prime of the form $4k+1$. Divide elements of $\mathbb{Z}_p \setminus \{0\}$ into sets $\{x, -x, x^{-1}, -x^{-1}\}$. Since $\{1, p-1\}$ is such a set, there is at least one other set of size 2. A check of the various possible equalities between the elements shows that this other set is of the form $\{m, -m^{-1}\}$ for some $m \in \mathbb{Z}$. Thus $m^2 \equiv -1 \pmod{p}$ and so $p \mid m^2 + 1$.

In $\mathbb{Z}[\mathrm{i}]$, the ring of Gaussian integers, $m^2 + 1 = (m+\mathrm{i})(m-\mathrm{i})$. The number p divides neither $m+\mathrm{i}$ nor $m-\mathrm{i}$ in $\mathbb{Z}[\mathrm{i}]$, so p is not a prime in this ring. In $\mathbb{Z}[\mathrm{i}]$, irreducibility and primality coincide, so p is not irreducible in $\mathbb{Z}[\mathrm{i}]$. Hence p factors as $p = xy$, where $x, y \in \mathbb{Z}[\mathrm{i}]$ are not units. Taking norms shows that $p^2 = N(p) = N(xy) = N(x)N(y)$. But, since x and y are not units of $\mathbb{Z}[\mathrm{i}]$, the norms $N(x)$ and $N(y)$ cannot be equal to 1. So the equality $p^2 = N(x)N(y)$ and the primality of p (in $\mathbb{Z}$) together imply that $p = N(x) = N(y)$, and so $p = N(x) = \Re(x)^2 + \Im(x)^2$. $\square$

Philip J. Davis & Reuben Hersh. In 1981, Philip J. Davis (1923–2018) & Reuben Hersh (1927–2020) considered two proofs of the irrationality of $\sqrt{2}$: the traditional 'pythagorean' proof and a second one that supposes $p/q = \sqrt{2}$, infers that $p^2 = 2q^2$, and deduces a contradiction by applying the uniqueness of decomposition into prime factors.[*] Davis & Hersh had no doubt that almost all professional mathematicians would judge that the second proof exhibits 'a higher level of aesthetic delight' and suggested that the second proof 'seems to reveal the heart of the matter [...] exposes the "real" reason'.[1]

Lawrence Stout. The mathematician Lawrence Stout (1948–2012) gave in a 1999 essay a number of principles that a beautiful proof should follow, which he illustrated by comparing three proofs of the binomial theorem.

First, it should make the result apparent: this was a quality that might depend on the beholder.[2] This rule was closely connected with the second principle Stout gave: 'It should explain why the result not only *is* true but *should be* true.'[3] This second principle seems to have required that the proof illuminate how the result sat in a broader mathematical context in which it was natural.

In isolation, these two principles clearly point to explanatoriness in proof. But, for Stout, they did not stand alone. Economy, one of the criteria in the Hardian triad,[†] was another principle,[4] as was unexpectedness. But the *type* of unexpectedness was distinct from Hardy's. Rather than being internal to the proof, per Hardy, unexpectedness for Stout came about through the revelation of connections to other parts of mathematics.[5] Related to this was Stout's fifth principle: that a proof that suggested a route towards new progress was 'more pleasing';[6] whether Stout meant more *beautiful* is not clear.

[*] ▸ p. 846

[1] Davis & Hersh, *Mathematical Experience*, pp. 299, 301.

[2] Stout, 'Aesthetic Analysis of Proofs of the Binomial Theorem', p. 11.

[3] *Ibid.*, p. 11, emphasis in orig.

[†] ▸ pp. 502

[4] *Ibid.*, p. 12.

[5] *Loc. cit.*

[6] *Loc. cit.*

Burton's study. In Leone Burton's (1936–2007) study of the practice of mathematicians,* published in 2004, several participants commented on the relationship between explanatoriness and beauty. One contrasted two papers they had read that both made a certain construction. The older paper simply made a calculation that did not explain why it worked. The more recent paper explained the existence of the constructed object, and, according to the participant, '[t]he explanation is the beautiful thing'.[1] Another participant made the related point that brevity might not be conducive to beauty, for a long proof might reveal more of 'what is going on', and so there could be beauty in the *details* of a long proof.[2]

Jonathan Borwein. In a public lecture delivered in 2001 and published in essay form in 2006, Jonathan Borwein (1951–2016) wrote that

> 'The mathematician's "aesthetic buzz" comes not only from simply contemplating a beautiful piece of mathematics, but, additionally, from achieving insight.'[3]

For Borwein, this insight was not necessarily manifested in a particular kind of proof, but could come about by visualization and experiment, perhaps aided by computer.[4]

Robert Osserman. In an interview reported in the same collection as Borwein's essay, the geometer Robert Osserman (1926–2011) pointed directly to understanding through proof: when he knew something to be true, he felt an aesthetic desire to have a proof so clear as to grant an immediate complete understanding, to let him *see* the truth.[5]

MINIMAL COMPLETENESS AND MAXIMAL APPLICABILITY

Jerry King (1935–2020) wrote in his 1992 book *The Art of Mathematics* that he believed he had identified two principles that he thought mathematicians accepted as standards for making aesthetic judgements:

Minimal completeness: 'A mathematical notion N satisfies the *principle of minimal completeness* provided that N contains within itself all properties necessary to fulfill its mathematical mission, but N contains no extraneous properties.'

Maximal applicability: 'A mathematical notion N satisfies the *principle of maximal applicability* provided that N contains properties which are widely applicable to mathematical notions other than N.'[6]

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* ▸ pp. 544–5

[1] Unnamed mathematician, quot. in Burton, *Mathematicians as Enquirers*, p. 66.

[2] *Ibid.*, p. 101.

[3] Borwein, 'Aesthetics for the working mathematician', p. 25.

[4] *Ibid.*, pp. 24–5.

[5] Sinclair, 'The Aesthetic Sensibilities of Mathematicians', pp. 99–100.

[6] King, *The Art of Mathematics*, p. 181;

King used the word 'notion' deliberately, to cover theorems, proofs, equations, or definitions: any of the loci where are seen properties such as beauty or elegance.[1]

Both principles hold for Euclid's proof of the infinitude of the primes and the proof of the irrationality of $\sqrt{2}$: these proofs were self-contained and efficient and thus minimally complete; the infinitude of the primes is a cornerstone of mathematics, while the proof of irrationality generalizes to $\sqrt{p}$ for any prime p, and thus both were maximally applicable.[2] In contrast, the Prime Number Theorem[*] was maximally applicable, but since its proof depended on complex analysis, it lacked self-containment and so it failed to be minimally complete.[3] A toy result, such as $x + 1/x \geqslant 2$ for all positive x, can be proven in a few lines of elementary algebra, and so was minimally complete, but was an isolated result with a very specific proof, so was not maximally applicable.[4] King thought that Euler's equation[†] was minimally complete and its embedding into complex analysis made it maximally applicable.[5]

THE FEELING OF ILLUMINATION

The book *Matière à Pensée* (1989) and its extended and edited English translation *Conversations on Mind, Matter, and Mathematics* (1995) comprise a number of dialogues between the neuroscientist Jean-Pierre Changeux (1936–) and the mathematician Alain Connes (1947–) on the relationship between the structure and function of the human brain and the practice and subject-matter of mathematics.[6] Both were preëminent in their respective subjects, as illustrated by the accolades the two received in 1982: Changeux was the Wolf laureate in medicine; Connes was one of the Fields medallists.

A larger part of their debate was given over to whether mathematics was independent of the humans who investigated it. Connes professed mathematical platonism, maintaining that mathematics was pre-existing and was discovered by the mathematician. Changeux was sceptical, believing that mathematics existed only as mental constructs.[7]

The part of the discussion most relevant here was devoted to 'illumination' as described by Henri Poincaré[‡] (1854–1912) and Jacques Hadamard[§] (1865–1963). While Connes found the verification process, which followed illumination, to be fraught with the fear that the intuition of correctness was wrong,[8] the response to illumination itself was overwhelmingly positive:

> '[CONNES:] the moment illumination occurs, it engages the emotions in such a way that it's impossible to remain passive or indifferent. On those rare occasions when I've actu-

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[1] King, *The Art of Mathematics*, pp. 181–2.

[2] *Ibid.*, pp. 188–9.

[*] ▸ p. 407

[3] *Ibid.*, pp. 189–90.

[4] *Ibid.*, pp. 190–1.

[†] ▸ pp. 839–43

[5] *Ibid.*, p. 191.

[6] Changeux & Connes, *Conversations*, pp. xi–xii.

[7] *Ibid.*, esp. pp. 3, 5, 9–32.

[‡] ▸ pp. 655–60

[§] ▸ pp. 662–4

[8] *Ibid.*, p. 76.

> ally experienced it, I couldn't keep tears from coming to my eyes.'[1]

Changeux and Connes agreed that this moment of illumination was not just emotional, but involved hedonic and epistemic feelings:

The feeling of illumination

> 'CHANGEUX: A kind of "pleasure alarm" goes off, in other words, rather than a danger alarm, signaling —
>
> CONNES: That what's been found works, is coherent and, one might even say, aesthetically pleasing. I'm certain that this pleasure is analogous to that experienced by painters the moment that they find a solution, the moment they see that a canvas is perfectly coherent and harmonious.'[2]

Thus the illumination was accompanied by a *feeling* of correctness and aesthetic satisfaction. For Connes, an important feature of illumination was that one was instantly convinced of the correctness of the revealed solution:

> 'CONNES: [...] In a fraction of a second one is convinced not only that it's plausible, but that it's *certain* that what one's found matches what one was looking for. [...]
>
> [...]
>
> CONNES: [...] one is able to *instantly* comprehend the harmony and the power of a new object independently of any specific problem.'[3]

Connes compared the speed to a physical reflex; Changeux compared it to the mental recognition or non-recognition of a face.[4]

Connes pointed out that the pleasure or even exhilaration of illumination was also accompanied by *relief* at opaqueness giving way to transparency, allowing conscious thought to access a new and no longer obscure area.[5] Changeux suggested that this description brought to mind Teresa of Ávila's (1515–82) ecstatic mystical experiences; Connes responded that the same parts of the brain were stimulated by mathematical illumination, by such ecstatic transport, and by aesthetic harmony.[6] He emphasized that 'a work of art, or a piece of music, can excite the limbic system in just the same way'; the difference between mathematics and art lay not in the moment of illumination itself, but in the verification stage, when the mathematician alone had to write out and check the details: 'the purely "mystical" phase has already long vanished by this point'.[7]

Changeux and Connes thought that the 'incubation' stage was when a Darwinian process of combination and selection of ideas took place.[8] Illumination could then come about when there was what Changeux called a 'mutual resonance of mental representations'.[9] Changeux pointed out that the frontal cortex of the brain, which is responsible for cognitive functions and thus, he thought, was likely the volume where the 'resonance' occurred, was closely connected

[1] Changeux & Connes, *Conversations*, p. 76.

[2] *Ibid.*, p. 81.

[3] *Ibid.*, p. 152, emphasis added.

[4] *Ibid.*, p. 152.

[5] *Ibid.*, p. 147.

[6] *Ibid.*, pp. 147–8.

[7] *Ibid.*, p. 148.

[8] *Ibid.*, pp. 81, 118.

[9] *Ibid.*, p. 118.

to the limbic system, which supported the production of emotional states.[1] The aesthetic pleasure in contemplating a work of art was explained by the same kind of resonance involving the cognitive and limbic systems, leading to the production of a new mental object that reveals previously hidden relationships.[2] This connection meant that cognition could have an emotional tenor that contributed to how it was evaluated. A good fit — a resonance or harmony — among mental representations could thus manifest as a pleasure or warning signal.[3]

AESTHETIC INDUCTION

The philosopher of science James McAllister (1962–) offered in his 1996 book *Beauty & Revolution in Science* a defence of a rationalist view of science that gave room to aesthetic judgement. He posited a model of *aesthetic induction* in which the empirical success of theories shaped scientists' aesthetic criteria and thus the aesthetic value that they saw in a theory, and applied this to concepts such as 'revolutions' in science.[4] Though aesthetic judgements in mathematics played a central role in his model, he cautioned that the mechanism by which beauty is attributed in mathematics might differ from the mechanism in science[5] Nevertheless, in a later essay he suggested that his model also applied to mathematics and used some parallels with the sciences to draw inferences about aesthetic judgements in mathematics.[6]

McAllister took a projectivist view in which beauty was a value that was projected into or attributed to an object by an observer, rather than being a property of an object.[7] Whether beauty was so projected depended not just on the intrinsic properties of the object but also on the observer's aesthetic criteria, which had the form:

$$\left.\begin{array}{l}\text{`Project beauty into an object if, other things being}\\ \text{equal, it exhibits property } P\text{.'}\end{array}\right\} \quad \text{(C)}$$

Intrinsic properties P used in (C) were what McAllister called 'aesthetic properties'.[8] In holding that beauty was projected into the object, McAllister did not deny that there occurred a genuine aesthetic response. He simply maintained that this response — whether the object was mathematical, scientific, artistic, or natural — was stimulated by the observer perceiving aesthetic properties and holding aesthetic criteria.

McAllister did not give a detailed account of what counted as an aesthetic property and could thus play the role of P, but he offered two criteria. First, a property was aesthetic if scientists publicly reacted to it as aesthetic, averred that it was aesthetic, or applied to it standard aesthetic terms. McAllister admitted that sometimes people used aesthetic terms in a non-aesthetic way, but thought that a 'principle of charity of interpretation' should be applied: purported expression of aesthetic evaluation should be interpreted as *actually* aesthetic unless evidence indicated otherwise.[1] Second, a property was aesthetic if an object having that property tended to be seen as having a high degree of aptness; McAllister drew here on theories of art in which beauty was bound up with aptness.[2] David Davies (1949–), in a response to McAllister, thought that this point required more careful delineation: 'an "apt" remark is "to the point", but it is strange to think of this as a matter of *aesthetic* merit'.[3] Davies also criticized[4] McAllister's view that metaphysical allegiance was an aesthetic property of scientific theories,[5] but this objection seems not to affect the application of McAllister's model to mathematics.

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[1] Changeux & Connes, *Conversations*, p. 118.

[2] *Ibid.*, pp. 150, 178.

[3] *Ibid.*, p. 118.

[4] McAllister, *Beauty & Revolution*, ch. 7.

[5] *Ibid.*, p. 59.

[6] McAllister, 'Mathematical Beauty'.

[7] McAllister, 'Mathematical Beauty', p. 15; McAllister, *Beauty & Revolution*, p. 34.

[8] McAllister, 'Mathematical Beauty', pp. 15–16; cf. McAllister, *Beauty & Revolution*, p. 34.

Thus, for McAllister, it was incorrect to say that a mathematical entity (which, for McAllister, encompassed processes, algorithms, objects, theorems, proofs, conjectures, diagrams[6]) *had* beauty. The beauty was projected into the entity by an observer as a consequence of their holding certain aesthetic criteria and of their perceiving aesthetic properties such as simplicity or symmetry. It was possible for different mathematicians to hold different aesthetic criteria, and this explained how there could be disagreements on beauty and how opinions on beauty could change over time.[7]

McAllister proposed an explanatory model called *aesthetic induction* to account for the evolution of aesthetic criteria held by mathematicians and scientists generally. According to this model, mathematicians and scientists assigned a weighting to each aesthetic property roughly corresponding to the success of previous proofs and theories that had that property. In science, as new theories displaced old ones, there was a concomitant gradual change in these weightings.[8] In mathematics, the process had to differ slightly, in that old proofs were (in general) not abandoned as incorrect, but reduced in importance through being seen as lacking rigour or being less easy to understand. And new proof *techniques* were introduced, such as computer proofs; such innovations led to proofs of previously inaccessible results. In either case, these developments resulted in a change of weightings assigned to aesthetic properties.

Mathematicians' aesthetic criteria had long been influenced by familiarity with classical proofs. They had thus come to hold brevity and simplicity as aesthetic properties with high weightings. In particular, such proofs could be grasped as a whole.[9] Thus there had been conflicting responses to long proofs and to computer proofs, neither of which could be so grasped.[10] McAllister made a comparison with the gradual acceptance of physical theories that did not offer visualization: rejected initially but accepted later, having demonstrated great empirical success.[11] (McAllister thought that a preference for visualizability was an aesthetic criterion, and suggested that a preference for 'grasping as a whole' was too.[12]) This parallel led him to predict that a change in the aesthetic criteria held by mathematicians would cause computer-assisted proofs to be accepted in future.[1]

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[1] McAllister, *Beauty & Revolution*, pp. 36–7.

[2] *Ibid.*, pp. 37–8.

[3] Davies, 'McAllister's aesthetics', p. 28, emphasis in orig.; cf. Todd, 'Unmasking the truth beneath the beauty', pp. 69–70.

[4] Davies, 'McAllister's aesthetics', pp. 28–9.

[5] McAllister, *Beauty & Revolution*, ch. 3, § 5.

[6] McAllister, 'Mathematical Beauty', p. 17.

[7] *Ibid.*, p. 16.

[8] *Ibid.*, pp. 28–30.

[9] *Ibid.*, p. 22.

[10] *Ibid.*, pp. 23–4.

[11] *Ibid.*, pp. 25–7.

[12] McAllister, *Beauty & Revolution*, pp. 52 sqq.; McAllister, 'Mathematical Beauty', pp. 24–5.

Jonathan Borwein was a proponent of experimental mathematics, in which computation is used to gather 'empirical' evidence about conjectures. He wrote in 2006 that:

> 'mathematical thought patterns also change with time and that these in turn affect aesthetic criteria — not only in terms of what counts as an interesting problem, but also what methods the mathematician can use to approach these problems, as well as how a mathematician judges their solutions. As mathematics becomes more "biological", and more computational, aesthetic criteria will continue to change.'[2]

Borwein's prediction did not precisely coincide with McAllister's predicted acceptance of computer-assisted proofs, but it is of a similar spirit, and was made by an active researcher, not a philosopher.

Whether McAllister's prediction has been borne out seems difficult to judge from the literature. Mathematics has grown familiar with computer proofs.[*] But is their apparent acceptance due, even in part, to a shift in aesthetic criteria? There is scant evidence from which to reach a verdict. The author is aware of only one explicit positive aesthetic evaluation of computer proofs, namely Robert S. Boyer's (1946–) view of the geometric computer proofs that came from the work of Shang-Ching Chou (1946–), Xiao-Shan Gao & Jing-Zhong Zhang (1936–).[†] (And Boyer's judgement was published in 1994, two years *before* McAllister's *Beauty & Revolution in Science*.)

Michael Aschbacher (1944–), in a 2005 discussion of highly complex proofs, such as that of the classification of finite simple groups, suggested that mathematicians would find it necessary to change what they considered to be good proofs.[3] He thought that elegance and simplicity were desirable, but that some fundamental results had only long and complicated proofs. Since mathematicians had to use these results, their 'standards of rigor and beauty must be sufficiently broad and realistic to allow [them] to accept and appreciate such results and their proofs'.[4] He seems here to have made a call for a *deliberate* modification of aesthetic criteria to accommodate complex proofs, in contrast to McAllister's model of an unconscious (or at least non-deliberate) evolution driven by empirical success.

AESTHETIC INDUCTION, REVISITED

Ulianov Montano (1973–) published in 2014 what he declared to be the first attempt 'to develop a rigorous and compre-

[1] McAllister, 'Mathematical Beauty', pp. 27–8.

[2] Borwein, 'Aesthetics for the working mathematician', p. 23.

[*] ▸ pp. 608–15

[†] ▸ pp. 612–14

[3] Aschbacher, 'Highly complex proofs', p. 2404.

[4] *Loc. cit.*

hensive theory of beauty in mathematics' that explained problems such as the experience of mathematical beauty being restricted to those with mathematical training and that encompassed the influence of beauty on the development of mathematics.[1] His book, *Explaining Beauty in Mathematics*, was a development of his 2010 doctoral thesis[2] and incorporated much of an essay he published in 2012 on ugly mathematics and computer-aided proofs.[3][*]

[The account here tries to present Montano's theory fairly, while making only limited use of the technical vocabulary he coined (or perhaps welded), and none of the formalisms he used.[4]]

Montano thought that McAllister's notion of aesthetic induction[†] was a good first approximation to a model explaining evolution of taste in mathematics, but that it had significant flaws. In particular, proofs by division into many cases, which seem to have been generally disparaged, were nevertheless successful — and had been since antiquity.[5] According to Montano, McAllister's aesthetic induction should have led to a higher aesthetic value being assigned to such proofs and so there should have been an aesthetic preference (or at least no aesthetic aversion) to proof by division into cases.[6] Such a development was perhaps implicit in McAllister's own prediction that computer-assisted proof, which presumably includes the automated checking of cases, would come to be seen as having aesthetic value.[7] To put it bluntly, McAllister's aesthetic induction could not account for successful but ugly proof methods.

Montano therefore introduced an enhanced, and much more complex, model of aesthetic induction.[8] One new feature was the addition of a notion of robustness to the aesthetic canon. In short, robustness was resistance to change, in the sense that a property tended to cause the same aesthetic response in spite of changes in the success of mathematics that possessed that property.[9] The degree of aesthetic preference for simplicity, for instance, seemed to be highly robust and did not vary much with the contingencies of the development of mathematics.[10]

The aesthetic-process view

The new model fed into Montano's development of an aesthetics of mathematics. His foundation was the interpretation of beauty not in terms of properties, but as a phenomenon that formed part of an *aesthetic-process*. This term labelled a series of events in which an individual who had an aesthetic experience became engaged.[11] Elements were aspects of 'the aesthetic' because of their relationship to an aesthetic-process; they could not be understood in isolation.[12] An idealized development of an aesthetic-process involved certain phases, which Montano termed *nodes*:[13]

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[1] Montano, *Explaining Beauty in Mathematics*, p. ix.

[2] Montano, 'Beauty in Mathematics'.

[3] Montaño, 'Ugly mathematics'.

[*] ▸ pp. 647–8

[4] See Pollard, Review of *Explaining Beauty in Mathematics*; Riggle, Review of *Explaining Beauty in Mathematics*, p. 418.

[†] ▸ pp. 793–5

[5] Montano, *Explaining Beauty in Mathematics*, pp. 49–50.

[6] *Ibid.*, p. 51.

[7] McAllister, 'Mathematical Beauty', pp. 27–8.

[8] Montano, *Explaining Beauty in Mathematics*, ch. 5, *passim*.

[9] *Ibid.*, pp. 65–6.

[10] *Ibid.*, p. 65.

[11] *Ibid.*, p. 75.

[12] *Ibid.*, p. 78.

[13] *Ibid.*, pp. 79–80.

Aesthetic induction, revisited

Cognitive or sensory input: A trigger started the process; in the context of mathematics, this stimulus would be seeing or hearing or thinking of some piece of mathematics.[1]

Intentional object build-up: The subject focused attention on certain features of the object depending on their own knowledge and attitudes. The subject created an inner representation of an *intentional object*: an object in the particular stance the subject took towards it. The intentional object was dependent on the subject in a way that the raw object was not. For instance, one mathematician might view a proof as establishing a curious but isolated result; another might view it as demonstrating a key result that linked diverse subjects.[2] The intentional object comprised the features that maintained the subject's attention and caused an affective response.[3] Montano held that the aesthetic judgements of mathematics never referred to its perceptual aspects. A judgement of beauty never referred to the written equation or drawn diagram. Therefore the intentional object comprised features found in the mental representation of some piece of mathematics.[4]

Active transformation: The subject 'processed' the intentional object by examining further its structural properties. For example, when aesthetically evaluating a proof, it was necessary to read through the proof in its entirety.[5]

Affective evaluation: When the subject had sufficient grasp of the object, they might feel a positive or negative affective response, governed by an 'aesthetic canon'.[6]

Articulation: As a consequence of the affective response, the subject's internal state was such that they were privately prepared to make a statement such as 'this is beautiful'. Such thoughts made the subject aware they were undergoing an aesthetic-process.[7]

Judgement: Following from the subject's internal state, they might actually utter a statement in the form of a proposition, such as 'this proof is beautiful'. This public act corresponded to the private articulation.[8]

Aesthetic criteria repository: The subject, as a consequence of the aesthetic-process they were involved in, might adjust the preferences that they used to make aesthetic evaluations.[9]

Value dynamics: The value assigned to the object might contribute to aesthetic induction.[10]

Montano's formulation of an intentional mathematical object seems adequate, but is not free from problems. In the specific instance of Euler's equation, which has been frequently cited in discussions of mathematical beauty,* he claimed that its simplicity ('the feature of involving a minimum of operations and no non-relevant constants'[11]) and composition ('the feature of being constituted by relevant items that are incorporated in a non-ad hoc manner'[12]) were relevant to the subject's appreciation, but that its being a special case of Euler's formula was not relevant.[1] Familiarity with complex analysis was necessary to understand the equation,[2] and the active transformation node of the aesthetic-process might involve enough mathematical context to bring into play new properties that were not immediately present in the initial intentional object.[3]

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[1] Montano, *Explaining Beauty in Mathematics*, p. 80.

[2] *Ibid.*, pp. 80–1.

[3] *Ibid.*, p. 87.

[4] *Loc. cit.*

[5] *Ibid.*, p. 81.

[6] *Ibid.*, pp. 81–2.

[7] *Ibid.*, p. 82.

[8] *Loc. cit.*

[9] *Ibid.*, pp. 82–3.

[10] *Loc. cit.*

* ▸ pp. 839–43

[11] *Ibid.*, p. 90.

[12] *Loc. cit.*

But the accounts of mathematicians suggest that it is not clear whether the aesthetic appreciation of Euler's equation can be so cleanly divorced from its mathematical context: one respondent to a survey on the beauty of theorems* linked its beauty to that of the whole *theory* of complex analysis.[4]

As Montano pointed out, the phenomenon of applying knowledge external to the intentional object, and so perceiving new properties, and thus having a different aesthetic experience, also occurred in the fine arts.[5]

Montano classed aesthetic experiences into three types, depending on the aesthetic appreciation response:

Basic: evoked by passive contemplation, where the intentional object was not actively mentally shaped but was immediately classified by a like/dislike response.[6]

Performative: evoked by active carrying-out of an intellectual activity, such as following a proof. The resulting intentional object, which had been actively constructed, was classified by a like/dislike response.[7]

Adaptive appreciation: evoked by acquired preferences, so that the passive or active character of the content was less relevant. The content could evoke an affective pleasure/displeasure response, and the mental activity involved could also evoke an affective pleasure/displeasure response.[8]

Montano argued that theorems, and in particular Euler's equation, required only contemplation and so the responses to them were basic.[9] Proofs and derivations required active attention and so the responses to them were performative.[10] Both could produce adaptive responses shaped by acquired preferences.[11]

Montano presented a short example derivation, showing how to deduce de Moivre's formula $(\cos x + \mathrm{i} \sin x)^n = \cos(nx) + \mathrm{i} \sin(nx)$ from Euler's formula $\mathrm{e}^{\mathrm{i}z} = \cos z + \mathrm{i} \sin z$. He argued that in following the proof, one constructed a (proof *qua*) intentional object that had the property of *step-parsimony*, in the sense of having few steps. This property evoked the affective judgement of elegance.[12]

Ultimately, for Montano, the aesthetic appreciation responses had at their core a pleasure/displeasure response. But this response was to a very particular constructed intentional object, and it was presumably this restriction that distinguished an aesthetic response from a general pleasure/displeasure response. (The philosopher Nick Riggle pointed out that Montano did not make clear the distinction between 'agreeable' and 'beautiful'.[13])

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[1] Montano, *Explaining Beauty in Mathematics*, p. 89.

[2] *Ibid.*, p. 91.

[3] *Ibid.*, pp. 92–6.

* ▸ pp. 834–7

[4] Wells, 'Are these the most beautiful?', p. 38.

[5] Montano, *Explaining Beauty in Mathematics*, pp. 94–5.

[6] *Ibid.*, pp. 98–9.

[7] *Ibid.*, pp. 101–5.

[8] *Ibid.*, pp. 107–9.

[9] *Ibid.*, p. 99.

[10] *Ibid.*, p. 102.

[11] *Ibid.*, pp. 107–9.

[12] *Ibid.*, pp. 102–5.

[13] Riggle, Review of *Explaining Beauty in Mathematics*, p. 419.

The core of Montano's book comprises a very complicated model of aesthetic value and judgements, with intricate terminology and frequent recourse to mathematical symbolism.[1] It attempted to assimilate and resolve all of the complexities Montano had found in the aesthetic-process perspective and the enhanced aesthetic induction he proposed. In line with Montano's desire for rigour, he used extensive formalisms for each aspect of his theory. These formalisms are of lesser importance here than Montano's approach, discussed above, and his conclusions, discussed below.

The aesthetic-process perspective gave an account of how background knowledge played a role in aesthetic experience. For Montano, it also showed that the idea of a mathematical value repository, in the sense of a canon of aesthetic criteria, made sense.[2] These accounts explained the inaccessibility of mathematical beauty to the layperson.[3] Further, mathematical truth was a precondition of mathematical beauty.[4] Mathematical ugliness was not quite as simple as 'the opposite of mathematical beauty', but it was grounded on properties (discerned in intentional objects) that were in some sense opposite.[5]

Case studies

Montano made three case studies examining how his account explained aesthetic experience:[6]

The function $y = \mathrm{e}^x$. In this case study, Montano argued that his theory could account for the beauty of this function, which, he wrote, Le Lionnais thought 'exemplified classical mathematical beauty'.[7] The corresponding intentional object included the background knowledge of analysis and the fact that e^x is invariant under differentiation — a kind of symmetry.[8] An aesthetic criterion that could be formulated as '[*i*]*f symmetry appears in object O, attach more aesthetic value to O*'[9] was at work (and this aesthetic criterion seemed very robust). The equation $y' = y = \mathrm{e}^x$ was thought beautiful in a similar way to a symmetrical visual object.[10]

But there are problems. It appears that Montano misunderstood Le Lionnais. Montano wrote that '[a]ccording to Le Lionnais, the fact that its derivative, y', is the same function, $y' = y$, is what makes it beautiful'.[11] Le Lionnais in fact wrote that the realization that it was unsurprising that the equation $y' = y$ had a solution was an example of how additional knowledge or a new perspective could render a mathematical fact banal.[12]* To the best of the present author's knowledge, no author besides Montano has left a judgement of $y = \mathrm{e}^x$ as beautiful.

Furthermore, $y = \mathrm{e}^x$ is not the unique solution to $y' = y$; the general solution is $y = c\mathrm{e}^x$, where c is an arbitrary constant. In a footnote, Montano resorted to special pleading:

> 'Functions like $\sin x$ and $\cos x$ appear again after two [*sic*[13]] iterations of the derivative. In this sense, $y = \mathrm{e}^x$ is not com-

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[1] Montano, *Explaining Beauty in Mathematics*, chs 7–8.

[2] *Ibid.*, pp. 161–2.

[3] *Ibid.*, p. 162.

[4] *Ibid.*, p. 163.

[5] *Loc. cit.*

[6] *Ibid.*, chs 12–14.

[7] Montano, *Explaining Beauty in Mathematics*, p. 169; cf. Le Lionnais, 'La Beauté en Mathématiques', p. 441.

[8] Montano, *Explaining Beauty in Mathematics*, p. 171.

[9] *Ibid.*, p. 174.

[10] *Ibid.*, pp. 171, 176.

[11] *Ibid.*, p. 169.

[12] Le Lionnais, 'La Beauté en Mathématiques', p. 441.

* ▸ p. 678

[13] Four, since $\mathrm{d}/\mathrm{d}x \sin x = \cos x$ and $\mathrm{d}/\mathrm{d}x \cos x = -\sin x$.

> pletely unique. Furthermore the function $y = ce^x$ has the quality of repeating itself under derivative. However, these kinds of repetition are less appealing than the immediate and more obvious repetition of e^x.'[1]

One can concede the point about $\sin x$ and $\cos x$; but *why* was $y = ce^x$ 'less appealing' than $y = \mathrm{e}^x$? It has precisely the same symmetry (a function invariant under differentiation). At the very least, some other aspect of the mental representation of the function had to be at work besides the 'kinds of repetition'.

Thus there seems to be a lack of evidence for the applicability of Montano's theory to aesthetic judgements of mathematical objects.

Cantor's diagonal argument. In this case study, Montano argued that his theory accounted for the judging of this proof as elegant. There are two ways in which attention was focused on a proof: *step-series* or *single-object*.[2]

In the step-series interpretation, the proof was viewed as a series of steps, each depending on and building on the preceding ones. Cantor's proof, viewed this way, had the properties of *simplicity* (for Montano, this means 'facilitating understanding and reducing the possibility of error') and *step-parsimony* (comprising a small number of steps).[3] This view of the proof was adopted when one read it for the first time, and involved the active engagement of one's attention.[4] Affective responses could be generated by unexpectedness in the proof, and it was the *route*, not the *destination*, that was the locus of unexpectedness.[5]

In the single-object interpretation, the intentional object was built by adding each step. In the case of Cantor's proof, the proof steps remained simple as they were combined. That is, the process was simplicity-conservative. Furthermore, in the finished object — the totality of the proof — the property of step-parsimony became perceptible.[6]

Montano then made what seems to be rather awkward argument about how the broad applicability of diagonalization in other proofs is brought to bear on the intentional object. Since these other proofs were not located in the same phenomenological space, Montano was forced to posit a 'meta-intentional' operation to incorporate this fact.[7] Another meta-intentional operation related to the general applicability of *reductio ad absurdum* proofs.[8] Such operations made perceptible a property that Montano called *method-unification*, which was a property *of the proof under consideration*, not the collection of proofs that use the method, nor the method itself.[9]

Appel–Haken proof of the four-colour theorem. In his last case study, Montano showed that an informal analysis of the perceived ugliness of computer proofs he had made earlier[10] could be incorporated into and explicated by his general theory.[11] (The earlier analysis used a parallel with narrative and so is discussed fully in Chapter 18.[*])

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[1] Montano, *Explaining Beauty in Mathematics*, p. 176, n. 6.

[2] *Ibid.*, p. 182.

[3] *Ibid.*, pp. 182–3.

[4] *Ibid.*, p. 183.

[5] *Ibid.*, pp. 186–7.

[6] *Ibid.*, p. 184.

[7] *Ibid.*, p. 185.

[8] *Loc. cit.*

[9] *Ibid.*, p. 188.

[10] *Ibid.*, ch. 3.

[11] *Ibid.*, p. 197.

In terms of Montano's general theory, the constructed intentional object of a proof with many cases lacked simplicity and unity. A computer proof was even worse, because it prevented construction of a complete intentional object, for the subject's attention could not be actively focused on this part of the proof.[1] Thus the aesthetic experience was not so much negative as *incomplete*.[2]

McAllister had predicted that a change in aesthetic criteria would lead to computer proofs coming to be accepted aesthetically.[†] But Montano's theory produced the opposite prediction: regardless of any change in aesthetic criteria, the aesthetic experience of a computer proof would remain incomplete.

There was nothing in this case study that bore particularly on the Appel–Haken proof.[‡] Precisely the same argument would apply to *any* computer proof large enough to prevent the construction of the intentional object.[3]

Conclusions

The weakness of Montano's study is its highly theoretical bent. In the introduction to *Explaining Beauty in Mathematics*, he warned off readers looking for examples of beauty in mathematics, assuring them of an abundant literature of that kind.[4] There is nothing wrong with a theoretical approach *per se*, but Montano's stands on shaky foundations because it rarely engaged at all with mathematicians or mathematics.[5]

Montano examined the aesthetic attitudes of mathematicians through the lens of a small number of authors,[6] such as Hardy,[§] Le Lionnais,[¶] and Paul Nahin (1940–), the author of some well-regarded popular mathematics books. In many places, Montano wrote of what 'mathematicians' think, without citing evidence. To take a single example: his discussion of Cantor's diagonal argument does not contain a single reference to a description of the proof as 'elegant'.

Viktor Blåsjö (1982–), a historian and philosopher of mathematics, detected in Montano's description of his own work as 'rigorous'[7] a device for excluding much of the literature written by mathematicians on the experience of mathematical beauty,[8] and observed that all Montano's efforts seemed to lead back to what mathematicians (including the present author) have said with rather less fuss.[9]

Montano's theory also has a paucity of examples showing how it explained aesthetic judgements. As mentioned above, he illustrated parts of his theory using Euler's equation and the derivation of de Moivre's formula, and presented three case studies; and that was the end of his direct mathematical engagement.

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[*] ▸ pp. 647–8

[1] Montano, *Explaining Beauty in Mathematics*, pp. 198–9.

[2] *Ibid.*, p. 199.

[†] ▸ pp. 794–5

[‡] ▸ pp. 602–8

[3] Cf. Pimm & Sinclair, Review of *Explaining Beauty in Mathematics*, p. 80.

[4] Montano, *Explaining Beauty in Mathematics*, p. x.

[5] See also Riggle, Review of *Explaining Beauty in Mathematics*, pp. 419–20.

[6] Pimm & Sinclair, Review of *Explaining Beauty in Mathematics*, p. 79.

[§] ▸ pp. 498–523

[¶] ▸ pp. 672–9

[7] Montano, *Explaining Beauty in Mathematics*, pp. ix–x, 73, 211.

[8] Blåsjö, 'Mathematicians Versus Philosophers', pp. 424–5.

[9] *Ibid.*, p. 425.

On such weak evidence it would be hasty to conclude Montano's theory can explain the beauty or elegance of theorems or of other proofs. Euler's equation is a very special result in that it is very concise; theorem statements can be much longer and more complex. Montano gave no argument to show that his theory explained the aesthetic judgements made of such results. Cantor's diagonal argument is a very unified proof that relies on one key idea, and it is a *reductio ad absurdum* proof. Montano did not demonstrate that his theory explained the aesthetic value assigned to longer proofs or direct proofs or proofs by induction.

BEAUTY AND SYMMETRY

John H. Conway (1937–2020), who was active in many areas of mathematics, including number theory and combinatorial game theory, was perhaps most explicit about beauty when discussing his work in group theory. The first of his many noteworthy contributions to this field was the construction of the group of symmetries of the Leech lattice. The Leech lattice is a highly-symmetric infinite lattice of points in 24-dimensional euclidean space, and has the property that the unit sphere centred on any point of the lattice is touched by 196 560 other spheres.[1] Conway showed that the group of symmetries of the Leech lattice has order $2^{22} \cdot 3^9 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 23 = 8\,315\,553\,613\,086\,720\,000$. It is not a simple group, but factoring out its two-element centre yields a simple group, now known as the first Conway group Co_1, which is one of the sporadic finite simple groups. Conway's investigations also yielded two other new sporadic groups, now known as the second and third Conway groups Co_2 and Co_3.[2] [Daniel Gorenstein (1923–92), who perhaps had a better overview than anyone else of the classification of finite simple groups, predicted that Conway's work on these sporadic groups would remain 'one of the most elegant achievements of mathematics'.[3]]

Conway thought of the Leech lattice as a beautiful object precisely because of its symmetry:

> 'The Leech lattice is fantastically symmetrical. I think of these things as Christmas tree ornaments, with [...] gorgeous symmetry appearing whenever you look. You look at them from one point of view and you see one kind of symmetry. Then it rotates and you are looking along some other axis and you see another kind of symmetry. And to my eye it is tremendously beautiful.'[4]

For Conway, the very essence of the search for simple groups was the desire to find and understand such symmetries.[5] He saw it as part of

[1] Leech, 'Notes on Sphere Packings', § 2.31.

[2] Thompson, *From Error-Correcting Codes Through Sphere Packings to Simple Groups*, pp. 119 sqq.

[3] Gorenstein, 'The Classification of Finite Simple Groups I', p. 106.

[4] Conway, quot. in Roberts, *Genius At Play*, ch. 6.

[5] *Ibid.*, ch. 12.

a quest for symmetry that could be traced back to Plato's admiration for the five regular solids.[1][*]

The symmetries form a property of a mathematical object such as the Leech lattice, and so the object is beautiful. But, for Conway, the group — the abstract collection of symmetries of the object — could itself be beautiful. He stated in 1980 that, for him, 'simple groups are beautiful things'.[2] The Monster group — the largest of the sporadic finite simple groups, with order

$$\begin{aligned}&2^{46}\cdot 3^{20}\cdot 5^{9}\cdot 7^{6}\cdot 11^{2}\cdot 13^{3}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71\\ &\quad = 808\,017\,424\,794\,512\,875\,886\,459\,904\,961\,710\,757\,005\\ &\qquad\quad 754\,368\,000\,000\,000\\ &\quad \approx 8.08\times 10^{53}\end{aligned}$$

— had aesthetic value deriving from exceptional symmetry:

> 'As far as the aesthetics of symmetry are concerned, the Monster is a very nice looking thing. It's a symmetrical object the likes of which you have never seen before, and will never see again.'[3]

And he hoped more would be found, though by 1980 the classification of finite simple groups seemed to be approaching its completion.[4]

The mathematical physicist Peter Goddard (1945–) recalled that he once suggested to Conway that there could be some class of structures whose symmetries were described precisely by the sporadic groups, which would give a unified way to understand them. In Goddard's recollection, Conway found the idea repugnant, since it would destroy the individuality of the objects. Conway denied having responded thus, and indeed said that he had once hoped that all sporadic groups would be unified as subgroups of one of their number.[5] He wrote that at various points during the discovery of the sporadic groups, a pattern seemed to emerge, such as with Co_1 and the Fischer group Fi'_{24}, both of which have as subquotients (that is, quotients of subgroups) several other sporadic groups. But then 'the discovery of some beautiful new group'[6] would destroy the pattern. To be precise, when Bernd Fischer (1936–2020) discovered Fi'_{24}, all the known or predicted sporadic groups except for the Janko groups J_1 and J_3 were subquotients of Co_1 and the Fischer group Fi'_{24}. During the 1970s, other sporadic groups were found that were subquotients of neither. In the end, all but six (or seven, if the Tits group is counted as sporadic) of the sporadic groups are subquotients of the Monster group.

Conway's aesthetic admiration of groups was always based on their symmetries, which were regrettably abstract. Calculation was the price one had to pay to perceive the symmetry, and thus the beauty, of mathematics objects. It was 'unfortunate' that the beauty could not be appreciated without climbing the 'scaffolding' of calculation.[7] Neither drawing, nor sculpture, nor even visual imagination could depict 'beautifully symmetric things' in higher dimensions; hence he 'reluctantly' turned to numbers.[8]

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[1] Roberts, *Genius At Play*, ch. 12.

[*] ▸ pp. 67–70

[2] Conway, 'Monsters and Moonshine', p. 165.

[3] Conway, quot. in Roberts, *Genius At Play*, ch. 12.

[4] Conway, 'Monsters and Moonshine', p. 165.

[5] Roberts, *Genius At Play*, ch. 15.

[6] Conway, 'Monsters and Moonshine', p. 165.

[7] Conway, quot. in Roberts, *Genius At Play*, ch. 6.

[8] Conway, quot. in *ibid.*, ch. 12.

BEAUTY AND EXPLANATORINESS, REVISITED

Marc Lange

The philosopher of science Marc Lange (1963–) suggested that explanatory proofs 'are beautiful (or, at least, not ugly)'.[1] Lange did not argue that the term *beauty* actually named explanatory power; rather, he thought that beauty and explanatory power had precisely the same cause: 'their exploiting precisely the feature in the setup that is salient in the theorem'.[2]

Lange pointed out that a proof by division into cases, unlike a unified proof, did not explain the theorem.[3] This lack of explanation was connected to the notion of *coincidence* in mathematics.[4] Lange initially characterized non-coincidental results in terms of their proofs as follows:

> 'Suppose we take that single component of the non-coincidence and make each step of the proof as logically weak as it can afford to be while still allowing the proof to explain that component. Then the weakened proof remains able to explain each of the non-coincidence's other components as well'.[5]

Although Lange refined this characterization, the connection with case enumeration is clear: a case-by-case treatment did not satisfy this criterion; fundamentally, such a proof could not demonstrate that the result was not a coincidence and was thus not explanatory. Thus, Lange's link between beauty and explanatory power was in agreement with the view maintained by Hardy and others that case-by-case proofs are dull.*

Marcus Giaquinto

In a 2016 essay, Marcus Giaquinto rejected the view that beauty was a mind-independent property of an object, but still thought that judgements of beauty could be correct or incorrect: it was common to admit that another person was a better judge, or to hope to become a better judge oneself.[6] He proposed a necessary (but not sufficient) condition for the *correctness* of judgements of beauty:

> 'A judgement of beauty about something of a particular genre is correct only if that thing has a propensity to give significant disinterested repeatable pleasure to connoisseurs of the genre from mentally apprehending it.'[7]

The key terms are *disinterested*, *repeatable*, and *connoisseurs*. The disinterestedness of aesthetic pleasure excluded experiences such as quenching a thirst or yielding some non-aesthetic value, or even the imagination of such experiences.[8] The pleasure had to be repeatable, in the sense that, at least after a sufficient interval, contemplation

[1] Lange, 'Explanatory Proofs and Beautiful Proofs', p. 45.

[2] *Loc. cit.*

[3] Lange, 'Depth and explanation in mathematics', p. 197.

[4] Lange, 'What Are Mathematical Coincidences?', pp. 337–8.

[5] *Ibid.*, p. 321.

* ▸ p. 502

[6] Giaquinto, 'Mathematical Proofs', p. 53.

[7] *Ibid.*, p. 55.

[8] *Ibid.*, p. 54.

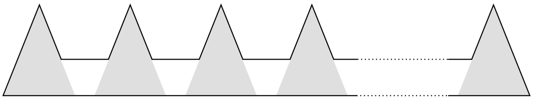
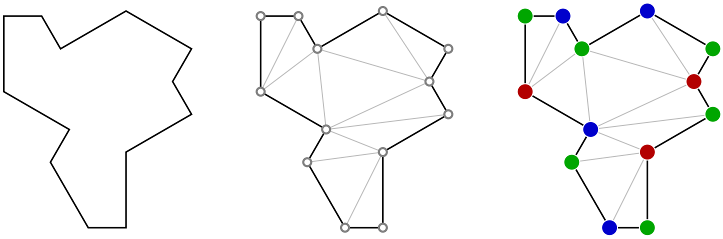


FIGURE 21.8
The worst case of the art gallery theorem: n sides may necessitate $\lfloor n/3 \rfloor$ guards.



The polygonal plan of a gallery.

A triangulation of the polygon.

A 3-colouring of the triangulation.

FIGURE 21.9
Proving the art gallery theorem for a gallery whose plan is the shown 13-gon. In this case, $3 \leqslant \lfloor 13/3 \rfloor$ guards suffice, placed at the vertices coloured red.

or experience of the same object had to grant the same pleasure; in contrast, the humorous pleasure produced by a joke declined with repetition.[1] Only certain individuals might enjoy a particular genre, and within the group of people who enjoyed a particular genre were the connoisseurs: those who have accumulated education and experience within that genre, and, consequently, an especially developed taste for instances of that genre. Within certain fields, there seemed to be a convergence of aesthetic judgments made by connoisseurs.[2]

He illustrated what he thought of as a beautiful proof by using the result that has become known informally — but almost universally — as the 'art gallery theorem'. Victor Klee (1925–2007) asked for the smallest number $f(n)$ such that any polygon P with n sides can be dominated by a set D of $f(n)$ points, in the sense that for any point p in P there is a straight line segment, lying entirely within P, from that point p to some point in the set D. In less technical and more picturesque terms, $f(n)$ is the minimum number of guards required to supervise any art gallery with n walls.[3]

Václav Chvátal (1946–) proved in 1975 that $f(n) = \lfloor n/3 \rfloor$.[4] The diagram in Figure 21.8 shows that $f(n) \geqslant \lfloor n/3 \rfloor$; the challenging part of the proof was to establish the opposite equality. Chvátal's proof was fairly short and elementary. It involves triangulating the polygon (see Figure 21.9 (left, middle)), which is possible by a straightforward induction, then proceeding to find a set of dominating points by induction. If $n \leqslant 5$ vertices, the result holds because all interior edges of the triangulation radiate from a single point. Otherwise, there is a way of distinguishing a particular edge of polygon to divide off a small part of the graph, then dividing into four cases to apply the induction hypothesis. The proof occupies slightly more than one page of the journal.

Steve Fisk (1946–2010) gave a proof whose original very terse presentation occupies less than six complete lines in the same journal.[5] In brief, this proof also starts by triangulating the polygon. One can show that the resulting graph admits a 3-colouring by the following

[1] Giaquinto, 'Mathematical Proofs', p. 54.

[2] *Ibid.*, p. 55.

[3] Chvátal, 'A Combinatorial Theorem in Plane Geometry', p. 39.

[4] *Ibid.*

[5] Fisk, 'A Short Proof of Chvátal's Watchman Theorem'.

induction: place two colours at the vertices connected by an internal diagonal; this diagonal splits the graph into two smaller graphs that can be 3-coloured by an extension of the colouring on these two vertices. In such a colouring, there is one colour that is assigned to at most $\lfloor n/3 \rfloor$ vertices (see Figure 21.9). Since each triangle contains a vertex with this colour, and since triangles are convex, these points dominate the whole polygon.[1]

Martin Aigner (1942–2023) & Günter Ziegler (1963–) included Fisk's proof in *Proofs from THE BOOK*,[*] calling it ‘truly beautiful’.[2] Giaquinto also thought that it was beautiful, and that anyone would judge it more beautiful than Chvátal's proof, which he said was ‘not bad at all’.[3]

For Giaquinto, certain properties of the proof contributed to the aesthetic judgement. One was clarity: the structure and key ideas of the proof were readily identifiable,[4] and there was neither division into cases nor ‘use of unintuitive tricks or technical handles’.[5]. The latter expression, which Giaquinto attributed to Manya Raman-Sundström, clearly denoted something closely related to the concept of *deus ex machina* in proof.[†] Division into cases had long been held to detract from beauty. The suggestion that it detracted specifically from clarity, which contributed to beauty, was a refinement due to Giaquinto. The presence of case division, thought Giaquinto, could explain in part the lesser aesthetic appeal of Chvátal's proof in relation to Fisk's.[6]

Brevity had also been identified as a property of beautiful proofs, but Giaquinto pointed out that the use of more advanced concepts and techniques could shorten a proof at the price of rendering it less accessible. An example of increased brevity and decreased clarity, he thought, was Don Zagier's (1951–) ‘one-sentence proof’ of Fermat's two-square theorem.[7] [This proof was a perhaps curious choice of example, because it does not use advanced techniques that require time and effort to master, but rather takes the form of a summary that must be unpacked into a longer, but still elementary, proof.[8] A better example might be Dedekind's proof of the same result, which is only a few lines long but relies on Gaussian integers and norms.[‡]] Thus, Giaquinto maintained, ‘a combination of brevity and simplicity of methods’ could contribute to beauty,[9] or at least a better balance of the two was aesthetically superior.[10] There is arguably an echo here of Hardy pointing to *economy* as a quality of beautiful proofs.[§]

Finally, Giaquinto suggested that being *imaginative* contributed to beauty. In the case of Fisk's proof, the idea of proving a *geometrical* result via the *combinatorial* concept of colouring was non-obvious.[11] *Imaginative* here was perhaps more restrictive than Hardy's *unexpected*, referring specifically to the proof's overall approach rather than its particular course.

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[1] Fisk, ‘A Short Proof of Chvátal's Watchman Theorem’; Aigner & Ziegler, *Proofs from THE BOOK* (6th edn), pp. 282–3.

[*] ▸ pp. 843–5

[2] Aigner & Ziegler, *Proofs from THE BOOK* (6th edn), p. 282.

[3] Giaquinto, ‘Mathematical Proofs’, p. 61.

[4] *Loc. cit.*

[5] *Ibid.*, p. 62.

[†] ▸ pp. 637–48

[6] *Loc. cit.*

[7] Giaquinto, ‘Mathematical Proofs’, p. 62; Zagier, ‘A one-sentence proof’, see also.

[8] See, for example, Aigner & Ziegler, *Proofs from THE BOOK* (1st edn), pp. 19–20.

[‡] ▸ pp. 788–9

[9] Giaquinto, ‘Mathematical Proofs’, p. 62.

[10] *Ibid.*, p. 63.

[§] ▸ pp. 502

[11] *Loc. cit.*

Beauty and explanatoriness, revisited

In examining a possible connection between beauty and explanatoriness in proofs, Giaquinto distinguished two problems:

1) whether explanatory proofs *tend to be* beautiful, and
2) whether beautiful proofs *tend to be* explanatory.

Giaquinto intended for the emphasized words to indicate neither anything so strong as implication, nor anything so weak as *mere* tendency; rather, they should exclude the possibility of a large class of proofs having the former attribute but not the latter.[1] He thought that a positive answer to problem 1 might be expected, because an explanatory proof granted understanding and thus gave pleasure through satisfying a 'major intellectual desire'.[2] 'Moreover, it is disinterested pleasure, independent of appetites or instrumental goals.'[3] Thus it was reasonable to think that the explanatoriness of the proof contributed to its aesthetic value.[4]

But Giaquinto thought that explanatoriness, even if it contributed to aesthetic value, did not guarantee beauty, which implied *great* aesthetic value. For instance, there is an explanatory proof of Pascal's rule for binomial coefficients:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

The explanatory proof is based upon distinguishing an element and interpreting the first summand as the number of ways of choosing k elements that include that distinguished element and the second summand as the number of ways of choosing k elements that do not include that distinguished element. It seems never to have been judged beautiful.[5] For Giaquinto, the plenitude of explanatory proofs that are 'too facile' to be beautiful suggested that that explanatory proofs do not tend to be beautiful.[6]

In Giaquinto's argument, there was here an implied distinction between 'an appetite' and 'an intellectual desire', which Giaquinto did not explore. Without this distinction, the pleasure in understanding would not be disinterested and so would not fit the characterization of aesthetic pleasure that Giaquinto adopted. Furthermore, his remark about 'too facile' suggested that depth, or at least difficulty, whether in the proof or in the result, was a prerequisite for beauty.

Regarding problem 2, Giaquinto considered a short elementary proof given by Paul Erdős (1913–96) of Leonhard Euler's (1707–83) theorem *The sum of the reciprocals of the prime numbers* $\sum_p 1/p$ *diverges.*[7] Erdős's proof proceeds via *reductio ad absurdum*, showing that from the supposition that the series converges follows an estimated count of the numbers less than an arbitrary N that have and do not have certain prime divisors, which leads to a contradiction for sufficiently large N.[8]

Giaquinto pointed out that in *Proofs from THE BOOK*, Aigner & Ziegler said the proof was 'of compelling beauty',[9] and that in his own

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[1] Giaquinto, 'Mathematical Proofs', p. 66.

[2] *Loc. cit.*

[3] *Loc. cit.*

[4] *Ibid.*, p. 67.

[5] *Loc. cit.*

[6] *Ibid.*, pp. 67–8.

[7] Euler, 'Variae observationes circa series infinitas', Thm 19.

[8] Erdös, 'Über die Reihe $\sum 1/p$', § 1; see also Aigner & Ziegler, *Proofs from THE BOOK* (6th edn), pp. 5–6.

[9] Aigner & Ziegler, *Proofs from THE BOOK* (6th edn), p. 5.

terms it had great aesthetic value grounded upon its clarity, brevity and simplicity, and imaginativeness. But it was *not* explanatory.[1]

Whether Erdős's proof was unrepresentative was a question that Giaquinto thought he could not answer. If there was a tendency to *see* beautiful proofs as explanatory, it might be based upon a bias towards exhibiting elementary proofs as exemplars of beauty. Thus a positive answer to problem 2 would be unreasonable without examining a collection of beautiful proofs that had no such bias.[2]

In his discussion of explanatory and non-explanatory proofs, Giaquinto maintained that

> 'it is *only* by thinking semantically, [...] rather than proceeding by rules of symbol-manipulation, that one could come to grasp why the conclusion is true.'[3]

If one accepts this assertion, then finding a large proportion of purely syntactic beautiful proofs would suggest that problem 2 has a negative answer.

Beauty and explanatoriness, revisited

Catarina Dutilh Novaes

In an essay published in 2019, the philosopher Catarina Dutilh Novaes (1976–) offered an account of beauty and ugliness in proofs which suggested that there was a way to reconcile I) the view that mathematicians' use of aesthetic terms should be taken literally, and II) the reductive view that such use of aesthetic vocabulary indicated a judgement of non-aesthetic — typically epistemic — properties.[4]

Dutilh Novaes supported a dialogical view of deduction, based upon its historical development.[5] In her and other such dialogical models, a Prover aims to convince a Sceptic of their argument. At each stage, the Sceptic may ask why a particular step holds, in response to which the Prover must offer clarification; or the Sceptic may give a local counterexample, in response to which the Prover must withdraw and take the deduction in a new direction.[6]. But deductive arguments such as proofs, at least as presented in text, usually — *pace* Imre Lakatos's (1922–74) *Proofs and Refutations*[*] — did not take a dialogical form. Dutilh Novaes argued that proofs in mathematical papers or books are a 'worked-out' form: they have already been perfected, at least to the degree required by a Sceptic internalized by the author or authors of the paper or book,[7] as in the description written by the philosopher Paul Ernest (1944–), which she quoted:

> 'Mathematical proof is a special form of text, which since the time of the ancient Greeks, has been presented in monological form. [...] But as it is an argument intended to convince, a listener is presupposed. The monologicality of proof tries to forestall the listener by anticipating all of his or her possible objections. So the dialectical response is condensed into the

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[1] Giaquinto, 'Mathematical Proofs', p. 70.

[2] *Loc. cit.*

[3] *Ibid.*, p. 65, emphasis added.

[4] Dutilh Novaes, 'The Beauty (?) of Mathematical Proofs', pp. 64–5.

[5] Dutilh Novaes, *The Dialogical Roots of Deduction*, p. 37.

[6] *Ibid.*, ch. 3.

[*] ▸ pp. 683–5

[7] *Ibid.*, p. 72.

> ideal perfection of a monologic argument, in which no sign of speaker or listener remain, except for the idealized and perfected utterance, the proof itself.'[1]

Based in part upon this dialogical perspective on proof, Dutilh Novaes thought that there was an analogy between mathematics and poetry, since both involved creativity within formal constraints: mathematics was restricted by logic, poetry by metre. This analogy, she suggested, with the parallel pleasurable responses to constrained creative novelty, could serve to illuminate the aesthetic aspect of proof.[2] In particular, the role of unexpectedness in proof seemed to mirror its role in genres such as classical music and poetry;[3] here Dutilh Novaes quoted the musical analogy by Gowers mentioned above, which focused in particular on this parallel.[*]

The notion of unexpectedness is central to Dutilh Novaes's argument. She annexed to the Hardian triad of unexpectness, inevitability, and economy[†] the additional properties of seriousness,[‡] generality, and depth, collectively terming them 'Hardy's Six', but cautioning that she made no claim about whether these six properties exhausted all aspects of the beauty of proofs.[4] Hardy had considered all six properties in the *Apology*, but it is unclear whether he considered the latter three to be *aesthetic* properties. (McAllister[§] also interpreted Hardy as having considered all six properties to be aesthetic.[5]) Dutilh Novaes treated *economy* as synonymous with *simplicity*, at least for the purposes of her argument.[6] She illustrated the six properties by analysing the degree to which they could be perceived in Cantor's diagonal argument:[7][¶]

Seriousness: The proof was serious: it concerned important mathematical ideas such as sets and infinity and cardinality.

Generality: The technique of diagonalization found application in a wide range of mathematics, such as the insolubility of the halting problem for Turing machines. (One might argue that the generality lies rather in the *proof method*, not the *proof*; but the proof strongly suggests the generalization to a proof that $|S| < |\mathbb{P}S|$, and certainly *exemplifies* the more general proof method.)

Depth: The concepts involved in the proof were foundational. (Certainly it serves to establish different 'levels of infinity'.)

Unexpectedness: There was a 'plot twist' when one found that the number produced by the diagonal argument was not part of the original enumeration. Dutilh Novaes thought that this realization often led to an 'aha!' experience, and thus the proof could be rated highly in terms of this property.[8]

Economy: The proof was concise and simple and used only the basic idea of the binary or other n-ary representation of real numbers.

Inevitability: This was the one property of 'Hardy's Six' that Dutilh Novaes judged was *prima facie* absent from the proof. She related that teaching had shown her that students found the proof to be

[1] Ernest, 'The Dialogical Nature of Mathematics', p. 38.

[2] Dutilh Novaes, 'The Beauty (?) of Mathematical Proofs', pp. 77–8.

[3] *Ibid.*, p. 78.

[*] ▸ pp. 782–3

[†] ▸ pp. 502

[‡] ▸ pp. 499–501

[4] *Ibid.*, pp. 66, 69.

[§] ▸ pp. 793–5

[5] McAllister, 'Mathematical Beauty', p. 17.

[6] Dutilh Novaes, 'The Beauty (?) of Mathematical Proofs', pp. 69, 71, 72.

[7] *Ibid.*, pp. 70–1.

[¶] ▸ p. 848

[8] *Loc. cit.*

> less than convincing.[1] [In 1998, Wilfrid Hodges (1941–) published a survey of attempts to refute the proof, which suggests that, at least among non-professionals, the proof could be difficult to grasp.[2]]

Dutilh Novaes thought that, like many other *reductio ad absurdum* proofs, Cantor's diagonal argument was not particularly explanatory; thus its beauty was not grounded on explanatoriness.[3] She cited a 2015 study by Matthew Inglis & Andrew Aberdein of mathematicians' use of aesthetic terms, which used statistical analysis to identify four different evaluative dimensions along which were distributed adjectives used to describe proofs.* She pointed out[4] that the results showed that *enlightening* was correlated strongly with beautiful ($r_s = .57$) but less strongly with *explanatory* and *simple* (respectively, $r_s = .2$ and $r_s = .08$).[5]

But was there any connection between the six properties and explanatoriness? Dutilh Novaes thought that there were connections to two different views of explanatory proofs. One was Steiner's criterion of 'characterizing properties'.† This account of explanatoriness was *local*: it concerned an individual proof of an individual theorem and the objects with which that theorem was concerned.[6] In contrast, Philip Kitcher's (1947–) was *global*: an explanatory proof, or more generally an explanatory derivation in science, was one of a set of derivations that best unified the set of statements accepted as true by the community.[7] Thus an explanatory proof served as a component that helped unify the theorems of mathematics. In both senses, a explanatory proof did more than merely demonstrate the truth of the theorem; it indicated *why* it is true.[8]

Steiner's local account and Kitcher's global view of explanation both bore upon some of the six properties, though from opposite directions. A proof that was explanatory in Steiner's sense would have generality via variations in the characterizing property; inevitability via the evident dependence on that property; and simplicity via the pivotal role of that property.[9] A proof that was deep or serious would usually link different ideas and concepts, perhaps fundamental, and so fulfilled at least to an extent the unifying role that was required to be explanatory in Kitcher's sense.[10] Thus five of the six properties were connected to explanatoriness via one of the two accounts.

The odd man out was *unexpectedness*. For Dutilh Novaes, in an explanatory proof, the way ahead and the journey thus far should be clear: 'such a proof is epistemically transparent throughout, and so there should be no trace of unexpectedness in it'.[11]

The fluency account of Topolinski & Reber suggested to Dutilh Novaes that if a proof that featured unexpectedness produced an 'aha!' experience, then there were two intertwined consequences: a positive affective response and a feeling of rightness.[12] If this suggestion was accepted — and Dutilh Novaes cautioned that she was not entirely convinced by the Topolinski–Reber account[13] — then in the particular

[1] Dutilh Novaes, 'The Beauty (?) of Mathematical Proofs', pp. 70–1.

[2] Hodges, 'An Editor Recalls Some Hopeless Papers'.

[3] Dutilh Novaes, 'The Beauty (?) of Mathematical Proofs', p. 71.

* ▸ pp. 858–9

[4] *Ibid.*, p. 68.

[5] Inglis & Aberdein, 'Beauty is not simplicity', Tbl. 4.

† ▸ pp. 778–9

[6] Dutilh Novaes, 'The Beauty (?) of Mathematical Proofs', p. 72.

[7] Kitcher, 'Explanatory Unification and the Causal Structure of the World', esp. § 4.

[8] Dutilh Novaes, 'The Beauty (?) of Mathematical Proofs', p. 71.

[9] *Ibid.*, p. 72.

[10] *Ibid.*, pp. 72–3.

[11] *Ibid.*, p. 73.

[12] *Ibid.*, p. 80.

[13] *Ibid.*, p. 81.

case of Cantor's proof, it seemed to be the unexpected 'plot twist' (of the number produced by the diagonal argument not being in the enumeration) that had for some people an accompanying 'aha!', and the plot twist and the 'aha!' together caused the proof to be judged beautiful. Those who did not have an 'aha!' experience seemed to find the proof neither beautiful nor quite convincing.[1]

These conclusions suggested that the aesthetic and epistemic judgements of the proof were bound.[2] Dutilh Novaes also cited Todd's account of aesthetic-epistemic feelings[*] to support the position that there was not a sharp division between aesthetic and epistemic judgements of mathematics.[3]

Starting from the idea that aesthetic and epistemic judgements of proofs are linked but not identical, Dutilh Novaes made five predictions that should be empirically testable in studies like that of Inglis & Aberdein:[†]

1) 'Beauty and explanatoriness are used as comparative, graded notions.' Unlike the other predictions, Dutilh Novaes did not offer an explanation of how this predication arose from the account she offered.[4] It seems uncontroversial: one proof may be more beautiful than another; one may be more explanatory than another. But comparisons of proofs in both respects deserve attention.

2) 'Judgments of beauty and explanatoriness correlate, but do not coincide.' This prediction emerged from the connection of five of the six aesthetic properties to both the local and global accounts of explanatoriness put forward by Steiner and by Kitcher. As Dutilh Novaes noted, the Inglis–Aberdein study found only a weak correlation of *beautiful* with *explanatory* but found a stronger correlation with *enlightening*; she thought that explanatoriness was closely related to enlightenment.[5] In fact, in the Inglis–Aberdein study, the term *enlightening* loaded onto the aesthetic dimension of evaluation; *explanatory* loaded on to *none* of the four identifiable dimensions.[6]

3) 'Judgments of beauty and unexpectedness/creativity strongly correlate.' This correlation was suggested by the idea that beauty in proofs was connected to creativity within constraints, as in classical music or poetry. It was supported by some of the adjectives that load onto the aesthetic dimension in the Inglis–Aberdein study and which strongly correlated with *beautiful*, such as *ingenious*, *striking*, *creative*, and *innovative*.[7]

4) 'Judgments of beauty and persuasiveness strongly correlate.' This correlation was predicted by aesthetic pleasure strengthening a feeling of rightness.[8]

5) 'Mathematicians disagree in their aesthetic judgments about proofs.' Both explanatoriness and beauty seemed to be subjective judgements that depend on the audience. However, Dutilh

[1] Dutilh Novaes, 'The Beauty (?) of Mathematical Proofs', p. 81.

[2] *Loc. cit.*

[*] ▸ pp. 823–4

[3] *Loc. cit.*

[†] ▸ pp. 858–9

[4] *Ibid.*, p. 82.

[5] *Ibid.*, pp. 82–3.

[6] Inglis & Aberdein, 'Beauty is not simplicity', Tbl. 3.

[7] Dutilh Novaes, 'The Beauty (?) of Mathematical Proofs', p. 83.

[8] *Loc. cit.*

Novaes suggested that there may be unanimity on the ugliness of certain proofs.[1]

BEAUTY IN PROOF BY INDUCTION

A graph is *n-list-colourable* if, for any assignment of lists of n colours to each vertex, it is possible to choose a colouring of the graph where the colour of each vertex is drawn from its assigned list. This property is more restrictive than being n-colourable: there is a planar graph with 63 vertices that is not 4-list-colourable;[2] famously, all planar graphs are 4-colourable.

THEOREM C. *Every planar graph is 5-list-colourable.* $\boxed{\text{C}}$

Theorem C was conjectured by Paul Erdős, Arthur L. Rubin (1956–) & Herbert Taylor in 1979.[3] In 1994, Carsten Thomassen (1948–) published the first proof.[4] Thomassen in fact proved the following stronger result on *near-triangulations*, which are planar graphs in which every bounded face is a triangle:

THEOREM T. *Let G be a near-triangulation. Let the colour list assigned to each vertex on the outer cycle of G have size at least* 3, *and let the colour list assigned to each vertex in the interior of G have size at least* 5. *Then any colouring of two adjacent vertices on the outer cycle can be extended to a colouring of the whole of G.*

Since any planar graph can be turned into a near-triangulation by adding edges, which can only restrict the choice of colouring, Theorem C is a corollary of Theorem T.

Proof of T. Proceed by induction on the number of vertices G. If G has 3 vertices, then it is a 3-cycle and the result is immediate: for any colouring of two of the vertices, there is a third colour available for the remaining vertex.

So suppose that G has at least 4 vertices, and that the result holds for all strictly smaller near-triangulations. Let the outer cycle of G be $v_1 v_2 \cdots v_p v_1$, where v_1 and v_2 are the vertices of the outer cycle of G whose colouring is to be extended to the whole of G. There are two cases, which are illustrated in Figure 21.10:

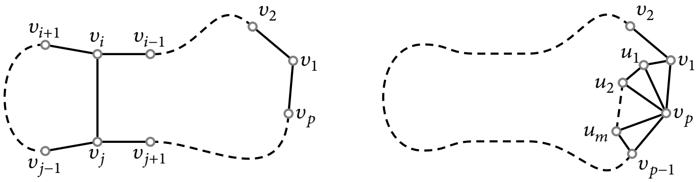
- Suppose G contains at least one chord. Assume without loss of generality that there is a chord between the vertices v_i and v_j, where $i \leqslant j - 2$ (see Figure 21.10 (left)). Reversing the ordering of the labels on the outer cycle of G if necessary, assume that $i \geqslant 2$. By the induction hypothesis, the colouring of v_1 and v_2 can be extended to the subgraph of G bounded by the cycle $v_1 v_2 \cdots v_i v_j \cdots v_p v_1$. In particular, this extension determines the colours assigned to v_i and v_j. Again by the induction hypothesis, this colouring of v_i

[1] Dutilh Novaes, 'The Beauty (?) of Mathematical Proofs', p. 83.

[2] Mirzakhani, 'A small non-4-choosable planar graph'.

[3] Erdős, Rubin & Taylor, 'Choosability in Graphs', p. 153.

[4] Thomassen, 'Every Planar Graph Is 5-Choosable'.



The graph has a chord along which it can be split to apply the induction hypothesis.

The graph has no chord, so a single vertex can be removed to apply the induction hypothesis.

FIGURE 21.10
The two cases in Thomassen's proof that all planar graphs are 5-list-colourable.

and v_j can be extended to a colouring of the subgraph bounded by the cycle $v_i v_{i+1} \cdots v_j v_i$. Together these subgraph colourings form a colouring of the whole of G.

- Suppose G contains no chord. Let $v_1, u_1, u_2, \ldots, u_m, v_{p-1}$ be the neighbours of v_p. Since G has no chord, the vertices $u_1, u_2, \ldots, u_m$ are in the interior. Since G is a near-triangulation, $v_1 v_2 \cdots v_{p-1} u_m \cdots u_2 u_1 v_1$ is a cycle; let G' be the subgraph bounded by this cycle (see Figure 21.10 (right)). Let c_1 and c_2 be two colours from the list assigned to v_p that are different from the colour assigned to v_1. Remove both colours from the lists assigned to each u_i; at least three colours remain in each list. By the induction hypothesis, using these modified lists, the colouring of v_1 and v_2 can extended to the whole of G' so that none of $u_1, u_2, \ldots, u_m$ has colour c_1 or c_2. Since neither of these is the colour assigned to v_1, and at least one of them is different from the colour assigned to v_{p-1} and so can be used to extend the colouring of G' to a colouring of G. $\boxed{\text{T}}$

Thomassen's proof was given by Lars-Daniel Öhman as an example of a beautiful proof by induction, since he thought such proofs were rarely used as examples of beauty.[1] Öhman said that the proof caused him 'a very strong positive emotion', and that since this reaction was both immediate and disinterested, he categorized it as an aesthetic response.[2] Other mathematicians he consulted agreed that the proof was aesthetically pleasing,[3] and it appeared in every edition of *Proofs from THE BOOK*.[4]

Although Theorem C is *prima facie* an easier result, it seems to be difficult to use in an inductive proof.[5] The more complicated statement of Theorem T turns out to be precisely what is required for the proof by induction. The initial colouring of two adjacent vertices allows the induction hypothesis to be used when the graph has a chord; the near-triangulation allows it when there is no chord. For Öhman, this exact fit of properties and proof indicated an 'artistic mastery'.[6] The proof also has the merit of giving an effective and efficient procedure to find a colouring, and this feature contributed to Öhman's perception of beauty.[7] He saw both the theorem and proof as lacking depth, since they did not connect ideas from different areas of mathematics or give any general illumination to a field. But he thought that this lack did not detract from their aesthetic value.[8]

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[1] Öhman, 'A Beautiful Proof by Induction', p. 73.

[2] *Ibid.*, p. 79.

[3] *Ibid.*, p. 81.

[4] Aigner & Ziegler, *Proofs from THE BOOK* (6th edn), pp. 278–9.

[5] Öhman, 'A Beautiful Proof by Induction', p. 77.

[6] *Ibid.*, p. 80.

[7] *Ibid.*, pp. 77, 82.

[8] *Ibid.*, p. 83.

Öhman saw 'an element of pleasant surprise' in the proof: its simplicity seemed surprising given the complexity of the examples that informed the original conjecture, and the vast number of cases required for the computer-aided proof of the four-colour theorem.[1]* The surprise was confined to the *simplicity* of the proof. The result itself was largely unsurprising, and Öhman seems not to have seen surprise in the course of the proof; in any case, he thought that surprise did not contribute to his aesthetic experience and so he found unconvincing Hardy's position that unexpectedness was a quality found in beautiful mathematics.[2]

ESSENTIAL SURPRISE

Writing in the same special journal issue as Öhman, Visala Rani Satyam took an opposing position on unexpectedness: she argued that surprise was an essential part of mathematical beauty. While simplicity and brevity in a piece of mathematics might be valuable, they alone did not make it beautiful. Rather, it was *unexpected* simplicity that gave rise to beauty;[3] Further, such surprise was a part of the experience of enlightenment, which was a 'feature' or 'quality' of mathematical beauty.[4]

Long equations and long calculations were thought of as ugly.[5] For Satyam, so were results that were isolated and did not serve any use in other mathematics.[6] An example of such a remote result was[7] the theorem *Every natural number greater than* 77 *can be expressed as a sum of distinct natural numbers whose reciprocals sum to* 1. For example,

$$78 = 2+6+8+10+12+40 \quad \text{and} \quad 1 = \frac{1}{2}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}+\frac{1}{40}.$$

The result is sharp, in the sense that 77 is the largest number that cannot be so expressed. [This result received a mean rating of 4.7 (on a scale of 0 to 10) from the respondents to David Wells's (1940–) questionnaire on beautiful theorems.†]

Ronald Graham's (1935–2020) original proof of this result applies only special and restricted techniques, and involves extensive calculation that is trivial for a computer but tedious and unenlightening for the mathematician. The proof starts from a table showing that all even integers from 78 to 168 and all odd integers from 79 to 333 can be expressed in this way, then uses two identities to show that from this basis the table can be extended arbitrarily far.[8]

Satyam seems to have adopted a novel position here. Hardy, for instance, would have disagreed, not about the ugliness of the result and proof (which he did not clearly distinguish[9]), but about the reasons for judging it so. The proof effectively begins with a division into 174

[1] Öhman, 'A Beautiful Proof by Induction', p. 79.

* ▸ pp. 602–8

[2] *Ibid.*, p. 81.

[3] Satyam, 'The Importance of Surprise in Mathematical Beauty', p. 197.

[4] *Ibid.*, pp. 197, 200.

[5] *Ibid.*, p. 198.

[6] *Loc. cit.*

[7] *Loc. cit.*

† ▸ pp. 834–7

[8] Graham, 'A Theorem on Partitions', pp. 435–7.

[9] Hardy, *Apology*, § 18.

cases, showing that all the numbers in the table are expressible in the required way, and two special identities crafted for the task at hand and nothing else. It certainly lacks the Hardian criteria for beauty of inevitability and economy, and arguably also unexpectedness.[*] For Hardy, the lack of connection to other parts of mathematics would be an indication that the result lacked *seriousness*.[†] But for Satyam, the absence of such connections, the fact that such results did not 'shed much light onto surrounding mathematics', seemed rather to render them ugly rather than non-serious.[1] The failure of a result to illuminate its mathematical context seems to reflect back and diminish the result itself.

Satyam observed that the responses to David Wells's (1940–) questionnaire on the beauty of theorems[‡] showed that there was disagreement over whether Euler's equation[§] was beautiful, despite it being so oft held up as an exemplar of mathematical beauty. François Le Lionnais (1901–84), whom Wells quoted, thought that it had lost its original brilliance because subsequent mathematical development had clarified it so thoroughly that it had become obvious.[¶] Satyam drew the (perhaps hasty) conclusion that this progress towards obviousness was a general reason for some respondents finding little beauty in it.[2] This kind of fading beauty, she thought, indicated the importance of surprise for beauty and how the passage of time could leave a result stripped of its unexpectedness and thus shorn of its beauty.[3] But she made the important point that, while a dull sense of obviousness could prompt a negative judgement, an *unexpected* obviousness — or, perhaps better, an unexpected *simplicity* — could evoke a 'sense of fit and order' and a positive aesthetic response.[4]

She quoted Rota's discussion of enlightenment as a 'light bulb' moment, and argued that such instantaneous transitions to understanding were relatively rare compared to a gradual process of coming to understand.[5] There was, at minimum, the preparatory work of assimilating the background that was necessary for understanding. More often, one had to make repeated attempts to solve a problem, leaving it after a failure and later revisiting it afresh. But whether enlightenment came about incrementally or suddenly, surprise was involved: a sequence of minor surprises as gradual enlightenment was gained, or a major surprise when enlightenment was achieved at a stroke.[6]

An example of immediate enlightenment was produced by the visual proof, derived from pythagorean pebble arithmetic, that a sum $1 + 3 + \ldots + (2k - 1)$ is a square number.[♠] For Satyam, this visual proof provided a 'geometric intuition' that was more 'powerful' than an algebraic proof, for it granted immediately enlightenment.[7]

In what seems to have been an instance of more gradual enlightenment, Satyam pointed to William Thurston's (1946–2012) sphere eversion as an example of mathematical beauty.[8] Informally, a sphere eversion is a way of turning a sphere inside-out in three-dimensional

[*] ▸ pp. 502

[†] ▸ pp. 499–501

[1] Satyam, 'The Importance of Surprise in Mathematical Beauty', p. 198.

[‡] ▸ pp. 834–7

[§] ▸ pp. 839–43

[¶] ▸ pp. 840–1

[2] *Ibid.*, p. 202.

[3] *Loc. cit.*

[4] *Loc. cit.*

[5] *Ibid.*, p. 204.

[6] *Loc. cit.*

[♠] ▸ pp. 46–7

[7] *Loc. cit.*

[8] *Loc. cit.*

euclidean space, allowing the surface of the sphere to pass through itself, but without cutting the surface or creating sharp creases; more formally, it is a regular homotopy between an immersion of S^2 in $\mathbb{R}^3$ and the negation of that immersion. Steven Smale (1930–) proved a highly abstract and technical theorem that implies that all immersions of S^2 in $\mathbb{R}^3$ are regularly homotopic.[1] This result was highly counterintuitive and elicited scepticism: notably, there is no such eversion of the circle in the plane.[2] While it was in principle possible to extract an explicit sphere eversion from Smale's existence proof, it would be made up of innumerable steps and would give no insight. A visualizable eversion was only found several years later by Arnold Shapiro (1921–62), and only became widely known in a 1980 presentation by George K. Francis (1939–) & Bernard Morin (1931–2018).[3]

Thurston devised a new sphere eversion, using a technique he called *corrugations*, which was later explained in the celebrated 1994 animation 'Outside In'.[4] Thurston's motive was his own intellectual satisfaction, because no previous method had been visualizable or had granted him insight.[5] In Satyam's terms, there was an aesthetic motive within this desire for enlightenment. Satyam explained why she herself found Thurston's eversion beautiful:

> 'it is a complete sense of surprise that triggers curiosity, a "How did they do that?" reaction. Even though it is not clear how or why this method works, the unexpectedness drives a person to want to know why, to seek further enlightenment.'[6]

Note that the surprise was not in the *existence* of the eversion: that was established by Smale. The surprise was elicited rather by the mystery of the particular eversion.

BEAUTY AND FIT

Linguists have traced the etymology of the word *art* back through the Latin *ars* to the reconstructed Proto-Indo-European *h_2*r-tí-*, meaning 'the fitting'.[7] It is therefore perhaps natural that connections between fit and beauty have formed a recurring theme in philosophy. According to Xenophon (c. 430–355/354 BCE), Socrates (c. 469–399 BCE) argued that a thing was beautiful insofar as it was fitted for a particular end. In one of his examples, a basket for carrying dung could be beautiful and a golden shield ugly, if the former was well-suited to its purpose and the latter was not.[8] When modern aesthetics took root in the eighteenth century, several writers examined this idea.[9] David Hume (1711–76) thought that beauty is seen whenever a thing is fitted for its purpose: 'where any object, in all its parts, is fitted to attain any agreeable end, it naturally gives us

[1] Smale, 'A Classification of Immersions of the Two-Sphere', Thms A, B.

[2] Francis, *A Topological Picturebook*, p. 104.

[3] Levy, *Making Waves*, p. 31; Francis & Morin, 'Arnold Shapiro's Eversion of the Sphere'.

[4] Levy, Maxwell & Munzner, 'Outside In'.

[5] Thurston, afterword to Levy, *Making Waves*, p. 40.

[6] Satyam, 'The Importance of Surprise in Mathematical Beauty', p. 205.

[7] Vaan, *Etym. Dict. Latin*, 'ars, artis' [p. 55].

[8] Xenophon, *Memorabilia*, § 1.8.4–7.

[9] Guyer, *Hist. Modern Aesthetics*, vol. 1, pp. 134–5, 143–5, 154–5, 162–3.

pleasure, and is esteem'd beautiful'.[1] For Hume, any made thing was thus beautiful if it was fitted for its purpose:

> 'This observation extends to tables, chairs, scritoires, chimneys, coaches, sadles, ploughs, and indeed to every work of art; it being an universal rule, that their beauty is chiefly deriv'd from their utility, and from their fitness for that purpose, to which they are destin'd.'[2]

Further, a thing was beautiful in the degree to which it had fitness to be used for its intended end.[3]

The artist William Hogarth (1697–1764) wrote a book on *The Analysis of Beauty*, whose introduction ends with a list of the principles that governed beauty:

> 'FITNESS, VARIETY, UNIFORMITY, SIMPLICITY, INTRICACY and QUANTITY; — *all which co-operate in the production of beauty, mutually correcting and restraining each other occasionally*.'[4]

Fitness, listed first, involved the suitability of the parts for the whole design, or of the performance of the whole in its purpose. The dimensions of a ship had to be suitable for sailing; if the ship sailed well, the sailors called it a beauty.[5]

Given the five other principles that Hogarth listed, it seems clear that he did not consider fitness alone to be sufficient for beauty. But he seems to have thought it a necessary but not sufficient condition for beauty: in discussing how parts are combined, he implied that that a judgement of beauty was contingent upon a basic fitness criterion being met.[6] Thus, although Edmund Burke (1729–97) offered numerous examples from nature of things that were ugly but fitting,[7] his counterexamples failed to refute Hogarth's argument.[8] Alexander Gerard (1728–95), on the other hand, made a much stronger association between fitness and beauty than Hogarth: 'UTILITY, or the *fitness* of things for answering their ends, constitutes another species of beauty, distinct from that of figure'.[9]

Fit has been taken to mean, or at least is closely connected to, suitability for a purpose. The fit of a thing seems to be related to how *natural* a thing seems in itself and in its situation. The number theorist Bryan John Birch (1931–) made a remark that bore upon the second point: 'mathematics is at its most beautiful when its truths are made to seem natural as well as certain'.[10] His statement pointed to a possible connection between fit and beauty in mathematics.

Scholarly exploration of this connection seems not to have begun until the twenty-first century, when Manya Raman-Sundström and Lars-Daniel Öhman studied the notion of *fit* as a mathematical value, and its potential connection to beauty, in three papers that appeared in 2013, 2016, and 2018.

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[1] Hume, *A Treatise of Human Nature*, § 3.3.1 para. 20.

[2] *Ibid.*, § 2.2.5, para. 17.

[3] *Ibid.*, § 3.3.1, para. 8.

[4] Hogarth, *The Analysis of Beauty*, Introduction, emphases in orig.

[5] *Ibid.*, ch. I.

[6] Guyer, *Hist. Modern Aesthetics*, vol. 1, pp. 144–5; see Hogarth, *The Analysis of Beauty*, ch. VII.

[7] Burke, *Philosophical Enquiry*, pt III, § VI ⟦pp. 95–6⟧.

[8] Guyer, *Hist. Modern Aesthetics*, vol. 1 pp. 154–5.

[9] Gerard, *An Essay on Taste*, p. 37; note markers removed.

[10] Birch, quot. in Cook, *Mathematicians*, p. 28.

Types of and criteria for fit

Raman-Sundström & Öhman distinguished three different types of fit that a proof may have, as well as certain aspects of or criteria for these types of fit:[1]

Direct fit. This quality concerned the proof and the theorem it established. It involved *coherence* and *specificity*. A proof was coherent if it used only the concepts that were invoked by the theorem. (Coherency of proof was clearly related to, and perhaps is barely distinguishable from, *purity* of proof.) A proof was specific if the tools it used were of an appropriate level of technical power. A brute-force approach would not be specific; perhaps a high-powered proof that used sophisticated techniques to prove a simple result would not be either.

Presentational fit. This concept referred to the communication of the proof to the reader in a text. Criteria for this property were *level of detail* and *transparency*. A proof text might contain too many details, covering calculations or parallel cases that were unnecessary for the reader; conversely, it might contain too few details, leaving the reader to fill in steps that they found difficult. A proof text with the correct level of detail gave enough support for rigour, but not so much as to obscure the key ideas. A proof text was transparent when its structure was natural and there was no *deus ex machina*.*

Familial fit. This property concerned the relationship between the particular proof and a class of proofs. It could involve *generality* and *connectedness*. A proof had generality if its fundamental ideas generalized to prove a larger class of theorems. It was connected if it was related to other proofs by shared ideas or techniques.

In a later single-author paper, Raman-Sunström distinguished two other forms of fit:[2]

Intrinsic fit. This property concerned the relationship between a proof and the ideas upon which it was based. A judgement of intrinsic fit required no knowledge except of the theorem and proof; it did not require familiarity with other proofs using similar techniques.[3]

Extrinsic fit. This quality concerned the relationship between a proof and a class of proofs to which it belonged. An extrinsically fitting proof made clear the place of the proof within this class.

Prima facie, intrinsic fit seems very closely related to the notion of *direct* fit that Raman-Sundström & Öhman had defined, but without the criteria they had used to describe it; similarly, extrinsic fit seems related to familial fit. However, there was a more complicated relationship between the direct/presentational/familial and intrinsic/extrinsic notions.

To illustrate their types of fit, Raman-Sundström and Öhman compared proofs of: I) the irrationality of $\sqrt{2}$; II) the pythagorean the-

[1] Raman-Sundström & Öhman, 'Mathematical Fit', pp. 185–7.

* ▸ pp. 637–48

[2] Raman-Sundström, 'The Notion of Fit', p. 274.

[3] *Ibid.*, p. 281.

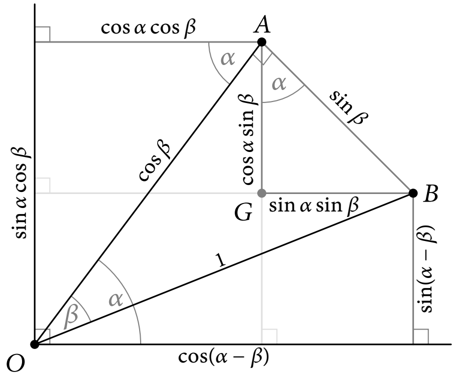

FIGURE 21.11
Derivation of sine and cosine subtraction formulae. The line AB has length 1; all the other lengths follow by direct application of trigonometry. The horizontal and vertical distances to point B give the cosine and sine formulae.

orem; III) the complex conjugate root theorem; IV) Pick's theorem.[1] This summary considers only the proofs of the pythagorean theorem and Pick's theorem, which are present in both Raman-Sundström & Öhman's paper and Raman-Sundström's single-author paper.

The first proof of the pythagorean theorem in Raman-Sundström & Öhman comparison was the similarity proof discussed on page 767. The second proof, by Jason Zimba (1969–), deduces the pythagorean theorem by establishing the trigonometric identity $\cos^2\vartheta + \sin^2\vartheta = 1$ (without recourse to the theorem itself, from which it follows as an easy consequence). First, the sine and cosine functions are defined in the range $(0, \pi/2)$ by dividing the side lengths of a right-angled triangle. Then the sine and cosine subtraction identities

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta, \tag{C}$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{S}$$

are derived geometrically (see Figure 21.11). Now let $\vartheta \in (0, \pi/2)$ be arbitrary and choose $\varphi \in (0, \vartheta)$, so that $\vartheta, \varphi, \vartheta - \varphi \in (0, \pi/2)$. Then

$$\begin{aligned}\cos\varphi &= \cos(\vartheta - (\vartheta - \varphi))\\ &= \cos\vartheta\cos(\vartheta-\varphi) + \sin\vartheta\sin(\vartheta-\varphi)\\ &= \cos\vartheta(\cos\vartheta\cos\varphi + \sin\vartheta\sin\varphi)\\ &\qquad + \sin\vartheta(\sin\vartheta\cos\varphi - \cos\vartheta\sin\varphi)\\ &= (\cos^2\vartheta + \sin^2\vartheta)\cos\varphi,\end{aligned}$$

from which the desired identity follows.[2]

Raman-Sundström & Öhman thought that the similarity proof had direct, presentational, and familial fit. It was coherent: it uses areas, which are the subject of the theorem. It was also specific, in the sense that the division of the triangle was exactly what is required to prove the theorem. It thus had direct fit. The proof had an appropriate level of detail, similar to the theorem itself, and was also transparent, using two key ideas that were combined in a natural way; it therefore had presentational fit. Finally, what of familial fit? Certainly the proof was general in the sense that it would apply to a theorem about any similar figures erected on the sides of a right-angled triangle. Connectedness was less clear, but Raman-Sundström & Öhman suggested that it satisfied this criterion because it sat in a class of other proofs from ancient Greek mathematics that use area preservation.[1]

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[1] Raman-Sundström & Öhman, 'Mathematical Fit', pp. 189–205; Raman-Sundström, 'The Notion of Fit', pp. 274–81.

[2] Zimba, 'On the possibility of trigonometric proofs of the Pythagorean theorem'.

For Raman-Sundström, the similarity proof intrinsically fitted the theorem. It directed attention to the fundamental idea of what the theorem is about: the relationship between similar figures on the three sides of a right-angled triangle. That the area of the largest is the sum of the areas of the smaller was made obvious by the dissection of the whole triangle into the two similar sub-triangles.[2]

Raman-Sundström & Öhman thought that the second proof of the pythagorean theorem, via trigonometric identities, lacked familial fit, and possessed direct and presentational fit only in a lesser degree than the first proof. It was not coherent: its approach was largely alien to the ideas in the theorem. It was specific, for the subtraction formulae were required to prove the theorem. It thus possessed less direct fit than the first theorem. Like the first proof, the level of detail was appropriate. It was, however, not transparent, because the direction of travel was not natural. Hence it had less presentational fit than the first theorem. It lacked generality, and had no connectedness, having been devised for the specific purpose of showing that trigonometry could be used to prove the theorem.[3] Thus the proof lacked familial fit.[4]

The second proof, thought Raman-Sundström, lacked intrinsic fit. The trigonometric manipulations establish the theorem without giving any inkling of its key ideas. In particular, starting with $\cos\varphi$ seemed like an unmotivated trick.[5] [It is, perhaps, a *deus ex machina*.*] Note, however, that the reasons for the lack of intrinsic fit paralleled the reasons for the lack of presentational fit; thus intrinsic fit did not simply coincide with direct fit.

To illustrate extrinsic fit, Raman-Sundström compared two proofs of Pick's theorem on lattice polygons (that is, a polygon whose vertices are points of the integer lattice), which says that *The area of a lattice polygon with* n_i *lattice points in its interior and* n_b *on its boundary is* $A = n_b/2 + n_i$. (See Figure 21.12 (left).) Both proofs involve taking an elementary triangulation of the polygon (that is, a triangulation where each triangle is a lattice polygon containing exactly three lattice points; see Figure 21.12 (right)). Each elementary triangle has area $\frac{1}{2}$.

The first proof proceeds by summing the angles of all the elementary triangles in two different ways. If there are N triangles, their angles sum to πN. The sum of angles at interior points is $2\pi n_i$; the sum of angles at boundary points is $\pi n_b - 2\pi$. Hence $\pi N = 2\pi n_i + \pi n_b - 2\pi$, and so $A = N/2 = n_i + n_b/2 - 1$.[6]

The second proof views the triangulation as a graph. Suppose there are e_b boundary edges and e_i internal edges; note that $e_b = n_b$. Then the graph has $v = n_i + n_b$ vertices, $e = e_i + e_b$ edges, and divides the plane into $f - 1$ bounded regions and one unbounded region.

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[1] Raman-Sundström & Öhman, 'Mathematical Fit', pp. 194–5.

[2] Raman-Sundström, 'The Notion of Fit', p. 275.

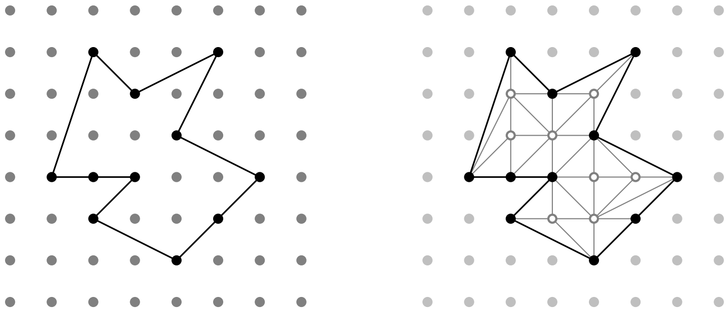
[3] Zimba, 'On the possibility of trigonometric proofs of the Pythagorean theorem', p. 275.

[4] Raman-Sundström & Öhman, 'Mathematical Fit', p. 196.

[5] Raman-Sundström, 'The Notion of Fit', pp. 276–7.

* ▸ pp. 637–48

[6] *Ibid*., pp. 278–9.


A lattice polygon

An elementary triangulation

FIGURE 21.12
Illustration of Pick's theorem and its proof using an elementary triangulation. The lattice polygon has $n_b = 11$ boundary lattice points, $n_i = 8$ interior lattice points, and area $n_b/2 + n_i - 1 = 12\frac{1}{2}$.

Counting the two sides of each edge in two different ways shows that $3(f-1) = e_b + 2e_i$. Then

$$\begin{aligned} A &= (f-1)/2 \\ &= \big(2(e-f) - e_b + 3 - 1\big)/2 \\ &= \big(2(v-2) - n_b + 2\big)/2 && \text{[by Euler's graph formula]} \\ &= (2n_i + n_b - 2)/2 \\ &= n_i + n_b/2 - 1. \end{aligned}$$

Raman-Sundström & Öhman considered the first proof to be non-coherent, because angle measurements are not involved in the theorem; to be specific; to have an appropriate level of detail, because the tools used were just enough to prove the result; to be non-transparent, in that there was no natural reason to consider angle sums of a triangulation; to be non-general, since angle sums are not preserved under transformations that do preserve the theorem; and to be weakly connected to other proofs that involve double-counting. It had limited direct and presentational fit, and very limited familial fit.[1]

Raman-Sundström thought that the first proof fitted extrinsically into the family of proofs that involve double-counting. The fact that the summed *angles* are double-counted was not important for this extrinsic fit. That angles are used was arguably an aesthetic defect, because the angles were not a key part of the theorem or the polygon; Raman-Sundström seemed to suggest, without quite stating it, that this proof was not intrinsically fitting.[2] Note the opposition between the evaluations of extrinsic fit and familial fit; the two types of fit thus have very little overlap.

For Raman-Sundström & Öhman, the second proof was also non-coherent, for it involves concepts from graph theory that are absent from the theorem. It had only limited specificity, because Euler's formula for planar graphs ($v - e + f = 2$) is a tool with wide applicability. It had an appropriate level of detail, but little transparency: treating a triangulation as a graph was not natural to the proof, though it would perhaps be natural for a reader who was familiar with graph theory. It was general, applying to the theorem for other lattices, and connected to many other theorems via Euler's formula. It thus had familial fit, limited direct fit, and very little presentational fit.[1]

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[1] Raman-Sundström & Öhman, 'Mathematical Fit', p. 202.

[2] Raman-Sundström, 'The Notion of Fit', pp. 279–80.

The second proof, said Raman-Sunström, fitted extrinsically into the class of proofs that apply Euler's formula for planar graphs. She held that here the double-counting of edges was just a means to the end of applying Euler's formula.[2]

Connection with beauty

Raman-Sundström thought that the first proof of the pythagorean theorem, using the triangle dissection and similarity, was beautiful, while the second, using trigonometry, although technically impressive, was not. She surveyed seminar participants on this question; the majority agreed with her assessment. The two participants who preferred the second proof did so for non-aesthetic reasons: they had only just become acquainted with the first proof and had not fully grasped it.[3]

For Raman-Sundström, both proofs of Pick's theorem were beautiful, and seminar participants agreed. But the second proof, using Euler's formula for planar graphs, was a little less beautiful than the first, because of the greater degree of algebraic manipulation required.[4] The second proof, however, did count among Aigner & Ziegler's *Proofs from THE BOOK*.[5]

What connection was there, if any, between fit and beauty in mathematics? Relational fit included the criterion of level of detail, which was related to brevity, which was often seen as a feature of beauty.

If a proof was presentationally or intrinsically fitting, then it was transparent or made the key idea of the theorem or the idea more accessible. Thus this kind of fit might evoke beauty because the proof was more easily grasped. Raman-Sundström & Öhman pointed out that this possibility was compatible with Rota's connection of mathematical beauty and enlightenment.[6][*]

Extrinsic fit was somehow connected with seeing the 'big picture', the interrelationship of different ideas. Raman-Sundström cited an unnamed mathematician who suggested that an individual proof could not be beautiful; it was the fit between things that created mathematical beauty.[7] In this case, extrinsic fit became essential for beauty. Similarly, familial fit and the criterion of connecteness might be related to the perception of unexpected relationships as well as to mental grouping of concepts that led to the kind of processing fluency that has been identified as a factor in judgements of beauty.[8][†]

[1] Raman-Sundström & Öhman, 'Mathematical Fit', p. 204.

[2] Raman-Sundström, 'The Notion of Fit', pp. 280–1.

[3] Raman-Sundström, 'The Notion of Fit', p. 277; see also Raman & Öhman, 'Beauty as fit', p. 200.

[4] Raman-Sundström, 'The Notion of Fit', p. 281.

[5] Aigner & Ziegler, *Proofs from THE BOOK* (6th edn), pp. 93–4.

[6] Raman-Sundström, 'The Notion of Fit', pp. 274, 282; Raman-Sundström & Öhman, 'Mathematical Fit', p. 207.

[*] ▸ pp. 742–5

[7] Raman-Sundström, 'The Notion of Fit', p. 282.

[8] Raman-Sundström & Öhman, 'Mathematical Fit', p. 207; see also Reber, Schwarz & Winkielman, 'Processing Fluency'.

[†] ▸ pp. 785–7

Feeling of Fit

In a 2018 paper, Cain S. Todd (1976–) considered the nature of the experience that mathematicians describe as aesthetic. It seems that his position had moderated since 2008, when he argued that aesthetic properties found in mathematics and science are 'masked' epistemic properties.* In 2018, he accepted that the demarcation of the aesthetic and the epistemic was a perhaps unresolvable question, and set it aside in favour of looking at the psychology of mathematicians' aesthetic judgements and how they related to epistemic judgements. He argued that aesthetic feelings in mathematics were rather 'aesthetic-epistemic' and were grounded in a 'feeling of fittingness'.[1]

Todd thought that aesthetic statements made by mathematicians could be grouped into three categories. *Appreciative* statements expressed bare judgements that a piece of mathematics was beautiful, elegant, ugly, and so on. *Criterial* statements were of two kinds: those that grounded a comparative judgement of alternatives in aesthetic judgement (for instance, holding a proof to be superior on the basis of its elegance); and those that grounded an aesthetic judgement in some *prima facie* non-aesthetic judgement (for instance, holding a proof to be elegant on the basis of its brevity and simplicity). *Discovery* statements were when the aesthetic value of the mathematical formulation of a theory was taken to lend weight to its truth.[2]

There were differences in the roles of criterial statements in mathematics and in art. In mathematics, criterial statements did not seem to be used to justify negative aesthetic judgements. There also seemed to be much less disagreement about aesthetic judgements in mathematics, and they applied only in a narrow range. Todd suspected that in mathematics the putative 'aesthetic' judgements might actually have an epistemic function,[3] especially since the criteria cited to *support* aesthetic judgements in mathematics seemed to have epistemic functions.[4]

In mathematics, it seemed to be difficult to demarcate a conceptual boundary between aesthetic and epistemic experience: what seemed to happen was that the same kind of experience was thought of as aesthetic or as epistemic, depending on the objects or activities that gave rise to it. Responses of both kinds could be positive or negative; they were immediate and the reasons for them were not easily accessed or articulated. The same properties, such as symmetry, simplicity, order, and clarity, were characteristic of both aesthetic and epistemic experiences.[5] Experiences that were initially reported as aesthetic might, on reflection, or on persistent questioning, be conceded to have been epistemic, and subsequently to have been aesthetic-epistemic.[6] Todd tentatively suggested that the initial classification of an experience as epistemic or aesthetic depended upon the domain to which its object belonged. Mathematics was curious because it seemed to have an intellectual/cognitive domain and yet many of the experiences were characterized as aesthetic.[1]

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* ▸ pp. 746–9

[1] Todd, 'Fitting Feelings and Elegant Proofs', p. 212.

[2] *Ibid.*, p. 214.

[3] *Ibid.*, p. 215.

[4] *Ibid.*, p. 216.

[5] *Ibid.*, p. 227.

[6] *Ibid.*, pp. 227, 229.

Todd pointed out that the study by Inglis & Aberdein found that there were some adjectives that scored highly on both the aesthetic and utility dimensions: *enlightening*, *fruitful*, *insightful*.[2][*]

One could draw the conclusion that aesthetic judgments in mathematics were in fact masked epistemic judgements, and that an aesthetic judgement could in fact be reduced to a statement about epistemic qualities.[3] At minimum, one could be sceptical about such judgements being unequivocally aesthetic, even while accepting that mathematicians have genuine affective responses to mathematics.[4]

Such a sceptical position would seem to be undermined by Inglis & Aberdein study's showing no correlation between judgements of a proof as beautiful and as simple, and by Semir Zeki (1940–) et al.'s study showing the same neural correlates of mathematical and other kinds of beauty.[5][†] But the same neural activity was also seen under other conditions, including pleasure and reward. Perhaps beauty of any kind was associated to pleasure. But Zeki et al.'s work at least suggested that there could be overlap between aesthetic experience and certain kinds of cognitive or epistemic experience.[6]

Todd suggested that epistemic feelings were affective responses to both the functioning of one's cognitive processes and the object of those processes. They were based upon the processes — for instance, their fluency[‡] or efficiency — and what those processes were directed towards — for instance, a theorem or proof.[7] But, as a matter of experience, the focus of epistemic feelings was the object, not the mental state, and thus aesthetic properties were assigned to the object.[8]

One element in aesthetic-epistemic feelings could be processing fluency,[9] but in mathematics there was a pleasure in understanding that seemed to be connected more to harmony and fit, such as in the 'aha!' experience when something was suddenly — and perhaps surprisingly — understood.[10] Todd gave the mutilated chessboard problem[§] as an example, and thought that the understanding it granted of *why* the statement was true was a 'genuine hybrid epistemic-aesthetic feeling'[11] that could not be reduced to simple fluency.

Todd hypothesized that all aesthetic-epistemic feelings were variations of a 'feeling of fittingness', which arose from a scrutinization and assessment of the relationship between certain cognitive processes and the objects to which those processes are directed. If this relationship had certain features, including coherence and simplicity and fluency, there was produced an aesthetic-epistemic feeling of fittingness.[12] (Todd emphasized that this hypothesis asserted neither that all aesthetic feelings were epistemic nor vice versa.[13]) The way this feeling was manifested, and whether the experience evoked pleasure, depended on the object, how that object was perceived, and the psychology of the subject.[14]

[1] Todd, 'Fitting Feelings and Elegant Proofs', pp. 229–30.

[2] Todd, 'Fitting Feelings and Elegant Proofs', p. 217; Inglis & Aberdein, 'Beauty is not simplicity', p. 3, Tbl. 3.

[*] ▸ pp. 858–9

[3] Todd, 'Fitting Feelings and Elegant Proofs', p. 216.

[4] *Ibid.*, pp. 216–17.

[5] *Ibid.*, pp. 217–18.

[†] ▸ pp. 760

[6] *Ibid.*, pp. 218–19.

[‡] ▸ pp. 785–7

[7] *Ibid.*, p. 222.

[8] *Ibid.*, pp. 222–3.

[9] *Ibid.*, p. 227.

[10] *Ibid.*, p. 228.

[§] ▸ pp. 774–5

[11] *Ibid.*, p. 229.

[12] *Loc. cit.*

[13] *Loc. cit.*

[14] *Ibid.*, p. 230.

VISUAL AND MATHEMATICAL BEAUTY

In a paper published in 2018, the philosopher of mathematics Irina Starikova noted the importance of visual representations in mathematical thought and communication,[1] and examined the question of whether different visual representations of the same mathematical object could vary in beauty, in the sense of one portrayal representing the object in a more beautiful way than another.[2]

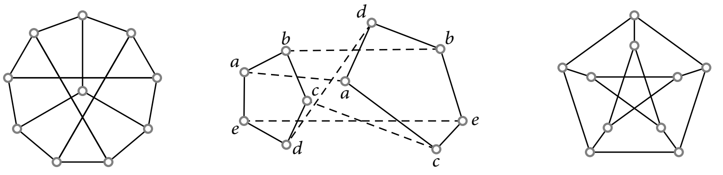
She proceeded from the assumptions that mathematical objects could be beautiful or ugly, and that they could be judged as having different degrees of beauty or ugliness.[3] She explicitly rejected Nick Zangwill's (1957–) position that there were no genuinely aesthetic judgements of mathematics.[4][*] Rather, the use of aesthetic terms indicated a mental response similar to that produced by music or landscapes.[5]

In this way, mathematical objects resembled sensible objects, and therefore an aesthetic judgement of a visual representation of a mathematical object might be a response either to the visible properties of the representation or to the abstract properties of the mathematical object.[6]

Starikova chose as an exemplar the visual diagrams representing graphs, with a particular focus on the Petersen graph, which was frequently described in aesthetic terms.[7] Such diagrams could be symmetric, and each diagrammatic symmetry is a symmetry — an automorphism — of the graph (*qua* mathematical object).[8] But in general only *some* of the automorphisms of the graph correspond to symmetries in a particular diagram.

Like all graphs, the Petersen can be drawn in various ways (see Figure 21.13): the diagram from its earliest appearance, due to Alfred Bray Kempe (1849–1922),[9] exhibits threefold rotational symmetry, plus reflectional symmetry, for a total of 6 symmetries; the diagram by the eponymous Julius Petersen[10] (1839–1910) is wholly asymmetric; the now-standard diagram shows fivefold rotational symmetry, plus reflectional symmetry, for a total of 10 symmetries. The symmetries of the Kempe and standard diagrams correspond to some of the 120 automorphisms of the Petersen graph.[11] In fact, it cannot be represented as a plane diagram with more than 10 symmetries.

According to Starikova's inquiries, when graph theorists were asked to explain why the Petersen graph was beautiful, they listed many properties it has.[12] In addition to being highly symmetric, it is



Kempe's diagram

Petersen's diagram

The standard diagram

FIGURE 21.13
Three views of the Petersen graph.

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[1] Starikova, 'Aesthetic Preferences in Mathematics', p. 164.

[2] *Ibid.*, p. 162.

[3] *Ibid.*, p. 163.

[4] *Ibid.*, pp. 175–6.

[*] ▸ pp. 749–54

[5] *Ibid.*, p. 174.

[6] *Ibid.*, p. 164.

[7] *Ibid.*, p. 174.

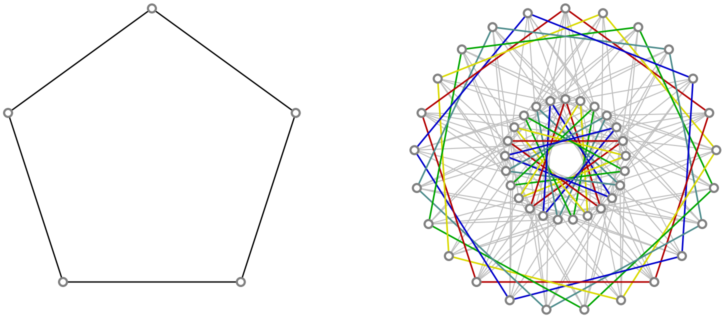
[8] *Ibid.*, p. 167.

[9] Kempe, 'A Memoir on the Theory of Mathematical Form', Fig. 13.

[10] Petersen, 'Sur le théorème de Tait', Fig. 5.

[11] Holton & Sheehan, *The Petersen Graph*, p. 249.

[12] Starikova, 'Aesthetic Preferences in Mathematics', pp. 169, 174–5.


The pentagon graph

The Hoffman–Singleton graph

FIGURE 21.14
Moore graphs. The pentagon graph is a cycle of odd length, with 5 vertices, each of degree 2, and has diameter 2. The Hoffman–Singleton graph has 50 vertices, each of degree 7, and diameter 2.

in fact 3-arc-transitive, meaning that for any pair of paths of length 3, some automorphism maps the first of those paths to the second. It is regular, meaning that each vertex has the same degree (*viz.*, 3); in fact, it is strongly regular, meaning that each pair of adjacent vertices has the same number of common neighbours and each pair of non-adjacent vertices has the same number of common neighbours (*viz.*, respectively 0 and 1).[1] Starikova maintained that symmetry contributed to the beauty of a mathematical object, because it aided in grasping the structure and its generality, and because it pointed towards other properties and connections.[2]

Among graphs with important and unusual properties, the Petersen graph is small, having relatively few vertices and edges.[3] It is, for instance, the smallest connected vertex-transitive graph that is not the Cayley graph of a group;[4] it is also the smallest snark.[5] (A *snark* is a graph in which every vertex has degree 3, but which is not 3-edge-colourable).

The Petersen graph is also one of the very restricted class of Moore graphs, and among Moore graphs of diameter 2 (which condition Starikova did not state explicitly) it is neither so trivial as the pentagon graph (see Figure 21.14 (left)) nor so complicated as the Hoffman–Singleton graph (see Figure 21.14 (right)), or any graph in the hypothetical remainder of the class, which would have 3250 vertices, each of degree 57.[6] The aesthetic response to the Petersen graph was perhaps partly due to this mean position.[7] [The *Moore bound* $M(\Delta, D)$ is an upper bound on the number of vertices in a graph with maximum degree Δ and diameter D, given by

$$M(\Delta, D) = \begin{cases} 1 & \text{if } \Delta = 1 \text{ (in which case } D = 1\text{);} \\ 2D + 1 & \text{if } \Delta = 2; \\ 1 + \Delta\frac{(\Delta-1)^D-1}{\Delta-2} & \text{if } \Delta \geqslant 3. \end{cases}$$

The *Moore graphs* are those that achieve this bound.]

But this balance of complexity and simplicity seemed to be closely connected to the visual representation, and so provoked the question of whether the beauty of the Petersen graph was dependent upon its being describable in a highly symmetrical and easily graspable diagram.[1] Starikova thought its beauty was not so dependent: even the most visually beautiful diagram did not seem to have beauty in such a degree to produce the response to the Petersen graph itself. More obviously, when mathematicians discussed the beauty of the graph, they did not refer to diagrams.[2]

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[1] Starikova, 'Aesthetic Preferences in Mathematics', pp. 169–170.

[2] *Ibid.*, p. 170.

[3] *Ibid.*, p. 168.

[4] *Ibid.*, p. 171.

[5] *Ibid.*, pp. 172–3.

[6] Dalfó, 'A survey on the missing Moore graph'.

[7] Starikova, 'Aesthetic Preferences in Mathematics', pp. 171, 175.

As in conceptual art, the diagram perceived through the senses had to be interpreted before an aesthetic judgement could be rendered. Again as in art, skill and sensitivity — innate or acquired — were necessary to make such an interpretation.[3] The rarely-seen 'star' variant diagram of the Petersen graph shown in Figure 21.15 has the same symmetries as the standard diagram (see Figure 21.13 (right)). Starikova reported that graph theorists judged the standard diagram more beautiful than the 'star' variant, sometimes distinguishing them as 'the beautiful' and 'the ugly', but she suggested that a mathematical layperson might find the 'star' variant more beautiful.[4] But, since both diagrams represent the Petersen graph, the mathematicians could not be responding *only* to the graph (*qua* mathematical object) when they made different aesthetic judgements.[5]

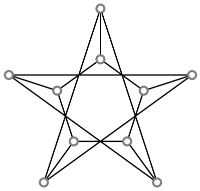

FIGURE 21.15
Variant diagram of the Petersen graph.

Starikova thought that 'the abstract beauty of the graph shines brightest through the "beautiful" drawing and is obscured in the "ugly"',[6] because the standard diagram showed much structural information about the Petersen graph that was obscured in the 'star' diagram. For instance, in the 'star' diagram there are more irrelevant edge-crossings. In the standard diagram, there are fewer intersections of edges, which made it clear that the graph is composed of two linked 5-cycles (the outer pentagon and the inner pentagram), which in turn yielded the immediate perception that it has the complete graph K_5 as a minor.[7] Starikova's argument here suggests that mathematicians' aesthetic preference for the standard diagram was not based on a sensory pleasure arising from its visual properties, but on an intellectual pleasure that came from the abstract properties of the graph to which the diagram more clearly pointed.[8] The aesthetic response might indeed be grounded in the greater fluency* in perceiving the mathematical structure via the standard diagram.[9] At the very least, the sensory beauty played only a partial or vestigial role. Rather, the standard diagram had an intellectual beauty that required the diagram to be interpreted mathematically.[10]

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[1] Starikova, 'Aesthetic Preferences in Mathematics', p. 174.

[2] *Ibid.*, pp. 174–5.

[3] *Ibid.*, p. 176.

[4] *Ibid.*, pp. 176–7.

[5] *Ibid.*, p. 177.

[6] *Ibid.*, p. 178.

[7] *Ibid.*, pp. 177–8.

[8] *Ibid.*, p. 178.

[*] ▸ pp. 785–7

[9] *Ibid.*, p. 180.

[10] *Ibid.*, pp. 178, 180.

22

People of The Book

Consensus and canon in mathematical beauty

> ‘You don’t have to believe in God,
> but you should believe in The Book.’
>
> — Paul Erdős, quot. in Spencer, ‘Erdős Magic’, p. 46.

> ‘No book is published without some discrepancy
> between each of the edition’s copies.’
>
> — Jorge Luis Borges, ‘The Lottery in Babylon’
> (trans. Andrew Hurley).

Paul Erdős (1913–96) famously told of ‘The Book’: an infinite — indeed, transfinite — book containing all the theorems of mathematics and the best proofs of each, held by a malicious Deity who rarely allows mortals a glimpse of its contents. (The earliest mention in print of The Book seems to have been in 1978.[1]) While ‘straight from The Book’ was a rhetorical device that Erdős used to praise proofs that are succinct, clever,[2] short, snappy, beautiful,[3] clear, insightful,[4] elegant and perfect,[5] it is important not to overemphasize its role: according to his friend Béla Bollobás (1943–), Erdős insisted that it was a joke and that it should not be taken seriously.[6]

Nevertheless, when Martin Aigner (1942–2023) and Günter Ziegler (1963–) proposed the project that became *Proofs from THE BOOK*, ‘a first (and very modest) approximation to The Book’, Erdős was enthusiastic and would have been a co-author were it not for his death in 1996.[7] The idea of The Book and the popularity of *Proofs from THE BOOK* — sufficient to motivate five subsequent editions — are *prima facie* evidence that there is a *canon* of proofs that are beautiful, elegant, or otherwise ‘best’; that there is at least some consensus on the aesthetic evaluation of proofs.

The present chapter considers the existence of a canon and the broader question of whether there is consensus on aesthetic value in mathematics during the later twentieth and early twenty-first centuries. There are two main themes: the extent to which the evidence of aesthetic judgements points to a consensus, and what mathematicians and philosophers have said about a consensus.

[1] Honsberger, *Mathematical Morsels*, p. vii.

[2] Baker, Bollobás & Hajnal, *A Tribute to Paul Erdős*, p. ix.

[3] Bollobás, ‘To Prove and Conjecture’, p. 220.

[4] Spencer, in Csicsery, ‘N is a Number’, 52:40 sqq.

[5] Hoffman, ‘The Man Who Loves Only Numbers’, p. 66; Hoffman, *The Man Who Loved Only Numbers*, p. 26.

[6] Bollobás, ‘To Prove and Conjecture’, p. 220.

[7] Aigner & Ziegler, *Proofs from THE BOOK* (6th edn), p. v.

A CONSENSUS ON CONSENSUS

Mathematicians seem generally to have agreed on there being a consensus on aesthetic value in mathematics. ¶ In 1968, Mark Kac (1914–84) & Stanisław Ulam (1909–84) wrote that 'there is usually considerable agreement among mathematicians concerning aesthetic values'.[1] ¶ Bollobás agreed, but held that it was surprising that there was any agreement on the value of mathematics, aesthetic or otherwise: 'It is a miracle of our subject that there seems to be a consensus as to what constitutes good mathematics. [...] you should work on problems that are beautiful and important'.[2] ¶ Lynn Arthur Steen (1941–2015) made a similar point about the unexpectness of this agreement: 'through some totally mysterious common element of the human mind, creative mathematicians share a strikingly similar set of aesthetic standards'.[3] Furthermore, he thought that these common aesthetic standards were independent of expertise: a mathematician who was not familiar with a particular field could still see beauty in its results.[4]

In his memoirs, Ulam wrote that there was a consensus about the value of new mathematics;[5] by 'value' he must have meant 'aesthetic value' or 'beauty', for he inferred from this consensus that there was 'something objective [...] about the feeling of beauty which mathematics offers', which might also depend on its applicability.[6] ¶ Armand Borel (1923–2003) said that, while there were 'differences of opinion and fluctuations in time in the evaluation of mathematical works',[7] there was a greater degree of agreement about aesthetic value in mathematics than there was in music or painting.[8] ¶ Jerry King (1935–2020) held that there was agreement among mathematicians on elegance[9] and went so far as so claim that he could not imagine that there existed a mathematician who could not see the elegance of the 'Pythagorean proof' of the irrationality of $\sqrt{2}$: 'A person who fails to see the beauty in this theorem and its proof could not be a mathematician in the first place.'[10] He inferred that there must exist some tacit aesthetic principles, commonly-held among mathematicians, that ground this broad agreement.[11] ¶ The philosopher of science Cain S. Todd thought that there was little disagreement about aesthetic claims in mathematics and science, unlike in art.[12] ¶ Phillip Griffiths (1938–), once director of the Institute for Advanced Study at Princeton, said that there was 'probably fairly good agreement' across the mathematical community about judging a piece of mathematics as beautiful.[13] ¶ Hyman Bass (1932–) thought that when it came to 'matters of taste and aesthetics' there was 'compared with other fields, a remarkable degree of consensus among mathematicians about such judgments'.[14]

Gian-Carlo Rota (1932–99) likewise agreed that there was a general consensus on aesthetic value, while allowing that there could be a change over time in what mathematics was considered beautiful:

[1] Kac & Ulam, *Mathematics and Logic*, p. vi.

[2] Bollobás, 'Advice to a Young Mathematician', p. 1004.

[3] Steen, 'Mathematics Today', p. 10.

[4] *Loc. cit.*

[5] Ulam, *Adventures of a Mathematician*, p. 277.

[6] *Loc. cit.*

[7] Borel, 'Mathematics', p. 16.

[8] *Ibid.*, p. 12.

[9] King, *The Art of Mathematics*, pp. 145, 180.

[10] *Ibid.*, p. 180.

[11] *Loc. cit.*

[12] Todd, 'Unmasking the truth beneath the beauty', pp. 71–2.

[13] Quot. in Cook, *Mathematicians*, p. 48.

[14] Bass, 'Mathematics and Teaching', p. 130.

> ‘The rise and fall of synthetic geometry […] shows that the beauty of a piece of mathematics is dependent upon schools and periods in history. […]
>
> However, given the historical period and the context, one finds substantial agreement among mathematicians as to which mathematics is to be regarded as beautiful.’[1]

Yet the examples Rota gave actually contradicted his thesis, for, as discussed later in this chapter, other mathematicians disagreed with his aesthetic judgements.[*]

The physicist Paul Dirac[†] (1902–84) thought that mathematical beauty was unlike other species of beauty, being both culturally and historically invariant:

> ‘It is quite clear that beauty does depend on one’s culture and upbringing for certain kinds of beauty, pictures, literature, poetry and so on … . But mathematical beauty is of a rather different kind. I should say perhaps it is of a completely different kind and transcends these personal factors. It is the same in all countries and at all periods of time.’[2]

The computer scientist Edsger Dijkstra (1930–2002), observing from outside mathematics — albeit from a closely related field — remarked on his perception of agreement between mathematicians on aesthetic values. Regarding the term *elegance*, he thought that:

> ‘the term is much more “technical” than most mathematicians suspect, much more “technical” in the sense that even among mathematicians of very different brands there exists a much greater consensus about what is a really elegant argument than they themselves seemed to be aware of.’[3]

In a collection of proofs he assembled, he wrote that he expected that ‘all mathematicians will agree that they are beautiful’,[4] and he reiterated the point in a collection of his papers: ‘among all sorts of mathematicians the consensus about what is mathematically elegant is much stronger than they themselves suspect’.[5]

The physicist David Gross (1941–) wrote, in the context of a discussion of mathematical beauty, that ‘there tends to be a large degree of consensus among mathematicians and physicists as to what is beautiful and what is not’,[6] though he seems to have indicated that the reasons mathematicians had for finding a theory beautiful could differ from those of physicists.[‡]

[There are various collections of beautiful theorems and proofs, but these must be treated with caution, for their contents tend to be chosen to be understandable to a wide audience.[7]]

[1] Rota, ‘Phenomenology of Mathematical Beauty’, p. 175.

[*] ▸ pp. 832–3

[†] ▸ pp. 900–13

[2] Dirac, unpublished lecture ‘Basic Beliefs and Fundamental Research’ (1972), quot. in Kragh, *Dirac*, p. 288; ellipsis in quot.

[3] Dijkstra, ‘My Hopes of Computing Science’, p. 1.

[4] Dijkstra, ‘A collection of beautiful proofs’, p. 174.

[5] Dijkstra, *Selected Writings on Computing*, p. xii.

[6] Gross, ‘Physics and mathematics at the frontier’, p. 8373.

[‡] ▸ p. 968

[7] See, for example, Dijkstra, ‘A collection of beautiful proofs’, p. 174; Polster, *Q.E.D.*, cover flap; Aigner & Ziegler, *Proofs from THE BOOK* (6th edn), p. v.

DISCORD ON CONSENSUS

Against this consensus on consensus, a few voices have been raised in discord. In response to G. H. Hardy's (1877–1947) *Apology*,[*] Nicholas Young (1943–) wrote that 'there is much less agreement among mathematicians than Hardy implies as to what is boring, what is beautiful and what is "real mathematics"'.[1] Wolfgang Krull (1899–1971) recognized that aesthetic disagreement arose between mathematicians and offered advice on how to react:

> 'As the proverb says, there's no disputing over tastes. Instead of pronouncing the work of the other side worthless, a mathematician should content himself with stating that the problems of the other party seem unattractive to him, not denying that these same problems may perhaps look very beautiful when seen from a different aesthetic point of view.'[2]

In his biography of Dirac, the historian of science Helge Kragh (1944–) took issue with Dirac's claim for universal agreement on mathematical beauty.[3][†] As concrete examples, he pointed to Minkowski space, which Dirac thought beautiful but Albert Einstein (1879–1955) had not liked when it first appeared,[4] and to Werner Heisenberg's (1901–76) and Einstein's differing judgements of quantum mechanics.[5] But these examples pointed to disagreement as to the beauty of scientific theories, not to their mathematical content, which may be an important difference.[‡]

Peter Lax[§] (1926–2025) said: 'What is beautiful is purely subjective. [...] Beauty is in the eye of the beholder.'[6] Some of the unnamed mathematicians interviewed by Leone Burton (1936–2007) as part of her study of mathematical practice[¶] agreed with Lax's position;[7] one in particular seemed to deny any role for beauty: 'Beauty doesn't matter. I have never seen a beautiful mathematical paper in my life.'[8]

Perhaps the discord is best illustrated by a discussion between the mathematicians Lipman Bers (1914–93) and Dennis Sullivan (1941–), which was part of a dialogue that ranged over the motivation for and evaluation of their subject. Their exchange about the putative consensus on aesthetic judgements in mathematics must surely be read as a model of irony:

> 'BERS: [...] I think mathematicians generally agree that simplicity and beauty are important, and there is no trouble in recognizing them when you see them.
>
> SULLIVAN: I agree with what Bers has said, that what is striking and beautiful in mathematics is pretty universally agreed upon. [...]
>
> [...]

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[*] ▸ pp. 498–523

[1] Young, *An introduction to Hilbert space*, p. 231.

[2] Krull, 'The aesthetic viewpoint in mathematics', p. 52.

[3] Kragh, *Dirac*, pp. 288–9.

[†] ▸ p. 830

[4] Pyenson, 'Hermann Minkowski'.

[5] Heelan, 'Heisenberg and radical theoretic change', esp. p. 130.

[‡] ▸ pp. 877–81

[§] ▸ pp. 535–6

[6] Lax, quot. in Albers, Alexanderson & Reid, *More Mathematical People*, pp. 155–6.

[¶] ▸ pp. 544–5

[7] Burton, *Mathematicians as Enquirers*, pp. 64, 99, 124.

[8] *Ibid.*, p. 64.

> BERS: Certainly historically the conception of what is beautiful has changed. And in general when an older man said something was not beautiful, he was wrong. Poincaré did not like Lebesgue integrals and space-filling curves, things like that.
>
> SULLIVAN: I don't like them either.
>
> BERS: Well, you're wrong.'[1]

From the starting-point of concurring on the universality of aesthetic judgements in mathematics, it took just two elided sentences to reach a point where Bers admitted that such agreement was dependent on historical context and for them to find a concept upon which they aesthetically disagreed.

CANON

Results

Rota held that Morley's trisector theorem* was neither beautiful in itself nor possessed of a beautiful proof,[2] and that the existence of non-equivalent differential structures on spheres of high dimension was never thought beautiful.[3] However, Rota made an unhappy choice of examples. Witness Cletus Oakley (1899–1990) & Justine Baker on Morley's theorem:

> 'It is one of the most astonishing and totally unexpected theorems in mathematics and, jewel that it is, for sheer beauty it has few rivals.'[4]

Leon Bankoff (1908–97) took a similar view, and also thought that few would disagree:

> 'If a committee of mathematicians were assembled to judge a beauty contest involving geometrical theorems, it is almost certain that one of the chief contenders would be Morley's Triangle Theorem. [...] it would be hard to find anyone who would deny that this elegant theorem deserves a high place of honor in the Geometrical Hall of Fame and Esthetic Excellence.'[5]

Douglas Hofstadter (1945–) called it 'absolutely stunning and beautiful';[6] William Dunham (1947–) said it was 'startling, difficult to prove, and utterly beautiful';[7] Marc Chamberland (1964–) proclaimed it a 'gorgeous result';[8] Martin Gardner (1914–2010) too thought it beautiful.[9]

Michail Monastyrsky (1945–) contradicted Rota's other example, mentioning

[1] Hammond, 'Mathematics', pp. 24–5.

* ▸ p. 3

[2] Rota, 'Phenomenology of Mathematical Beauty', p. 172.

[3] *Loc. cit.*

[4] Oakley & Baker, 'The Morley trisector theorem', p. 738.

[5] Bankoff, 'The Beauty and the Truth of the Morley Theorem', p. 294.

[6] Quot. in Roberts, *Genius At Play*, ch. 14.

[7] Dunham, *Euler*, p. 148.

[8] Chamberland, *Single Digits*, p. 78.

[9] Gardner, *New Mathematical Diversions*, p. 198.

> ‘[Milnor’s] beautiful construction of the different differential structures on the seven-dimensional sphere. [...] The original proof of Milnor was not very constructive but later E. Briscorn [*sic*[1]] showed that these differential structures can be described in an extremely explicit and beautiful form.’[2]

Rota also thought that ‘[a]n example of mathematical beauty upon which *all* mathematicians agree is Picard’s theorem’,[3] or more precisely Picard’s little theorem: *Any entire function that misses at least two points of the complex plane is constant.* Furthermore, Rota held that the beauty of the result was matched by that of Émile Picard’s (1856–1941) original proof, which applies the modular lambda function to deduce the result from Liouville’s theorem: *Any bounded entire function is constant.*

Rota’s assertions lead naturally to the contentious question of whether it is possible to judge aesthetically a theorem separately from its proof or its broader theoretical context. Hardy included proofs with the theorems when considering aesthetic qualities.[4] The philosopher C. D. Broad (1887–1971) thought that they should be considered separately: in his review of Hardy’s *Apology*,* he wrote that he felt that in Hardy’s examples the beauty resided in the proofs and the *seriousness* in the theorems, but thought that in more complicated examples, both qualities could be found in both the theorem and the proof.[5]

Some mathematicians have said that they felt unable to judge aesthetically a theorem in isolation from the theory in which it lies.[6] If proofs are in some sense the ‘inside story’ of theorems, to borrow a phrase,[7] then an awareness of the proof or proofs of the theorem is likely to alter the aesthetic value perceived in the theorem. This dependency would undermine the putative agreement on aesthetic value, at least as it applies to theorems, for two mathematicians having different levels of knowledge of the theoretical context of a result would see in it different aesthetic value.

Return to Rota’s example of Picard’s little theorem, on whose beauty, he averred, all mathematicians agreed. A degree of mathematical knowledge is required to understand the statement of the theorem: in particular, the notion of entirety sits atop a tower of supporting concepts (holomorphism, differentiability, limit, neighbourhood) that must be mastered in order to understand the theorem, but these are all covered in a first course in complex analysis. Yet understanding Picard’s original proof requires deeper knowledge that professional mathematicians specializing in other fields may not have. So a close connection between the beauty of a theorem and the beauty of its proof would tend to weaken, at least in terms of *degree*, Rota’s posited agreement on beauty between mathematicians.

Other examples count against any supposed agreement between mathematicians and non-mathematicians on theorems that the latter can understand and aesthetically appreciate. Take, for instance, Fer-

[1] Egbert Brieskorn.

[2] Monastyrsky, ‘Some trends’, p. 4.

[3] Rota, ‘Phenomenology of Mathematical Beauty’, p. 173, emphasis added.

[4] Hardy, *Apology*, § 18.

* ▸ pp. 520–1

[5] Broad, Review of *A Mathematician’s Apology*, p. 325.

[6] See the remarks by Artmann and Drucker in Wells, ‘Are these the most beautiful?’, p. 39.

[7] See Rav, ‘Why do we prove theorems?’, p. 22.

mat's two-square theorem.[*] Hardy was able to exhibit this result to the general reader to whom he addressed the *Apology* as an example of a beautiful theorem;[1] although there was at the time no proof accessible to a general audience. His decision suggests that he thought that his readers would be able to appreciate its beauty to some extent. But the result can be proven in many different ways,[2] and presumably knowledge of some of these proofs adds to its beauty. For instance, there is a short proof due to Richard Dedekind (1831–1916) that uses Gaussian integers, the notions of primality, irreducibility, and norms, and a number-theoretic result of Lagrange that can be proved quickly using finite fields.[3][†] This proof easily satisfies the Hardian triad:[‡] it is unexpected, since it invokes the complex numbers and finite fields to prove a result about the integers; inevitable, for the course of the proof, once established, is direct and natural; and economical, for it uses only basic facts about complex numbers and finite fields. Likewise, there is a proof by D. R. Heath-Brown (1952–) which is unexpected and economical, since it utilizes involutions, and is inevitable once it starts. The Heath-Brown proof was included in *Proofs from THE BOOK* from the second edition onwards.[4] The Heath-Brown proof is accessible to the mathematical layman; the proof using Gaussian integers is accessible to a mathematical undergraduate who has encountered Gaussian integers, norms, the distinction between primality and irreducibility, and finite fields. If knowledge of these proofs influences the beauty seen in the theorem, then that perceived beauty will vary depending on which proofs one knows or is capable of understanding.

The case of the four-colour theorem is perhaps instructive. The Appel–Haken proof, dependent (even after various simplifications) on computer assistance to deal with its many cases, has been viewed with distaste and disappointment by some mathematicians.[§] But the theorem itself was still accepted as beautiful by many: it was the ninth-ranked theorem in the survey by Wells,[5] which is examined in more detail below. However, for Reuben Hersh (1927–2020) & Philip J. Davis (1923–2018) other mathematicians, the theorem was marred by its having led to such a proof: 'it just goes to show, it wasn't a good problem after all'.[6]

Questionnaire: theorems

In 1989, David Wells (1940–) published a questionnaire in *The Mathematical Intelligencer* in which he asked readers to rate twenty-four theorems on a scale of 0 to 10.[7] Wells wondered whether professional mathematicians would agree either in their assessment of what comprised mathematical beauty, or in their judgements of beauty, or in how their own experience of beauty interacts with their work.[8] Wells emphasized that he only asked his respondents to judge *theorems*, not different proofs of theorems, or between theorems and

[*] ▸ p. 7

[1] Hardy, *Apology*, § 13.

[2] Dickson, *Hist. Theory of Numbers*, vol. II, pp. 227 sqq.

[3] Dedekind, 'Supplemente', p. 444.

[†] ▸ pp. 788–9

[‡] ▸ pp. 502

[4] Aigner & Ziegler, *Proofs from THE BOOK* (2nd edn), pp. 20–1.

[§] ▸ pp. 602–8

[5] Wells, 'Are these the most beautiful?', p. 37.

[6] Davis & Hersh, *Mathematical Experience*, p. 426.

[7] Wells, 'Which is the most beautiful?'

[8] *Ibid.*, p. 30.

proofs. He conceded, however, that respondents' judgements would be influenced by their knowledge of proofs of the theorems.[1]

The results of this survey and Wells's analysis thereof, together with a selection of respondents' comments, were published in 1990.[2] These results and comments form a unique snapshot of how a section of the mathematical community viewed mathematical beauty; no such survey has been carried out before or since. As Wells himself admitted, the survey was crude.[3] In particular, there was a self-selection bias in such a survey, which included the possibility that some people did not respond because they found the beauty of theorems difficult or impossible to separate from the beauty of proofs.[4] Furthermore, different respondents interpreted the survey in different ways. At least one respondent indicated that he rated the theorems based on the impression he had when he first encountered them.[5] Another assigned a lower rating when a result had no simple proof; Wells stated that he himself would take the lack of a simple proof as an indicator of depth and would thus rate the result higher.[6]

The complete results of the survey are listed in an appendix to this chapter; see Table 22.2 on pages 868–9. Euler's equation $e^{i\pi} = -1$ took the crown, with a mean score of 7.7; this outcome was an instance of a widespread aesthetic exaltation of Euler's equation, which will be discussed later.[*]

Regrettably, Wells did not publish the distribution of ratings for each theorem, only the mean rating. Nevertheless, he did observe that some theorems that had a high mean ranking also received a notable number of low rankings.[7] Mirroring this situation, the theorem ranked twentieth, *At any party, there is a pair of people who have the same number of friends present*, received a score of more than 7 from more than a quarter of respondents.[8]

Wells raised the possibility that social factors might have had an influence: some respondents might have assigned high ratings to theorems because they knew that those theorems were generally recognized beautiful, rather than because they felt a strong sense of beauty when contemplating them. This social influence was certainly plausible, for several of the highest-ranked theorems had been given as examples of beautiful results. For instance, Hardy's two principal examples from *A Mathematician's Apology*, namely the infinitude of the primes and the irrationality of $\sqrt{2}$, were ranked third and seventh, respectively; Hardy's authority may have firmly established these results as 'known' to be beautiful.

However, the mere authority of great mathematicians seems to have been insufficient to establish a theorem as beautiful. In choosing a result representative of Ramanujan's[†] (1887–1920) work in the theory of partitions for his obituary, Hardy wrote

> 'It would be difficult to find more beautiful formulæ than the 'Rogers–Ramanujan identities' [...]; but here Ramanujan must

[1] Wells, 'Which is the most beautiful?', p. 30.

[2] Wells, 'Are these the most beautiful?'

[3] *Ibid.*, p. 40.

[4] *Ibid.*, p. 38.

[5] *Ibid.*, p. 39.

[6] *Loc. cit.*

[*] ▸ pp. 839–43

[7] *Ibid.*, p. 38.

[8] *Ibid.*, p. 39.

[†] ▸ pp. 667–72

> take second place to Prof. Rogers; and […] I would agree with Major MacMahon in selecting
>
> $$\left.\begin{aligned} &p(4)+p(9)x+p(14)x^2+\dots\\ &\qquad = 5\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\cdots\}^5}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^6}\text{,’[1]}\end{aligned}\right\}\quad \text{(P)}$$

where $p(n)$ is the number of partitions of n.[2] J. E. Littlewood (1885–1977) agreed: in his review of Ramanujan's *Collected Papers*, he described this formula as being 'of supreme beauty';[3] H. P. F. Swinnerton-Dyer (1927–2018) too thought it beautiful.[4] So at least four first-rate mathematicians — Hardy, MacMahon, Littlewood, Swinnerton-Dyer — had indicated that (P) was beautiful.

Yet, in Wells's questionnaire, where it appeared as Theorem (C), (P) came *last*, with a mean rating of 3.9. [Another theorem has a mean rating of 3.9 but Wells explicitly referred to (P) as 'last';[5] presumably the printed mean ratings were rounded to one decimal place.] Indeed, Wells was astonished at his respondents' rating of it: he had been curious as to where it would rank, but never imagined it would be bottommost.[6] However, R. P. Lewis, in his response to Wells, rated it highest and added an unofficial marginal score of 12; this respondent was presumably Richard P. Lewis (1942–2007), a number theorist and combinatorialist, and (P) would have been close to his area of research. This situation perhaps illustrates that the background of a mathematician might influence their judgements of beauty, a point that Wells and some of the respondents recognized.[7]

[In the questionnaire, Wells showed the term $(1 - x^4)$ in the denominator, placed the multiplier 5 in the numerator, and displayed the equation with the left- and right-hand sides switched.[8] It is just conceivable that these changes made a difference to the perceived beauty of the formula.]

Of course, perhaps the judgement of Hardy, MacMahon, Littlewood, and Swinnerton-Dyer of (P) was simply not widely known among the respondents to Wells's questionnaire. Although Hardy's obituary of Ramanujan was reprinted in the very *Collected Papers* Littlewood was reviewing, and although Littlewood's review was reprinted in his *Miscellany*, the circulation of these works cannot compare with, say, Hardy's *A Mathematician's Apology*, or other popular works about mathematics.

Finally, one is forced to wonder how many of Wells's respondents recognized (P) as a result of Ramanujan; if more had, would it have received higher ratings, since Ramanujan's work was 'known' to be beautiful?

One of the most interesting of the top-rated theorems was the sixth, which is a special case of Brouwer's fixed-point theorem: *A continuous mapping of the closed unit disc into itself has a fixed point.* This theorem seems to require greater mathematical training to understand than any higher-rated theorem.

[1] Hardy, 'Srinivasa Ramanujan', pp. xxxiv–xxxv.

[2] Originally published in Ramanujan, 'Some properties of $p(n)$', Eq. (17).

[3] Littlewood, Review of *Collected papers*, p. 425.

[4] Swinnerton-Dyer, 'Congruence Properties of $\tau(n)$', p. 289.

[5] Wells, 'Are these the most beautiful?', p. 39.

[6] *Loc. cit.*

[7] *Ibid.*, p. 40.

[8] Wells, 'Which is the most beautiful?', p. 31.

The nineteenth-ranked result, with a mean rating of 4.7, was *There is no equilateral triangle whose vertices are plane lattice points.* This result is equivalent to *There is no rational number whose square is* 3. But the corresponding theorem about the number 2 was ranked seventh, with a mean rating of 6.7, which perhaps hints that beauty in theorems was not invariant across logical equivalence, though of course implicational opacity means that such equivalence might not be noticed. In the end, Wells drew the conclusion that 'the idea that mathematicians largely agree in their aesthetic judgements is at best grossly oversimplified'.[1]

Questionnaire: formulae

As part of their study of the neural activity that accompanies the experience of mathematical beauty,[*] Semir Zeki (1940–), John Romaya, Dionigi Benincasa & Michael Atiyah (1929–2019) asked fifteen test subjects to answer a questionnaire asking them to evaluate sixty formulae on a scale of −5 (ugly) to +5 (beautiful).[2] The formulae and the mean ratings are listed in an appendix to this chapter; see Table 22.3 on pages 869–73.

Euler's equation $1 + e^{i\pi} = 0$ scored highest, with a mean score of +3.6, in agreement with Well's questionnaire;[†] again, this was in agreement with a general admiration of Euler's equation.[‡] Although there may be a distinction between the beauty of a formula viewed *qua* theorem and *qua* equation,[3] Euler's equation seems to have been supreme in either case.

Table 22.1 shows the distribution of ratings assigned by the test subjects, sorted by mean rating. Broadly speaking, there seems to have been more agreement on formulae that were on average considered more beautiful. But this may be an artefact of the formulae being more generally rated beautiful than ugly: the lowest mean rating was −1.80, but there were sixteen formulae (more than a quarter of the entire list) whose mean ratings were higher than +1.80. In the highest-rated formulae, assigned rating were almost entirely positive. But the Bianchi identity (labelled (60)) had a mean rating of 1.60 and yet six subjects rated it −1. A nice example of disagreement is the arithmetic–geometric mean inequality (labelled (38)), which had a mean rating of +0.53 but whose ratings were clustered in $\{-2, -1\}$ and $\{+1, +2, +3\}$ (slightly ugly and moderately beautiful?).

The formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}} \tag{R}$$

(label (14) in the questionnaire) had the lowest mean rating of −1.80, perhaps because the numbers it contains (9801, 1103, 26390, 396) seem unnatural. Yet (R), which is due to Ramanujan,[4] has very pleasing computational properties, for the series on the right-hand side

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[1] Wells, 'Are these the most beautiful?', p. 40.

[*] ▸ pp. 760–2

[2] Zeki, Romaya et al., 'The experience of mathematical beauty', pp. 1–2.

[†] ▸ pp. 834–7

[‡] ▸ pp. 839–43

[3] King, *The Art of Mathematics*, p. 181.

[4] Ramanujan, 'Modular Equations and Approximations to π', Eq. (44).

Survey label	Mean rating	Number of each rating assigned										
		−5					0					5
(1)	3.60	0	0	0	0	1	0	2	0	2	3	7
(2)	3.27	0	0	0	0	0	0	2	2	3	6	2
(54)	3.13	0	0	0	0	0	1	0	5	2	4	3
(5)	3.00	0	0	0	0	0	1	0	3	6	4	1
(6)	2.93	0	0	0	0	0	1	2	3	3	3	3
(58)	2.93	0	0	0	0	1	1	1	1	3	7	1
(18)	2.80	0	0	0	1	1	0	2	2	2	2	5
(34)	2.73	0	0	0	0	1	2	0	2	6	0	4
(31)	2.67	0	0	0	0	1	1	2	2	2	6	1
(10)	2.53	0	0	1	0	0	1	2	2	4	2	3
(16)	2.47	0	0	0	0	0	2	4	1	3	3	2
(20)	2.27	0	0	0	0	0	3	2	3	3	3	1
(55)	2.20	0	1	0	0	0	1	0	5	6	1	1
(17)	2.00	0	0	0	0	0	3	2	5	3	1	1
(22)	1.93	0	0	0	0	2	2	0	5	3	3	0
(36)	1.87	0	0	0	0	0	5	0	4	4	2	0
(32)	1.73	0	0	0	1	1	3	1	3	3	2	1
(29)	1.67	0	1	0	1	1	1	2	3	3	0	3
(8)	1.60	1	0	0	0	0	3	2	4	2	2	1
(23)	1.60	0	0	1	0	0	3	2	4	3	2	0
(60)	1.60	0	0	0	0	6	0	0	4	1	1	3
(52)	1.53	0	0	0	1	2	3	0	4	2	2	1
(13)	1.47	0	0	0	2	1	3	1	3	2	1	2
(3)	1.40	0	0	0	1	2	5	1	1	0	3	2
(25)	1.40	0	0	0	1	3	2	2	1	3	2	1
(24)	1.33	0	0	0	0	1	4	4	2	3	1	0
(43)	1.27	0	2	0	0	3	0	2	2	2	2	2
(49)	1.27	0	0	1	0	1	2	3	5	2	1	0
(44)	1.13	0	0	1	0	2	5	0	2	3	1	1
(26)	1.07	0	0	2	1	1	4	1	0	2	2	2
(37)	1.07	0	0	0	0	3	3	4	2	1	2	0
(46)	1.07	0	2	0	1	0	2	1	5	2	1	1
(56)	1.07	0	0	0	0	2	3	5	3	1	1	0
(9)	1.00	1	0	0	1	0	1	6	2	4	0	0
(57)	1.00	0	0	0	1	3	2	3	3	2	0	1
(4)	0.87	0	0	1	1	3	2	3	1	2	0	2
(21)	0.80	1	0	1	0	1	4	1	4	0	3	0
(40)	0.80	1	0	0	2	0	4	2	0	5	1	0
(30)	0.73	0	0	3	1	2	1	2	2	0	2	2
(48)	0.73	0	0	1	0	1	5	4	2	1	1	0
(7)	0.67	0	0	1	2	3	3	0	2	1	2	1
(19)	0.60	0	0	2	1	1	3	4	1	0	3	0
(27)	0.60	0	1	0	2	3	3	0	2	1	2	1
(53)	0.60	0	1	0	1	2	3	3	2	2	1	0
(12)	0.53	0	0	2	1	1	5	1	2	1	1	1
(38)	0.53	0	0	0	5	1	0	2	4	3	0	0
(47)	0.40	0	0	1	0	3	7	1	1	0	1	1
(41)	0.20	2	1	2	0	0	1	1	4	2	2	0
(11)	0.13	0	1	0	1	2	7	2	0	0	2	0
(42)	0.07	1	0	0	1	3	6	0	1	3	0	0
(35)	0.00	0	0	2	2	2	1	5	2	1	0	0
(50)	−0.20	1	0	2	1	0	6	2	1	2	0	0
(59)	−0.27	0	3	1	2	1	1	1	4	1	1	0
(33)	−0.33	0	1	2	2	2	4	1	1	1	0	1
(45)	−0.33	1	0	1	2	0	5	5	1	0	0	0
(39)	−0.40	0	2	0	0	3	7	2	0	1	0	0
(51)	−0.53	2	0	3	0	1	4	1	2	1	1	0
(15)	−1.13	2	1	4	1	1	1	2	1	1	0	1
(28)	−1.13	2	1	3	1	1	2	3	0	2	0	0
(14)	−1.80	4	3	0	3	1	0	1	1	0	1	1
		−5					0					5

TABLE 22.1
Distribution of ratings awarded to the formulae in the questionnaire by Zeki et al. The results are ordered by mean rating; parenthesized numbers are the original labels in the questionnaire. (Prepared from Zeki, Romaya et al., 'The experience of mathematical beauty', data sheet 3.)

converges very rapidly, yielding eight additional decimal places of π for each term. In particular, Jonathan Borwein (1951–2016) saw (R) as having beauty like that which G. N. Watson (1886–1965) compared to Michelangelo's sculptures in the Sagrestia Nuova.[1]* The distribution of ratings assigned to (R) is intriguing: most subjects rated it very negatively, but four subjects rated it positively and there was one rating of +5 and one of +4. The subject who gave (R) a rating of +5 scored their understanding of the formula as 1 on a scale of 0 to 3; the only subject who scored their understanding as 3 gave it a rating of +2. These scores suggest that the variation of ratings for (R) was not due to the level of understanding of the formula (and knowledge of its computational properties?). One could speculate that a subject who rated (R) highly might have recognized it as Ramanujan's and concluded that *it must be beautiful even if I cannot see it*, just as one might think the same, regardless of one's own response, about a work that one recognized as Leonardo's or Mozart's.

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[1] Borwein, 'Aesthetics for the working mathematician', p. 31.

* ▸ p. 5

Euler's equation

Among results praised as beautiful, there is one that stands out in terms of the sheer frequency with which it has been cited, namely Euler's equation

$$e^{i\pi} + 1 = 0. \tag{E1}$$

Sometimes, in place of (E1), the discussion has pertained to the equivalent equation

$$e^{i\pi} = -1, \tag{E2}$$

or to the more general Euler's formula:

$$e^{i\vartheta} = \cos\vartheta + i\sin\vartheta. \tag{F}$$

(It is conceivable that commuting the i and π in (E1–E2) or the i and ϑ in (F) might make a difference to the beauty of the equation, but for brevity this distinction is elided here.)

Before turning to its earlier history, it is worthwhile to make a brief chronological survey of the judgements made or reported by mathematicians of Euler's equation from the early 1990s to the present day. Its length serves to justify the special attention the result will receive here.

Eli Maor (1937–) wrote that (E1) was '[o]ne of the most beautiful results in all of mathematics'[1] and that (E2) 'must surely rank among the most beautiful formulas in all of mathematics'.[2] ¶ Robert Kanigel (1946–) described (E2) as 'Euler's elegant relationship [...] a single, strange, beautiful statement'.[3] ¶ Jerry King wrote that (E1) had 'great beauty'.[4] ¶ Calvin Clawson (1941–) thought 'many mathematicians' thought that (E2) was 'the most elegant mathematical expression ever discovered'.[5] ¶ Piergiorgio Odifreddi (1950–) said that (E2) was 'what many consider to be the most beautiful mathematical formula'.[6] ¶ Doris Schattschneider (1939–) simply said of (E1): 'That's beautiful!'[7] ¶ Yuri Manin (1937–2023) said that (E2) was '[a]rguably [...] the most beautiful single formula in all of mathematics'.[8] ¶ Paul Nahin (1940–) said that it was 'of exquisite beauty' and 'the gold standard for mathematical beauty', and that it would 'still appear, to the arbitrarily advanced mathematicians ten thousand years hence, to be beautiful and stunning and untarnished by time'.[9] ¶ The philosopher Mark Steiner (1942–2020) called (E1) a 'beautiful identity'.[10] ¶ David Orrell (1962–) wrote that, '[l]ike many mathematicians', he found (E1) 'very beautiful', and pointed to its 'unity and symmetry'.[11] ¶ Charles Sandifer (1951–) did not distinguish forms (E1) and (E2) of the equation and called it 'one of the most beautiful formulas in all of mathematics',[12] although it is not quite clear whether he was merely reporting a generally-held view or endorsing it himself. ¶ In the title of his book, David Stipp called (E1) '*A Most Elegant Equation*'; in the text itself, he repeatedly called it beautiful.[1] ¶ Robin Wilson (1943–) used the subtitle of his book on the history of Euler's equation to proclaim the forms (E1) and (E2) '*The most beautiful theorem in mathematics*'.[2]

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[1] Maor, *To Infinity and Beyond*, p. 53.

[2] Maor, e, p. 160.

[3] Kanigel, *The Man Who Knew Infinity*, p. 208.

[4] King, *The Art of Mathematics*, p. 86.

[5] Clawson, *Mathematical Mysteries*, p. 104.

[6] Odifreddi, *The Mathematical Century*, p. 43.

[7] Schattschneider, 'Beauty and Truth in Mathematics', p. 46.

[8] Manin, 'Mathematical Knowledge', § 1.2.3 [p. 9].

[9] Nahin, *Dr. Euler's Fabulous Formula*, p. xxxii.

[10] Steiner, 'Penrose and Platonism', p. 136.

[11] Orrell, *Truth or Beauty*, p. 193.

[12] Sandifer, *How Euler Did Even More*, p. 87.

The evidence of questionnaires fits with these assessments. The form (E2) was the highest-rated theorem in Wells's 1988 questionnaire, with a mean rating of 7.7 (on a scale of 0 to 10), although one of his correspondents confessed to awarding it a rating of 10 not for the formula itself, but for complex analysis as a whole. The (unusual) slight modification $1 + e^{i\pi} = 0$ received the highest mean rating of +3.6 (on a scale of −5 to +5) in the 2014 survey carried out by Zeki et al.[3][*] [In 2004, readers of the journal *Physics World*, when asked for their 'greatest equations',[4] placed (E1) a close second behind Maxwell's equations.[5][†]]

When was Euler's equation first recognized as beautiful? In part, this depends on when it was first discovered. Leonhard Euler (1707–83) himself derived the eponymous formula (F) in his *Introduction to Analysis of the Infinite* (1748),[6] but he did not state (E1) or (E2). The earliest known explicit statement of Euler's equation seems to have been in a 1813–14 paper by Jacques Français (1775–1933) on the geometrical representation of complex numbers, where it appears in the form (E2) (or, to be exact, $-1 = e^{\pm\pi\sqrt{-1}}$).[7]

According to a reminiscence written by his former student W. E. Byerly (1849–1935), when the mathematician Benjamin Peirce (1809–80) deduced the equivalent equation

$$e^{\pi/2} = \sqrt[i]{i}$$

while lecturing, he paused in contemplation and declared it 'absolutely paradoxical',[8] so it *may* have been considered noteworthy or mysterious since at least the late 1860s.

But the earliest written aesthetic judgement of Euler's equation that the present author has located is in the book *Mathematics and the Imagination*, first published in 1940, where Edward Kasner (1878–1955) & James Newman (1907–66) wrote of (E1):

> 'Elegant, concise and full of meaning, we can only reproduce it and not stop to inquire into its implications. It appeals equally to the mystic, the scientist, the philosopher, the mathematician.'[9]

[The modifier phrase '[e]legant, concise and full of meaning' should presumably apply to the equation ('it'), rather than to Kasner & Newman ('we'), as grammar would suggest.]

And the earliest explicit judgements of *beauty* that the author has located date only to the latter half of the 1940s. They appear in the writings of the chemist and writer François Le Lionnais[‡] (1901–84).

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[1] Stipp, *A Most Elegant Equation*, title, 'Introduction', chs 4, 9–11.

[2] Wilson, *Euler's Pioneering Equation*, title, p. 3.

[3] Zeki, Romaya et al., 'The experience of mathematical beauty', Data Sheet 3.

[*] ▸ pp. 837–8

[4] Crease, 'The greatest equations ever (i)'.

[5] Crease, 'The greatest equations ever (ii)', p. 15.

[†] ▸ pp. 884–5

[6] Euler, *Introduction to Analysis of the Infinite*, § 138 ⟦vol. 1, p. 112⟧.

[7] Français, 'Nouveaux principes de géométrie de position', Corol. 2 to Def. 4 [p. 64].

[8] Eliot et al., 'Benjamin Peirce', p. 6.

[9] Kasner & Newman, *Mathematics and the Imagination*, p. 103.

[‡] ▸ pp. 672–9

Perhaps curiously, he gave primacy neither to (E1), nor to (E2), but to the equivalent equation

$$\sqrt{-1}^{\sqrt{-1}} = e^{-\pi/2}, \tag{E3}$$

which seems to make Le Lionnais unique among the authors who have discussed Euler's equation in aesthetic terms. In an article he published in 1946, he gave as an example 'on the level of elementary mathematics, Euler's formula, which was once called "the most beautiful formula of mathematics"'; the handwritten formula (E3) was inserted alongside as an illustration.[1] In a 1948 essay on mathematical beauty, he commented again on the beauty of Euler's equation primarily in the form (E3) and only secondarily in the form (E1):

> 'Euler's formula: $\sqrt{-1}^{\sqrt{-1}} = e^{-\pi/2}$ (which can be written equivalently: $e^{\pi\sqrt{-1}} + 1 = 0$)[2] establishes between the most important numbers in mathematics: 1, π, e, a bond that seemed fantastic in its time. It was generally considered as THE MOST BEAUTIFUL FORMULA OF MATHEMATICS.'[3]

The qualifier phrases 'was once called' and 'was generally considered' in the quoted passages, which suggest that the equation was more beautiful in the past, will be considered further below.

After Kasner & Newman judged it elegant in 1940 and Le Lionnais praised its possibly fading beauty in 1946–8, and before the publication of Wells's questionnaire in 1988, there seems to have been a paucity of written attestations of aesthetic value in Euler's equation. It seems to have been praised more in *non*-aesthetic terms. ¶ For example, in his celebrated 1961–4 course of lectures for physics undergraduates, Richard Feynman (1918–88) called (F) 'the most remarkable formula in mathematics'.[4] [A description of (F) as 'our jewel' appears in *The Feynman Lectures on Physics*, but not in the audio recording of the original lecture.] ¶ H. E. Huntley (1892–1975) mentioned (E2) in his 1970 study of mathematical beauty,[*] but did not explicitly describe it as beautiful, only as 'remarkable' and indirectly as 'economical'.[5] ¶ Davis & Hersh thought that (E1) had 'an aura of mystery'.[6]

In the four decades between Le Lionnais's second essay and Wells's questionnaire, the only example of aesthetic praise of Euler's equation that the author has located is in the book *From Zero to Infinity* by the popular mathematics author Constance Reid (1918–2010), where (E1) is praised as 'simple and elegant'[7] starting with the 1964 third edition.

There is also a passing reference in Arthur C. Clarke's (1917–2008) novel *The Fountains of Paradise* (1979), in which a character considers (E1) to be 'profound yet beautifully simple'.[8] But that Clarke incorporated this reference suggests that he thought it would be meaningful to many of his readers, which in turn suggests that the *idea* of (E1) as (aesthetically?) praiseworthy was 'in the air'.

[1] Le Lionnais, 'Paysages Mathématiques', p. 16, n. 3; trans. C.

[2] Strangely, the translation in Le Lionnais, 'Beauty in Mathematics', p. 128, has the incorrect operation '−'.

[3] Le Lionnais, 'La Beauté en Mathématiques', pp. 442–4, emphasis in orig.; trans. C.

[4] Feynman, Leighton & Sands, *Lectures on Physics* (book), vol. 1, § 22-6.

[*] ▸ pp. 563–71

[5] Huntley, *The Divine Proportion*, pp. 37, 38.

[6] Davis & Hersh, *Mathematical Experience*, p. 199.

[7] Reid, *From Zero to Infinity* (3rd ed.), p. 165.

[8] Clarke, *The Fountains of Paradise*, ch. 22.

A quality that seems to have elevated the formula (E1) above its equivalents is that, in this form, the equation relates the five constants 0, 1, e, π, and i.

Even in the form (E3), Le Lionnais thought that the 'brilliance of this expression is due to the almost perfect elimination of every element foreign to the three numbers' 1, e, and π.[1] ¶ Davies & Hersh associated the 'aura of mystery' of (E1) to its linking of 'the five most important constants in the whole of analysis'.[2] ¶ Two of Wells's correspondents declared (E1) superior to the form (E2) (which appeared in his questionnaire), because it combines the five constants.[3] ¶ King said that because it involves the five constants, the 'most important in mathematics', the fundamental operations of addition, multiplication, and exponentiation, and the fundamental relation of equality, and 'nothing extraneous', (E1) satisfied what he called the '*aesthetic principle of minimal completeness*'.[4]* On the other hand, King found (E2) 'merely interesting', and thought it lacked completeness because it omits 0.[5] ¶ Schattschneider described (E1) as beautiful precisely because it included 'five of the most important numbers' in 'one incredibly spare equation'.[6] ¶ Steiner supported his judgement of (E1) as beautiful in itself by noting that it related the five constants in a 'simple, beautiful manner',[7] and by emphasizing the different origins of the constants. ¶ Orrell wrote that his aesthetic appreciation was of the precise form (E1); in contrast, he thought that (E2) 'just looks like a calculation'.[8] ¶ The presence of the five constants seems to be consistently identified as the most important point, with little importance assigned to the possible swapping of the summands $e^{i\pi}$ and 1, or commuting i and π in the exponent.

As discussed above, Le Lionnais seems to have thought that, by the time he wrote in 1946–8, the beauty of Euler's equation had faded. Although his preferred form (E3) related 1, e, and π,

> 'Nowadays the intrinsic reason for this connection has become so clear that the same formula seems to us, if not insipid, at least wholly natural.'[9]

Wells's questionnaire elicited a notable number of low ratings for generally highly-ranked theorems and quoted from Le Lionnais's discussion of Euler's equation;[10] one correspondent in particular marked down (E3) as too simple.[11] Nathalie Sinclair (1970–) & David Pimm (1953–) interpreted Wells as having meant that some of the low ratings applied in particular to Euler's equation, and for the reason Le Lionnais suggested.[12]

Is Euler's equation obvious? In a sense, yes: one learns early in the theory of complex analysis that $ae^{i\vartheta}$ represents the complex number with magnitude a and argument ϑ, and the number -1 has magnitude 1 and argument π. These facts become familiar, and habituation can breed ennui. And yet this interaction of exponentiation and angles in the complex plane was historically surprising: π was derived from the geometry of antiquity; i was first treated as a number subject to formal manipulation in the algebra of the Renaissance; e was first investigated in the context of compound interest by Jakob I Bernoulli (1655–1705) in the early modern period. Today the student encounters π as a concept from geometry; e in the context of logarithms; and i in solving quadratic equations. The historical surprise surely echoes in the student's response to seeing how they combine in Euler's formula and equation.

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[1] Le Lionnais, 'La Beauté en Mathématiques', p. 444; trans. C.

[2] Davis & Hersh, *Mathematical Experience*, p. 199.

[3] Wells, 'Are these the most beautiful?', p. 40.

[4] King, *The Art of Mathematics*, p. 86, emphasis in orig.

* ▸ pp. 790–1

[5] *Ibid.*, pp. 86, 191.

[6] Schattschneider, 'Beauty and Truth in Mathematics', p. 46.

[7] Steiner, 'Penrose and Platonism', p. 136.

[8] Orrell, *Truth or Beauty*, pp. 193–4.

[9] Le Lionnais, 'La Beauté en Mathématiques', p. 444; trans. C.

[10] Wells, 'Are these the most beautiful?', p. 38.

[11] *Ibid.*, p. 39.

[12] Sinclair & Pimm, 'A Historical Gaze', p. 14.

Proofs

Proofs from THE BOOK

Although the very notion of 'The Book' suggests a canon of perfect proofs, Martin Aigner and Günter Ziegler, the authors of *Proofs from THE BOOK*, did not believe in such a canon. Indeed, Ziegler said that

> 'For some theorems, there are different perfect proofs for different types of readers. I mean, what is a proof? A proof, in the end, is something that convinces the reader of things being true. And whether the proof is understandable and beautiful depends not only on the proof but also on the reader: What do you know? What do you like? What do you find obvious?'[1]

Aigner & Ziegler openly stated that they had 'no definition or characterization of what constitutes a proof from The Book'.[2] Their choices were to a large extent influenced by the suggestions of Erdős, and they only chose proofs that were understandable using only basic undergraduate mathematics. They also only chose proofs that were short. Partly, this criterion was entailed by the practical issue of gathering proofs into a single volume, but also by their view that brevity is an aspect of perfect proofs, at least for humans. Ziegler thought that it was possible for a hundred-page perfect proof to exist, but that only a superhuman intellect could discover it.[3] He asked:

> 'why shouldn't there be a proof in God's Book that goes over a hundred pages, and on each of these hundred pages, makes a brilliant new observation — and so, in that sense, it's really a proof from The Book?'[4]

For human mathematicians, thought Ziegler, a 'Book' proof could not be more that ten pages long. 'God — if he exists — has more patience.'[5] In sum, there were no proofs that are perfect in an absolute sense, only proofs that were perfect for a given person or hypothetical superhuman intellect.

[The titles of some of the translations of *Proofs from THE BOOK* hint at perfect proofs descending from a platonic heaven. French: *Raisonnements divins*, '*Divine reasonings*'; Japanese: *Tensho no shōmei*

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[1] Ziegler, quot. in Klarreich, 'In Search of God's Perfect Proofs'.

[2] Aigner & Ziegler, *Proofs from THE BOOK* (6th edn), p. v.

[3] Klarreich, 'In Search of God's Perfect Proofs'.

[4] Ziegler, quot. in *ibid.*

[5] Ziegler, quot. in *ibid.*

[天書の証明], ‘*Proofs from Heaven’s Book*’; Chinese: *Shùxué tiānshū zhōng de zhèngmíng* [数学天书中的证明], ‘*Proofs from Mathematical Heaven’s Book*’.]

There have been six editions of *Proofs from THE BOOK*, with various changes between them. Most of these changes consisted of the addition of new chapters dealing with new areas, some suggested by the authors’ correspondents, while the chapters themselves were largely internally stable. Comparing the first appearance of each chapter with the sixth edition, one sees that:

- 32 underwent only minor changes after the edition in which they were introduced (including the chapter ‘Some irrational numbers’, from which Euler’s result $\sum_{n\geqslant 1} 1/n^2 = \pi^2/6$ was removed to be the subject of a separate chapter);
- 4 chapters gained extra results after their introduction;
- 3 acquired additional proofs of previously-given results;
- 4 received modified proofs, where the change was relatively small or simply a matter of a fuller exposition;
- 2 received replacement proofs;
- 1 was deleted.

Naturally, the replacement of proofs, and the one chapter that was deleted in its entirety, deserve attention.

In the first edition, there was a chapter on ‘The problem of the thirteen spheres’, giving John Leech’s (1926–92) proof that it is impossible for thirteen unit spheres to touch a given unit sphere in three-dimensional space. Aigner & Ziegler admitted that this proof required explicit calculations, making it unique among the proofs they included, although they still proclaimed it a ‘Book Proof’.[1] Yet they removed it for the second edition, issuing a *nostra culpa* that suggests they had changed their minds: ‘[it] turned out to need details that we couldn’t complete in a way that would make it brief and elegant’;[2] this statement suggests not a change in the underlying criteria they used to judge the elegance of the proof, but a realization that the proof did not satisfy those criteria.

The first edition also contained Don Zagier’s (1951–) ‘one-sentence proof’ of Fermat’s two-square theorem: *Every prime of the form* $4k+1$ *is a sum of two square numbers.** Zagier’s proof was again explicitly proclaimed to be the ‘Book Proof’[3] of this result, and was explained in detail over almost two pages. For the second edition, this proof was replaced by two proofs by Axel Thue (1863–1922) and by D. R. Heath-Brown (1952–); the latter proof served as the inspiration for Zagier.[4] There was no explicit discussion of this change in the second edition. Although the Zagier proof can be summed up in a single sentence, it has to be unpacked into a two-page explanation that still leaves some steps to the reader. The Heath-Brown proof, written in full, is actually shorter, and the involutions it uses are easier to

[1] Aigner & Ziegler, *Proofs from THE BOOK* (1st edn), p. 67.

[2] Aigner & Ziegler, *Proofs from THE BOOK* (2nd edn), p. vi.

* ▸ p. 7

[3] Aigner & Ziegler, *Proofs from THE BOOK* (1st edn), p. 19.

[4] See Heath-Brown, ‘Fermat’s two squares theorem’; Zagier, ‘A one-sentence proof’.

understand than those used by the Zagier proof. The sixth edition added a geometrical interpretation, due to Alexander Spivak (1966–), of an otherwise-mysterious involution in the Heath-Brown proof.[1]

For the fourth edition, a different solution was given for Hilbert's third problem, of giving two tetrahedra with equal bases and equal heights that cannot be decomposed into congruent polyhedra, even when combined with other congruent polyhedra. The new solution used a simplification due to Benjamin Kagan (1869–1953),[2] apparently unknown to Aigner & Ziegler at the time of the third edition, which decreased the required amount of linear algebraic machinery.[3] This change has little significance for the question of a canon.

None of the changes discussed here poses a serious challenge to the idea of a canon, for each seems to have come about from a realization that the old proof was mistakenly thought to be from The Book, rather than a fundamental change in what constitutes a Book proof.

Particular proofs

Euclid's proof of the infinitude of the primes. This proof[*] seems to receive universal aesthetic acclaim. It was held up by Hardy as one of his two examples of beautiful proofs.[4] It is the only proof to appear in both Dijkstra's[5] and Burkard Polster's[6] (1965–) collections of beautiful proofs, and was the first proof given by Aigner & Ziegler in all six editions of *Proofs from THE BOOK*.[7] C. F. Gardiner[8] (1930–) and John Allen Paulos[9] (1945–) both gave it as an example of beauty, with Gardiner openly echoing Hardy. Saunders Mac Lane (1909–2005) thought it elegant.[10] Chandler Davis (1926–2022), Marjorie Senechal (1939–) & Jan Zwicky (1955–) asserted that '*all mathematicians* know that Euclid's proof of the infinitude of primes is in [The Book]'.[11]

But there is substantial variation in the proofs of this result that are subsumed under Euclid's moniker. Some of these variations are due to changes in the standards of notation and language, but there is a fundamental dichotomy that cannot be avoided. Despite Hardy considering it an exemplar of *reductio ad absurdum* ('one of a mathematician's finest weapons'[12] [†]), Euclid's original proof of the theorem he gave as *Prime numbers are more than any assigned multitude of prime numbers*[13] is not a *reductio*. R. B. Braithwaite (1900–90), in his review of Hardy's *Apology*, specifically preferred the non-*reductio* version on aesthetic grounds.[‡]

Thus the unanimous aesthetic exaltation of 'Euclid's proof' has in fact been directed towards two intrinsically different proofs. Besides Hardy, *reductio* formulations were given by Polster, Gardiner, and Paulos; non-*reductio* versions by Dijkstra (who applied Euclid's strategy to show that there is a prime outside the collection of primes up to some prime *P*) and Aigner & Ziegler.

[1] Aigner & Ziegler, *Proofs from THE BOOK* (6th edn), pp. 23–5.

[2] Aigner & Ziegler, *Proofs from THE BOOK* (4th edn), p. 54.

[3] Compare Aigner & Ziegler, *Proofs from THE BOOK* (3rd edn), pp. 47 sq.

[*] ▸ p. 103

[4] Hardy, *Apology*, § 12.

[5] Dijkstra, 'A collection of beautiful proofs', § 5.

[6] Polster, *Q.E.D.*, p. 36.

[7] Aigner & Ziegler, *Proofs from THE BOOK* (6th edn), p. 3.

[8] Gardiner, 'Beauty in Mathematics', Theorem 1.

[9] Paulos, *Beyond Numeracy*, 'Prime Numbers'.

[10] Mac Lane, *Mathematics, Form and Function*, p. 456.

[11] Davis, Senechal & Zwicky, 'Introduction', p. xiii, emphasis added.

[12] Hardy, *Apology*, § 12.

[†] ▸ p. 503

[13] Euclid, *Elements*, Prop. IX.20.

[‡] ▸ p. 520

[Jean Dieudonné (1906–92) thought that the *result* was 'the most beautiful theorem of Greek arithmetic',[1] but, unusually, his judgement seems to have been entirely independent of the proof, for he followed the quoted description by giving a very different proof that made use of the Fundamental Theorem of Arithmetic and the formula for the partial sum of a geometric series.[2]]

Irrationality of $\sqrt{2}$. In 1986, Tommy Dreyfus (1946–) & Theodore Eisenberg (1942–) published the result of a survey of five alternative proofs of the irrationality of $\sqrt{2}$. The five proofs are as follows (with some given only in outline):[3]

i) *Proof.* Suppose for *reductio ad absurdum* that $\sqrt{2} = p/q$ for some relatively prime $p, q \in \mathbb{Z}$. Then $2q^2 = p^2$. Hence $2 \mid p^2$, and so $2 \mid p$. That is, $p = 2r$ for some $r \in \mathbb{Z}$. Thus $2q^2 = 4r^2$ and so $q^2 = 2r^2$. Hence $2 \mid q$. This contradicts the relative primality of p and q. $\square$

[This proof was one of Hardy's examples of beautiful mathematics in his *Apology*.[4]]

ii) *Proof outline..* Suppose that the polynomial $x^2 - 2$ has a rational solution p/q. Using general results about how such a rational solution divides into coefficients, deduce that $p \in \{\pm 1, \pm 2\}$ and $q = \pm 1$. A contradiction follows by showing that in none of these cases does p/q satisfy the polynomial. $\square$

iii) *Proof.* Suppose that $\sqrt{2} = p/q$ for some $p, q \in \mathbb{Z}$. Then $2q^2 = p^2$. By the Fundamental Theorem of Arithmetic, $p = 2^{\alpha}_{2}3^{\alpha}_{3}\cdots s^{\alpha}_{s}$ and $q = 2^{\beta}_{2}3^{\beta}_{3}\cdots t^{\beta}_{t}$, where every term in each product is a distinct prime. Then

$$2^{2\beta_2+1}3^{2\beta_3}\cdots t^{2\beta_t} = 2q^2 = p^2 = 2^{2\alpha_2}3^{2\alpha_3}\cdots s^{2\alpha_s}.$$

In particular, the exponent on 2 is odd on the left-hand side and even on the right hand side, which contradicts the Fundamental Theorem of Arithmetic's assertion of the uniqueness of decomposition as a product of primes. $\square$

iv) *Proof.* Suppose that $\sqrt{2} = p/q$ for some $p, q \in \mathbb{Z}$; deduce a contradiction from the fact that $2q^2 = p^2$ is insoluble in the integers because in base 3 the last non-zero digit of a square is 1 but the last non-zero digit of twice a square is 2. $\square$

[This proof is due to R. J. Gauntt.[5]]

v) *Proof outline..* Suppose that $\sqrt{2} = p/q$ and so $p^2 = 2q^2$ for some $p, q \in \mathbb{Z}$. Noting that $p > q$, write $p = q + b$ and deduce by elementary algebra that $q > b$. Write $q = b + a$ and deduce by more elementary algebra that $a^2 = 2b^2$. A contradiction follows because iterating this process would yield an infinite descending chain $p > q > b > a > \ldots$. $\square$

On the criteria of both elegance and suitability to teach to high-school students, proofs i and iii were consistently preferred in the survey.[6]

---

[1] Dieudonné, *Mathematics*, p. 87.

[2] *Ibid.*, pp. 87–9.

[3] Dreyfus & Eisenberg, 'On the Aesthetics of Mathematical Thought', pp. 7–8.

[4] Hardy, *Apology*, § 13.

[5] Gauntt & Rabson, 'The irrationality of $\sqrt{2}$'.

[6] Dreyfus & Eisenberg, 'On the Aesthetics of Mathematical Thought', p. 8.

Dreyfus & Eisenberg did not discuss the reasons for the aesthetic preference, but proofs I and III have merits such as inevitability and economy, and unlike proof II do not require enumeration of cases. Proof IV also seems somehow unnatural: why should base 3 be important? Proof V requires a certain amount of algebraic manipulation that impedes understanding.

Either Davis or Hersh or both wrote in 1981 that nine out of ten mathematician would find in proof III 'a higher level of aesthetic delight'[1] than in proof I, because it 'seems to reveal the heart of the matter, [...] exposes the "real" reason'.[2]

[Missing from Dreyfus & Eisenberg's survey is a geometric proof published by Apostol in 2001,[3] but which is actually a rediscovered variant of an earlier proof which implicitly uses Euclid's 'repeated subtraction',[4] and which the historian T. L. Heath (1861–1940), based on an argument of Hieronymus Zeuthen (1839–1920), suggested could have been known to Theodorus (c. 465 BCE–after 399 CE):[5]

Proof. Suppose for *reductio ad absurdum* that $\sqrt{2}$ is rational. Then there is a right-angled isosceles triangle with integer side lengths. Consider the smallest such triangle.

Draw an arc centred at one of the acute angles from the right-angle to the hypotenuse. From the intersection point, draw a perpendicular from the hypotenuse to the leg (see Figure 22.1). This yields a smaller right-angled isosceles triangle, whose sides have integer length since their lengths are differences between integers. This is a contradiction. ☐

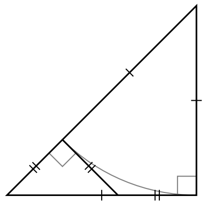

FIGURE 22.1
A proof that $\sqrt{2}$ is irrational.

This proof seems beautiful; in an essay about aesthetics, Borwein called it 'lovely'.[6]]

An irrational power of an irrational may be rational. Consider the following proof that *There exist irrational numbers a and b such that a^b is rational*:

Proof. If $\sqrt{2}^{\sqrt{2}}$ is rational, take $a = b = \sqrt{2}$; otherwise take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ so that $a^b = \sqrt{2}^2 = 2$. ☐

Imre Leader (1963–) described this proof as 'elegant'[7] and J. Roger Hindley called it 'neat',[8] but it has also been cited (not without disagreement) as an exemplar of an ugly proof,[9] precisely because it is non-constructive and openly acknowledges ignorance about whether $\sqrt{2}^{\sqrt{2}}$ is rational. [The earliest appearance of this proof seems to be[10] in a 1953 paper of Dov Jarden (1911–86).[11] In fact, $\sqrt{2}^{\sqrt{2}}$ is irrational and indeed transcendental by the Gelfond–Schneider theorem.[12] An elementary and constructive proof takes $a = \sqrt{2}$ and $b = \log_2 9$, so that $a^b = 3$. (It is straightforward to deduce the irrationality of b from the Fundamental Theorem of Arithmetic.)]

[1] Davis & Hersh, *Mathematical Experience*, p. 299.

[2] *Ibid.*, pp. 299–301.

[3] Apostol, 'Irrationality of the square root of two'.

[4] Euclid, *Elements*, Prop. X.2.

[5] Heath, *Hist. Greek Math.*, vol. I, p. 206–9; Artmann, *Euclid*, § 24.3.

[6] Borwein, 'Aesthetics for the working mathematician', pp. 27–8.

[7] Leader, 'The Axiom of Choice', p. 157.

[8] Hindley, 'The Root-2 Proof', p. 1.

[9] `math.stackexchange.com/a/433262`

[10] *Loc. cit.*

[11] Jarden, 'A Simple Proof'.

[12] Niven, *Irrational Numbers*, ch. 10.

Cantor's diagonal argument. Georg Cantor (1845–1918) introduced his now-famous 'diagonal argument' to re-prove the existence of uncountable sets and, more generally, to prove that the cardinality any set S is less than that of its power set $\mathbb{P}S$.[1] The proof of the uncountability of the interval $(0,1]$ presents the key idea and strategy:

Proof. Suppose there is an enumeration $\alpha_1, \alpha_2, \ldots$ of the real numbers in the interval $(0,1]$. Write each of these real numbers in its unique infinite decimal expansion without an infinite tail of digits 0 (so that, for example, 0.1 is written 0.099 999 …) and consider the 'diagonal' formed by the i-th decimal digit $\alpha_{i,i}$ of the i-th number:

$$\begin{array}{lcl}
\alpha_1 & = & 0.\,\alpha_{1,1}\alpha_{1,2}\alpha_{1,3}\alpha_{1,4}\alpha_{1,5}\cdots \\
\alpha_2 & = & 0.\,\alpha_{2,1}\alpha_{2,2}\alpha_{2,3}\alpha_{2,4}\alpha_{2,5}\cdots \\
\alpha_3 & = & 0.\,\alpha_{3,1}\alpha_{3,2}\alpha_{3,3}\alpha_{3,4}\alpha_{3,5}\cdots \\
\alpha_4 & = & 0.\,\alpha_{4,1}\alpha_{4,2}\alpha_{4,3}\alpha_{4,4}\alpha_{4,5}\cdots \\
\alpha_5 & = & 0.\,\alpha_{5,1}\alpha_{5,2}\alpha_{5,3}\alpha_{5,4}\alpha_{5,5}\cdots \\
\vdots & & \vdots \qquad\qquad\qquad\quad \ddots
\end{array}$$

Let $\beta = 0.\,\beta_1\beta_2\beta_3\beta_4\beta_5\cdots$ be defined by letting $\beta_i \in \{1,2\}$ be unequal to $\alpha_{i,i}$. Then $\beta \in (0,1]$ and β differs from each α_i in the i-th place. Thus $\alpha_1, \alpha_2, \ldots$ was not an enumeration of the real numbers in the interval $(0,1]$. ☐

Cantor had earlier given two proofs that the set of real numbers $\mathbb{R}$ is uncountable; both proofs make use of the structure of $\mathbb{R}$ via real analysis.[2] The diagonal argument requires no such structure. As Cantor himself recognized, the diagonal argument was 'much simpler'.[3]

It has been given by philosophers as an example of a beautiful[4] or elegant[5] proof. There seems to be no dissent among mathematicians: it is one of the *Proofs from THE BOOK*;[6] Hardy (who did not separate the beauty of theorems and proofs[7]) said it was among the 'beautiful theorems' of set theory;[8] Gardiner listed it as an example of beautiful mathematics;[9] Paulos said that proofs connected with countability were 'particularly beautiful'[10] shortly before presenting the diagonal argument.

Kelly's proof of the Sylvester–Gallai theorem. In 1893, James Joseph Sylvester* (1814–97) conjectured an elementary geometrical result that remained unproven for forty years; Erdős independently made the same conjecture in 1933 and his friend Tibor Grünwald (1912–92) supplied the first proof.[11] Grünwald later changed his surname to Gallai; hence the modern name for the theorem:

SYLVESTER–GALLAI THEOREM. *In any finite set of points in the plane, not all collinear, there is a line that contains exactly two of the given points.*

[1] Cantor, 'On an Elementary Question'.

[2] Franks, 'Cantor's Other Proofs that $\mathbb{R}$ is Uncountable'.

[3] Cantor, 'On an Elementary Question', p. 920.

[4] For example Dutilh Novaes, 'The Beauty (?) of Mathematical Proofs', pp. 69 sqq.

[5] Montano, *Explaining Beauty in Mathematics*, ch. 12.

[6] Aigner & Ziegler, *Proofs from THE BOOK* (6th edn), p. 128.

[7] Hardy, *Apology*, § 18.

[8] *Ibid.*, § 13.

[9] Gardiner, 'Beauty in Mathematics', Theorem 4.

[10] Paulos, *Beyond Numeracy*, 'Infinite Sets'.

* ▸ pp. 438–46

[11] Erdős, 'Personal Reminiscences', p. 208.

The proof of the Sylvester–Gallai theorem due to Leroy M. Kelly[1] (1914–2002) was included in *Proofs from THE BOOK* through all six editions:[2]

Proof (*Kelly*). Let X be the given set of points. Then there is at least one pair (ℓ, P) where ℓ is a line through a pair of points in X and P is a point in X that does not lie in ℓ. There are only finitely many such pairs since X is finite. Among all such pairs, fix one where the distance from ℓ to P is minimal. Suppose for *reductio ad absurdum* that ℓ contains three points in X. Let Q be the point on ℓ closest to P. (Note that Q may or may not be in X.) Consider the two segments into which Q divides ℓ, considering Q itself to belong to both segments. Since there are at least three points of X on ℓ, there are at least two points $R, S \in X$ in one of these segments. Assume that S lies between Q and R. (See Figure 22.2.) Then S lies closer to the line PR than P does to ℓ, which contradicts the choice of (ℓ, P). Hence ℓ contains only two points. ☐

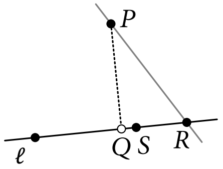

FIGURE 22.2
Kelly's proof of the Sylvester–Gallai theorem.

The statement of the Sylvester–Gallai theorem uses only the notions of incidence and collinearity. Kelly's proof, although certainly brief and elementary, makes use of parallelism and distance, which are not involved in the statement of the result. H. S. M. Coxeter (1907–2003) thought these concepts were 'foreign to the problem'[3] and compared Kelly's proof to 'using a sledge hammer to crack an almond'.[4] Coxeter's own 'nutcracker',[5] as he called it, was a proof that only used incidence, collinearity, and *intermediacy* (or 'betweenness')[6]

Proof (*Coxeter*). Let X be the given set of points. Choose three points $A, B, C \in X$ that form a triangle. Pick a line ℓ through A that does not meet any other point in X. Let P' be the intersection of ℓ with the line BC. Consider all intersections of ℓ with lines through pairs of points in X, and let P be the first such intersection along the segment AP' after A itself (possibly $P = P'$).

Suppose for *reductio ad absurdum* that every line through a pair of points in X contains at least three points from X. In particular, P lies on a line containing three points $Q, R, S \in X$. Relabelling these points if necessary, assume that the segment PR contains Q but not S. Since $A, R \in X$, by the supposition there is a third point $O \in X$ on the line AR. (See Figure 22.3.) There are now two cases: if O lies in the segment AR (marked (1) in the diagram), then the line SO intersects the segment AP (regardless of which side of the segment PR the point S lies); if O lies outside the segment AR (marked (2a) and (2b) in the diagram), then the line QO intersects the segment AP. In either case, there is an intersection of a line through a pair of points in X with the segment AP, which contradicts the choice of P. ☐

Coxeter did not evaluate his own proof in aesthetic terms, but his objection put a question mark over Kelly's proof truly being the best. This is not to say that Coxeter's proof was beautiful: it requires

[1] Coxeter, 'A problem of collinear points', p. 28.

[2] Aigner & Ziegler, *Proofs from THE BOOK* (1st edn), p. 45; Aigner & Ziegler, *Proofs from THE BOOK* (6th edn), p. 77.

[3] Coxeter, 'A problem of collinear points', p. 27.

[4] Coxeter, *Introduction to Geometry*, p. 181.

[5] *Loc. cit.*

[6] Coxeter, 'A problem of collinear points', pp. 27–8; Coxeter, *Introduction to Geometry*, § 12.2.

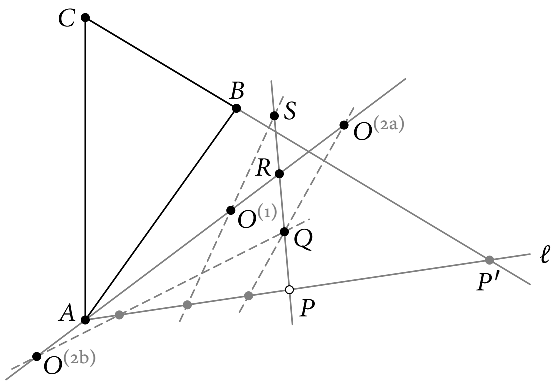

FIGURE 22.3
Coxeter's proof of the Sylvester–Gallai theorem.

the auxiliary line ℓ and two cases (one of which arguably divides into subcases).

In 2016 Matthew Inglis & Andrew Aberdein published a study of the diversity of mathematician's appraisals of Kelly's proof of the Sylvester–Gallai theorem on four independent dimensions of evaluation that had been identified in the same authors' 2015 study of the correlation of evaluative terms, which is discussed later in this chapter.[*] The four dimensions of evaluation were *aesthetic*, *intricacy*, *precision*, *utility*.

Participants in the study, all research-active mathematicians, were asked to score on a scale of 1 to 5 how well a set of sixteen adjective, four applying to each dimension, described Kelly's proof, which was chosen because it was non-trivial but accessible. The adjectives for each dimension were obtained by choosing those that had the highest loading onto that dimension in the previous study; for the aesthetic dimension, these were *ingenious*, *inspired*, *profound*, *striking*. (There were adjustments to the adjectives chosen for the intricacy and utility dimensions.)[1] The common aesthetic terms *beautiful* and *elegant* had less strong loadings onto this dimension; they both had a moderate negative loading onto the intricacy dimension. This result fitted with anecdotal evidence that these terms suggest some ease of understanding. In the earlier study, the four aesthetic dimension adjectives were found to have correlations with *beautiful* of .6, .58, .46, and .59, respectively,[2][†] but a degree of caution is required in drawing conclusions about judgements of beauty alone.

The scores for each set of four adjectives were combined to produce a score in the range of 4 to 20 for each dimension.[3] For each of the four dimensions, there was great disagreement amongst the ratings assigned by participants.[4] Furthermore, neither career stage nor research area were found to be correlated with participants' assessments.[5] Such a wide distribution tended to suggest that proof evaluations on *any* of the dimensions were not shared broadly amongst research mathematicians.

[*] ▸ pp. 858–9

[1] Inglis & Aberdein, 'Diversity in Proof Appraisal', pp. 175–8.

[2] Inglis & Aberdein, 'Beauty is not simplicity', Tbl. 4.

[†] ▸ p. 859

[3] Inglis & Aberdein, 'Diversity in Proof Appraisal', pp. 169–70.

[4] *Ibid.*, p. 173.

[5] *Loc. cit.*

In particular, this study counted against the existence of an aesthetic canon of proofs, or at least indicated disagreement on whether this particular proof should be part of the canon. It is significant that 60.4 % of participants scored it below the midpoint on the aesthetic dimension, especially since it was selected by Aigner & Ziegler for inclusion in *Proofs from THE BOOK*.

Reductio ad absurdum

There seems to have been some disagreement on whether *reductio ad absurdum* proofs are beautiful, and it is fortunate for the historian that several authors expressed a view on this during a narrow period in the mid-twentieth century. In 1940, Hardy praised *reductio* proof in *A Mathematician's Apology*;[*][1] the following year, Braithwaite expressed an aesthetic preference for proofs that avoid it.[†][2]

In 1944, Raymond Wilder (1896–1982) presented a nuanced view of the aesthetics of *reductio ad absurdum* generally: he averred that it was valued 'for its elegance and its incisive character',[3] and that it played a necessary role in proofs of non-existence. However, he maintained that many mathematicians thought that to prove existence, one should avoid *reductio* in favour of a constructive argument that actually builds an element of the class in question. In this situation, a *reductio* argument should be preferred only if it offers a major increase in simplicity.[4] As an illustration, he considered replacing the 'extremely elegant proofs of the fundamental theorem of algebra which use the contra-positive method'[5] by intuitionistic proofs. [Presumably the 'elegant proofs' he had in mind were those that show, for a complex polynomial $p \in \mathbb{C}[z]$, that if the minimum of $|p(z)|$ is non-zero, then a contradiction follows.] Wilder thought that a decision like this would ultimately depend on one's foundational convictions.[6]

[Wilder took some amusement from modern mathematicians being so used to *reductio* proofs that they 'automatically take an argument that is ideally suited to the constructive form of proof, and twist it into a reductio ad absurdum'.[7]]

Finally, in 1948, François Le Lionnais wrote that *reductio ad absurdum* proofs had romantic beauty.[8][‡]

Long proofs

> ‘I fear that as they grow in length
> our demonstrations will lose that appearance of harmony’
>
> — Henri Poincaré, Poincaré, 'The Future of Mathematics'
> (trans. F. Maitland).

In 2005, Borwein said that 'several of the recent very long proofs are neither simple nor beautiful'.[9] Without going into the question of how to measure the length of a proof or what constitutes 'long', it has been generally accepted that the proof of the classification of finite

[*] ▸ p. 503

[1] Hardy, *Apology*, § 12.

[†] ▸ p. 520

[2] Braithwaite, Review of *A Mathematician's Apology*, p. 421.

[3] Wilder, 'The nature of mathematical proof', p. 312.

[4] *Loc. cit.*

[5] *Ibid.*, p. 313.

[6] *Loc. cit.*

[7] *Ibid.*, p. 312.

[8] Le Lionnais, 'La Beauté en Mathématiques', p. 452.

[‡] ▸ p. 677

[9] Borwein & Stanway, 'Knowledge and community in mathematics', p. 12.

simple groups is the canonical example of a long proof. The original version is scattered across journal articles estimated at a total of ten to fifteen thousand pages,[1] and the expectation was that a revised, consolidated, and simplified version would occupy several thousand monograph pages.[2] By any measure, or by comparison with any other proof, it is gargantuan.

Michael Atiyah (1929–2019) admitted to having 'mixed feelings' about this result, because it seemed to grant only a limited amount of understanding.[3] A proof, for Atiyah, was important as a verification that one had understood. A proof that lacked coherence, that just hung together, that one did not understand as a whole, that one worried might contain a flaw, could not carry the same stamp of truth.[4]

Davis & Hersh observed that there was no clear division between very long proofs such as that of the classification and proofs by computer.[5] Their point seems to have been primarily epistemological: that there was no reason for accepting a proof too long for any one person to read, but rejecting a proof by computer. There seems, however, to be an important aesthetic difference. A reader can delve into a long proof and perhaps enjoy the beauty of subsidiary results or their proofs or the constructions devised. A reader who delves into the computer-aided part of the Appel–Haken proof of the four-colour theorem would encounter the monotony of minor variations on a single theme.

Theories

There seems to have been little work on the aesthetic comparisons of theories, and so there is little to say about consensus here.

Rota claimed that 'most mathematicians are likely to agree' that the theory of finite fields and the Galois theory of equations are beautiful.[6] The author has not located any opposition to this assertion, unlike the fate of Rota's pronouncements on proofs and theorems.*

Rota also made the interesting observation that there may be a divide between the views of mathematicians and non-mathematicians on aesthetic value in mathematics. In the specific case of theories, he suggested that classical euclidean geometry was considered to be a beautiful theory by non-mathematicians (as the poet said, 'Euclid alone has looked on beauty bare'[7]), but that he did not know of it being considered beautiful by mathematicians.[8]

Bourbakism

Nicolas Bourbaki was the pseudonym chosen by a group of mostly French mathematicians who, in 1935, set about reformulating mathematics in a unified way, aiming at a high level of rigour and formalization, and taking as fundamental the notion of mathematical structure,[9]

[1] Gorenstein, Lyons & Solomon, *The Classification of the Finite Simple Groups*, p. 3.

[2] *Loc. cit.*

[3] Atiyah & Minio, 'An Interview With Michael Atiyah', p. 18.

[4] *Ibid.*, p. 17.

[5] Davis & Hersh, *Mathematical Experience*, p. 390.

[6] Rota, 'Phenomenology of Mathematical Beauty', p. 173.

* ▸ pp. 832–3

[7] Millay, *The Harp-Weaver and Other Poems*, pt 4, sonnet XXII.

[8] Rota, 'Phenomenology of Mathematical Beauty', p. 171.

[9] Senechal, 'The Continuing Silence of Bourbaki', p. 22; Dieudonné, 'The Work of Nicholas Bourbaki', pp. 137–8.

with the end of creating a self-contained set of texts that did not depend on any others.[1]

There appears to have been a rather wide range of opinions on the Bourbakian approach. Some of the negative views may be due to the books being misused as teaching textbooks. Pierre Cartier (1932–2024) considered the first part of the Bourbaki corpus to be an encyclopaedia: 'as a textbook, it's a disaster',[2] he said, although he admitted that, owing to an absence of other textbooks, he himself had learned from each new Bourbaki volume.[3] But others thought highly of the style. Arjeh Cohen (1949–) thought Bourbaki evoked 'glamour, rigor, and elegance',[4] and that the chosen approach had produced elegance that he and others had enjoyed.[5]

There were also some aesthetic influences on the very development of Bourbakian mathematics. For Claude Chevalley (1909–84), a founding member of the Bourbaki group,[6] rigour consisted in crystallizing, rendering inert and unchanging, a mathematical object he admired. Lack of rigour in a proof was like wading through mud; such a proof was *unclean*. He was described by his daughter Catherine (1951–) as 'probably the only member of Bourbaki who thought of mathematics as a way to put objects to death for esthetic reasons'.[7] For Chevalley, the rigour of mathematics was thus ultimately aesthetic, a kind of conservation in preparation for exhibition.[8]

Lang, Mordell, Siegel

An aesthetic disagreement on the exposition of theories arose between Serge Lang (1927–2005) on the one hand and Louis Mordell (1888–1972) and Carl Ludwig Siegel (1896–1981) on the other. In 1962, Lang published *Diophantine Geometry* (a book that opens with an aesthetic motivation: 'Diophantine problems represent some of the strongest aesthetic attractions to algebraic geometry'[9]).

Mordell's review of *Diophantine Geometry* was complimentary about its comprehensiveness and system, but critical on the grounds of style, exposition, clarity.[10] He noted that Lang seemed not to care about the accessibility of his works, noting[11] that Lang thought that advanced monographs were about presenting 'one's vision of some beautiful part of mathematics'.[12] Mordell sent a copy of his review to Siegel, who responded with a letter that made clear his views of the book:

> 'I was disgusted with the way in which my own contributions to the subject had been disfigured and made unintelligible […].
>
> The whole style of the author contradicts the sense for simplicity and honesty which we admire in the works of the masters in number theory — Lagrange, Gauss or, on a smaller scale, Hardy, Landau. Just now Lang has published another book on algebraic numbers which, in my opinion, is still worse

[1] Senechal, 'The Continuing Silence of Bourbaki', p. 27.

[2] Cartier, quot. in *ibid.*, p. 24.

[3] *Ibid.*, p. 25.

[4] Cohen, 'Communicating Mathematics Across the Web', p. 296.

[5] *Loc. cit.*

[6] Guedj, 'Nicholas Bourbaki', p. 18.

[7] Chevalley, 'Claude Chevalley'; trans. Senechal, 'The Continuing Silence of Bourbaki', p. 26.

[8] Bell, 'Reflections on Mathematics and Aesthetics', p. 161.

[9] Lang, *Diophantine Geometry*, p. v.

[10] Mordell, Review of *Diophantine Geometry*, *passim*, esp. p. 498.

[11] *Ibid.*, p. 493.

[12] Lang, *Short Calculus*, p. v.

> than the former one. I see a pig broken into a beautiful garden and rooting up all flowers and trees.’[1]

In a later exchange of letters with Mordell, with whom he seems to have remained on good terms personally, Lang conceded that while tastes may vary, a mathematical author must follow his own artistic feeling: ‘just as a composer of music [...], I have to take my responsibility as to what *I* consider to be beautiful, and write my books accordingly’.[2]

Lang later wrote that what Siegel and others saw as ‘senseless abstraction’[3] actually formed a necessary prelude to a unification of topology, complex differential geometry, and algebraic geometry.[4] But he also said that, for all their skill as mathematicians, the ‘lack of vision and understanding’[5] of Mordell and Siegel had a detrimental effect on the development of mathematics, especially in the places they wielded influence:

> ‘Mordell used to pull out Siegel’s letter from his wallet to show people, to my knowledge without receiving comments that both his and Siegel’s attitudes were parochial and blind (if not worse).
>
> Thus some members of the mathematical community behaved with “collegiality” and bowed to authority’.[6]

[Regardless of whether one considers correct Lang’s judgement that Mordell’s and Siegel’s influence was unintentionally malign, one aspect of this affair rankles. Lang published his account[7] in 1995. Mordell died in 1972; Siegel in 1981. ‘τὸν τεθνηκότα μὴ κακολογεῖν’.[8]]

Problems

One of the most famous anecdotes in mathematics points to an assumed aesthetic agreement about problems. In his obituary of Ramanujan, Hardy recorded the following exchange:

> ‘I had ridden in taxi-cab № 1729, and remarked that the number $(7\cdot 13\cdot 19)$ seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. “No,” he replied, “it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways.” I asked him, naturally, whether he knew the answer to the corresponding problem for fourth powers; and he replied, after a moment’s thought, that he could see no obvious example, and thought that the first such number must be very large.’[9]

In fact, Euler had discovered[10] that $635\,318\,657 = 133^4 + 134^4 = 158^4 + 59^4$, and this is the smallest such number. But notice Hardy’s use of ‘*naturally*’: he must have thought that his question was the one that most mathematicians would have chosen as the next step. Others have agreed: ‘There is a sense, however, in which Hardy’s

[1] Siegel, letter to Mordell, dated 3 Mar. 1964, facs. in Lang, ‘Mordell’s review’, p. 18.

[2] Lang, letter to Mordell, dated Nov. 1966, repr. in Lang, Review of *Diophantine Equations*, p. 1234.

[3] See Siegel, letter to Mordell, dated 3 Mar. 1964, facs. in Lang, ‘Mordell’s review’, p. 18.

[4] *Ibid.*, p. 21.

[5] *Ibid.*, p. 31.

[6] *Ibid.*, pp. 31–2.

[7] *Ibid.*

[8] Diogenes Laertius, ‘Chilon’, MEI 70.

[9] Hardy, ‘Srinivasa Ramanujan’, p. xxxv.

[10] Dickson, *Hist. Theory of Numbers*, vol. II, pp. 644–7.

generalization is “the most mathematician-like”.[1] Yet there are other ways to generalize the problem. For example, for a given k, there are numbers expressible as a sum of cubes in k different ways,[2] and so one could ask for the smallest such number. (The answer is known for $k \leqslant 6$.[3])

META-CANON

Regardless of whether there is agreement on the aesthetic judgement of mathematics, there remains the related question of whether there is agreement on the standards or factors that influence aesthetic judgements. For instance, the loci of beauty, the terminology used, the criteria that should be applied. Kenneth R. Conklin (1942–) thought that there was such a meta-canonical set of standards and factors:

> ‘Just as there are standards of aesthetic judgment for works of art, music, and drama, so we should expect there to be standards of aesthetic judgment for mathematical products — and indeed *there are such standards*.’[4]

Ziegler disagreed, at least with regard to criteria for beauty:

> ‘We’ve always shied away from trying to define what is a perfect proof. […] actually, *there is no definition and no uniform criterion*.’[5]

The issue certainly deserves critical examination, but only a very limited number of works have touched upon it.

Loci

There seems to be a tension between the beauty of a mathematical theory and that of its components — the individual theorems, problems, definitions, objects, and constructions that it comprises. Does the beauty of a theorem depend on or supervene on the theory in which it sits? Or does the beauty of a theory depend on or supervene on its theorems?

Gardiner certainly saw beauty in both theorems and theories:

> ‘Not just equations, nor isolated theorems, but whole mathematical theories have considerable beauty. The exposition is important here. For this reason I link the theories with specific books. For example, we have Gauss’s treatise *Disquisitiones Arithmeticae* on number theory, *Galois theory* by Artin, *Quantum Mechanics* by Dirac, *Fundamental Theory* by Eddington, *Principia* by Newton, and so on. Would the latter

[1] Hofstadter, *Gödel, Escher, Bach*, p. 564.

[2] Hardy & Wright, *Theory of Numbers*, Theorem 412.

[3] OEIS, A011541.

[4] Conklin, ‘The Aesthetic Dimension of Education’, p. 29, emphasis added.

[5] Ziegler, quot. in Klarreich, ‘In Search of God’s Perfect Proofs’, emphasis added.

> be more beautiful if written in the language of the calculus? […]
>
> These are works of art on a grand scale like symphonies and quartets in music. […] But there are also beautiful results on a smaller scale — the "songs" of mathematics.'[1]

Atiyah said that he did not take seriously questions about 'a favourite theorem or problem', instead seeing theorems as mere markers in the body of mathematics.[2] But he said that a theory was interesting if it enabled one to solve special problems and place them in context,[3] which suggests that he thought at least that the beauty of a theorem could not be evaluated independently from its situation in a theory.

Timothy Gowers (1963–) wrote about 'the two cultures of mathematics',[4] namely that of *solving problems* versus that of *building and understanding theories*. The problem-solver and theory-builder cultures are not truly divorced from one another, as C. P. Snow (1905–80) saw the scientific and literary cultures of his time as being.[5] Rather, there was a difference of emphasis: was the point of solving problems to build and understand theories, or was the point of building and understanding theories to solve problems? Alexander Grothendieck (1928–2014) seemed to think of the same division when he distinguished the 'hammer-and-chisel' and the 'rising sea' approaches to problem-solving: the former was a direct attack on the problem, perhaps striking it from many angles until it cracked open; the latter left the problem to become immersed and dissolved.[6] Ernst Straus (1922–83) seems to have had a similar, though not quite identical, distinction in mind when he wrote of dividing mathematicians into 'Euler types and Gauss types'.[7]

There seems to be some connection, albeit little explored, between this division and disagreement on the loci of mathematical beauty. Erdős himself was a problem-solver *par excellence*. He was attracted to beautiful problems and had little or no interest fitting his work into the context of a theory. Straus identified Erdős as an example of the 'Euler type',[8] and also once wrote that Erdős was a 'Don Juan' rather than a 'Sir Galahad' like Einstein, who chose physics instead of mathematics precisely because there were so many beautiful problems in mathematics that could distract scientists from their duty to pursue the central questions.[9] And Erdős's pursuit of beautiful problems put him at odds with Hermann Weyl (1885–1955), whom Straus thought a 'Gauss type'.[10] *

A question of identity

One can become acquainted with the same theorem, proof, example, or other mathematical artefact, via different media. The same proof, for instance, could communicated via a text, lecture, or a less formal oral presentation. And even considering only printed media, the same proof might be presented very differently

[1] Gardiner, 'Beauty in Mathematics', p. 79.

[2] Atiyah & Minio, 'An Interview With Michael Atiyah', p. 18.

[3] *Loc. cit.*

[4] Gowers, 'Two Cultures'.

[5] Snow, *The Two Cultures*.

[6] McLarty, 'The Rising Sea', pp. 301–2.

[7] Straus, 'Note on the Controversy', p. 22.

[8] *Loc. cit.*

[9] Straus, 'Paul Erdős at 70', p. 246; cf. Einstein, 'Autobiographical Notes', pp. 15, 17.

[10] Straus, 'Note on the Controversy', p. 22.

* ▸ pp. 679–80

in a textbook, a monograph, and a research article. But such near-banal observations prompt the fundamental question of when two pieces of communicated mathematics can be regarded as expressing *the same* theorem or proof or example, and in particular when they can be considered *the same* for the purposes of aesthetic evaluation. This question seems to be profoundly important for the aesthetics of mathematics, but has been, perhaps surprisingly, little studied.

The issue remains even in the restricted context of the textual presentation of formalized proofs. In proof theory, two formalized proofs are regarded as the same if and only if they comprise identical strings of symbols. But even then, proofs can be more similar or less similar. David Hilbert's (1862–1943) 'twenty-fourth problem'[*] included the question of exploring the area between two proofs, and in his notes he suggested that *syzygies* offered a means of comparing proofs.[1] Loosely, first-level syzygies are relations between sequences of rewriting. Second-level syzygies comprise relations between first-level syzygies, and so on to higher levels. What Hilbert seems to have been thinking was that the 'distance' between two formal proofs — which are sequences of rewriting — could correspond to the number of second-level syzygies required to connect them.[2] But Hilbert excluded this problem from his Paris lecture and so it is unsurprising that it had no impact on the subsequent discourse about aesthetic evaluation.

It seems that the only extended discussion of the question of identity of mathematical artefacts from an aesthetic perspective was by Ewa Bigaj, who devoted much of the second part of her 2020 doctoral thesis to aesthetic comparisons of proofs. Bigaj pointed out that the proof-theoretic conception of the identity of formal proofs was too strict for general use. If, for instance, one wished to compare the computational effectiveness of proofs, identity seemed to involve a higher conceptual level. Different purposes led to different assessments of sameness.[3]

Bigaj illustrated the question with three variants on Euclid's proof of the infinitude of the prime numbers.[†] Two of the variants mirrored the same global and local *reductio* as in Hardy's account and in Euclid's original presentation.[‡] But *on some level* the variants were 'the same'. Bigaj made an analogy with music: two people might become familiar with a musical work through different interpretations or performances. When one of them made an aesthetic evaluation, it was necessary to look to the *reasons* given for the evaluation to see whether two performances or interpretations were 'the same'.[4] An aesthetic evaluation of a performance of *Die Zauberflöte* might be based on Mozart's score or Schikaneder's libretto, the staging, the conductor's interpretation, or the singing. If the basis of the evaluation was Mozart's score, then any two performances could be regarded as 'the same', for they shared this core. If the basis was the quality of the *coloratura* in 'Der Hölle Rache', then (assuming that different sopranos sang the part, or that the same soprano sang it differently),

[*] ▸ p. 466

[1] Hilbert, quot. in Thiele, 'Hilbert's Twenty-Fourth Problem', p. 2.

[2] Malheiro & Reis, 'Identification of proofs via syzygies', esp. § 3.

[3] Bigaj, 'Three Essays', pp. 73–5.

[†] ▸ pp. 103, 845

[‡] ▸ p. 503

[4] *Ibid*., pp. 76–9.

the performances had to count as different. For Bigaj, making such distinctions was similarly important for deciding whether to regard two communicated proofs as the same. When two people exchanged aesthetic evaluations of proofs, whether they could be regarded as making an aesthetic comparison of the *same* proof depended on the reasons for their evaluations: if the reasons were grounded in the core ideas, then they could be considered aesthetic evaluations of the same proof even though they had read different presentations.[1]

Elegance, simplicity, beauty: a single evaluative braid?

> ‘There seem to be some “inevitable” combinations of aesthetic words that are mathematically invoked as if conjoined: for instance, beauty *and* elegance, perfection *and* beauty.’
>
> — Nathalie Sinclair & David Pimm, ‘A Historical Gaze’, p. 16, emphases in orig.

Horace Romano ‘Rom’ Harré (1927–2019) wrote, as part of his discussion of ‘quasi-aesthetic appraisals’,[*] that the terms *simple* and *elegant* appeared together often, and in that order, but never in the opposite order. Certainly the terms — and indeed the concepts — have often occurred together. Mac Lane causally linked them:

> ‘Euclid’s proof that there are infinitely many primes or the proof of the Cauchy–Goursat theorem […] are said to be elegant because each gets a “deep” result with a minimum of complexity.’[2]

John Bell (1945–) thought beauty in mathematics dependent (or at least supervenient) on properties including simplicity and elegance:

> ‘The beauty of a mathematical concept often rests on the contrast between the simplicity and elegance of the concept itself and the richness and variety of the mathematical structures embodying it.’[3]

Marc Chamberland described as ‘deceptively simple and beautiful’[4] Steve Fisk’s (1946–2010) proof of the ‘Art Gallery theorem’ (which is one of the *Proofs from THE BOOK*[5]).[†]

On the other hand, Rota wrote that the combinatorial account of the representations of the symmetric group by Alfred Young (1873–1940) was ‘the simplest though not the most elegant’.[6] Bollobás called ‘difficult and beautiful’[7] Gowers’s proof of the theorem *There is a constant* C *such that any subset of the first* N *integers of size at least* $\delta N > 0$*, and such that* $N \geqslant \exp\exp\exp((1/\delta)^C)$ *contains an arithmetic progression of length* 4. (The lower bound on N in this proof was a huge improvement on the previously-known bound.) These comments suggest that simplicity did not always accompany beauty or elegance.

[1] Bigaj, ‘Three Essays’, pp. 77–9.

[*] ▸ pp. 734–6

[2] Mac Lane, *Mathematics, Form and Function*, p. 456.

[3] Bell, ‘Reflections on Mathematics and Aesthetics’, p. 174.

[4] Chamberland, *Single Digits*, p. 97.

[5] Aigner & Ziegler, *Proofs from THE BOOK* (6th edn), p. 266.

[†] ▸ pp. 805–6

[6] Rota, ‘Combinatorics, Representation Theory and Invariant Theory’, p. 42.

[7] Bollobás, ‘The Work of William Timothy Gowers’, p. 663.

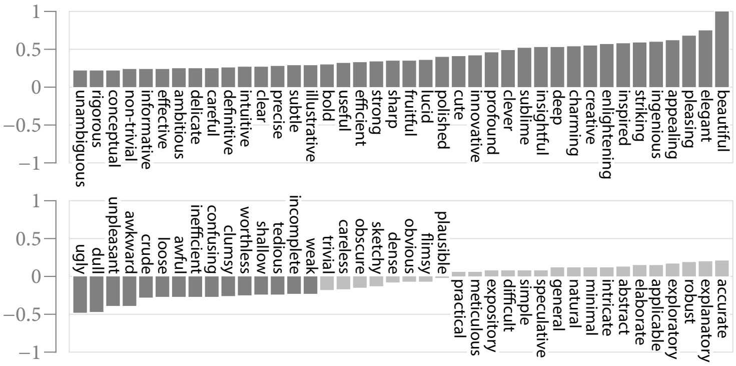

FIGURE 22.4
Graph showing the Spearman rank correlations between each of eighty adjectives and 'beautiful' when used to describe proofs. The coefficients categorized as significantly different from zero (greater than ±0.21) are shown in dark grey. (Prepared from Inglis & Aberdein, 'Beauty is not simplicity', Tbl. 4)

Do simplicity, elegance, beauty appear together in mathematics? The putative connection of simplicity to beauty was undermined by a 2015 study conducted by Inglis & Aberdein in which 255 mathematicians, including doctoral students, post-doctoral researchers, and faculty members, were asked to fix in mind a proof they had recently read and, for each of eighty supplied adjectives, rate how well that adjective described the chosen proof using the scale {very inaccurate, inaccurate, neither accurate nor inaccurate, accurate, very accurate}.[1] (There is a potential weakness in this study, in that the adjectives were chosen by the researchers, so it is unclear whether the full range would have been applied by mathematicians themselves.[2])

Statistical analysis of the results suggested that the adjectives loaded onto four independent dimensions of evaluation:

- an *aesthetic* dimension: *beautiful*, *elegant*, *charming*, *dull*, …;
- an *intricacy* dimension: *dense*, *intricate*, *simple*, *clear*, …;
- a *utility* dimension: *practical*, *informative*, *efficient*, …;
- a *precision* dimension: *careful*, *precise*, *meticulous*, ….[3]

(Note that both positive and negative terms appear in the same group, for example, *charming* and *dull*.) This result meant that evaluations of proofs seemed to vary independently along these dimensions: a proof could be seen as beautiful and intricate, or as simple and dull. Furthermore, if one looked only at the correlation of adjectives with *beautiful* (see Figure 22.4), then *elegant* was strongly correlated (r_s = .75), but *simply* only weakly (r_s = .21).[4]

Note that there was a fairly strong correlation with *enlightening* with beauty (r_s = .57).[5] Alone this is not evidence in favour of Rota's reinterpretation of mathematical beauty as enlightenment,* but it does suggest that being enlightening could be a property of proofs that contributes to beauty. There was a weaker correlation with *explanatory* (r_s = .2),[6] which points to a more complicated relationship between explanatoriness and beauty.†

[An analogous 1983 study of attitudes to mathematics among eighth-grade students in Japan (second year of middle school, ages 13–14) reached a comparable conclusion with regard to school mathematics: it suggested that the evaluation of school mathematics on a beautiful–ugly range was independent of its evaluation on other ranges such as boring–interesting, valuable–worthless, love–hate, and enjoyable–unenjoyable (which appear to load onto an evaluative dimension), heavy–light, difficult–easy and painful–painless (a potency dimension), and clear–obscure and unclear–plain (a clarity dimension).[1]]

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[1] Inglis & Aberdein, 'Beauty is not simplicity', pp. 95–6.

[2] Todd, 'Fitting Feelings and Elegant Proofs', p. 217.

[3] Inglis & Aberdein, 'Beauty is not simplicity', pp. 96–100.

[4] *Ibid.*, Tbl. 4.

[5] *Loc. cit.*

* ▸ pp. 742–5

[6] *Loc. cit.*

† ▸ pp. 787–90, 804–12

Judging mathematics, judging art

Three studies by Samuel G. B. Johnson & Stefan Steinerberger, published in 2019, examined whether people made consistent aesthetic categorizations in comparing short mathematical arguments with paintings and music, and tried to understand how they make such comparisons.[2] The studies used the following four arguments, here given with the names Johnson & Steinerberger assigned them for brevity:

GEOMETRIC: A visual proof that the geometric series $\sum_{i=1}^{\infty} 1/2^i$ sums to 1 (see Figure 22.5).

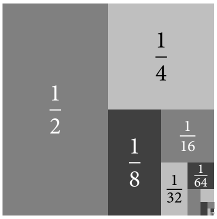


FIGURE 22.5
A proof of $\sum_{i=1}^{\infty} 1/2^i = 1$.

GAUSS: The trick for summing the arithmetic series $1 + 2 + \ldots + 100$, attributed to Carl Friedrich Gauss[*] (1777–1855), presented in a visual form.[†]

PIGEONHOLE: A proof that *In a simple graph with five vertices, there are two vertices of the same degree.* This result and its proof were stated in terms of a group of five people and the (implicitly symmetrical) relation of friendship. The proof, in graph theoretic terms, is as follows: If some vertex is adjacent to all other vertices, then each vertex has degree in the set $\{1, 2, 3, 4\}$. On the other hand, if no vertex is adjacent to all others, then each vertex has degree in the set $\{0, 1, 2, 3\}$. In both cases, there are five vertices and four possible degrees. Hence, by the pigeonhole principle, at least two have the same degree.

FAULHABER: The 'pebble arithmetic' proof that a series of consecutive odd numbers starting at 1 sums to a square number.[‡]

It was perhaps a weakness of the studies that two of the arguments used, GEOMETRIC and FAULHABER, were visual proofs[§] and indeed 'proofs without words'.[¶] The choice of arguments was influenced by the necessity of making them accessible to all the participants. In each study, there was a large group of participants, only some of whom had taken university-level mathematics courses. Two of the three studies also included small groups with relatively greater knowledge: professional mathematicians for the first study, and undergraduates studying mathematics in the second.[3]

The first study asked participants to rate, on a scale of 0 to 10, the similarity of each the four arguments to each of the following four landscape paintings (see Figure 22.6):

[1] Minato, 'Some mathematical attitudinal data'.

[2] Johnson & Steinerberger, 'Intuitions about mathematical beauty', p. 244.

[*] ▸ p. 401

[†] ▸ p. 778

[‡] ▸ pp. 46–7

[§] ▸ pp. 780–2

[¶] ▸ pp. 781–2

[3] *Ibid.*, pp. 245, 246, 248.

YOSEMITE: Albert Bierstadt's (1830–1902) *Looking Down Yosemite Valley, California*.

ROCKIES: Bierstadt's *A Storm in the Rocky Mountains, Mt. Rosalie*.

SUFFOLK: John Constable's (1776–1837) *The Hay Wain*.

ANDES: Frederic Edwin Church's (1826–1900) *Heart of the Andes*.

The ratings assigned for argument–painting similarities were in general low: the highest mean rating was 3.51 for the GEOMETRIC–YOSEMITE pairing; the lowest was 1.96 for the GAUSS–ANDES pairing.[1] Various statistical analyses showed that the similarity judgements were not random. There was some consensus about the similarity of arguments and paintings.[2] In particular, participants reliably believed that YOSEMITE was most similar to GEOMETRIC, and SUFFOLK most similar to GAUSS and FAULHABER, and ANDES to PIGEONHOLE.[3] There was relatively little variance in similarity rankings based on different levels of mathematics education within the large group, but there was only a low correlation of these rankings with the ratings assigned by the smaller group of professional mathematicians.[4]

The second study again asked participants to rate, on a scale of 0 to 10, the similarity of each these arguments to an excerpt (of duration 20 s) of each of the following four pieces of piano music:

---

[1] Johnson & Steinerberger, 'Intuitions about mathematical beauty', Tbl. 1.

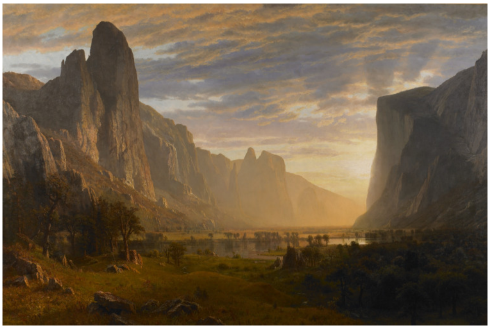
[2] *Ibid.*, pp. 245–6.

[3] *Ibid.*, p. 246.

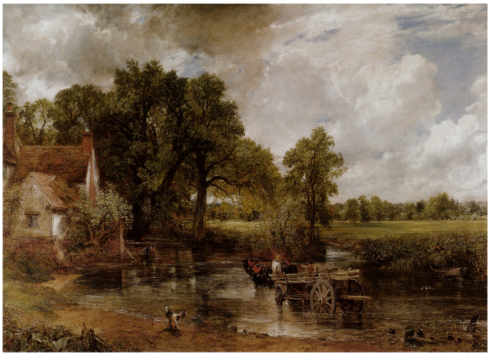
[4] *Loc. cit.*

FIGURE 22.6

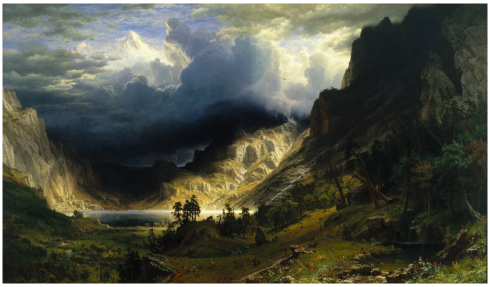
The four landscape paintings used in the first study by Johnson & Steinerberger, with the shorthand names used for each.



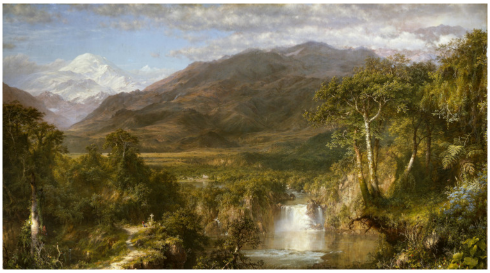
Albert Bierstadt, [YOSEMITE]
Looking Down Yosemite Valley, California



John Constable, [SUFFOLK]
The Hay Wain



Albert Bierstadt, [ROCKIES]
A Storm in the Rocky Mountains, Mt. Rosalie



Frederic Edwin Church, [ANDES]
The Heart of the Andes

SCHUBERT: Franz Schubert (1797–1828), Moment Musical № 4, Moderato in C♯ minor (Op. 94, D 780, № 4), performed by David Fray (1981–).

BACH: Johann Sebastian Bach (1685–1750), Fugue from Toccata in E minor (BWV 914), performed by Glenn Gould (1932–82).

BEETHOVEN: Ludwig van Beethoven (1770–1827), Diabelli Variations (Op. 120), performed by Grigory Sokolov (1950–).

SHOSTAKOVICH: Dmitri Shostakovich (1906–75), Prelude in D♭ major (Op. 87, № 15), performed by Adrian Brendle (1990–).

The average ratings participants gave for argument–music similarities were all higher than for argument–painting similarities: the highest mean rating was 5.11 for the GAUSS–BACH pairing; the lowest was 4.31 for the GAUSS–SHOSTAKOVICH pairing.[1] But statistical analysis showed that, though the rating were again non-random, there was a more moderate consensus on the ratings.[2]

In the third study, participants rated, on a scale of 0 to 10, the degree to which the four paintings and the four arguments were described by each of ten adjectives:

I) beautiful, II) serious,
III) universal, IV) profound, V) novel, VI) clear,
VII) simple, VIII) elegant, IX) intricate, X) sophisticated.

These adjectives were derived from Hardy's discussion in *A Mathematician's Apology*:[*] terms III–VI were chosen as near-synonyms, thought more suitable for evaluating paintings, of Hardy's *general*, *deep*, *unexpected*, *inevitable*; terms VII–X were chosen to express different aspects of *economy*.[3] [One could quibble about some of the choices: clarity and inevitability are not necessarily connected; elegance is surely something more than a mere aspect, positive or negative, of economy. But the relation of Johnson & Steinerberger's terms to Hardy's is of limited importance to their study.]

The analysis showed that there was consistency among participants in their ratings of the applicability of terms to both paintings and arguments.[4] The three terms that were most closely associated to *beautiful* for both paintings and arguments were *elegant*, *profound*, and *clear*. For arguments, but not for paintings, *elegant* was by far the most significant predictor of *beautiful*.[5] Ranking the similarity of painting and arguments using the ratings for *beautiful* and for the three terms *elegant*, *profound*, and *clear* produced broadly parallel orderings.[6] But the similarity in the rankings was due more to the participants who had taken university-level mathematics courses: Johnson & Steinerberger concluded from this dependency that 'intuitions about mathematical beauty sharpen with experience in higher mathematics'.[7]

For Johnson & Steinerberger, the results of these studies was evidence that the aesthetic judgements of mathematical laypersons were consistent and bore at least some relationship to those of experts. Further, they relied on the same aesthetic terms to evaluate both paintings

[1] Johnson & Steinerberger, 'Intuitions about mathematical beauty', Tbl. 2.

[2] *Ibid.*, p. 247.

[*] ▸ pp. 498–523

[3] *Ibid.*, p. 248.

[4] *Loc. cit.*

[5] *Ibid.*, p. 249.

[6] *Loc. cit.*

[7] *Loc. cit.*

and arguments, and similarities in the applicability of these terms predicted the 'raw' similarity judgements of the first study. Thus, they held, '[m]athematical beauty [...] does not appear to be solely in the eye of the beholder but appears to have deep psychological roots.'[1]

SUGGESTED BASES FOR A CANON

A social basis?

In her study of the practice of mathematics, published in 2004,[2] Leone Burton (1936–2007) concluded that that social factors shaped and propagated an aesthetic canon for mathematics. One participant in her study said that the existence of a 'culture' was a prerequisite for appreciating beauty, whether in mathematics or in art. It was this culture that led to agreement about beauty within a field.[3]

Another participant seemed to be aware of having their aesthetic taste moulded to fit the culture upon their entry into the mathematical community. The learning processes that new members underwent served in part to allow existing members to 'induct' them into an aesthetic canon, or to 'persuade' them that 'certain things are aesthetic'.[4] But, for this participant, the very dependence of aesthetic judgements upon the culture implied that there might be variations in what was thought beautiful in mathematics. Just as for art, the aesthetic judgements of mathematics would vary between cultures and across time.[5]

A biological basis?

Following Zeki et al.'s study of the neural correlates of the beauty of formulae,* Zeki, Oliver Chén & Romaya wondered whether the appreciation of mathematical beauty fit better into the category of biological or of artefactual sensory experience.[6] Biological experiences are those mediated by inherited brain concepts, and are thus resistant to change as a result of experience. Artefactual experiences are grounded upon concepts that are acquired through experience, and are thus modifiable by further experience.[7] Loosely, biological experiences depend upon 'built-in' factors; artefactual experiences do not. Biological sensory experiences, and concomitant aesthetic judgements, seem to apply to natural things like bodies, faces, and landscapes; artefactual sensory experiences apply to culturally-influenced objects like architecture, artwork, and clothes. One might expect, therefore, that there would be a higher degree of consensus between individuals about aesthetic judgements of the objects of biological than of artefactual experience,[1] and empirical studies seems to have borne out this expectation.[2]

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[1] Johnson & Steinerberger, 'Intuitions about mathematical beauty', p. 250.

[2] Burton, *Mathematicians as Enquirers*.

[3] *Ibid.*, p. 72.

[4] *Ibid.*, pp. 72, 123–4.

[5] *Ibid.*, p. 124.

* ▸ pp. 760–2

[6] Zeki, Chén & Romaya, 'The Biological Basis of Mathematical Beauty', p. 1.

[7] Zeki, Chén & Romaya, 'The Biological Basis of Mathematical Beauty', pp. 1–2; Zeki & Chén, 'The Bayesian-Laplacian brain', p. 2.

Prima facie, mathematics would obviously be an object of artefactual experience. It was an extreme example of an artefact of culture and learning, and was accessible only to a few.[3][*] But if, on the other hand, the beauty of mathematics were biological in foundation, then there should be less variability among what were judged to be more beautiful formulae.[4] This diminished variation was what Zeki, Chén & Romaya found in their statistical analysis of the ratings assigned in the preliminary questionnaire[†] by the subjects in the study of neural correlates of beauty of formulae: there was a significant negative association of mean ratings with their standard deviation.[5] That is, high mean ratings were associated to low standard deviations. Informally, the more beautiful the formula, the more agreement there was on its beauty. The analysis also showed that this association could not be explained by the level of understanding of the formulae,[6] and it was unlikely that spatial form played any role, give the very different appearance of the top-rated formulae.[7] (The association persisted even after controlling for a possible 'ceiling effect', whereby some formulae may have been rated −5 or +5 since no lower or higher ratings were available.[8])

As Zeki, Chén & Romaya observed, this situation was curious: to *access* mathematics depends upon advanced culturally-shaped learning, but once such access is gained, '*the same formulae can be experienced as beautiful by mathematicians belonging to different races and cultures*'.[9] They hypothesized that this common aesthetic experience arose because the practice of mathematics, although dependent on training, engaged the logical-deductive system of the brain, which was an inherited mechanism that was similar for all humans. Thus the experience of mathematical beauty was the result of an application of this logical-deductive system, and so this experience would be similar for all people, regardless of how culture has influenced them.[10]

Another support for a biological grounding of judgements of mathematical beauty came from evidence of their resistance to change when confronted with apparent expert judgements. In 2022, Haoxuan Zhang & Zeki made a further study in which each participant was presented, in a random order, with the same sixty equations as used in the study of neural correlates. They were asked to rate each equation in terms of how beautiful they found it and in terms of how well they understood it; they were then shown what they were told was an average beauty rating by experts and asked to re-rate the equation. In fact, the expert beauty rating was computer-generated: a low initial rating from the participant could prompt a higher or equal 'expert rating' according to specified probabilities; a high initial rating could prompt a lower or equal 'expert rating'; a middling initial rating could produce either.[11]

[1] Zeki & Chén, 'The Bayesian-Laplacian brain', pp. 4–5.

[2] Chen & Zeki, 'Frontoparietal Activation'; Vessel et al., 'Stronger shared taste for natural aesthetic domains'.

[3] Zeki, Chén & Romaya, 'The Biological Basis of Mathematical Beauty', pp. 1, 7.

[*] ▸ pp. 574–600

[4] *Ibid.*, p. 2.

[†] ▸ pp. 837–8

[5] *Ibid.*, pp. 2–4.

[6] *Ibid.*, pp. 4–5.

[7] *Ibid.*, pp. 6–7.

[8] *Ibid.*, p. 5.

[9] *Ibid.*, p. 7, emphasis added.

[10] *Ibid.*, p. 7.

[11] Zhang & Zeki, 'Judgments of mathematical beauty are resistant to revision', pp. 742–3.

Analysis of the results showed that while there *was* a change in assigned ratings as a result of the 'expert rating', it was *small*, indicating that judgements of mathematical beauty were resistant to external influence.[1] Such resistance was to be expected from aesthetic appreciation with a biological basis, which would be little affected by experience.

In 2014, the same year as the work of Zeki, Chén & Romaya appeared, the neurologist Anjan Chatterjee (1958–) published his book *The Aesthetic Brain*, on the functioning and origin of the appreciation of beauty and art. In this work, he considered mathematical beauty among other forms of beauty. He thought that 'if mathematics can be beautiful, its beauty must be very different from that of a sexy body'.[2] He argued, on the basis of the sensory processes being different, that responses to 'beauty' in people, places, and proofs were not truly responses to the same property.[3] The emotional response to beauty lay on a spectrum from *hot*, evoking passion, to *cold*, maintaining detachment.[4] His example of the latter was appreciating an elegant proof.[5] In this characterization, he was doubtless influenced by Bertrand Russell's (1872–1970) famous description of the beauty of mathematics as 'cold and austere',[*] which he quoted in full.[6]

Chatterjee pointed out that the idea of aesthetic experiences being disinterested, which goes back to Shaftesbury[†] (1671–1713), undermined the notion that there could be evolutionary selective pressure in favour of aesthetic appreciation.[7] But he speculated that there was an evolutionary advantage to taking pleasure in mathematics. The ability to abstract and manipulate structural relationships from a complex world would increase the probability of fulfilling such needs as nourishment, shelter, and safety, and thus the probability of continuing survival and opportunity to reproduce. Human ancestors who enjoyed playing with and simplifying such abstract relationships would thus be more likely to beget descendants.[8] This hypothesis was compatible with Zeki, Chén & Romaya's idea that an inherited logical neural system undergirded the practice and appreciation of mathematics.

CAVEATS

Individual consistency

A question more fundamental even than the putative agreement about beauty among mathematicians is that of consistent judgements of beauty by an individual mathematician across time. As Hofstadter remarked:

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[1] Zhang & Zeki, 'Judgments of mathematical beauty are resistant to revision', pp. 744–5.

[2] Chatterjee, *The Aesthetic Brain*, p. 5.

[3] *Ibid*., pp. 65–6.

[4] *Ibid*., p. 66.

[5] *Loc. cit.*

[*] ▸ p. 479

[6] *Ibid*., p. 55.

[†] ▸ pp. 353–6

[7] *Ibid*., p. xx.

[8] *Ibid*., p. 63.

> ‘Everyone has had the experience of finding something beautiful at one time, dull another time — and probably intermediate at other times. So is beauty an attribute which varies in time? One could turn things around and say that it is the beholder who has varied in time.’[1]

Experience can doubtless shape one’s affective response, particularly if factors like unexpectedness play a role. Davis & Hersh gave the particular example of work in the theory of difference equations so accustoming one to previously unexpected concepts that they became incorporated into one’s intuition.[2]

Alan Laverty, one of the respondents to Wells’s questionnaire, wrote that:

> ‘The scores I gave to [several] would fluctuate according to mood and circumstance. Extreme example: at one point I was considering giving (13) a 10, but I finally decided it just didn’t thrill me very much.’[3]

The theorem (13) is *A regular icosahedron inscribed in a regular octahedron divides the edges in the Golden Ratio.* Laverty in the end rated it 2. This change illustrates the degree to which individual judgement of beauty may be inconstant.

Social influence

Whatever consensus has existed on mathematical beauty may have been shaped by the social structures of mathematics as a pursuit. No mathematician is an island, entire of itself. Mathematicians have been inspired by others and have sought validation in the appreciation of others,[4] and have ultimately depended upon the approval of others for their bread. And the agreement of others on the aesthetic value of their work has been a part of this:

> ‘There is not much fun in deciding for yourself that a particular area of mathematics is beautiful and spend your life on it if you are the only person who finds it beautiful.’[5]

Thomas Tymoczko (1943–96) noted that aesthetics in mathematics differs from aesthetics in art because there are no ‘mathematics critics’ debating the aesthetic merits of mathematics and guiding aesthetic appreciation of mathematics.[6] And Imre Lakatos (1922–74) — or at least the character GAMMA in his dialogue *Proofs and Refutations* — thought that critics could ‘develop mathematical taste’ and perhaps ‘stem the tide of pretentious trivialities’.[7] But critics would have little or no role were there broad agreement in aesthetic judgements in mathematics, so hypothesizing such a calling was a tacit admission that there *was* no such agreement.

It is also important to note that aesthetic views may be shaped by the authority of influential mathematicians. Freeman Dyson (1923–

[1] Hofstadter, *Gödel, Escher, Bach*, p. 583.

[2] Davis & Hersh, *Mathematical Experience*, p. 171.

[3] Laverty, quot. in Wells, ‘Are these the most beautiful?’, p. 40, insertion in orig.

[4] Thurston, ‘On proof and progress in mathematics’, p. 171.

[5] Lenstra, interview quot. in Sinclair, ‘The Aesthetic Sensibilities of Mathematicians’, p. 101.

[6] Tymoczko, ‘Value Judgments in Mathematics’, pp. 72–3.

[7] Lakatos, *Proofs and Refutations*, p. 98.

2020) complained that some of what he saw as beautiful mathematics was not pursued because it was outside of current tastes and anyone pursuing it would not prosper in their career:

> 'Roughly speaking, unfashionable mathematics consists of those parts of mathematics which were declared by the mandarins of Bourbaki not to be mathematics. A number of very beautiful mathematical discoveries fall into this category. To be mathematics according to Bourbaki, an idea should be general, abstract, coherent, [...]. Unfashionable mathematics is mainly concerned with things of accidental beauty, special functions, particular number fields, exceptional algebras, sporadic finite groups.'[1]

[Sometimes, though, influential persons not only have tastes that differ, but have an attitude of dismissal of aesthetic value: Solomon Lefschetz (1884–1972) was contemptuous of elegant proofs of results that he considered obvious,[2] and was dismissive of a 'pretty proof' that Donald C. Spencer (1912–2001) and Kunihiko Kodaira (1915–97) had discovered of one of his own deep theorems.[3]]

Some data on the influence of authority on aesthetic judgements emerged following the unexpected disagreement that Inglis & Aberdein found between the range of opinions expressed about Kelly's proof of the Sylvester–Gallai theorem and its inclusion in *Proofs from THE BOOK*.[*] This disagreement led them to make a subsequent study, published in 2020, of the possible effect of social conformity on such value judgements. As before, participants scored the applicability of a set of twenty adjectives (four from each of the axes their original study identified, and four that loaded onto no axis) to the proof, but with the crucial difference that, on a random basis, participants were either informed or not informed that the proof they were evaluating was taken from the fifth edition of *Proofs from THE BOOK*.[4]

The heterogeneity of participants' evaluations mirrored the earlier study.[5] But there was a significant difference in the evaluations on the aesthetic dimension: the mean aesthetic rating given by pure mathematicians who were informed about the proof being included in *Proofs from THE BOOK* was 12.69 (on the scale of 4 to 20); for those who were not so informed, the mean aesthetic rating was 9.69. No such significant effect was found for applied mathematicians and statisticians.[6] Inglis & Aberdein concluded that this social conformity effect only manifested among pure mathematicians because they were more familiar with Erdős's concept of The Book.[7]

[1] Dyson, 'Unfashionable Pursuits', p. 52.

[2] Nasar, *A Beautiful Mind*, p. 59.

[3] Rota, 'Fine Hall in its Golden Age', p. 235.

[*] ▸ pp. 850–1

[4] Inglis & Aberdein, 'Are aesthetic judgements purely aesthetic?', pp. 1132–3.

[5] *Ibid.*, p. 1134.

[6] *Loc. cit.*

[7] *Loc. cit.*

APPENDIX: QUESTIONNAIRE RESULTS

This section contains the results of the two questionnaires (on the beauty of theorems and of formulae) discussed in this chapter.

TABLE 22.2
Results of the questionnaire on the beauty of theorems set by Wells (Wells, 'Which is the most beautiful?'; Wells, 'Are these the most beautiful?', pp. 37–8). For discussion, see pages 834–7.

Rank	Mean rating	Survey label	
1)	7.7	(W)	$e^{i\pi} = -1$.
2)	7.5	(A)	Euler's polyhedron formula: $V + F = E + 2$.
3)	7.5	(D)	The number of primes is infinite.
4)	7.0	(X)	There are 5 regular polyhedra.
5)	7.0	(G)	$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{6}$.
6)	6.8	(Q)	A continuous mapping of the closed unit disk into itself has a fixed point.
7)	6.7	(E)	There is no rational number whose square is 2.
8)	6.5	(I)	π is transcendental.
9)	6.2	(P)	Every plane map can be coloured with 4 colours.
10)	6.0	(F)	Every prime of the form $4n+1$ is the sum of two integral squares in exactly one way.
11)	5.3	(V)	The order of a subgroup divides the order of the group.
12)	5.2	(B)	Any square matrix satisfies its characteristic equation.
13)	5.0	(S)	A regular icosahedron inscribed in a regular octahedron divides the edges in the Golden Ratio.
14)	4.8	(H)	$\frac{1}{2 \times 3 \times 4} - \frac{1}{4 \times 5 \times 6} + \frac{1}{6 \times 7 \times 8} - \ldots = \frac{\pi - 3}{4}$.
15)	4.7	(O)	If the points of the plane are each coloured red, yellow, or blue, there is a pair of points of the same colour of mutual distance unity.
16)	4.7	(N)	The number of partitions of an integer into odd integers is equal to the number of partitions into distinct integers.
17)	4.7	(J)	Every number greater than 77 is the sum of integers, the sum of whose reciprocals is 1.
18)	4.7	(T)	The number of representations of an odd number as the sum of 4 squares is 8 times the sum of its divisors; of an even number, 24 times the sum of its odd divisors.
19)	4.7	(L)	There is no equilateral triangle whose vertices are plane lattice points.
20)	4.7	(M)	At any party, there is a pair of people who have the same number of friends present.

(Continues)

TABLE 22.2 Results of Wells's questionnaire on the beauty of theorems (*Continued*)

Rank	Mean rating	Survey label	
21)	4.2	(R)	Write down the multiples of root 2, ignoring fractional parts, and underneath the numbers missing from the first sequence. $\begin{array}{ccccccccc} 1 & 2 & 4 & 5 & 7 & 8 & 9 & 11 & \dots \\ 3 & 6 & 10 & 13 & 17 & 20 & 23 & 27 & \dots \end{array}$ The difference is $2n$ in the n-th place.
22)	4.1	(U)	The word problem for groups is insoluble.
23)	3.9	(K)	The maximum area of a quadrilateral with sides a, b, c, d is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where $s = (a+b+c+d)/2$ is half the perimeter.
24)	3.9	(C)	Let $p(n)$ be the number of partitions of n. Then $\frac{5\{(1-x^5)(1-x^{10})(1-x^{15})\dots\}^5}{\{(1-x)(1-x^2)(1-x^3)(1-x^4)\dots\}^6} = p(4) + p(9)x + p(14)x^2 + \dots.$

TABLE 22.3
Results of the preliminary questionnaire on the beauty of formulae for the participants in the study by Zeki et al. (Zeki, Romaya et al., 'The experience of mathematical beauty'). For discussion of the results, see pages 837–8; for their distribution, see Table 22.1 on page 838.

Rank	Mean rating	Survey label	
1)	+3.60	(1)	$1 + e^{i\pi} = 0.$
2)	+3.27	(2)	$\cos^2 \vartheta + \sin^2 \vartheta = 1.$
3)	+3.13	(54)	The Cauchy–Riemann equations, which characterize the differentiable complex functions: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$
4)	+3.00	(5)	$e^{ix} = \cos x + i \sin x.$
5)	+2.93	(6)	$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$
6)	+2.93	(58)	The Fermat equation, which has no solution in the positive integers by the Fermat–Wiles theorem: $a^n + b^n = c^n, \quad n > 2.$
7)	+2.80	(18)	Cauchy's residue theorem: $\oint_\gamma f(z)\, dz = 2\pi i \sum \operatorname{Res}(f, a_k).$
8)	+2.73	(34)	Stokes's theorem: $\int_{\partial M} \omega = \int_M d\omega.$

(*Continues*)

TABLE 22.3 Results of Zeki et al.'s questionnaire on the beauty of formulae (*Continued*)

Rank
Mean rating
Survey label

9) +2.67 (31) $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

10) +2.53 (10) $\mathrm{e} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$.

11) +2.47 (16) The pythagorean equation: $a^2 + b^2 = c^2$.

12) +2.27 (20) The diffusion equation: $\frac{\partial\rho}{\partial t} = D\frac{\partial^2\rho}{\partial x^2}$.

13) +2.20 (55) Laplace equation: $\Delta\varphi = 0$.

14) +2.00 (17) The Fundamental Theorem of Calculus:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(s)\,\mathrm{d}s = f(x).$$

15) +1.93 (22) $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x$.

16) +1.87 (36) The quadratic variation of a one-dimensional Brownian motion up to time t is t: $\langle B, B\rangle_t = t$.

17) +1.73 (32) $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}, \quad |r| < 1$.

18) +1.67 (29) Cauchy's integral formula:

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n} f(z) = \frac{n!}{2\pi\mathrm{i}}\oint_c \frac{f(w)}{(w-z)^{n+1}}\,\mathrm{d}w\,.$$

19) +1.60 (8) Maclaurin series for the exponential function:

$$\exp(X) = \sum_{n=0}^{\infty}\frac{X^n}{n!}.$$

20) +1.60 (23) Maclaurin series of an analytic function:

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n.$$

21) +1.60 (60) Bianchi identity:

$$\nabla_\mu\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) = 0,$$

where $R^{\mu\nu}$ is the Ricci tensor and R is the scalar curvature.

22) +1.53 (52) A formulation of the spectral theorem expressing the operator A as an integral of the coördinate function over the operator's spectrum:

$$A = \int_{\sigma(A)} \lambda\,\mathrm{d}E_\lambda$$

23) +1.47 (13) An identity involving the Dirac delta:

$$f(x) = \int_{-\infty}^{\infty}\delta(x-y)f(y)\,\mathrm{d}y\,.$$

(*Continues*)

TABLE 22.3 Results of Zeki et al.'s questionnaire on the beauty of formulae (*Continued*)

Rank	Mean rating	Survey label	
24)	+1.40	(3)	Euler's polyhedron formula: $V - E + F = 2$.
25)	+1.40	(25)	Triangle inequality for a normed vector space: $$\|x + y\| \leqslant \|x\| + \|y\|.$$
26)	+1.33	(24)	Formula satisfied by each eigenvalue λ and corresponding eigenvector $\vec{x}$ of a matrix A: $A\vec{x} = \lambda\vec{x}$.
27)	+1.27	(43)	Formula for the volume of an n-dimensional sphere: $$V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} r^n.$$
28)	+1.27	(49)	An identity involving the Riemann zeta function: $$1 = \sum_{n=2}^{\infty} (\zeta(n) - 1).$$
29)	+1.13	(44)	Relationship between the 2-sphere (that is, the Riemann sphere, representing the complex numbers augmented by a point ∞), the complex projective line (the extension of a line with a 'point at infinity'), and the special orthogonal groups SO(3) and SO(2): $$S^2 \cong \mathbb{CP}^1 \cong \mathrm{SO}(3)/\mathrm{SO}(2).$$
30)	+1.07	(26)	The Prime Number Theorem: $\pi(x) \sim \dfrac{x}{\log x}$.
31)	+1.07	(37)	An identity involving the generating function for the number of partitions of a number into distinct parts: $$\prod_{k=1}^{\infty} (1 + x^k) = \sum_{n=0}^{\infty} p(n) x^n.$$ [In the original questionnaire, the product range incorrectly starts at $k = 0$.]
32)	+1.07	(46)	The second Bianchi identity of the Riemann tensor: $$R^{\alpha}_{\beta[y\delta;\lambda]} = 0.$$
33)	+1.07	(56)	Pell's equation: $x^2 - ny^2 = 1$.
34)	+1.00	(9)	The Fourier transform of a Gaussian function is also a Gaussian function: $$\mathcal{F}_x[\mathrm{e}^{-ax^2}](k) = \sqrt{\frac{\pi}{a}}\,\mathrm{e}^{-\pi^2 k^2/a^2}.$$
35)	+1.00	(57)	The sine-Gordon differential equation: $\varphi_{tt} - \varphi_{xx} + \sin\varphi = 0$.

(*Continues*)

TABLE 22.3 Results of Zeki et al.'s questionnaire on the beauty of formulae (*Continued*)

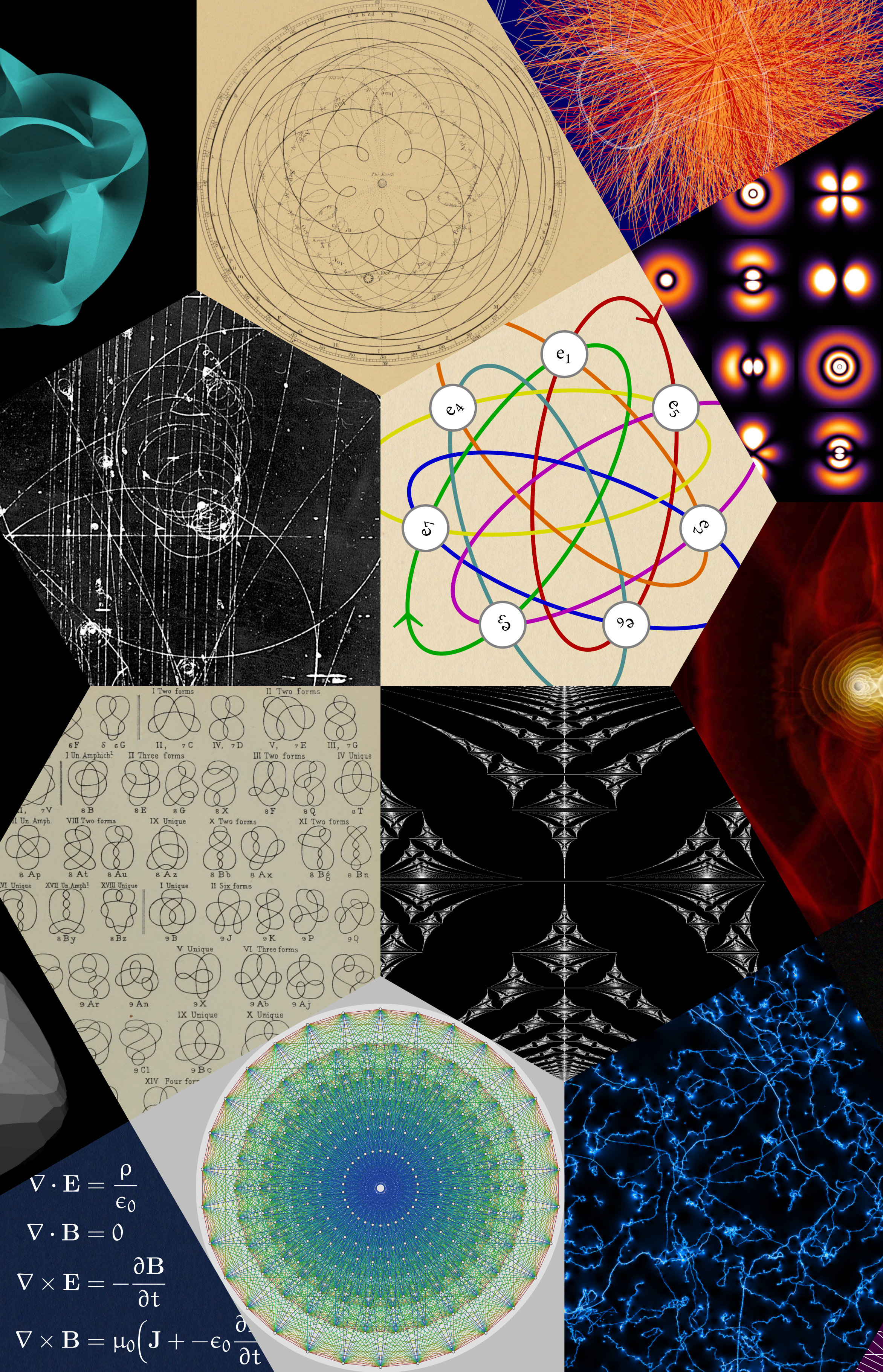
Rank	Mean rating	Survey label	
36)	+0.87	(4)	The Gauss–Bonnet theorem, which links the geometry and topology of Riemannian manifolds: $\displaystyle \int_M K\,\mathrm{d}A + \int_{\partial M} k_g\,\mathrm{d}s = 2\pi\chi(M),$ where M is a compact two-dimensional Riemannian manifold with Gaussian curvature K and geodesic curvature is k_g, and with Euler characteristic $\chi(M)$.
37)	+0.80	(21)	The definition of π as the ratio of a circle's circumference and diameter: $\pi = \frac{c}{d}$.
38)	+0.80	(40)	The Cartan structure equations: $T = \mathrm{d}e + \omega \wedge e, \quad R = \mathrm{d}\omega + \omega \wedge \omega.$
39)	+0.73	(30)	$\displaystyle \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots.$
40)	+0.73	(48)	The defining relations for the Clifford algebra $Cl_{1,3}(\mathbb{R})$: $\{\gamma_i, \gamma_j\} = 2\eta_{ij}.$
41)	+0.67	(7)	The expression of the Riemann zeta function as a Dirichlet series using the Möbius function: $\displaystyle \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad s \in \mathbb{C}, \mathbb{R}(s) > 1.$
42)	+0.60	(19)	The Lotka–Volterra predator-prey equations: $\displaystyle \frac{\mathrm{d}x}{\mathrm{d}t} = x(\alpha - \beta y), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -y(\gamma - \delta x).$
43)	+0.60	(27)	The Euler product formula for the Riemann zeta function: $\displaystyle \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$
44)	+0.60	(53)	Properties defining Berezin integration: $\displaystyle \int \vartheta\,\mathrm{d}\vartheta = 1, \quad \int \mathrm{d}\vartheta = 0.$
45)	+0.53	(12)	The iteration used to define the Mandelbrot set: $z_{n+1} = z_n^2 + c$.
46)	+0.53	(38)	The arithmetic–geometric mean inequality: $\displaystyle \frac{1}{n}\sum_{k=1}^{n} a_k \geqslant \left(\prod_{k=1}^{n} a_k\right)^{1/n}, \quad a_k > 0.$
47)	+0.40	(47)	Definition of a Möbius transform: $\displaystyle f(z) = \frac{az+b}{cz+d}.$

(*Continues*)

TABLE 22.3 Results of Zeki et al.'s questionnaire on the beauty of formulae (*Continued*)

Rank	Mean rating	Survey label	
48)	+0.20	(41)	Stirling's approximation for factorials: $$n! = n^n e^{-n\sqrt{2\pi n}}(1 + o(n)).$$
49)	+0.13	(11)	Cantor's theorem: $2^{\lvert S\rvert} > \lvert S\rvert$.
50)	+0.07	(42)	A formula for the character of an irreducible representation of a Lie group: $$\chi_\Omega(\exp X) = \int_\Omega e^{i\langle F,X\rangle+\sigma(F)}.$$
51)	0.00	(35)	A special case of Poisson summation: $$\sqrt{\pi}\sum_{n=-\infty}^{\infty} e^{-n^2} = \sum_{n=-\infty}^{\infty} e^{-n^2/4}.$$
52)	−0.20	(50)	$A \cap A^{\complement} = \varnothing$.
53)	−0.27	(59)	Riemann's functional equation: $$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$
54)	−0.33	(33)	Birkhoff's ergodic theorem: $$\int f\,d\mu = \lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n} f\circ T^k.$$
55)	−0.33	(45)	An example of an exact sequence, in which the image of one morphism equals the kernel of the next: $$\mathbb{Z} \overset{2}{\hookrightarrow} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}.$$
56)	−0.40	(39)	$\lvert\varnothing\rvert = 0$.
57)	−0.53	(51)	$U^{\complement} = \varnothing$.
58)	−1.13	(15)	$1729 = 1^3 + 12^3 = 9^3 + 10^3$.
59)	−1.13	(28)	$3^2 + 4^2 = 5^2$.
60)	−1.80	(14)	$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}$.

* * *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$$\nabla \cdot \mathbf{B} = 0$$

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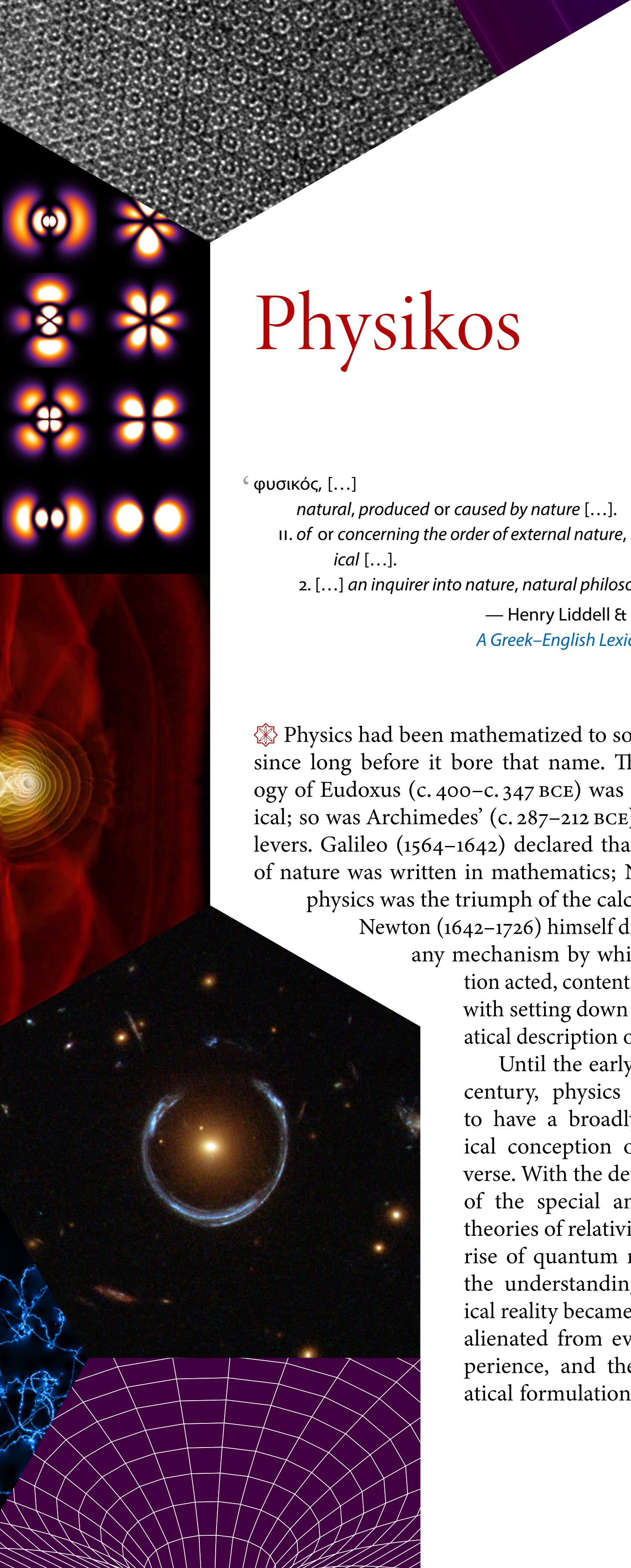


III

Physikos

> ‘ φυσικός, […]
> *natural*, *produced* or *caused by nature* […].
> ɪɪ. *of* or *concerning the order of external nature*, *natural*, *physical* […].
> 2. […] *an inquirer into nature*, *natural philosopher* […]. ’
>
> — Henry Liddell & Robert Scott,
> *A Greek–English Lexicon*, ‘φυσικός’.

Physics had been mathematized to some degree since long before it bore that name. The cosmology of Eudoxus (c. 400–c. 347 BCE) was mathematical; so was Archimedes’ (c. 287–212 BCE) theory of levers. Galileo (1564–1642) declared that the book of nature was written in mathematics; Newtonian physics was the triumph of the calculus. Isaac Newton (1642–1726) himself did not offer any mechanism by which gravitation acted, contenting himself with setting down a mathematical description of its action.

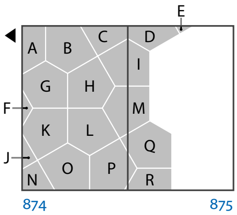
Until the early twentieth century, physics continued to have a broadly mechanical conception of the universe. With the development of the special and general theories of relativity, and the rise of quantum mechanics, the understanding of physical reality became ever more alienated from everyday experience, and the mathematical formulation of natural laws became the rock upon which stood physics. And physicists admired, and sometimes saw as a guide, the beauty of that mathematics.

The mathematical formulation of physical laws began to involve concepts developed for purely mathematical purposes. These concepts were so far removed from the elementary notions of number and magnitude that a very simple question arose: *Why?* Why did work that mathematicians had pursued with no application in mind, and often for aesthetic reasons, turn out to be applicable in physics?

In the later twentieth century, the quest for a grand unified theory — a final theory of physics — led theoretical physics to begin to detach from experiment. Purely aesthetic concerns, such as beauty and elegance of the mathematical formulation, started to play a role in the evaluation of theories that were far from what could then — or perhaps realistically ever, or perhaps even in principle — be confronted with empirical testing.

As discussed in Chapter 23, the aesthetic value of a scientific theory can involve far more than its mathematical formulation. Thus the chapters in this part, focusing on mathematical beauty, offer only a limited perspective on the broader history of the aesthetics of modern science. A comprehensive history of the aesthetics of science would be a volume in itself — and remains to be written.

> ‘ The mathematician will study form;
> the *physikos* will, as he always has, study matter. ’
>
> — James G. Lennox, “As if we were investigating snubness”, p. 151.



A Cross-section of a Calabi–Yau manifold. **B** Pentagram of Venus. **C** Lead–lead collision in the ATLAS detector at the Large Hadron Collider. **D** Natural quasi-crystal with decagonal symmetry, composition $\mathrm{Al}_{71}\mathrm{Ni}_{24}\mathrm{Fe}_{5}$, found in a meteorite. **E** Hydrogen spectrum. **F** The de Sitter universe. **G** Bubble chamber photograph of particle interactions including e^+. **H** Fano plane showing the multiplication rule for the basis of the octonions. **I** Atomic orbitals of the electron in a hydrogen atom at different energy levels. **J** Model of the asteroid 2001 Einstein. **K** P. G. Tait’s classification of knots of knottiness up to 7. **L** Hofstadter’s butterfly. **M** Simulation of two black holes merging. **N** Maxwell’s equations. **O** Orthogonal projection of polytope with E_8 symmetry. **P** Simulation representing cosmic string signals in the cosmic microwave background radiation. **Q** Einstein ring produced by gravitational lensing of the light from a distant galaxy around a closer one. **R** Flamm paraboloid describing the Schwarzschild metric solution to the Einstein field equations.

C © 2015 CERN. [CC-BY-4.0] **D** © 2015 Luca Bindi et al. [CC-BY-4.0] **J** © 2008–24 Astronomical Institute of the Charles University, Josef Ďurech, Vojtěch Sidorin. [CC-BY-4.0] **L** © 2018 ‘colt_browning’. [CC-BY-4.0] **O** © 2007 Claudio Rocchini. [CC-BY-4.0] **A, B, E, G, I, K, M, P, Q** Public domain.

Pythagoras Reborn

Mathematical beauty and natural science

23

> ‘ “ […] I observe the universe.
> Like you, I also have some criteria
> for evaluating whether the observed objects are sound:
> harmony and beauty, in the mathematical sense.” ’
>
> — Liu Cixin, *Remembrance of Earth’s Past* 3: *Death’s End*, pt II
> (trans. Ken Liu).

> ‘ We cannot truly account for our acceptance
> of such theories without endorsing our acknowledgement of
> a beauty that exhilarates and a profundity that entrances us. ’
>
> — Michael Polanyi, *Personal Knowledge*, p. 16.

On 30 April 1932, while replying to a toast to ‘Science’ proposed by William Llewellyn (1858–1941), the President of the Royal Academy of Arts,[1] Ernest Rutherford (1871–1937), the 1908 Nobel laureate in Chemistry, said:

> ‘I think that a strong claim can be made that the process of scientific discovery may be regarded as a form of art. This is best seen in the theoretical aspects of Physical Science. The mathematical theorist builds up on certain assumptions and according to well understood logical rules, step by step, a stately edifice, while his imaginative power brings out clearly the hidden relation between its parts. A well constructed theory is in some respects undoubtedly an artistic production. A fine example is the famous Kinetic Theory of Maxwell […]. The theory of relativity by Einstein, quite apart from any question of its validity, cannot but be regarded as a magnificent work of art’.[2]

Rutherford’s remarks indicated that physical theories could be subject to aesthetic appreciation, and hinted at a connection between their aesthetic value and their mathematical formulation. But there is a distinction to be made between the aesthetic value of a theory and the aesthetic value of its expression in mathematics. The philosopher of science James McAllister* (1962–) noted that the two were sometimes conflated: it seemed to be widely accepted that the symmetry of a theory was made most manifest when the theory was stated in mathematical terms, and that from this axiom some had drawn the inference that mathematical beauty was all there was to beauty

[1] Eve, *Rutherford*, p. 352.

[2] Rutherford, quot. in *ibid.*, p. 353.

* ▸ pp. 793–5

in physical theories.[1] The physicist and cosmologist Max Tegmark (1967–) made a perhaps even more blunt assertion, suggesting that aesthetic judgements of a theory depended *only* on its mathematical formulation, and not on its metaphysical commitments: 'I judge the elegance and simplicity of a theory not by its ontology, but by the elegance and simplicity of its mathematical equations'.[2]

BEAUTY IN SCIENCE, MATHEMATICAL AND NON-MATHEMATICAL

A glance at history shows that beauty in science can be entirely non-mathematical. For instance, the physicist Oliver Lodge (1851–1940), in a 1882 lecture at the London Institution, spoke of the vortex theory of the atom[3] in the following terms:

> 'Now does not the idea strike you that atoms of matter may be vortices like these — vortices in a perfect fluid, vortices in the ether. [...] It is not yet proved to be true, but is it not *highly beautiful*? a theory about which one may almost dare to say that it deserves to be true. The atoms of matter according to it are not so much foreign particles imbedded in the all-pervading ether as portions of it differentiated off from the rest by their vortex motion'.[4]

Lodge's argument for the beauty of the vortex atom hypothesis was unconnected with any mathematical formulation, but rested rather on its metaphysical unity: it postulated only one kind of stuff, namely the 'perfect fluid' of the aether. Matter was aether in vortex motion; light was carried by aether in wave motion.

A more modern non-mathematical aesthetic judgement in fundamental physics concerned the muon and tau families of particles. Both families echo the electron family, except that their members are much more massive and decay into particles in the electron family. They seem superfluous: Anthony Zee (1945–) wrote that '[t]he universe would function perfectly well if the muon and tau families were not even included'.[5] The existence of these families violates no principle of *mathematical* beauty, but rather a principle of parsimony; a seemingly simpler design is encumbered by apparently purposeless embellishments.[6] Similarly, non-mathematical aesthetic judgements have concerned theories outside of physics: according to James Watson (1928–), Rosalind Franklin (1920–58) accepted that DNA formed a double helix because 'the structure was too pretty not to be true'.[7]

At the 1927 Solvay conference, George Lemaître (1894–1966) discussed the notion of an expanding universe with Albert Einstein (1879–1955); Lemaître recalled Einstein's reaction as follows:

[1] McAllister, *Beauty & Revolution*, p. 53.

[2] Tegmark, *Our Mathematical Universe*, ch. 13.

[3] See Kragh, 'The Vortex Atom'.

[4] Lodge, 'The Ether and its Functions II', p. 329, emphasis added.

[5] Zee, *Fearful Symmetry*, p. 259.

[6] *Loc. cit.*

[7] Watson, *The Double Helix*, ch. 28.

> 'After some favorable technical remarks, he concluded by saying that it seemed to him *from the physical point of view* quite abominable.'[1]

That is, Einstein's initial rejection of the expanding universe idea as 'abominable' seems not to have been based on a judgement of its mathematical statement. (Like in English, the original French *abominable* has aesthetic connotations. Even if Lemaître's word-choice did not reflect precisely Einstein's reaction, *Lemaître* here admitted an aesthetic judgement not grounded on mathematics.) Arthur Eddington* (1882–1944) also seems to have had a non-mathematical distaste for the related idea that the universe had a beginning:

> 'Philosophically, the notion of a beginning of the present order of Nature is repugnant to me. [...] I see no way round it; but whether future developments of science will find an escape I cannot predict.'[2]

In 1955, the sociologist and philosopher Lewis S. Feuer (1912–2002) cautioned against the idea that mathematical elegance and the simplicity of a theory were 'grounded on the same aesthetic motive', which he thought had confused the discussion of simplicity in science.[3] For Feuer, even if Occam's razor pointed to the simplest hypothesis compatible with the phenomena, mathematical elegance was concerned with the 'aesthetic economy' of the formulation.[4] The two were distinct, for the same theory might admit formally equivalent but aesthetically distinct mathematical statements.[5]

The physicist Leonard Susskind (1940–) said that 'elegance and simplicity can sometimes be found in principles that don't at all lend themselves to equations';[6] as examples he gave the principles upon which Darwinian evolution is grounded: random mutation and competitive selection. The physicist Robert Mills (1927–99) thought that

> 'physicists [...] regard a fundamental theory as elegant if it springs from a principle that is concise and easy to grasp, and yet has profound and far-reaching consequences that *may* require the most complex mathematics to work out.'[7]

Note the word 'may': the elegance of the theory had no dependence on the complexity of mathematics; Mills thought that for physicists, elegance had a purely conceptual grounding. Similarly, Marcelo Gleiser (1959–) wrote that

> 'In the eyes of a physicist, a "beautiful" theory is one that uses a minimum amount of conceptual input for a maximal amount of explanatory power.'[8]

The physicist Heinz Pagels (1939–88) thought that there had been a shift in the type of beauty physicists saw in theories. Some saw visual beauty in geometry; some saw abstract beauty in symbolism.[9] In physics, there had been a shift from visual to abstract beauty:

Beauty in science, mathematical and non-mathematical

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[1] Lemaître, 'Rencontres avec A. Einstein', p. 129, emphasis added; trans. C.

* ▸ pp. 893–4

[2] Eddington, 'The End of the World', p. 450.

[3] Feuer, 'The Principle of Simplicity', p. 115.

[4] *Loc. cit.*

[5] *Ibid.*, p. 117.

[6] Susskind, *The Cosmic Landscape*, pp. 379–80.

[7] Mills, 'Beauty and Truth', p. 200, emphasis added.

[8] Gleiser, *The Simple Beauty of the Unexpected*, p. 64.

[9] Pagels, *The Cosmic Code*, p. 305.

Beauty in science, mathematical and non-mathematical

> ‘In modern quantum physics the aesthetic sense is conceptual in contrast to an earlier time when physicists’ ability to visualize the natural world played an important role. Instead of pictures we have symmetries described by mathematics. The quantum world of elementary particles is organized according to complex and beautiful symmetry principles.’[1]

McAllister’s classes of aesthetic properties

McAllister listed five classes of aesthetic properties of scientific theories, which form a useful framework in which to discuss the difference between aesthetic properties of the mathematical and non-mathematical aspects of a theory:[2]

Form of symmetry. Symmetry could be a feature of the mathematics of a scientific theory, such as the near-perfect symmetry of Maxwell’s equations (see (M1–M4) on page 884). But symmetry could also be seen in the concepts that grounded a theory or the entities to which it applied, such as the similarity between space and time posited in the theory of relativity, which Paul Dirac (1902–84) found beautiful,[3] or wave-particle duality, which Louis de Broglie (1892–1987) hypothesized on the grounds of symmetry.[4]

Form of simplicity. There has been little agreement among philosophers of science as to the nature of the simplicity considerations scientists use. Simplicity might be connected to the empirical adequacy of the theory; it might involve the explanatoriness of a theory; it might be observer-relative. Certainly it could be perceived in the mathematical expression of a theory: fewer fundamental formulae, containing fewer variables, using standard constants, all entailed greater simplicity. But it could also be perceived in the non-mathematical aspects, such as positing the existence of fewer fundamental concepts.[5] Furthermore, there could be a difference in simplicity between a mathematical and a non-mathematical statement of the same phenomenon. As J. B. S. Haldane (1892–1964) put it, in a slightly sardonic way:

> ‘In scientific thought we adopt the simplest theory which will explain all the facts under consideration and enable us to predict new facts of the same kind. The catch in this criterion lies in the word “simplest.” It is really an aesthetic canon such as we find implicit in our criticisms of poetry or painting. The layman finds such a law as
>
> $$\frac{\partial x}{\partial t} = \kappa \frac{\partial^2 x}{\partial y^2}$$
>
> less simple than “it oozes,” of which it is the mathematical statement. The physicist reverses this judgement.’[6]

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[1] Pagels, *The Cosmic Code*, p. 305.

[2] McAllister, *Beauty & Revolution*, pp. 40–58, 105–24.

[3] Dirac, ‘The Relation between Mathematics and Physics’, p. 128.

[4] Mehra & Rechenberg, *Hist. Devel. Quantum Theory*, vol. 1.1, p. 586–7.

[5] McAllister, *Beauty & Revolution*, pp. 105–24.

[6] Haldane, ‘Science and Theology as Art Forms’, p. 227.

Invocation of a model. This term referred to a theory positing an analogy between the phenomena it aimed to describe and some other, presumably better understood model.[1] An example was Pierre-Simon Laplace's (1749–1827) caloric theory, which presented heat as a fluid. Aesthetic value might be perceived in the model, but a precise mathematical formulation of a theory was usually not dependent on the model. Thus any aesthetic value seen in the model would seem to be entirely non-mathematical.[2]

Visualizability or abstractness. A visualizable structure was one that could be understood and manipulated with the aid of a mental image.[3] McAllister suggested that the instantiation of visualization was non-mathematical, and thus that visualizability was not manifested in the mathematical components of the theory.[4] On the other hand, mathematicians' self-reporting has indicated a very heavy bias in favour of using visual images in their thought, so this kind of aesthetic property seems likely to be found in mathematical content.[5]

Metaphysical allegiance. The metaphysics that underlay a physical theory had historically included concepts like atomism, action at a distance, and determinism.[6] Some of these concepts, like action at a distance, seem only tenuously connected to the mathematics. But determinism (as opposed to probabilistic theories) was a commitment that was potentially bound very tightly to the mathematical expression of a theory. So some properties in this class seem potentially applicable to mathematics.

David Davies (1949–) and Cain S. Todd (1976–) questioned whether metaphysical allegiance and invocation of a model could really be considered kinds of aesthetic properties.[7] But regardless of whether one accepts all of McAllister's classes as aesthetic, it is clear that in *some* of them there are aesthetic properties that can potentially be discerned in non-mathematical aspects of a theory. A perhaps more precise formulation of this conclusion is the following: a scientific theory may have aesthetic properties that supervene only on non-mathematical properties of that theory.

Effects on scope

The aesthetics of mathematics is not a special case of the aesthetics of the natural sciences. The epistemological concerns of the natural sciences and mathematics are different. Roughly, mathematics, as a pursuit, is concerned only with the correctness of reasoning in the sense of *internal* conceptual consistency and rigour, perhaps with direct reference to explicitly-given axioms. Natural science is concerned with empirical adequacy: whether the predictions made by its theories agree with observation. This empirical adequacy can sometimes be in tension with mathematical correctness: there are many instances, particularly in fundamental physics, where the mathem-

[1] McAllister, *Beauty & Revolution*, p. 44.

[2] *Ibid.*, p. 45.

[3] *Ibid.*, p. 40.

[4] *Ibid.*, pp. 52–3.

[5] Hadamard, *Psychology of Invention*, pp. 84–6.

[6] McAllister, *Beauty & Revolution*, pp. 54–5.

[7] Davies, 'McAllister's aesthetics', p. 28; Todd, 'Unmasking the truth beneath the beauty', p. 70.

atics is not rigorous by the standards of pure mathematics.[1] Richard Feynman[*] (1918–88) made a memorable distinction between what he called 'Babylonian' and 'Greek' traditions in mathematics.

> 'The Babylonian attitude [...] is that you know all of the various theorems and many of the connections in between, but you have never fully realized that it could all come up from a bunch of axioms.'[2]

Thus, unlike Euclid's *Elements*, mathematics in this Babylonian mode was not — and did not claim to be — a rigorous structure erected upon an axiomatic foundation. But physics, he thought, required the Babylonian kind of mathematics.[3]

The historian therefore faces a problem of demarcation: what aspects of the history of beauty in the physical sciences count as part of the history of *mathematical* beauty as it has been relevant to physical science? A description of Einstein's field equations as *beautiful* is surely relevant to this history; the *elegance* of its sparse metaphysical commitments may not be.

For the purposes of the present chapter, and of Chapters 24 and 25, the author has adopted a narrow focus on those parts of the discourse on aesthetic value in science that are more-or-less explicitly mathematical. For instance, a discussion of symmetry is only considered relevant when the symmetry is discussed with at least some reference to a mathematical formulation.

NUMBER-PATTERNS

At various times, the search for order in nature has led scholars to postulate relationships that are purely pleasing numerical patterns, in the sense that they are not founded upon any deeper explanation.

Titius–Bode law. The most famous example is probably the Titius–Bode law, which states that the semi-major axes of the orbits of the planets in the Solar system are approximated by

$$r_n = 4 + 3 \cdot 2^n,$$

for $n = -\infty, 0, 1, 2, \ldots$, where the semi-major axis of the Earth's orbit around the Sun is $r_1 = 10$,[4] so that (in later terms) r_n denotes a distance very close to $(r_n/10)$ au.

The law was first formulated by the astronomer Johann Daniel Titius (1729–96) and silently inserted into his 1766 German translation of Charles Bonnet's (1720–93) *Contemplation de la Nature*.[5]

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[1] Davey, 'Is Mathematical Rigor Necessary in Physics?'

[*] ▸ pp. 920–3

[2] Feynman, *The Character of Physical Law* (book), p. 46.

[3] *Ibid.*, pp. 47 sqq.

[4] Nieto, *The Titius–Bode Law*, p. 1.

[5] *Ibid.*, p. 9.

Titius derived the law, which he called an 'admirable relationship [bewundernswürdige Verhältniss]',[1] from a combination of a remark of Christian Wolff (1679–1754) about the normalized distances from the sun of the orbits of the then-known planets and a speculation of Johann Heinrich Lambert (1728–77) about undiscovered planets orbiting between Mars and Jupiter.[2] The astronomer Johann Elert Bode (1747–1826) saw the second edition of Titius's translation, in which the inserted passage was removed to a footnote, and was immediately convinced of the truth of the law. He failed to give credit to Titius when he first mentioned the law in the second edition of his *Deutliche Anleitung zur Kenntniss des gestirnten Himmels*, only crediting him in later editions and in other writings. Because of this omission, because of Bode's championing of the law, and because of his greater scientific fame, the law was long credited to him alone and still bears his name.[3]

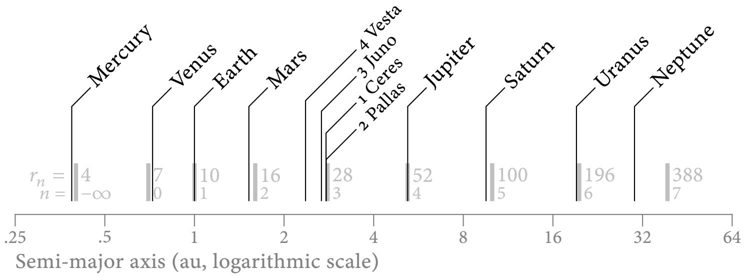

FIGURE 23.1
The semi-major axes predicted by the Titius–Bode law, shown in grey (1 au ≈ 10), with the semi-major axes of the orbits of the eight planets and the first four discovered minor planets.

At the time the law was formulated, it fitted well with the orbits of the then-known planets (see Figure 23.1). The term $r_3 = 28 = 4 + 3 \cdot 2^3$ also predicted a planet with an orbit with semi-major axis 2.8 au, between the orbit with semi-major axis 1.6 au (per r_2) predicted for Mars (compared with the observed 1.524 au) and the predicted orbit with semi-major axis 5.2 au (per r_4) of Jupiter (compared with the observed 5.203 au). When Uranus was discovered in 1781 and its orbit calculated by Laplace later that decade, it fitted the law: the orbit of Uranus has a semi-major axis of 19.191 au, close to the predicted 19.6 au (per r_6). Similarly, the orbit of the asteroid Ceres, discovered in 1801, has semi-major axis 2.767 au, close to the predicted 2.8 au for a planet orbiting between Mars and Jupiter.[4] The discovery of Pallas and other asteroids posed a problem, but it was resolved by suggesting that a single planet had once occupied the appropriate orbit and had been disrupted.[5] The discovery of Neptune in 1846 dealt a blow to the Titius–Bode law, even though both John Couch Adams (1819–92) and Urbain Le Verrier (1811–77) made use of it in predicting the position of Neptune in the sky:[6] Neptune's orbit has semi-major axis 30.07 au, far from the predicted 38.8 au.[7] [8]

Numerical coincidences. There are countless numerical coincidences in nature that were noticed but held less attention. For instance, the

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[1] Titius, ins. in Bonnet, *Betrachtung über die Natur*, pp. 7–8, trans. C.; cf. Bonnet, *Contemplation de la Nature*, p. 7.

[2] Nieto, *The Titius–Bode Law*, pp. 7–8.

[3] *Ibid.*, pp. 11–12, 14–15.

[4] *Ibid.*, pp. 16–17.

[5] *Ibid.*, p. 18.

[6] *Ibid.*, p. 21.

[7] *Ibid.*, pp. 20–1.

[8] Observed semi-major axes are for epoch J2000. Planetary data from Standish et al., 'Orbital Ephemerides of the Sun, Moon, and Planets', Tbl. 5.8.1 [p. 316]; minor planet data from the JPL Small-Body Database: `https://ssd.jpl.nasa.gov/tools/sbdb_lookup.html#/`

chemist James E. Mills (1876–1950) published in 1932 a curious paper that deduced myriad coincidences between various combinations of physical constants;[1] he seemed to find it insignificant that the coincidences were dependent on the units in which the constants were expressed.[2]

In the journal *Nature*, which he edited, John Maddox (1925–2009) lamented the intellectual effort devoted to numerical coincidences, publishing what he perhaps ironically characterized as an 'elegant' example that expressed the ratios of masses of mesons and baryons in terms of π.[3]

Balmer formula. In spite of the disdain for what can seem no more than an aesthetic-numerical game of trying to find pleasing combinations of numbers, there is at least one instance where a numerical law was inferred from observation and was only later secured by a theoretical underpinning. Johann Balmer (1825–98) discovered that that the frequencies of the observed lines of the spectrum of the hydrogen atom obey the formula[4]

$$\lambda = B\left(\frac{n^2}{n^2 - 4}\right) \qquad \text{where } B = 3.6456 \times 10^{-7}\ \text{m.}$$

(More exact values of B have since been derived.) This equation helped, and was explained by, the formulation of the Bohr model of the atom.

JAMES CLERK MAXWELL

Though James Clark Maxwell's[*] (1831–79) contributions ranged over physics, both theoretical and experimental, and, though he also made a revolutionary study of the kinematic theory of gases, it is his part in the development of the theory of electromagnetism that is most relevant here. This theory is centred on four equations named for Maxwell, which, as will be shown in this chapter, seem to be universally and consistently held up as exemplars of mathematical beauty in physical law.[5] Expressed in modern notation as differential equations, they are:

$$\text{Gauss's law} \qquad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{M1}$$

$$\text{Gauss's magnetic law} \qquad \nabla \cdot \vec{B} = 0 \tag{M2}$$

$$\text{Maxwell–Faraday eq.} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{M3}$$

$$\text{Ampère's law} \qquad \nabla \times \vec{B} = \mu_0\left(\vec{J} + -\varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \tag{M4}$$

[1] Mills, 'Relations Between Fundamental Physical Constants'.

[2] *Ibid.*, p. 1101.

[3] Maddox, 'The temptations of numerology'; Stanbury, 'The alleged ubiquity of π'.

[4] Maier, 'Balmer, Johann Jakob', p. 425.

[*] ▸ pp. 434–6

[5] See also Frenkel, *Love and Math*, ch. 16.

James Clerk Maxwell

Even a reader who is unaware of the physical interpretation of the symbols can see clear symmetries in the equations, which has doubtless been part of their aesthetic appeal.

Given the aesthetic acclaim that Maxwell's equations (M1–M4) have received, it is tempting to assign to Maxwell an aesthetic sense that guided him in formulating them. Henri Poincaré wrote that Maxwell's reformulation of electromagnetic theory was in part due to seeing how the equations would become more symmetrical, although the effect of the new formulation remained undetectable experimentally for many years.[1] Poincaré explained how he thought Maxwell could make this leap so early and yet be correct:

> 'It was because Maxwell was profoundly steeped in the sense of mathematical symmetry; would he have been so, if others before him had not studied this symmetry for its own beauty?'[2]

But on the origin of his taste for science, Maxwell wrote:

> 'I always regarded mathematics as the method of obtaining the *best* shapes and dimensions of things; and this meant not only the most useful and economical, but chiefly the most harmonious and the most beautiful.'[3]

What Maxwell said here does not indicate using beauty *in* mathematics; the mathematics was but a means to a beautiful end.

Maxwell's development of his electromagnetic theory seems to have been dependent less on aesthetic sense and more on visual imagination. He had this latter faculty in a degree equal to Michael Faraday (1791–1867), whose theory of lines of force he admired.[4] Maxwell could describe and explore mathematically theories of this type, but he always seemed to intend to have a mechanical model of electrodynamic actions.[5]

And Maxwell did not write his theory in the concise modern notation of (M1–M4), and indeed did not group the four equations together. They first appeared in his paper 'On Physical Lines of Force', published in four parts during 1861–2. Gauss's magnetic law and the Maxwell–Faraday law appeared in part II, as equations (56) and (54); Gauss's law and Ampère's law appeared in part III, as equations (115) and (112). And evidence suggests that Maxwell originally intended the paper to end with part II.[6]

Beauty in the mathematical formulation of the theory is never mentioned in Maxwell's book *A Treatise on Electricity and Magnetism*, a record of the state of his thought, first published in 1873. Elegance is mentioned precisely once.[7] Maxwell's effect on the history of mathematical beauty was a delayed one, and largely due to the reformulation of his equations by Oliver Heaviside (1850–1925) into (M1–M4), which brought out their symmetry. As Heaviside's contemporary George Francis FitzGerald (1851–1901) put it: 'Every mathematician can appreciate the value and beauty of this.'[8]

[1] Poincaré, 'Analysis and Physics', § II.

[2] *Loc. cit.*

[3] Maxwell, quot. in Galton, *English Men of Science*, pp. 155–6, emphasis in orig.; attributed to Maxwell in Hilts, 'A Guide to *English Men of Science*', p. 59.

[4] Whittaker, *A History of the Theories of Aether and Electricity*, vol. 1, p. 242.

[5] *Ibid.*, vol. 1, pp. 246 sqq.

[6] Everitt, 'Maxwell, James Clerk', p. 209.

[7] Maxwell, *A Treatise on Electricity and Magnetism*, vol. I, p. 281.

[8] FitzGerald, 'Heaviside's Electrical Papers', p. 294.

ALBERT EINSTEIN

> ‘ If he possessed […] the impelling desire
> to express himself in terms of beauty,
> Einstein could write the most intoxicating lyrics
> about relativity and the pleasures of pure mathematics. ’
>
> — Aldous Huxley, ‘Subject-Matter of Poetry’, pp. 29–30.

There seems to be a consensus that Albert Einstein (1879–1955) was guided by aesthetic judgement in formulating his physical theories and in particular the special and general theories of relativity. The true situation is more complicated than this consensus suggests, particularly because of a historiographical conflation, when it comes to interpreting Einstein’s thought, of beauty, elegance, and simplicity.

Mathematical beauty and elegance

There is no doubt at all that Einstein was sensitive to mathematical beauty and elegance throughout his life, both in terms of pure mathematics and in its application to physics. A few examples will suffice here. While working towards the general theory of relativity, Einstein thought his rigorous mathematical derivation of the theory of gravitation in a static field[1] was ‘stunningly beautiful and amazingly simple’.[2] After he published the general theory of relativity in 1915, he wrote that it was ‘of incomparable beauty’[3] (although he did not refer specifically to *mathematical* beauty).

The year after Einstein published the general theory, Karl Schwarzschild (1873–1916) and Johannes Droste (1886–1963) independently discovered the first non-trivial solution of Einstein’s field equations. This solution, now called the Schwarzschild metric, describes the gravitational field around a spherical mass under certain assumptions. Einstein wrote that ‘Droste’s paper is beautiful; he calculates truly gracefully’.[4] In 1918, he wrote with reference to Hermann Weyl’s (1885–1955) book *Space·Time·Matter* [*Raum·Zeit·Materie*], which included a systematic presentation of the general theory of relativity and of which he was reading the printer’s proofs, ‘I constantly admire anew the beauty and elegance of your derivations.’[5]

In a letter to the *New York Times* in memory of the then recently deceased Emmy Noether (1882–1935), Einstein wrote:

> ‘Pure mathematics is, in its way, the poetry of logical ideas. One seeks the most general ideas of operation which will bring together in simple, logical and unified form the largest possible circle of formal relationships. In this effort toward logical beauty spiritual formulas are discovered necessary for the deeper penetration into the laws of nature.’[6]

[The text of this letter has in the past been attributed to Weyl, and he may have had a degree of input or influence, but Einstein composed

[1] See Einstein, ‘Lichtgeschwindigkeit und Statik des Gravitationsfeldes’.

[2] Einstein, letter to Hopf, dated after 20 Feb. 1912, repr. in Einstein, *Collected Papers*, vol. 5, [ltr 364] p. 418; vol. 5(supp.), p. 266.

[3] Einstein, letter to Zangger, dated 26 Nov. 1915, repr. in *ibid.*, vol. 8A, [ltr 152] p. 151; trans. C.

[4] Einstein, letter to Ehrenfest, dated 7 Nov. 1916, repr. in *ibid.*, vol. 8A, [ltr 276] p. 362; trans. vol. 8(supp.), p. 263.

[5] Einstein, letter to Weyl, dated 18 May 1918, repr. in *ibid.*, vol. 8B, [ltr 511] p. 724; trans. vol. 8(supp.), p. 531.

[6] Einstein, ‘The Late Emmy Noether’.

it himself in German and it was translated into English by or at the behest of Abraham Flexner (1866–1959).[1]]

[Einstein also discussed beauty in elementary geometry in correspondence with Max Wertheimer (1880–1943).[*]]

Beauty and truth

There has been a tendency to assign to Einstein a great reliance on aesthetic value in mathematics as a guide in theoretical physics. For instance, in a 1979 panel discussion, the physicist Chen-Ning Yang (1922–2025) said that part of his contribution was based on 'a statement made and emphasized repeatedly by Einstein, that beautiful mathematics is the language of fundamental physics';[2] in 1971, the astrophysicist Subrahmanyan Chandrasekhar (1910–95) wrote that Einstein 'arrived at his field equations by qualitative arguments of a physical nature combined with an unerring sense for mathematical elegance and simplicity'.[3] On the other hand, Roger Penrose (1931–), while recognizing the 'profoundly beautiful mathematical structure' of the theory, thought that Einstein was guided by physical considerations, notably the principle of equivalence.[4]

Hermann Bondi (1919–2005) said that a discussion of unified field theories with Einstein in 1947, when he was a young scientist and Einstein the doyen of physics, made it clear that Einstein evaluated mathematical statements of physical theories in terms of their aesthetic value:

> 'when I put down a suggestion that seemed to me cogent and reasonable, he did not in the least contest this, but he only said, "Oh, how ugly." As soon as an equation seemed to him to be ugly, he really rather lost interest in it and could not understand why someone else was willing to spend so much time on it. He was quite convinced that beauty was a guiding principle in the search for important results in theoretical physics.'[5]

* * *

Max Abraham (1875–1922) was a staunch opponent of relativity, a champion of the aether, and a tenacious promoter of the reduction of all phenomena to electromagnetism.[6] Starting in January 1912, he published a theory of gravitation whose core was a four-dimensional equation for the gravitational potential:[7]

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Phi}{\partial u^2} = 4\pi\gamma\nu,$$

where γ is the gravitational constant and ν is the density of matter. He also derived a relation between the speed of light and gravitational

[1] Roquette, 'Emmy Noether and Hermann Weyl', § 8.

[*] ▸ pp. 640–1

[2] Yang, quot. in Bethe et al., 'Einstein and the Physics of the Future', p. 501.

[3] Chandrasekhar, 'Beauty and the quest for beauty in science', p. 70.

[4] Penrose, *The Road to Reality*, p. 888.

[5] Bondi, quot. in Whitrow, *Einstein*, p. 82.

[6] Cattani & De Maria, 'Max Abraham and the Reception of Relativity in Italy', pp. 160–1.

[7] Abraham, 'Zur Theorie der Gravitation', p. 1.

potential that, in the limit of weak fields, reduced to a relation Einsten had previously formulated.[1]

This approach initially appealed to Einstein, but by the end of January 1912 he had concluded that Abraham had made serious errors, which led to an increasingly fractious debate in journals and private correspondence.[2] Einstein's initial reaction was at least partly due to the mathematical beauty of Abraham's approach, and he seems to have concluded that this beauty was at least partly to blame for Abraham having being misled:

> 'Abraham's theory has been created out of thin air, i.e., out of nothing but considerations of mathematical beauty, and is completely untenable. How this intelligent man could let himself be carried away with such superficiality is beyond me. To be sure, at the first moment (for 14 days!) I too was totally "bluffed" by the beauty and simplicity of his formulas.'[3]

* * *

Einstein did not think that one could choose between two theories on grounds of superior mathematical beauty or elegance alone. For instance, in 1916–18 there was a debate between Einstein and Willem De Sitter (1872–1934) about relativity, in the course of which De Sitter proposed a cosmological model which was homogeneous and matter-free. This model was in conflict with Einstein's view that the metric of space-time (from another perspective, the gravitational field) should be fully determined by matter. Einstein thus thought there must be a flaw in De Sitter's solution, and sought to find it. There so developed a correspondence between Einstein, De Sitter, Hermann Weyl (1885–1955), and Felix Klein (1849–1925).[4]

Einstein's objection to the De Sitter solution was primarily metaphysical rather than aesthetic. He had a preference for a certain *cause* for the metric of space-time. Einstein in fact admitted that in terms of mathematical elegance, the De Sitter solution was superior, but maintained that it was not the correct model of the universe:

> 'I believe I can assert very definitely that this, because it is four-dimensionally uniform, *mathematically more elegant conception of the world*, does not correspond to reality.'[5]

Ultimately, Einstein came to accept De Sitter's solution as homogeneous, fully regular, matter-free, but thought it physically irrelevant because it did not yield a globally static universe, as he thought a correct theory should.

Einstein did not see the general (not specifically mathematical) beauty of a theory as necessarily pointing to its truth. In 1922, in the context of Eddington's and Weyl's efforts to formulate a unified theory of gravitation and electromagnetism, Einstein starkly separated truth and beauty:

[1] Klein et al., 'Einstein on the Static Field', pp. 124–5.

[2] Klein et al., 'Einstein on the Static Field', pp. 125–7; Cattani & De Maria, 'Max Abraham and the Reception of Relativity in Italy', pp. 162–4.

[3] Einstein, letter to Besso, dated 26 Mar. 1912, repr. in Einstein, *Collected Papers*, vol. 5, [ltr 377] p. 436–7; trans. vol. 5(supp.), p. 278.

[4] For a survey, see *ibid.*, vol. 8A, pp. 351–7.

[5] Einstein, letter to Klein, dated before 3 Jun. 1918, repr. in *ibid.*, vol. 8B, [ltr 556] p. 786; trans. vol. 8(supp.), p. 577, emphasis added.

> 'What Weyl and Eddington did in this regard is indeed beautiful [zwar schön] but not true.'[1]

Similarly, in August 1925 he wrote to Paul Ehrenfest (1880–1933) that his own latest effort to formulate a unified theory of gravitation and electromagnetism was 'very beautiful but doubtful'.[2]

Leaving aside the aesthetic value of mathematics, Einstein's view on the relative roles of experimental investigation and mathematical theory-building seems to have shifted during his lifetime. Early in his work, he valued experimental inquiry as a guide towards choosing between approaches that, from a mathematical perspective, seemed to him equally plausible.[3] What he seems to have been unwilling to do was to make *ad hoc* modifications to an already-formulated theory in the light of contrary evidence; rather, he would be prepared to abandon the theory completely instead of trying to tweak it. The overall mathematical consistency of the theory was too valuable.[4]

Einstein wrote in 1917 that mathematics alone could not lead to physical truth:

> 'It does seem to me that you are very much overestimating the value of purely formal approaches. The latter are certainly valuable when it is a question of formulating definitively an *already established* truth, but they almost always fail as a heuristic tool.'[5]

He seems to have still held this view in 1922, when he wrote that '[t]ruth cannot be found by pure speculation; God Almighty goes His own ways'.[6]

But later, perhaps starting around 1928, he seems to have become more enamoured of the possibility of wholly mathematical reasoning yielding truth about the universe.[7] He declared this position in a lecture he delivered in 1933:

> 'I am convinced that we can discover by means of purely mathematical constructions the concepts and the laws connecting them with each other, which furnish the key to the understanding of natural phenomena. Experience may suggest the appropriate mathematical concepts, but they most certainly cannot be deduced from it. Experience remains, of course, the sole criterion of the physical utility of a mathematical construction. But the creative principle resides in mathematics. In a certain sense, therefore, I hold it true that pure thought can grasp reality, as the ancients dreamed.'[8]

The physicist and logical positivist philosopher Philipp Frank (1884–1966) thought that the last sentence of the preceding quotation suggested that Einstein thought that statements of physics could be proved by purely mathematical means, but only '[i]n a certain sense'. This 'certain sense', thought Frank, referred to the heuristic method of seeking 'mathematical simplicity and beauty'.[9] Frank himself thought

[1] Einstein, letter to Zangger, dated 18 Jun. 1922, repr. in Einstein, *Collected Papers*, vol. 13, [ltr 241] p. 370; trans. mod. vol. 13(supp.), p. 198.

[2] Einstein, letter to Ehrenfest, dated 18 Aug. 1925, repr. in *ibid.*, vol. 15, [ltr 49], p. 101; trans. vol. 15(supp.), p. 63.

[3] Hentschel, 'Einstein's Attitude Towards Experiments', pp. 597–8.

[4] *Ibid.*, p. 594.

[5] Einstein, letter to Klein, dated 17 Dec. 1917, repr. in Einstein, *Collected Papers*, vol. 8A, [ltr 408] p. 569; trans. vol. 8(supp.), p. 418, emphasis in orig.

[6] Einstein, letter to Zangger, dated 18 Jun. 1922, repr. in *ibid.*, vol. 13, [ltr 241] p. 370; trans. mod. vol. 13(supp.), p. 198.

[7] Hentschel, 'Einstein's Attitude Towards Experiments', p. 616.

[8] Einstein, 'On the Method of Theoretical Physics', p. 274.

[9] Frank, 'Einstein, Mach, and Logical Positivism', p. 283.

that it was an empirically justified fact that this heuristic method was successful in theoretical physics.[1]

Towards the end of his life Einstein wrote that '[e]xperience alone can decide on truth'.[2] This statement is not inconsistent with his view that pure mathematical reasoning could lead to physical truths: it simply required that the conclusions reached by mathematics had to be tested empirically.

◆

Did Einstein actually seek mathematical beauty or elegance in formulating physical theories? He seems only to have thought of mathematical *simplicity* as an apparent guide, and he grounded this view on its previous success: 'Our experience hitherto justifies us in believing that nature is the realization of the simplest conceivable mathematical ideas.'[3] In what did this simplicity consist? Some pointers lie in his characterization of the simplest possible explanation as 'the one which contains the fewest possible mutually independent postulates or axioms; since the content of these logical, mutually independent axioms represents that remainder which is not comprehended'.[4] That is, postulates or axioms were what one had to accept as given; comprehension was restricted to how these foundations yielded the theory and so the observed phenomena.

This belief in simplicity as a guide naturally raises the question of whether, for Einstein, simplicity was an aesthetic value. Einstein's biographer Abraham Pais (1918–2000), himself a physicist, thought so.[5] The historian and philosopher of science Yehuda Elkana (1934–2012) went further: 'for Einstein, simplicity was equivalent to beauty'.[6] Some later writers seem to have agreed with this assessment,[7] but others have thought it too unsubtle.[8]

Elegance, tailors, and cobblers

In the introduction to his little book *Relativity*, a popular account of the special and general theories of relativity, Einstein wrote that

> 'In the interest of clearness, it appeared to me inevitable that I should repeat myself frequently, without paying the slightest attention to the elegance of the presentation. I adhered scrupulously to the precept of that brilliant theoretical physicist L. Boltzmann, according to whom matters of elegance ought to be left to the tailor and to the cobbler.'[9]

His reference to the adage he attributed to Ludwig Boltzmann (1844–1906) has been a source of some historiographical confusion. It has sometimes been misquoted, like when Leonard Susskind (1940–) attributed to Einstein the epigraph 'If you are out to describe the truth, leave elegance to the tailor',[10] or when Steven Weinberg (1933–2021)

[1] Frank, 'Einstein, Mach, and Logical Positivism', pp. 283–4.

[2] Einstein, 'On the Generalized Theory of Gravitation', p. 17.

[3] Einstein, 'On the Method of Theoretical Physics', p. 274.

[4] Einstein, 'Address at Columbia University'.

[5] Pais, *Subtle is the Lord*, p. 140.

[6] Elkana, 'The Myth of Simplicity', p. 222.

[7] See, for example, McAllister, *Beauty & Revolution*, p. 109.

[8] Osborne, 'Mathematical beauty and physical science', p. 299, n. 15.

[9] Einstein, *Relativity*, p. ix.

[10] Susskind, *The Cosmic Landscape*, p. 111.

wrote that 'Einstein has been quoted as saying that scientists should leave elegance to tailors'.[1] But these were more than misquotations or misattributions: they were *misinterpretations* of Einstein's point. Both Susskind and Weinberg cited Einstein in discussions of the aesthetics of physical theories. As the context makes clear, Einstein was referring not to the elegance of the *theory*, but that of the *presentation*; specifically, he was making the point that in exposition, he accepted the textual inelegance of repetition as the price of clarity.

At least one contemporary of Einstein also seems to have thought that he was making a point about theoretical elegance. The mathematician Eduard Study (1862–1930) said in a letter to Einstein, dated two years after *Relativity*'s publication, that '[t]he "elegance" of the formulas in the theory of invariants, for example, is a *characteristic of natural laws* whose existence must not be ignored'.[2] But Study's conception of elegance involved, at least in part, the expositional quality of notational choices: he complained that Boltzmann would use separate symbols for x, πx, $\pi^2 x$, and $\frac{3}{2}\pi x$.[3]

On the other hand, Einstein's friend Maurice Solovine (1875–1958) understood that Einstein was making a point about elegance in writing: he wrote in 1921 that famous men such as Einstein could ignore elegance, but that mortals such as Solovine himself could not, especially when rendering Einstein's prose into French.[4]

In any case, Einstein may have been ambivalent about Boltzmann's adage. In correspondence with Gaston Moch (1859–1935), who had offered to translate *Relativity* into French, Einstein referred to Boltzmann as having 'jokingly [scherzhaft]' advised on leaving elegance to the tailor and cobbler.[5]

General relativity

Einstein's general theory of relativity seems to have been, and still is, universally esteemed for its beauty. Lev Landau (1908–68) & Evgeny Lifshitz (1915–85) wrote that it was 'probably the most beautiful of all existing physical theories'.[6] Wolfgang Pauli (1900–58) drew specific attention to the beauty of its mathematical statement: 'it will always remain the pattern of a theory of consummate beauty of the mathematical structure'.[7] In a 1926 lecture at the University of Munich, the mathematician Constantin Carathéodory (1873–1950) pointed to it having purely mathematical value, independent of its physical relevance, precisely because of its beauty:

> 'From the point of view of a mathematician, the construction of Einstein's ideas is one of the most beautiful ever. Even if it did not have any physical background it would be worth going through.'[8]

The physicist Brian Greene (1963–) described the general theory of relativity as 'mathematically elegant', 'conceptually powerful', and 'fully consistent', in that order.[9] And Max Tegmark wrote that it 'arguably broke the record as the most mathematically beautiful theory', because it made gravity a consequence of geometry.[1]

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[1] Weinberg, *Dreams of a Final Theory*, p. 134.

[2] Study, letter to Einstein, dated 27 Sep. 1918, repr. in Einstein, *Collected Papers*, vol. 8B, [ltr 627] p. 897; trans. vol. 8(supp.), p. 657, emphasis in orig.

[3] Study, letter to Einstein, dated 27 Sep. 1918, repr. in *ibid.*, vol. 8B, [ltr 627] p. 897.

[4] Solovine, letter to Einstein, dated 28 Feb. 1921, repr. in *ibid.*, vol. 12, [ltr 69] p. 106.

[5] Einstein, letter to Moch, dated 10 Jul. 1920, repr. in *ibid.*, vol. 10, [ltr 78] p. 340; trans. vol. 10(supp.), p. 211.

[6] Landau & Lifshitz, *The Classical Theory of Fields*, p. 228.

[7] Pauli, 'Albert Einstein and the Development of Physics', p. 120.

[8] Carathéodory, quot. in Georgiadou, *Constantin Carathéodory*, p. 105.

[9] Greene, *The Fabric of the Cosmos*, p. 72.

The beauty of the theory seems to have been taken either as an indication of or a reason to desire its correctness: Hendrik Lorentz (1853–1928) wrote that 'Einstein's theory has the very highest degree of aesthetic merit: every lover of the beautiful must wish it to be true.'[2] And Weinberg said:

> 'I remember that, when I learned general relativity in the 1950s, before modern radar and radio astronomy began to give impressive new evidence for the theory, I took it for granted that general relativity was more or less correct. [...] I believe that the general acceptance of general relativity was due in large part to the attractions of the theory itself — in short, to its beauty.'[3]

Paul Dirac was very forceful and direct in arguing that it was the *mathematical* beauty of the general theory of relativity that led to its acceptance.*

◆

An early success of the general theory of relativity was that it explained the thitherto anomalous advance of the perihelion of Mercury. The orbit of the planet Mercury had resisted accurate prediction using Newtonian physics; in particular, forecasts of its transits across the face of the Sun, in principle calculable to the second, failed to coincide with observation. Just as observations of Uranus departing from its predicted track led to the suggestion that there was another, still more distant, planet whose gravity pulled Uranus from its calculated orbit, so the observed behaviour of Mercury led to predictions of a planet even close to the Sun than Mercury.[4] This hypothetical planet was named Vulcan, and was thought to be always lost in the glare of the Sun and only observable in transit or during total eclipses. Sustained efforts to obseve Vulcan continued until 1915, when Einstein calculated that Mercury's observed orbit around the Sun was as predicted by general relativity, without the effect of the gravity of hypothetical Vulcan.[5]

But before general relativity, and with Vulcan unobserved, the astronomer Asaph Hall (1829–1907), who had discovered Mars's two moons Phobos and Deimos,[6] suggested altering the inverse square law of gravitation so that the exponent was 2.000 000 16; this small change would produce the observed motion.[7] His idea seems to have gained little traction, presumably at least in part because of the mathematical ugliness of such an exponent.

[1] Tegmark, *Our Mathematical Universe*, ch. 13.

[2] Lorentz, *The Einstein Theory of Relativity*, p. 23.

[3] Weinberg, *Dreams of a Final Theory*, p. 98.

* ▸ p. 904

[4] Baum & Sheehan, *In Search of Planet Vulcan*, chs 6, 9.

[5] *Ibid.*, chs 10–16.

[6] Meo, 'Hall, Asaph'.

[7] Hall, 'A Suggestion in the Theory of *Mercury*'.

ARTHUR EDDINGTON

The astronomer, physicist, and philosopher of science Arthur Eddington (1882–1944) made a number of fundamental contributions to both theoretical and experimental physics, including the hypothesis that some kind of nuclear reaction was the source of stellar energy and the observations during the total solar eclipse of 29 May 1919 that confirmed the gravitational deflection of light passing near the Sun, as predicted by Einstein's general theory of relativity.[1]

Eddington also wrote extensively on the philosophy of science. He saw a clear difference in the metaphysical character of explanation in physics between the nineteenth and twentieth centuries. A physicist of the nineteenth century desired to understand the world by means of some model made of levers or wheels or idealized fluid (such as the lumniferous aether).[2] The behaviour of such a model might be governed by mathematical expressions, but the model was a key intellectual component of explanation. But, by Eddington's time, the model was no longer necessary: mathematical laws simply described the world.[3]

For Eddington, the aesthetic value of the mathematical formulations of physical theories depended on how closely the mathematical formulations paralleled the physical concepts they governed:

> 'It is part of the aesthetic appeal of the mathematical theory of relativity that the mathematics is so closely adapted to the physical conceptions.'[4]

For instance, the bending of light due to gravitational deflection is simply the result of the curvature of space-time as described by the mathematics: there is nothing extraneous to the mathematics. But this close parallel did not exist in all theories: Eddington gave the example of the motions of the Moon. It is possible to predict by calculation very precisely and accurately the future position of the Moon. But the mathematics does not closely follow the physics:

> 'aesthetically the lunar theory is atrocious; it is obvious that the moon and the mathematician use different methods of finding the lunar orbit.'[5]

Similarly, Eddington was slightly dubious about perceived elegance where he thought the mathematics and physics had separated: 'I cannot avoid the suspicion that the mathematical elegance is obtained by a short cut which does not lead along the direct route of real physical progress'.[6]

◆

A key point in Eddington's thought was that in formulating the mathematical expression of a physical theory, one should not freely insert the constants of nature. He preferred that these numbers be '*exuded*

[1] Douglas, *Life of Eddington*, esp. pp. 39–43, 60–4.

[2] Eddington, *The Nature of the Physical World*, p. 209.

[3] *Ibid.*, pp. 209–10.

[4] *Ibid.*, pp. 257–8.

[5] *Ibid.*, p. 258.

[6] Eddington, *The Mathematical Theory of Relativity*, p. 257.

from the symbols.'[1] Although Eddington did not express it explicitly, there seems to have been an aesthetic component to some of his work in this regard, notably in his attempts to deduce the fine structure constant, which characterizes the strength of the electromagnetic interaction. This constant, denoted α, is defined as

$$\alpha = \frac{e^2}{2\varepsilon_0 hc},$$

e = charge on the electron
= $1.602\,176\,634 \times 10^{-19}$ C;
ε_0 = vacuum permittivity
$\approx 8.854\,187\,812\,8(13) \times 10^{-12}$ F m^{-1};
h = Planck constant
= $6.626\,070\,15 \times 10^{-34}$ J Hz^{-1}
c = Speed of light in a vacuum
= $299\,792\,458$ m s^{-1}

where e is the charge on the electron, ε_0 is the vacuum permittivity, h is the Planck constant, and c is the speed of light in a vacuum. (The table of values uses modern SI units.) The fine structure constant is dimensionless, as one can see by replacing the units with their base unit equivalents and cancelling:

$$\frac{\mathrm{C}^2}{\mathrm{F\,m^{-1}\cdot J\,Hz^{-1}\cdot m\,s^{-1}}} = \frac{(\mathrm{s\,A})^2}{\mathrm{kg^{-1}\,m^{-2}\,s^4\,A^2\cdot m^{-1}\cdot kg\,m^2\,s^{-2}\cdot s\cdot m\,s^{-1}}} = 1$$

In 1929, Eddington tried to evaluate the fine-structure constant by theoretical means, arguing that it had to be the reciprocal of an integer, namely the number of symmetrical degrees of freedom of two electrons; thus he concluded that $\alpha^{-1} = 136$.[2] He noted that the best available experimental evidence said its value was $\alpha^{-1} = 137.1$, and admitted the discrepancy was about three times the probable experimental error, but said that he was unable to convince himself that there was a theoretical error.[3] He subsequently located an error: there was an additional degree of freedom, which led him to conclude in 1930 that $\alpha^{-1} = 137$.[4] [At the time of writing, the most recent experimental measurement has obtained $\alpha^{-1} = 137.035\,999\,206(11)$.[5]]

A kindred argument led Eddington to declare:

> 'I believe there are 15 747 724 136 275 002 577 605 653 961 181 555 468 044 717 914 527 116 709 366 231 425 076 185 631 031 296 protons in the universe, and the same number of electrons.'[6]

(The number is 136×2^{256}.)

Leading physicists greeted Eddington's attempts to derive the fine-structure constant with scepticism and disdain.[7] Arnold Sommerfeld (1868–1951), who received many honours for his work on theoretical physics,[8] took perhaps the most positive view, calling Eddington's approach 'exceedingly beautiful and satisfying.'[9]

[1] Eddington, *The Nature of the Physical World*, p. 210.

[2] Eddington, 'The Charge of an Electron'.

[3] *Ibid.*, esp. 358.

[4] Eddington, 'The Interaction of Electric Charges'.

[5] Morel et al., 'Determination of the fine-structure constant'.

[6] Eddington, *The Philosophy of Physical Science*, p. 170.

[7] Kragh, 'Magic Number', p. 416.

[8] Forman & Hermann, 'Sommerfeld'.

[9] Sommerfeld, 'Über die Anfänge der Quantentheorie', p. 482; trans. C.

[There has also been a tendency to notice numerological coincidences involving 137. For instance, in Hebrew, Kabbalah is קַבָּלָה.[1] Using the standard method of *gematria*, the *Mispar Hekhreḥi*, ק is 100, ב is 2, ל is 30, and ה is 5.[2]

Such coincidences are doubtless instances of the sharpshooter's fallacy. Searching through his notes, the present author found one of his own: the physicist and mathematician Edmund Whittaker (1873–1956) wrote in 1945 an article surveying and praising Eddington's work on deducing mathematical constants; it was published in the *Mathematical Gazette* starting on page 137.[3] 'A *fine* coincidence, no?'[4]]

HERMANN WEYL

According to Freeman Dyson (1923–2020), Hermann Weyl (1885–1955) once humorously said that when faced with the choice between the true and the beautiful, he usually chose the latter.[*][5] Dyson used this anecdote to illustrate his view that Weyl had a 'profound faith in an ultimate harmony of Nature, in which the laws should inevitably express themselves in a mathematically beautiful form'.[6]

[There is a third-hand account, via Dyson and Chandrasekhar, that Weyl gave as an explicit example of choosing beauty over truth the gauge theory of gravitation he presented in *Raum·Zeit·Materie*.[7]]

Weyl certainly favoured unity and mathematical simplicity in theoretical physics: In 1918 he wrote:

> 'I am bold enough to believe that the whole of physical phenomena may be derived from one single universal world-law of greatest mathematical simplicity.'[8]

For Weyl, beauty was closely connected to symmetry,[9] and this connection extended to mathematics. For instance, he thought that Pythagoras' aesthetic preference for the circle and the sphere[†] were due to their perfect rotational symmetry.[10] It is arguably implicit in his remark that 'all a priori statements in physics have their origin in symmetry'[11] that beauty was in some way connected to the search for such *a priori* statements.

Weyl's view of the role of symmetry fitted with his assessment that the most beautiful instance of the universe being governed by mathematical laws was Maxwell's theory of the electromagnetic field, which is expressed in his four equations.[12][‡]

[1] Miller, *137*, ch.

[2] Derovan, Scholem & Idel, 'Gematria', p. 424.

[3] Whittaker, 'Eddington's Theory of the Constants of Nature'.

[4] Moriarty, 'The Secret of Psalm 46', emphasis in orig.

[*] ▸ p. 679

[5] Dyson, 'Prof. Hermann Weyl', p. 458.

[6] *Loc. cit.*

[7] Chandrasekhar, 'Beauty and the quest for beauty in science', p. 27.

[8] Weyl, 'Reine infinitesimalgeometrie', p. 385, n. 4; trans. Goenner, 'On the History of Unified Field Theories', p. 35.

[9] Weyl, *Symmetry*, p. 3.

[†] ▸ p. 44

[10] *Ibid.*, p. 5.

[11] *Ibid.*, p. 126.

[12] Weyl, 'The Open World', p. 48.

[‡] ▸ pp. 884–5

WERNER HEISENBERG

The physicist Werner Heisenberg (1901–76), who was awarded the 1932 Nobel Prize in physics, had a keen sense of the mathematical beauty of physical theories. This beauty certainly seems to have been a motivation for him: in his memoirs, published in German in 1969, he recalled his state of mind in the late spring of 1925, as he made progress towards a mathematical expression that would explain the hydrogen spectrum:[1]

> 'I had the feeling that, through the surface of atomic phenomena, I was looking at a strangely beautiful interior, and felt almost giddy at the thought that I now had to probe this wealth of mathematical structures nature had so generously spread out before me.'[2]

What was this beauty? In a lecture he delivered to the Bavarian Academy of Fine Arts (Bayerische Akademie der Schönen Künste) in 1970, Heisenberg made an attempt to describe beauty, drawing on two definitions from antiquity, one from Plotinus, the other seemingly recalling Vitruvius:[*] 'beauty as the proper conformity of the parts to one another, and to the whole'.[3] He then gave an example from mathematics that seems to fit better the Vitruvian definition: the fitting-together in a logically consistent way of Euclidean geometry and of number theory.[4]

Heisenberg continued his lecture by making a brief survey of the attitudes to mathematical beauty of some past thinkers, and he tried in particular to interpret their views in accordance with the Vitruvian definition of beauty. He attributed to Pythagoras the discovery that vibrating strings produce consonant musical intervals if their lengths are in small-number ratios,[5] which he considered to be 'one of the most momentous discoveries in the history of mankind'.[6] He twice indicated that he considered the mathematical relationship to be the *source* of the harmony or beauty.[7] That is, the mathematical relation was not simply describing or characterizing the configurations of strings that produced beautiful sounds, but was the *cause* of that beauty. As in the Vitruvian definition, the mathematical relation fitted together two parts into a whole, and thus beauty was produced.[8]

The Vitruvian definition 'applies in the highest degree'[9] to Newtonian physics:

> 'The parts are the individual mechanical processes […]. And the whole is the unitary principle of form which all these processes comply with and which was mathematically established by Newton in a simple system of axioms.'[10]

He pointed out that unity and simplicity are not exactly the same, but he thought that far-reaching unification obtained within a theory led to it being perceived as simple and beautiful.[11]

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[1] Heisenberg, *Physics and Beyond*, pp. 60–1.

[2] *Ibid.*, p. 61.

[*] ▸ pp. 129–30

[3] Heisenberg, 'The Meaning of Beauty', p. 167.

[4] *Ibid.*, pp. 167–8.

[5] *Ibid.*, p. 169.

[6] *Loc. cit.*

[7] *Loc. cit.*

[8] *Ibid.*, pp. 169–70.

[9] *Ibid.*, p. 174.

[10] *Loc. cit.*

[11] *Loc. cit.*

Werner Heisenberg

Heisenberg seems to have repeatedly, though implicitly, linked beauty and simplicity. He continued in this same lecture by mentioning that the significance of beauty in discovering truth and then quoting the Latin phrases *simplex sigillum veri* ('the simple is the mark of the true') and *pulchritudo splendor veritatis* ('beauty is the splendour of truth').[1] [The first phrase was inscribed on the wall of the physics audiorium at the University of Göttingen,[2] and had also been prescribed by George Pólya (1887–1985) as a guide in mathematical problem-solving.[*]] In private correspondence, he seems to have made the same connection and to have pointed more explicitly to the beauty and simplicity of the mathematical formulation:

> 'That these relationships display, in all their mathematical abstraction, an incredible degree of simplicity, is a gift we can only accept humbly. Not even Plato could have believed them to be so beautiful.'[3]

Towards the end of his 1970 lecture, he quoted Pauli's essay on 'The Influence of Archetypal Ideas on the Scientific Theories of Kepler'. Pauli had projected back to Kepler a Jungian conception of an archetype as a primordial image, and had interpreted the development of Kepler's cosmology in such terms. For Pauli, archetypes were a 'bridge between the sense perceptions and the ideas and are, accordingly, a necessary presupposition even for evolving a scientific theory of nature'.[4] Heisenberg agreed with this statement and specifically emphasized the role of archetypes in the sciences in those periods between the discovery of a phenomenon and that phenomenon being rationally understood. The experience of beauty was a part of this:

> 'Kepler says "*Geometria est archetypus pulchritudinis mundi*"; or, if we may translate this in more general terms — "Mathematics is the archetype of the beauty of the world."'[5]

The quotation from Kepler was not exact,[†] though the meaning was unchanged. Heisenberg's 'more general' translation reflected the less geometrical conception of the physical world that had taken hold, at least on the quantum scale.

Therefore it is perhaps unsurprising that Heisenberg thought that beauty in the mathematical formulation of a theory could be an indicator of truth. In his memoirs (in which he freely admitted that he reconstructed conversations and did not claim to quote literally[6]), he recalled certain points he made to Einstein in a discussion that took place in the spring of 1926:

> '"[...] If nature leads us to mathematical forms of great simplicity and beauty — by forms I am referring to coherent systems of hypotheses, axioms, etc. — to forms that no one has previously encountered, we cannot help thinking that they are 'true,' that they reveal a genuine feature of nature. [...]

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[1] Heisenberg, 'The Meaning of Beauty', p. 174.

[2] *Loc. cit.*

[*] ▸ pp. 681–2

[3] Heisenberg, letter to Edith Kuby (sister of his wife), dated Jan. 1958, quot. in Heisenberg, *Inner Exile*, p. 144.

[4] Pauli, 'The Influence of Archetypal Ideas', p. 221.

[5] Heisenberg, 'The Meaning of Beauty', p. 182.

[†] ▸ p. 280

[6] Heisenberg, *Physics and Beyond*, p. xvii.

> “You may object that by speaking of simplicity and beauty I am introducing aesthetic criteria of truth, and I frankly admit that I am strongly attracted by the simplicity and beauty of the mathematical schemes with which nature presents us. […]”’[1]

But in a 1965 discussion with Dirac, he also pointed out that, while the beauty of an equation could suggest its truth, it was necessary to check whether it fitted the experimental evidence:

> ‘*Heisenberg*: I do agree that the beauty of an equation is a very important point and that one can get already a lot of confidence from the beauty of an equation. On the other hand, you have to check whether it fits or whether it doesn’t. It’s only physics when it really fits with nature. But that may turn out much later.
>
> *Dirac*: And if it doesn’t fit you’d hold up publication would you? Just like Schrödinger?
>
> *Heisenberg*: I’m not sure whether I would. In at least one case I have not done so.’[2]

The last exchange refers to Schrödinger’s hesitation to publish a wave equation he had discovered for the electron, publishing only a modified, non-relativistic, version, because the relativistic version did not agree with experimental evidence.[*] This episode was a recurring theme in Dirac’s written discussions of mathematical beauty.[†]

ERWIN SCHRÖDINGER

Erwin Schrödinger (1887–1961), who shared the 1933 Nobel Prize in physics with Paul Dirac,[‡] certainly seems to have evaluated mathematics in aesthetic terms,[3] but wrote little about it himself. Aesthetic appreciation was implicit when he wrote:

> ‘I need not here speak of the quality of the pleasures derived from pure knowledge: those who have experienced it will know that it contains a strong æsthetic element and is closely related to that derived from the contemplation of a work of art.’[4]

Another piece of evidence from his own hand for aesthetic appreciation is contained in an 1925 letter to the physicist Wilhelm Wien (1864–1928), although one could make a case that the beauty to which he referred might not have been located in the mathematics:

> ‘If only I only knew more mathematics! I am very optimistic about this thing and expect that if I can only … solve it, it will be … beautiful.’[5]

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[1] Heisenberg, *Physics and Beyond*, pp. 68–9.

[2] BBC Horizon programme ‘Certain of Uncertainty / State of Nature’, 11 Aug. 1965, quot. in Farmelo, *The Strangest Man*, ch. 27.

[*] ▸ p. 899

[†] ▸ pp. 906–7

[‡] ▸ pp. 900–13

[3] Schrödinger, letter to Einstein, dated 3 Nov. 1925, repr. in Einstein, *Collected Papers*, vol. 15, [ltr 101] p. 182.

[4] Schrodinger, ‘Science, Art and Play’, p. 25.

[5] Schrödinger, letter to Wien, dated 27 Dec. 1925, quot. in Moore, *Schrödinger*, ch. 6, emphasis in orig.

But the most explicit — althought indirect — evidence of Schrödinger's aesthetic appreciation of mathematics is a testimonial Dirac made in a lecture in a 1972 summer school, over a decade after Schrödinger's death:

> 'of all the physicists that I met, I think SCHRÖDINGER was the one that I felt to be most closely similar to myself. I found myself getting into agreement with SCHRÖDINGER more readily than with anyone else. I believe the reason for this is that SCHRÖDINGER and I both had a very strong appreciation of mathematical beauty, and this appreciation of mathematical beauty dominated all our work. It was a sort of act of faith with us that any equations which describe fundamental laws of Nature must have great mathematical beauty in them. It was like a religion with us. It was a very profitable religion to hold, and can be considered as the basis of much of our success.'[1]

Thus Dirac held that Schrödinger had not only appreciated mathematical beauty, but that he had thought that the universe had to be governed by laws that were mathematically beautiful.

In a 1963 essay, Dirac gave the explicit example of Schrödinger having been guided towards formulating his equation governing wave functions 'simply by looking for an equation with mathematical beauty'.[2] But when Schrödinger applied his equation to the electron in the hydrogen atom, he was led to results disagreed with experiment. Discouraged, he abandoned the topic for some months, but then noticed that he could formulate a non-relativistic version of the equation, removing refinements required by relativity. For the non-relativistic equation, the results agreed with experiment. The initial disparity between theory and experiment was because, at the time, no-one knew that the electron had a spin. Later, this could be taken into account, and the disparity vanished.[3]

Schrödinger's having told Dirac this story[4] somewhat increases the likelihood that Schrödinger had told him about seeking mathematical beauty, rather than Dirac having made his own interpretation of the episode.[5] Dirac, however, drew a varying moral from it.*

[1] Dirac, 'Recollections of an Exciting Era', p. 136.

[2] Dirac, 'The Evolution of the Physicist's Picture of Nature', p. 53.

[3] Mehra & Rechenberg, *Hist. Devel. Quantum Theory*, vol. 5.2, §§ III.2–3; see also Dirac, 'The Evolution of the Physicist's Picture of Nature', p. 47.

[4] Dirac, 'The Evolution of the Physicist's Picture of Nature', p. 47; Dirac, 'My Life as a Physicist', pp. 740–1.

[5] See Dirac, 'The Evolution of the Physicist's Picture of Nature', pp. 46–7, 53.

* ▸ pp. 906–7

PAUL DIRAC

> ‘The classical mind, it seems to me, has some of these qualities —
> lucidity, austerity, […] but mixed with this austerity […]
> this absolutely strong and prevailing aesthetic sense.
> It is innate in Dirac.
> It is innate in all real classical minds.’
>
> — C. P. Snow, ‘The Classical Mind’, p. 811.

In the Department of Theoretical Physics at Moscow State University, there is a tradition that distinguished visitors are invited to write an inscription of their choice on a blackboard; the text is never erased. On 3 October 1956, Paul Dirac (1902–84) wrote:

‘Physical law should have mathematical beauty’.[1]

[This statement is often misquoted. For example, the versions given by Dirac’s biographers Helge Kragh and Graham Farmelo were, respectively, ‘A physical law must possess mathematical beauty’[2] and ‘PHYSICAL LAWS SHOULD HAVE MATHEMATICAL BEAUTY’.[3] Both misquotations shift the meaning, albeit only subtly in the second.]

Dirac became famous — and, partly through misquotation and misunderstanding, to a degree infamous — for his insistence on the role of mathematical beauty in guiding theoretical physics. Not only did he discuss it extensively in lectures and articles, but he explained in reminiscences how his taste for mathematical beauty had evolved.

But a modicum of caution is required. Dirac’s most scientifically productive period was from 1925, when he started working in the then-new field of quantum mechanics, to 1933, when his achievements were recognized by the award of the Nobel Prize in physics, which he shared with Schrödinger. In the pioneering papers that he wrote during this time, he did not explicitly mention mathematical beauty. Only after 1935 did he start openly proselytizing for mathematical beauty and emphasizing its importance in his own work.[4] It is natural to suspect a unconscious colouring of his memories of his own intellectual development.

Development of taste

Dirac originally trained as an engineer; he later wrote that this led him to tolerate approximation and so to be able to see beauty in theories based on approximation.[5] After he had finished his engineering degree, he was unable to find work in the post-First World War economic slump in Britain. Because of his recognized ability in theoretical subjects, he was offered the chance to do a second undergraduate degree in mathematics in only two years, without having to pay tuition fees.[6] He singled out for praise one of his teachers,[7] Peter Fraser (1880–1958), who never wrote a research paper but whose interests ranged over the whole of pure mathematics, and whose approach tended to be geometrical.[8] He credited Fraser with teaching him mathematical rigour, which had been absent from his training in

[1] `theorphys.phys.msu.ru/about/dirac.jpg`

[2] Kragh, *Dirac*, p. 275.

[3] Farmelo, *The Strangest Man*, ch. 26.

[4] Kragh, ‘Paul Dirac’, p. 31; Farmelo, *The Strangest Man*, ch. 31.

[5] Dirac, ‘Recollections of an Exciting Era’, p. 112.

[6] Kragh, *Dirac*, p. 6; Farmelo, *The Strangest Man*, chs 3–4.

[7] Dirac, ‘Recollections of an Exciting Era’, p. 113.

[8] Hodge, ‘Peter Fraser’.

engineering mathematics, and projective geometry, which he said had a 'profound influence' on him 'because of the mathematical beauty involved in it'.[1] Dirac thought that mathematicians divided into two classes; those who preferred algebra and those who preferred geometry. A good mathematician had to be able to work in both areas and pass between them, but would always maintain a preference for one or the other. His own preference was always for geometry.[2]

After Dirac completed his second degree in mathematics, he pursued postgraduate research at Cambridge. During his doctoral studies, Dirac's supervisor Ralph Fowler (1889–1944) held Saturday-afternoon tea-parties at which someone would give a talk about the geometry of euclidean space, often higher-dimensional, using the methods of projective geometry.[3] Dirac recalled the influence of these events:

> 'These tea parties did very much to stimulate my interest in the beauty of mathematics. The all-important thing there was to strive to express the relationships in a beautiful form'.[4]

The first recorded mention of beauty by Dirac is in a manuscript, probably dating to 1924, of a talk he delivered at one of these tea-parties:

> (beauty)?
> 'the inverse square law of force is of no more interest to the pure mathematician than any other inverse power of distance'.[5]

The typesetting of the quotation reflects the manuscript: the word 'beauty' was inserted, parenthesized and with a question mark, above the word 'interest'.[6] *Beauty*, therefore, was not Dirac's first choice of terminology.

His first mention *in print* of the beauty of a physical theory seems to have been in 1930.[7] In the first chapter of the first edition of his book *The Principles of Quantum Mechanics*, he wrote:

> 'Classical electrodynamics forms a self-consistent and very elegant theory, and one might be inclined to think that no modification of it would be possible which did not introduce arbitrary features and completely spoil its beauty.'[8]

But in the second edition (1935), there remains only a ghost of this sentence, lacking any explicit mention of beauty:

> 'The underlying ideas and the laws governing their application form a simple and elegant scheme, which one would be inclined to think could not be seriously modified without having all its attractive features spoilt.'[9]

In later life, Dirac described himself as following in Einstein's footsteps:

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[1] Dirac, 'Recollections of an Exciting Era', pp. 113–14.

[2] *Ibid.*, pp. 111–12.

[3] *Ibid.*, pp. 115–16.

[4] *Ibid.*, p. 116.

[5] Dirac, MS notes for a talk on relativity [1924], Archives for the History of Quantum Physics, quot. in Darrigol, *From c-Numbers to q-Numbers*, p. 301.

[6] Darrigol, *From c-Numbers to q-Numbers*, p. 302.

[7] Kragh, *Dirac*, p. 282.

[8] Dirac, *The Principles of Quantum Mechanics*, § 1.

[9] Dirac, *The Principles of Quantum Mechanics* (2nd ed.), § 1.

> 'Einstein introduced the idea that something which is beautiful mathematically is very likely to be valuable in describing fundamental physics. This is really a more fundamental idea than any previous idea. I think we owe it to Einstein more than to anyone else, that one needs to have beauty in mathematical equations which describe fundamental physical theories.'[1]

One should perhaps be wary of seeing Einstein's influence in Dirac's early development towards mathematical beauty. His biographer Kragh suggested Weyl as the greater influence, noting Weyl's declared preference for choosing the beautiful over the true.[2]* Weyl certainly impressed Dirac with his mathematical approach to physics,[3] but, given that Dyson reported Weyl as being only semi-serious in this assertion, the conclusion that Weyl excelled Einstein in influencing Dirac is at best tentative.

Dirac equation

Although Dirac had entered Cambridge hoping to study relativity, the main interest of Fowler, his supervisor, was in quantum theory.[4] Dirac had little knowledge of this area, but quickly learned it and found it interesting. He published seven papers, including works on relativity and quantum theory, within two years.[5]

The defining event of his doctoral career came when Fowler sent him the proof-sheets of Heisenberg's article that contained the first statement of the fundamental ideas of quantum mechanics.[6] Dirac later described his intellectual situation at that moment in the following terms:

> 'The background was essentially an intense interest in mathematical beauty which had been especially built up through studying projective geometry.'[7]

Dirac's discovery of the relativistic equation that describes the electron and all other spin-$\frac{1}{2}$ massive particles was an important episode that he later recalled in terms of the role of mathematical beauty. By the end of 1926, the importance of the concept of the spin of particles had been accepted, but the relationship between spin, relativity, and quantum mechanics was not yet understood.[8]

Dirac described how in 1927, he had considered complex matrices satisfying the equations

$$\alpha_\mu^2 = 1 \qquad \alpha_\mu \alpha_\nu = -\alpha_\nu \alpha_\mu \quad (\text{for } \mu \neq \nu). \tag{D}$$

Condition (D) is satisfied by the three Pauli spin matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

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[1] Dirac, 'Why We Believe in the Einstein Theory', p. 6.

[2] Kragh, *Dirac*, p. 287.

* ▸ pp. 679, 895

[3] *Loc. cit.*

[4] *Ibid.*, p. 8.

[5] *Ibid.*, p. 11.

[6] *Ibid.*, p. 14.

[7] Dirac, 'Recollections of an Exciting Era', pp. 119–20.

[8] Kragh, *Dirac*, p. 55.

There is no larger collection of 2×2 complex matrices that satisfy (D). Dirac noticed that if one multiplied them into the three components of a momentum, the resulting quantity $\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3$ has the property that its square was $p_1^2 + p_2^2 + p_3^2$, as can be proved easily from (D).[1] For Dirac, this property was a 'very pretty mathematical result',[2] but if the quantity was used as the Hamiltonian in a Schrödinger equation, it could only apply to a particle with zero rest mass. What one needed for a particle with non-zero rest mass was a set of *four* distinct matrices satisfying (D), and no such set of 2×2 matrices existed. Dirac therefore put aside the idea until it occurred to him some weeks later that the restriction to 2×2 matrices was unnecessary: 4×4 matrices could be used, and there did exist sets of four distinct 4×4 complex matrices satisfying (D).[3]

This realization allowed him to formulate the equation, but also meant that the wave *function* described by the equation had to have four components. Pauli had originally posited a two-component wave function to accommodate spin, but there was no physical justification for the move to four components. Dirac, however, was confident that physical reality would correspond to where mathematics had taken him.[4] The equation itself was described as 'achingly beautiful' by the physicist Frank Wilczek* (1951–), who gave it in the form:

$$\left[\gamma^\mu\left(\mathrm{i}\frac{\partial}{\partial x^\mu} - eA_\mu(x)\right) + m\right]\psi(x) = 0.$$

A potential problem with Dirac's equation was that it allowed for negative energy levels for a free electron. While some might have seen in this implication an indication of a fundamental flaw, Dirac held fast to his equation, suggesting first that almost all negative-energy states were already occupied by electrons, so that the usual positive-energy electrons could not occupy these states unless one of them was vacated, creating a 'hole'. This 'hole' would appear to be a particle of positive enery and positive charge, and Dirac suggested that it be identified with the proton.[5] Weyl pointed out that the mass of the 'negative-energy' electron should be the same as that of the electron itself, and so it could not be the proton.[6] Undaunted, in 1931 Dirac postulated the existence — or at least, the existence subject to confirmation of any physical phenomenon indicating a negative-energy electron — of an 'anti-electron': a particle with the electron's mass and opposite electric charge.[7] This particle, the positron, was detected the following year.[8]

Beauty and truth

In a 1939 lecture on 'The Relation between Mathematics and Physics', Dirac pointed out that there was no logical reason why mathematical reasoning should allow us to predict physical phenomena, and so this ability had to be ascribed to some deeply-rooted mathematical quality in nature.[9]

[1] Dirac, 'Pretty Mathematics', p. 603.

[2] Dirac, 'Recollections of an Exciting Era', p. 142.

[3] Dirac, 'Pretty Mathematics', pp. 603–4.

[4] Kragh, *Dirac*, p. 59.

* ▸ pp. 985–7

[5] Dirac, 'Quantised Singularities', p. 61.

[6] Weyl, *The Theory of Groups and Quantum Mechanics*, pp. 262–3.

[7] Dirac, 'Quantised Singularities', p. 61.

[8] Mehra & Rechenberg, *Hist. Devel. Quantum Theory*, vol. 6.2, p. 793.

[9] Dirac, 'The Relation between Mathematics and Physics', p. 122.

For Dirac, there seemed to be a *principle of simplicity*, by which the equations that described the fundamental laws of motion should have a simple form. This principle seemed to work: if rough experimental results fitted approximately to some simple equations, more precise experimental results fitted the equations better.[1] But twentieth-century developments had necessitated changes to this principle. Newton's law of gravitation takes the form of a very simple equation: the gravitational force F acting between two objects is given by

$$F = G\frac{m_1 m_2}{r^2}; \tag{N}$$

where G is the gravitational constant, m_1 and m_2 are the masses of the objects, and r is the distance between their centres of mass. But (N) was superseded by Einstein's law of gravitation, which requires the development of a substantial body of mathematics before it can even be written down. Dirac thought that, unless simplicity was given a 'rather subtle meaning' that only made sense from the perspective of higher mathematics, Einstein's law was far less simple.[2]

> 'What makes the theory of relativity so acceptable to physicists in spite of its going against the principle of simplicity is its great *mathematical beauty*. This is a quality which cannot be defined, any more than beauty in art can be defined, but which people who study mathematics usually have no difficulty in appreciating. The theory of relativity introduced mathematical beauty to an unprecedented extent into the description of Nature.'[3]

He made an arguably even stronger claim in a 1968 lecture: 'the *main reason* why the theory of relativity is so universally accepted is its mathematical beauty'[4] What form did this mathematical beauty take? Specifically, in Newtonian physics, the symmetries of space-time are the transformations that constitute the Galileo group; in the special theory of relativity, they are the transformations that make up the Lorentz group. Of the two groups, the latter was, for Dirac, a far more beautiful object. The Galileo group can be viewed mathematically as a subgroup of the Lorentz group (or, as Dirac put it in his 1939 lecture, a 'degenerate special case' of the latter).[5] But the beauty he saw was not limited to the group structure: 'the Lorentz transformations are beautiful transformations from the mathematical point of view'.[6] The reason for believing in the special theory of relativity was 'because it puts importance on these Lorentz transformations which are beautiful mathematically'.[7] But the step from special to general relativity involved a smaller increase in mathematical beauty than the step from Newtonian physics to special relativity; this difference explained, for Dirac, why general relativity was not quite so accepted as special relativity.[8]

It was not only relativity where mathematical beauty was indicative of correctness: in a similar vein, he said in 1942 that he thought that non-relativistic quantum mechanics rested upon a mathematical formalism that was 'so natural and beautiful as to make one feel sure of its correctness as the foundation of the theory'.[1]

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[1] Dirac, 'The Relation between Mathematics and Physics', pp. 122–3.

[2] *Ibid.*, p. 123.

[3] *Ibid.*, p. 123, emphasis in orig.

[4] Dirac, 'Methods in Theoretical Physics', p. 21, emphasis added.

[5] Dirac, 'The Relation between Mathematics and Physics', p. 123; see also Dirac, 'Quantum Mechanics and the Aether', p. 143.

[6] Dirac, 'Why We Believe in the Einstein Theory', p. 6.

[7] *Loc. cit.*

[8] Dirac, 'The Relation between Mathematics and Physics', p. 123.

For Dirac, the conclusion to be drawn was that the principle of simplicity should be modified into a *principle of mathematical beauty*. When seeking to express fundamental laws of nature, the scientist should aim for mathematical beauty. Simplicity still played a role, but took second place to beauty.[2] The expectation, by analogy with the principle of simplicity, was that when experimental data fitted a mathematically beautiful theory, further data should also match it.

In a 1954 article in *The Scientific Monthly*, Dirac said that in spite of the changes it required to fundamental scientific concepts, the special theory of relativity was quickly accepted not only because of its agreement with experiment, but because of the 'beautiful mathematical theory underlying it, which gives it a strong emotional appeal', and he thought this second reason the stronger one.[3] In spite of all the changes that our knowledge of physics endured, there was for Dirac one fundamental principle:

> 'there is just one rock which weathers every storm, to which one can always hold fast — the assumption that the fundamental laws of nature correspond to a beautiful mathematical theory. This means a theory based on simple mathematical concepts that fit together in an elegant way, so that one has pleasure in working with it.'[4]

This principle has implications for the way one should *do* theoretical physics. If a theory had mathematical beauty, it should enjoy the physicist's confidence. If there was a disagreement between such a theory and limited experimental evidence, the physicist's reaction should be to hold fast to the theory and suspect experimental error. Only after a substantial body of experimental evidence had been built up should the physicist accept that the theory had to be changed, and any such change should be in the direction of greater mathematical beauty.[5]

Dirac thought that quantum mechanics was marred by Heisenberg's uncertainty principle, which was 'an ugly and rather artificial limitation'.[6] However, there was a mathematically beautiful theory underlying it. Again, it was the beauty of this theory, together with the agreement with experiment, that led to its general acceptance.[7]

For Dirac, quantum field theory, an attempt to combine the special theory of relativity, classical field theory, and quantum mechanics, was only an intermediate step towards a future theory that would supersede it. A major reason for his belief was that the theory, as he saw it, was

> 'complicated and ugly. It has none of the simplicity and beauty which are characteristic of a good physical theory. These qualities occur to a marked extent in relativistic mechanics alone,

[1] Dirac, 'The physical interpretation of quantum mechanics', p. 3.

[2] Dirac, 'The Relation between Mathematics and Physics', pp. 123–4.

[3] Dirac, 'Quantum Mechanics and the Aether', pp. 142–3.

[4] *Ibid.*, p. 143.

[5] *Loc. cit.*

[6] *Ibid.*, p. 144.

[7] *Loc. cit.*

> or in quantum mechanics alone, but disappear with our present methods of combining the two.'[1]

In a letter responding to this article, the philosopher Armand Lowinger (1909–?) wrote, specifically with regard to Dirac's argument about the mathematical beauty underlying quantum mechanics, that '[t]he beauty of the theory had absolutely nothing to do with the acceptance of the theory by the physicists',[2] and pointed out that Dirac had apparently contradicted himself by saying that '[a]ll that a physicist really wants of his theory is a definite set of rules enabling him to obtain results that can be compared with experiment'.[3]

William Ernest Hocking (1873–1966), another philosopher, queried the absoluteness of Lowinger's view, wondering how he could make an assertion that implied a knowledge of *all* factors motivating the acceptance of a theory by *all* those who might be classed as 'the physicists'. Hocking suggested that subjective factors could not be ruled out. He further suggested that the term *beauty* in mathematics was something like *power* or *elegance* or *simplicity*, and that the conciseness and efficiency of a theory could play some role — together with a degree experimental confirmation — in theory acceptance. He also said that he would not think less of scientists who saw the beauty of a theory as an additional confirmation of success.[4]

Dirac's response to Lowinger was that there was no contradiction. The ultimate goal in theoretical physics was agreement with experiment, but experience had shown that beautiful rules tended to be highly successful and ugly ones of limited use.[5] The initial acceptance of quantum mechanics came, Dirac thought, from its beauty; only later came a substantial body of experimental evidence.[6] A current theory of limited beauty should be superseded by a later theory that would from the beginning strike physicists as more beautiful.[7]

He then turned to a chapter in the history of physics to which he would return in his famous 1963 article 'The Evolution of the Physicist's Picture of Nature' and in a 1968 lecture on 'Methods in Theoretical Physics', namely Schrödinger's discouragement with his first formulation of his equation — which Dirac thought a 'very beautiful equation'[8] — because it did not agree with experiment.[9]

In his 1954 response to Lowinger, in his 1963 article, and in his 1968 lecture, Dirac stated a moral to be drawn from this story. In 1954, he gave it as:

> 'The moral of the story is that one should have faith in a theory that is beautiful. If the theory fails to agree with experiment, its basic principles may still be correct and the discrepancy may be due merely to some detail that will get cleared up in the future.'[10]

The 1963 version is:

> 'I think there is a moral to this story, namely that it is more important to have beauty in one's equations than to have them

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[1] Dirac, 'Quantum Mechanics and the Aether', p. 145.

[2] Lowinger, 'Logic or Beauty?'

[3] Dirac, 'Quantum Mechanics and the Aether', p. 144.

[4] Hocking, 'Logic or Beauty?'

[5] Dirac, 'Logic or Beauty?'

[6] *Ibid.*

[7] *Ibid.*

[8] Dirac, 'The Evolution of the Physicist's Picture of Nature', p. 47.

[9] Dirac, 'Logic or Beauty?'; Dirac, 'The Evolution of the Physicist's Picture of Nature', p. 47; Dirac, 'Methods in Theoretical Physics', p. 23.

[10] Dirac, 'Logic or Beauty?'

> fit experiment. If Schrödinger had been more confident of his work, he could have published it some months earlier, and he could have published a more accurate equation. […] It seems that if one is working from the point of view of getting beauty in one's equations, and if one has really a sound insight, one is on a sure line of progress. If there is not complete agreement between the results of one's work and experiment, one should not allow oneself to be too discouraged, because the discrepancy may well be due to minor features that are not properly taken into account and that will get cleared up with further developments of the theory.'[1]

The 1968 moral bears little resemblance to the previous ones:

> 'The moral of this story is that one should not try to accomplish too much in one stage. One should separate the difficulties in physics one from another as far as possible, and then dispose of them one by one.'[2]

Note the differences. The 1954 version is concise and clearly indicated that, for Dirac, the mathematical beauty of a theory was a reason to hope that it might prove to be correct, in spite of present contrary experimental evidence, when some further insight or datum brought the theory and evidence back into alignment. The 1963 version is much more prolix than the first version and is arguably less clear, since the first sentence's 'fit experiment' must be interpreted in light of the later 'complete agreement', and the statement 'more important to have beauty' should be understood as not being discouraged about the validity of a theory found by seeking beauty. The entirely different 1968 moral has no aesthetic component.

The verbosity of the 1963 version has proved unfortunate, because it has created a tendency to quote in isolation the clause 'it is more important to have beauty in one's equations than to have them fit experiment' from the first sentence.[3]

As recounted in their book *Mathematical Experience*, first published in 1981, Philip J. Davis (1923–2018) & Reuben Hersh (1927–2020) asked a certain pseudonymous 'Professor Taylor', 'an international authority in Engineering Science',[4] how he felt about 'the often quoted remark of a certain theoretician that he would rather his theories be beautiful than be right'.[5] This statement almost certainly refers to Dirac; but if it does, and if it is an accurate description of the view on which 'Taylor' was asked to comment, then 'Taylor' was reacting to what was at best an incomplete representation of Dirac's position. The response was unsurprisingly negative:

> 'This cuts close to the bone. It really does. But as I see it, *mere aesthetics* doesn't pay dividends. In my experience, I should be inclined to replace the word "beautiful" by the word "analyzable." I should like my models to be beautiful, effective, and

[1] Dirac, 'The Evolution of the Physicist's Picture of Nature', p. 47.

[2] Dirac, 'Methods in Theoretical Physics', p. 23.

[3] See, for example, Davis & Hersh, *Mathematical Experience*, p. 169; Merrell, *Unthinking Thinking*, p. 88; Tauber, *The Elusive Synthesis*, p. 309.

[4] Davis & Hersh, *Mathematical Experience*, p. 44.

[5] *Ibid.*, p. 47.

> predictive. But the real goal is the understanding of a situation. Therefore the models must be analyzable because understanding can come only through analyzability. If one has all of these things, then this is a great and rare achievement, but I should say that my immediate goal is analyzability.'[1]

But Dirac did not promote 'mere aesthetics' in either the 1954 or 1963 morals.

However, Dirac seems to have been inconsistent in the degree to which he thought mathematical beauty should command confidence in a theory's correctness. In a conversation that took place in 1968–9, Dirac seemed to say that he called only for a certain resistance to abandoning a mathematically beautiful theory: 'There are occasions when mathematical beauty should take priority over [temporary] agreement with experiment.'[2] But in 1979, he wrote:

> 'Let us now face the question, that a discrepancy has appeared, well confirmed and substantiated, between the theory and the observations. How should one react to it? […] Should one then consider the theory to be basically wrong?
>
> I would say that the answer to the last question is emphatically No. Anyone who appreciates the fundamental harmony connecting the way nature runs and general mathematical principles must feel that a theory with the beauty and elegance of Einstein's theory *has* to be substantially correct. If a discrepancy should appear in some application of the theory, it must be caused by some secondary feature relating to this application which has not been adequately taken into account, and not by a failure of the general principles of the theory.'[3]

This statement suggests a far higher level of commitment to the theory, or at least to its mathematical aspects.

Dirac seemed to encourage a similar degree of commitment to a mathematically beautiful theory in a discussion at a 1979 symposium for the centenary of Einstein's birth. Walter Kaufmann (1871–1947) had obtained in 1905 experimental results for the electron that seemed to contradict the Lorentz–Einstein theory and thus the special theory of relativity.[4] Hendrik Lorentz (1853–1928) felt crushed and was willing to abandon the theory;[5] Einstein held fast, proposing an alternative experiment and suggesting that the difficulty of Kaufmann's experiment meant that the agreement with the theory was adequate.[6] In time, further problems with Kaufmann's experiments were noted and the Lorentz–Einstein theory vindicated.[7] In the discussion following talks at the symposium that considered this episode, Chen-Ning Yang recalled Dirac's remarks about choosing beauty over agreement with experiment,[8] and Dirac observed that 'that's what people ought to have done when Kaufmann obtained his results'.[9] In the proceedings of a different symposium that same year, Dirac wrote that Einstein had a faith in mathematical beauty over agreement with experiment:

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[1] Davis & Hersh, *Mathematical Experience*, pp. 47–8, emphasis added.

[2] Dirac, conversation with Mehra, quot. in Mehra, '"The golden age of theoretical physics"', p. 59, n. 78, insertion in orig.

[3] Dirac, 'The Test of Time', p. 21.

[4] Miller, *Albert Einstein's Special Theory of Relativity*, § 7.4.2.

[5] *Ibid.*, pp. 315–16.

[6] *Ibid.*, § 12.4.3.

[7] *Ibid.*, §§ 12.4.4–5.

[8] Woolf, *Some Strangeness in the Proportion*, p. 92.

[9] *Loc. cit.*

> 'Einstein seemed to feel that beauty in the mathematical foundation was more important, in a very fundamental way, than getting agreement with observation.'[1]

This view is at odds with Einstein's own assessment,[*] and seems more likely to be a projection of Dirac's position.

Mathematical beauty and the approach to physics

On separate occasions, Dirac outlined two dichotomies in approaches to theoretical physics; each has a prominent role for aesthetic concerns in the mathematics:

Priority of physical/mathematical ideas: In a lecture course he delivered in summer 1942 at the Dublin Institute for Advanced Studies,[2] Dirac sketched the role of mathematical beauty in formulating theories, contrasting his own method with Eddington's. Eddington, said Dirac, wanted first to get the physical ideas clear and then to devise a mathematical system. Dirac's own method was to set up the mathematics and then to seek a physical interpretation, and he thought this approach should be easier than Eddington's, because the easier task was tackled first. Creating the mathematical system was easier because the number of required choices was small:

> 'The scheme, to be acceptable, must be neat and beautiful, and the number of such schemes which pure mathematics can provide is very limited.'[3]

Experimental/mathematical direction: In later life, he said that one could work from an experimental basis, reading about all the latest results and trying to fit them into a theory, or work from a mathematical basis, examining the existing theory and attempting to remove its flaws while preserving its successes.[4] Using either approach, one should seek mathematical beauty. Simplicity should be sought as well, but secondarily: 'It often happens that the requirements of simplicity and of beauty are the same, but where they clash the latter must take precedence.'[5] Working from a mathematical basis, the two main methods involved trying to remove inconsistencies and to unify previously separate theories,[6] He characterized the aim of much research on the mathematics of quantum mechanics in 1963 as aiming 'to understand the theory better and to make it more powerful and more beautiful'.[7]

Towards the end of his life, Dirac characterized his own work as

> 'examining mathematical quantities of a kind that physicists use and trying to fit them together in an interesting way re-

[1] Dirac, 'The Early Years of Relativity', p. 83.

[*] ▸ pp. 887–90

[2] Kragh, *Dirac*, p. 175.

[3] Dirac, *Quantum Electrodynamics*, p. 1.

[4] Dirac, 'Methods in Theoretical Physics', p. 19.

[5] Dirac, 'The Relation between Mathematics and Physics', p. 124.

[6] Dirac, 'Methods in Theoretical Physics', p. 21.

[7] Dirac, 'The Evolution of the Physicist's Picture of Nature', p. 53.

> gardless of any application that the work may have. It is simply a search for pretty mathematics.'[1]

It was 'good luck' if the mathematics so developed had some application to nature.[2] In the same paper, he put forward a development of an entirely theoretical wave function that he considered pretty mathematics, but he recognized that its consistency conditions were so restrictive that it would be difficult to introduce interactions with other particles or fields.[3] Later, N. Mukunda (1939–), C.-C. Chiang & E. C. G. Sudarshan (1931–2018) showed how to overcome this defect in a simple way.[4] Dirac thought highly of this development, believing that there was again hope that the equation described a particle in nature.[5]

The foundation of physics on mathematical beauty

The fact that 'fundamental physical laws are described in terms of a mathematical theory of great beauty and power'[6] was somehow baffling. Why should the universe obey mathematical laws at all? Dirac thought that one just had to accept that the universe was so, as a matter of faith, and wrote that the situation could be described in the following terms: 'God is a mathematician of a very high order, and He used very advanced mathematics in constructing the universe.'[7] Thus, as our knowledge of mathematics advanced, so we could hope to increase our knowledge of the universe.[8] But more: the Creator did not just use advanced mathematics, but *beautiful* mathematics. There was no logical reason for this credo; one just accepted as 'an article of faith' that the universe was constructed so that the fundamental ideas could be expressed using beautiful mathematics.[9] [Dirac's references to a creator-god seem to have been purely rhetorical; he was not religious, varying between atheist and agnostic during his lifetime, and thought that the question of the existence of God could be settled scientifically.[10]]

On a purely pragmatic level, experience had shown that it was *profitable* to seek mathematical beauty in physical laws.[11] Thus, although Dirac wrote that for him (and Schrödinger), belief in the mathematical beauty of the laws of nature was 'like a religion', he immediately pointed out the reason for this belief: its profitability.[*] [12]

Dirac also suggested that there was an ongoing process of unification between pure mathematics and physics, and that perhaps eventually every branch of pure mathematics would lend itself to physics.[13] Thus he posited that it should be possible to choose an area of pure mathematics, develop it, and look for a physical interpretation. He wrote about this idea for the first time in his paper on magnetic monopoles in 1931,[14] and discussed it at greater length in his 1939 lecture on 'The Relation between Mathematics and Physics'.[15] The choice should be largely governed by 'considerations of mathematical beauty'; Dirac seems to have thought doomed to failure one previ-

[1] Dirac, 'Pretty Mathematics', p. 603.

[2] *Loc. cit.*

[3] Dirac, 'Pretty Mathematics', pp. 604–5; Dirac, 'A positive-energy relativistic wave equation', esp. p. 455.

[4] Mukunda, Chiang & Sudarshan, 'Dirac's new relativistic wave equation in interaction with an electromagnetic field'.

[5] Dirac, 'Pretty Mathematics', p. 605.

[6] Dirac, 'The Evolution of the Physicist's Picture of Nature', p. 53.

[7] *Loc. cit.*

[8] *Loc. cit.*

[9] Dirac, conversation with Mehra, quot. in Mehra, '"The golden age of theoretical physics"', p. 59, n. 78; Dirac, 'Methods in Theoretical Physics', p. 21.

[10] Kragh, *Dirac*, pp. 256–7.

[11] Dirac, conversation with Mehra, quot. in Mehra, '"The golden age of theoretical physics"', p. 59, n. 78; Dirac, 'Methods in Theoretical Physics', p. 21.

[*] ▸ p. 899

[12] Dirac, 'Recollections of an Exciting Era', p. 136.

[13] Dirac, 'The Relation between Mathematics and Physics', p. 124.

[14] Dirac, 'Quantised Singularities', p. 60; cf. Kragh, *Dirac*, p. 282.

[15] Dirac, 'The Relation between Mathematics and Physics', p. 125.

ous attempt to proceed along these lines, for the chosen area (*viz.*, non-associative algebra) lacked beauty.[1] A better choice might be the theory of functions of one complex variable, which Dirac thought 'of exceptional beauty' and which is connected with the Lorentz group. This beauty and this connection led Dirac to suspect some deep connection between this theory and relativity.[2] He revisited the idea of the unification of pure mathematics and physics in a 1963 lecture:

> 'We may try to make progress by [...] setting up theories which are of interest, in the first place, only because of the beauty of their mathematics. We may then hope that such equations will ultimately prove their value in physics, basing this hope on the belief that Nature demands mathematical beauty in her laws.'[3]

But Dirac also admitted that mathematical beauty did not entail existence of a physical realization. He had shown that the formal mathematics of quantum mechanics permitted wave equations that could only describe the motion of an electron in the field of a magnetic monopole. Because these wave equations required no alteration in the formalism, he suggested that it would be surprising if they did not arise in nature; that is, if magnetic monopoles did not exist.[4] He later described the mathematics behind this conclusion as 'pretty'.[5] But he then added, with regard to the existence of monopoles:

> 'Many attempts to find them have been made, but all have been unsuccessful. *One should conclude that pretty mathematics by itself is not an adequate reason for nature to have made use of a theory.* We still have much to learn in seeking for the basic principles of nature.'[6]

Ugliness and falsity

As quantum electrodynamics was developed, problems emerged when infinite quantities emerged in calculation. By the 1950s, theoretical physicists including Hans Bethe (1906–2005), Julian Schwinger (1918–94), Sin-Itiro Tomonaga (1906–79), Richard Feynman (1918–88), and Freeman Dyson had developed renormalization as way to eliminate these infinities and allow calculations that accurately predicted experimental results.[7]

Dirac was unimpressed. In the obituary he wrote, Dyson reported that when he asked Dirac in 1950 for his thoughts on the new developments, Dirac replied: 'I might have thought that the new ideas were correct if they had not been so ugly'.[8] Dyson dryly noted that '[n]ature disagrees with Dirac's aesthetic judgment and agrees with Schwinger and Feynman'.[9]

But in an January 1980 lecture entitled 'The Engineer and the Physicist', Dirac gave a non-aesthetic heuristic argument for his opposition to renormalization. Recalling his original training as an engineer,

[1] Dirac, 'The Relation between Mathematics and Physics', p. 125.

[2] *Loc. cit.*

[3] Dirac, 'Hamiltonian Methods and Quantum Mechanics', p. 59.

[4] Dirac, 'Quantised Singularities', p. 71.

[5] Dirac, 'Pretty Mathematics', p. 604.

[6] *Ibid.*, p. 604, emphasis added.

[7] Cao, *Conceptual Developments of 20th Century Field Theories*, § 7.4.10–15.

[8] Dirac, quot. in Dyson, 'Paul Dirac'.

[9] *Ibid.*

he argued that engineers would only neglect terms that were small and that did not greatly influence the result; this was diametrically opposed to the exclusion of infinities that was involved in renormalization.[1] But the lack of rigour was only part of Dirac's objection: there was a genuinely aesthetic component. Dirac was willing to make non-rigorous arguments in his own work. Notably, he used what is usually called the *Dirac delta* $\delta(x)$, which is a quantity dependent on x such that

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1, \qquad \delta(x) = 0 \text{ for } x \neq 0. \tag{D}$$

The Dirac delta is not a function, although it can be thought of as the limit of a sequence of functions. But $\delta(x)$ can be manipulated like a function, and Dirac pragmatically used it as such, confining himself to situations where it was clear that the lack of rigour would not lead to incorrect results.[2] John von Neumann (1903–57) admitted that Dirac's presentation of quantum mechanics was 'scarcely to be surpassed in brevity and elegance', but thought that Dirac's approach lacked rigour because of its use of such '"improper" functions'.[3]

Dirac maintained this fundamental objection to renormalization up until the end of his life: though it gave good agreement with experiment, he said firmly that he saw no beauty in it;[4] it was an artificial way of removing the infinities that cropped up in the theory.[5] He thought that the foundation of quantum mechanics was correct and beautiful; the problem was that the wrong Hamiltonian was being used.[6] This lack of beauty in quantum electrodynamics he saw as an obstacle to using it as an approach to a theory of everything:

> 'The ultimate goal is to obtain suitable starting equations from which the whole of atomic physics can be deduced. [...] One way of proceeding towards it is first to perfect the theory of low-energy physics, which is quantum electrodynamics [...]. However, the present quantum electrodynamics does not conform to the high standard of mathematical beauty that one would expect for a fundamental physical theory, and leads one to suspect that a drastic alteration of basic ideas is still needed.'[7]

It was not only renormalization that Dirac was ready to reject on aesthetic grounds; he objected to other mathematical formulations in the same way. For instance, in response to Heisenberg's work on developing a unified field theory for elementary particles, Dirac wrote:

> 'My main objection to your work is that I do not think your basic (non-linear field) equation has sufficient mathematical beauty to be a fundamental equation of physics.'[8]

[1] Farmelo, *The Strangest Man*, ch. 24.

[2] Dirac, *The Principles of Quantum Mechanics* (2nd ed.), § 20.

[3] von Neumann, *Mathematical Foundations of Quantum Mechanics*, pp. viii–ix.

[4] Dirac, interview in Buckley & Peat, *A Question of Physics*, p. 39.

[5] Dirac, 'The inadequacies of quantum field theory', esp. pp. 195–6.

[6] *Ibid.*, pp. 197–8.

[7] Dirac, 'Methods in Theoretical Physics', p. 30.

[8] Dirac, letter to Heisenberg, dated 6 Mar. 1967, quot. in Brown & Rechenberg, 'Paul Dirac and Werner Heisenberg', p. 148.

Psychology

While Dirac wrote extensively on the importance of mathematical beauty in physics, he said relatively little about the subjective experience of it. One of the few accounts he gave is the following:

> 'One can't just make random guesses. [...] You're solving a problem [...], and you conclude you've made some mistakes. Suddenly you think of corrections and everything fits. You feel great satisfaction. The beauty of the equations provided by nature is much stronger than that. It gives one a strong emotional reaction.'[1]

Notice that Dirac described the experience in the generic *second* person (as the switch from 'one' to 'you' indicated), almost as if he thought anyone could experience it. He seems never to have imagined that he had an especial aesthetic sense that others might have lacked.[2]

Legacy

Dirac's biographer Kragh made an important observation about Dirac and mathematical beauty. All of Dirac's work, including his successes, were guided by his belief in mathematical beauty (and his own great *sense* for this beauty). But the mid-1930s was a watershed in his work: his great and lasting contributions all came *before*; and his explicit promotion of mathematical beauty as a standard and a strategy came *after*.[3]

Yoichiro Nambu (1921–2015), one of the 2008 Nobel laureates in physics, said that the history of particle physics since the 1930s has functioned in two modes: the Yukawa mode, meaning the introduction of new particles corresponding to new effects, named for Hideki Yukawa's (1907–81) postulation of what are now called pions to explain the strong interaction, and the Dirac mode, 'the search for beautiful equations'.[4]

LEOPOLD INFELD

The physicist Leopold Infeld (1898–1968) thought the De Sitter universe mathematically more attractive than Einstein's, because it could (loosely) be represented as a sphere, which automatically entailed the requirements of isotropy and homogeneity.[5] Yet, for all 'the mathematical beauty of the de Sitter universe',[6] it had a fundamental problem: the density of particles in the De Sitter universe is zero; it is empty.[7] One could attempt to salvage the De Sitter universe as a hypothesis by changing the dynamical equations so that

[1] Dirac, interview in Buckley & Peat, *A Question of Physics*, p. 39.

[2] Cf. Penrose, 'The Rôle of Aesthetics', p. 267.

[3] Kragh, *Dirac*, p. 292.

[4] Brown, Brout et al., 'Spontaneous Breaking of Symmetry', p. 479.

[5] Infeld, 'General Relativity and the Structure of Our Universe', p. 488.

[6] *Ibid.*, p. 490.

[7] *Ibid.*, p. 491.

the density of matter, defined in a new way, was no longer zero. Why not make such a change? Because

> 'such a procedure would be ugly, and it is doubtful whether a logically consistent theory could be built in this way. Einstein's work is characterized by a rare search for logical simplicity, beauty and clarity. Within the framework of the relativity theory there is no place for *ad hoc* assumptions and artificial hypotheses. They would spoil not only the beauty but also the self-consistency of the general theory of relativity.'[1]

When Infeld was writing, in 1949, it was unknown whether the universe was closed, in the sense that a beam of light sent from a point would in principle eventually return to that point, or open, in the sense that such a beam would never return to its origin. He thought that mathematical aesthetic judgement would incline anyone towards the former possibility:

> 'every mathematician — if given the choice — would rather see our universe closed than open. There is mathematical beauty in such a universe which reveals itself when we consider any mathematical problem on such a cosmological background. In such a closed universe we have simple boundary conditions and do not need to worry about infinities in time and space. Compared with the closed universe the open one of Einstein–de Sitter appears to be dull and uninspired.'[2]

FREEMAN DYSON

The mathematical physicist Freeman Dyson (1923–2020) said in a 2014 interview that he developed an interest in numbers before he became interested in 'the real world'; specifically, he had an early fascination with pure number-theoretic problems, 'which are amazingly subtle and difficult and beautiful.'[3]

But he fully appreciated mathematical beauty in physics:

> 'The black-hole solution of Einstein's equations is also a work of art. The black hole is not as majestic as Gödel's proof, but it has the essential features of a work of art: uniqueness, beauty, and unexpectedness. Oppenheimer and Snyder built out of Einstein's equations a structure that Einstein had never imagined.'[4]

According to Dyson, '[t]he older Einstein and the older Oppenheimer were blind to the mathematical beauty of black holes, and indifferent to the question whether black holes actually exist.'[5] Based on discussions with J. Robert Oppenheimer (1904–67), Dyson attributed

[1] Infeld, 'General Relativity and the Structure of Our Universe', p. 491.

[2] *Ibid.*, p. 496.

[3] Lin, 'A "Rebel" Without a Ph.D.'

[4] Dyson, 'The Scientist as Rebel', p. 14.

[5] *Ibid.*, p. 12.

this blindness to a view that first-rate theoretical physicists should only concern themselves with establishing the fundamental equations. Studying particular solutions was an exercise for students or a research topic for lesser minds. Beauty and simplicity were located in the Dirac equation or the Einstein field equations, not in the particular solutions. Hence Oppenheimer tended to disparage his own work on the formation of black holes, for he found the black-hole solution of general relativity to be 'ugly, complicated, and lacking in fundamental significance'.[1]

Another topic on which Dyson wrote about mathematical beauty in physics was the theory of the stability of matter. Dyson & Andrew Lenard (1927–2020) were the first to solve the theoretical problem of how charged particles can form macroscopic matter in spite of electromagnetic repulsion, but Dyson wrote that they 'began with mathematical tricks and hacked our way through a forest of inequalities without any physical understanding', which yielded reasoning that was complicated and not enlightening.[2] But Elliot Lieb (1932–) & Walter Thirring (1927–2014) started from physical understanding and expressed it in rigorous mathematics, which, for Dyson, was the foundation of 'a grand and beautiful mathematical structure'.[3]

For Dyson, the beauty that was evident in the mathematical formulation of physical laws could be expected to continue as physics progressed:

> 'It is true that the fundamental equations of physics are simple and beautiful, and that we have good reason to expect that the equations still to be discovered will be even more simple and beautiful.'[4]

But he did not venture an explanation for why beauty and simplicity should so endure, and indeed seems to have considered it a mystery. In his obituary of Dirac, he asked, perhaps only rhetorically:

> 'Why should Nature care about our feelings of beauty? Why should the electron prefer a beautiful equation to an ugly one? Why should the universe dance to Dirac's tune?'[5]

Dyson recognized that there were limits to the effectiveness of mathematical beauty. In 1964, he wrote that group theory was a better approach to particle physics than *S*-matrix theory or quantum field theory, and he specifically stated that its superiority to *S*-matrix theory was in part aesthetic: 'it has an elegant and impeccably rigorous mathematical basis'.[6] However, while group theory described beautifully those aspects that could be abstracted into pure symmetry, that was its limit: it did not explain the 'messier' facts such as the experimentally observed masses and decay times of particles.[7]

Dyson also warned that '[m]athematical intuition is more often conservative than revolutionary, more often hampering than liberating'; here, 'intuition' included a sense of what is aesthetically correct.

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[1] Dyson, 'The Scientist as Rebel', p. 13.

[2] Dyson, preface to Lieb, *The Stability of Matter*.

[3] Dyson, preface to Lieb, *The Stability of Matter*; cf. Urquhart, 'Mathematics and Physics', p. 421.

[4] Dyson, 'The World on a String', p. 225.

[5] Dyson, 'Paul Dirac'.

[6] Dyson, 'Mathematics in the Physical Sciences', p. 146.

[7] *Loc. cit.*

He pointed to the aesthetic preference for circles as being a component of the 'worst of all the historic setbacks of physical science',[1] namely the adherence to the circle-based geocentric cosmologies that prevailed from ancient Greece until Copernicus and Kepler.

He also thought that intellectual conservatism about the metaphysical commitments of a theory could influence aesthetic judgement. Maxwell's equations,[*] thought Dyson, were only seen as beautiful if one had let go of a mechanical conception of the universe.[2] Similarly, while the beauty of the mathematics of quantum mechanics was, for Dyson, 'transcendent and overwhelming',[3] it could only be perceived by first letting go of classical thinking. The symmetries of quantum systems were not what were classically thought of as symmetries: rather, they were abstract.[4]

CHEN-NING YANG

Chen-Ning Yang (1922–2025) shared the 1957 Nobel Prize in physics with Tsung-Dao Lee (1926–2024) for their theoretical discovery, later experimentally confirmed, that the weak interaction did not conserve parity symmetry. But in his Nobel lecture, he expressed an appreciation for the power of symmetry and the mathematics that governs it:

> 'nature seems to take advantage of the simple mathematical representations of the symmetry laws. When one pauses to consider the elegance and the beautiful perfection of the mathematical reasoning involved and contrast it with the complex and far reaching physical consequences, a deep sense of respect for the power of the symmetry laws never fails to develop.'[5]

This appreciation dated from an early age: his father, the mathematician Ko-Chuen Yang (1896–1973), introduced his son to basic group theory, and the younger Yang said he had been 'fascinated by the beautiful diagrams' in Andreas Speiser's (1885–1970) book on finite group theory; he presumably referred to the pictures in the chapter on symmetries of ornaments.[6][†] It was reading about group characters in a book by L. E. Dickson[7] (1874–1954) that showed him 'the incredible beauty and power of group theory'.[8]

In the commentary to his *Selected Papers*, published in 1982, Yang wrote that most physicists took a 'utilitarian view of mathematics'. He thought his father's influence had shaped his own greater appreciation of mathematics, which he expressed in a peculiarly military mood:

> 'I appreciate the value judgment of the mathematicians, and I admire the beauty and power of mathematics: there are in-

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[1] Dyson, 'Mathematics in the Physical Sciences', p. 132.

[*] ▸ pp. 884–5

[2] Dyson, Review of *Nature's Numbers*, p. 610.

[3] *Loc. cit.*

[4] *Loc. cit.*

[5] Yang, 'The Law of Parity Conservation', pp. 237–8.

[6] Speiser, *Die Theoerie der Gruppen*, ch. 6.

[†] ▸ pp. 688–9

[7] See Dickson, *Modern Algebraic Theories*, ch. XIV.

[8] Yang, *Selected Papers*, p. 5.

> genuity and intricacy in tactical maneuvers, and breathtaking sweeps in strategic campaigns.'[1]

Yang was particularly fascinated by the coincidence between the independently-developed theories of gauge fields and of fibre bundles, which he found beautiful.[2] Gauge fields had been developed in theoretical physics, and fibre bundles in pure mathematics, and there is an almost one-to-one correspondence of the main concepts developed in each field under different terminologies.[3] Indeed, Yang recalled that when he and Robert Mills (1927–99) wrote their first paper on non-abelian gauge theory in 1954, they thought the work 'elegant'; Yang realized its importance in physics during the 1960s and 1970s, and its relationship to mathematics in 1974.[4] In 1975, Yang mentioned this relationship to the mathematician Shiing-Shen Chern (1911–2004), who had been one of his professors in the early 1940s and who had been one of the pioneers in the study of fibre bundles:[5]

> 'That non-Abelian gauge fields are conceptually identical to ideas in the beautiful theory of fiber bundles, developed by mathematicians *without reference to the physical world*, was a great marvel to me. In 1975, I discussed my feelings with Chern, and said, "This is both thrilling and puzzling, since you mathematicians dreamed up these concepts out of nowhere." He immediately protested, "No, no, these concepts were not dreamed up. They were natural and real."'[6]

Around 1955, Yang and Lee tried to develop a field theory based on quaternions. The idea was to take the usual wave functions ψ_{n} and ψ_{p} for the neutron and proton respectively, which are complex-valued, and write $\Psi = \psi_{\mathrm{n}} + \mathrm{j}\psi_{\mathrm{p}}$ to obtain *one* quaternion field.[7] (Since $\mathrm{ji} = -\mathrm{k}$ in the quaternion algebra, if $z = x + \mathrm{i}y$ and $z' = x' + \mathrm{i}y'$, then $z + \mathrm{j}z' = x + \mathrm{i}y + \mathrm{j}x' + \mathrm{k}(-y')$.) One of the benefits of quaternion fields was that SU(2) transformations and electromagnetic gauge transformations were expressible very simply in this language.[8]

[The group SU(n) and its overgroup U(n) will appear again in this chapter and in Chapter 25: U(n), the *unitary group*, is the group of $n \times n$ unitary matrices, and SU(n), the *special unitary group*, is the group of $n \times n$ unitary matrices with determinant 1. (A matrix is unitary if its conjugate transpose is equal to its inverse). So U(1) is the circle group made up of all complex numbers with absolute value 1, and SU(1) is the trivial group.]

Yang and Lee did not succeed in progressing beyond rewriting the usual theory using quaternions. But Yang wrote in the commentary to his *Selected Papers*, published in 1982, that he still thought that the basic direction was correct, for he held that the SU(2) symmetries found in nature had to arise from somewhere: 'the explanation is most likely in quaternion algebra: its symmetry is exactly SU_2'.[9] That is, the group of *unit* quaternions under multiplication are isomorphic to SU(2).

[1] Yang, *Selected Papers*, p. 74.

[2] Wu & Yang, 'Concept of nonintegrable phase factors', p. 460; Yang, 'Magnetic Monopoles, Fiber Bundles, and Gauge Fields', p. 86.

[3] Wu & Yang, 'Concept of nonintegrable phase factors', p. 3851, Tbl. 1.

[4] Yang, quot. in Zhang, 'C.N. Yang and Contemporary Mathematics', p. 15.

[5] Yang, 'Einstein's impact on theoretical physics', p. 49.

[6] Yang, 'Magnetic Monopoles, Fiber Bundles, and Gauge Fields', p. 97, emphasis in orig.; cf. Yang, 'Einstein's impact on theoretical physics', p. 49.

[7] Yang, *Selected Papers*, p. 22.

[8] *Loc. cit.*

[9] *Ibid.*, p. 23.

But there was also an aesthetic motivation: Yang thought that 'the quaternion algebra is a *beautiful* structure'.[1] Given that non-commutative algebras were already known to be — in Yang's idiom — the language nature had chosen, one should suspect that the only such 'nice' algebra would appear there:[2]

> 'Nature *does* choose the most elegant and unique mathematical structures to build the universe with, and the quaternion [*sic*] as a number system is elegant and unique.'[3]

Yang acknowledged that there was some difficulty in finding a theory based on quaternions, which he put down to some lack of understanding of the quaternions or the absence of some concept.[4]

Yang's faith in the aesthetic aspect of his belief in a quaternion-based universe should be tempered by a statement he made in a March 1979 panel discussion, which stands in contrast to Dirac's belief in an ongoing unification of pure mathematics and physics:[*]

> 'Not all beautiful mathematics is useful to physics. [...] if one starts by scanning all beautiful mathematics in order to find that part of it that fits physics, then one is most likely going to fail.'[5]

While physics has imported much 'beautiful and profound mathematics', Yang supposed that only a few percent of the 'profound' mathematics would ever find use in physics;[6] the context suggests that he was also referring to beautiful mathematics.

But Yang still defended the idea that mathematical beauty could play a role in theoretical physics. Using diagrams and arrows to indicate the conceptual flow (see Figure 23.2),[7] Yang described two routes to finding mathematical concepts useful to physics. In the first route, one tried to invent mathematics that organized the experimental information; in the second route one began with beautiful mathematics and tried to relate it to experiment. In both cases there was feedback between theory and experiment, but in the first route the dominant conceptual flow is from experiment to theory; the opposite holds in the second route.[8] This division seems akin to the dichotomies Dirac had put forth,[†] and Yang thought that the second route would turn out to be of increasing importance. In both routes, said Yang, 'theory is strongly coupled to mathematics and mathematical beauty',[9] but — and this may simply be an artefact of Yang's explanation — he seems to have placed a greater role on mathematical beauty in the second route, when the conceptual flow is greater from theory to experiment.

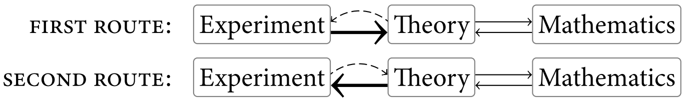

FIGURE 23.2
Yang's different conceptual flows between theory, experiment and mathematics.

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[1] Yang, *Selected Papers*, p. 23, emphasis in orig.

[2] *Ibid.*, p. 23.

[3] *Ibid.*, p. 23, emphasis in orig.

[4] *Ibid.*, p. 23.

[*] ▸ pp. 910–11

[5] Yang, quot. in Bethe et al., 'Einstein and the Physics of the Future', p. 501.

[6] *Loc. cit.*

[7] *Ibid.*, p. 502, Fig. XII.1.

[8] *Ibid.*, p. 501.

[†] ▸ p. 909

[9] *Ibid.*, pp. 501–2.

SUBRAHMANYAN CHANDRASEKHAR

The theoretical physicist Subrahmanyan Chandrasekhar (1910–95), who shared the 1983 Nobel Prize in physics for his work on stellar evolution and structure, clearly had a lively appreciation for beauty in science, and published in 1987 a collection of his papers and lectures on the subject entitled *Truth and Beauty: Aesthetics and Motivations in Science*. His contributions to the discussion of the role of aesthetic value in science were primarily those of a great synthesizer and popularizer.

In explicit opposition to Dirac's view that mathematical beauty could not be defined,[*] Chandrasekhar posited two criteria for beauty.[1] One was taken from Francis Bacon (1561–1626): 'There is no excellent beauty that hath not some strangeness in the proportion.'[2] The other criterion was adopted from Heisenberg: 'beauty is the proper conformity of the parts to one another and to the whole'.[3] (Heisenberg, in turn, had drawn it from Vitruvius.[†]) Chandrasekhar thought that 'this definition touches the essence of what we may describe as "beautiful": it applies equally to *King Lear*, the *Missa Solemnis*, and the *Principia*'.[4] He also agreed with Heisenberg on the significance of the discovery of numerical ratios as a source of harmony, which 'established for the first time a profound connection between the intelligible and the beautiful'.[5] For Chandrasekhar, the term 'strangeness' in Bacon's phrase was to be interpreted as 'exceptional to a degree that excites wonderment and surprise'.[6] For instance, the greatest strangeness in the general theory of relativity was the altered view of space-time.[7]

Chandrasekhar thought that 'the most shattering experience' of his professional career was the discovery by Roy Kerr (1934–) of the eponymous metric, which is an exact solution of the Einstein field equations and which describes the geometry of space around black holes.[8] That a discovery sought for its mathematical beauty should describe a phenomenon in nature produced in Chandrasekhar a reaction he described as 'shuddering before the beautiful';[9] here he alluded to Plato: 'he who is newly initiated, who beheld many of those realities, when he sees a godlike face or form which is a good image of beauty, shudders at first'.[10] Chandrasekhar's conclusion was that 'beauty is that to which the human mind responds at its deepest and most profound'.[11] He agreed with Heisenberg's Latin mottos *simplex sigillum veri* and *pulchritudo splendor veritatis*.[‡]

But for all the delight that Chandrasekhar took in seeing beautiful mathematics describe physical reality, he thought that the beauty of the mathematical formulation of a theory was insufficient reason to trust in its validity. With regard to the general theory of relativity, he said:

> 'Why then do we have faith and confidence in the theory? One should respond more explicitly than merely to say, as some

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[*] ▸ p. 904

[1] Chandrasekhar, 'The General Theory of Relativity', p. 1255.

[2] Bacon, 'Of Beauty', p. 479.

[3] Chandrasekhar, 'Shakespeare, Newton, and Beethoven', p. 1197; Chandrasekhar, 'The General Theory of Relativity', p. 1255; quoted slightly freely from Heisenberg, 'The Meaning of Beauty', pp. 167, 169, 170, 174, 177.

[†] ▸ p. 896

[4] Chandrasekhar, 'Shakespeare, Newton, and Beethoven', p. 1197.

[5] *Ibid.*, p. 1198.

[6] Chandrasekhar, 'The General Theory of Relativity', p. 1255.

[7] *Loc. cit.*

[8] Chandrasekhar, 'Shakespeare, Newton, and Beethoven', pp. 1199–1200.

[9] *Ibid.*, p. 1200.

[10] Plato, *Phaedrus*, STE 251a–b.

[11] Chandrasekhar, 'Shakespeare, Newton, and Beethoven', p. 1200.

[‡] ▸ pp. 896–7

> have, that our confidence derives from the "beauty of the mathematical description of nature which the theory provides".[1]

The confidence in the theory stood rather upon its internal consistency, its agreement with what were believed to be the physical requirements of such a theory, and the absence of contradiction with other parts of physics.[2]

RICHARD FEYNMAN[3]

> ‘Feynman's physics is about simplicity, beauty, unity and analogy’
>
> — Rob Phillips, 'In Retrospect: The Feynman Lectures on Physics', p. 30.

Richard Feynman (1918–88) shared the 1965 Nobel Prize in physics with Julian Schwinger and Sin-Itiro Tomonaga for his contributions to the development of quantum electrodynamics, which Dirac found so mathematically ugly.[*]

It has sometimes been claimed that Feynman disparaged mathematics as a pursuit,[4] but this story appears to be a mythological excrescence that grew out of a good-natured exchange of jibes between Feynman and his friend Mark Kac (1914–84):

> *Feynman*: 'It is true, as some of us have been saying, that if mathematics had never been invented, the progress of physics would have been delayed by about a week.'
>
> *Kac*: 'Absolutely true, by exactly the week in which the Lord God created the universe.'[5]

[The author has been unable to determine whether the quip that 'physics is to mathematics as sex is to masturbation' (or some variation thereof) has been accurately attributed to Feynman; it seems to have been ascribed to him in writing for the first time in 1992, as an unsourced epigraph in Lawrence Krauss's (1954–) book *Fear of Physics*.[6]]

In fact, Feynman seems to have appreciated beauty in pure mathematics. He said that the Fundamental Theorem of Algebra was 'fantastic' and had proofs which were 'very beautiful and very interesting'.[7] Abstraction seems to have been part of what he found beautiful in mathematics: a manoeuvre that was 'characteristic' of what made mathematics beautiful was to take an expression

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial x'} \cos\vartheta - \frac{\partial T}{\partial y'} \sin\vartheta,$$

---

[1] Chandrasekhar, 'The Aesthetic Base of the General Theory of Relativity', p. 1229.

[2] *Ibid.*, pp. 1229–32.

[3] Feynman's books cited here were based on lectures and were edited by others (see, for example, Feynman, Leighton & Sands, *Lectures on Physics (book)*, vol. 1, foreword; Feynman, *QED*, p. xi). The original lectures are thus cited first, and the published versions second. Except as noted, quotations match both.

[*] ▸ pp. 911–12

[4] See, for example, Zee, 'The Effectiveness of Mathematics', p. 310; Bangu, 'On the Pythagoreanism of Modern Physics', p. 73, n. 74; p. 93, n. 97; Bangu, 'Pythagorean Heuristic', p. 392, n. 11;p. 405, n. 33.

[5] Quot. in Boehm, 'Mark Kac's Beautiful World of Mathematics', p. 8.

[6] See Krauss, *Fear of Physics*, p. 27.

[7] Feynman, 'Lectures on Physics', lect. 23, 38:54 sqq.; Feynman, Leighton & Sands, *Lectures on Physics (book)*, vol. 1, § 22-5.

note that the argument that led to it did not depend on the particular scalar field T, and so omit it, leaving the equation of operators

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \cos \vartheta - \frac{\partial}{\partial y'} \sin \vartheta,$$

in which the operators were, as the physicist and astronomer James Jeans (1877–1946) said, and as Feynman slightly misquoted, ‘hungry for something on which to operate’.[1]

Mathematical beauty in theory

For Feynman, there was a simplicity and beauty in the very fact that all the laws of theoretical physics can be written mathematically.[2] He certainly thought that mathematics was the only thing that could make accessible the beauty of a physical theory:

> ‘every one of our laws is a purely mathematical statement in rather complex and abstruse mathematics [...]it is impossible to answer [...] the challenge of explaining, in a way that a person can feel, the beauties of the laws of nature without their having some deep understanding of mathematics. I’m sorry, but it seems to be the case.
>
> [...]
>
> [...] I would use the words of Jeans who said that [...] “the Great Architect seems to be a mathematician” [*sic*[3]], and for you who don’t know mathematics it’s really quite difficult to get a real feeling across of the beauty, of the deepest beauty, of nature.’[4]

He also came very close to saying that he accepted mathematical beauty as a guide to theoretical truth, but did not explicitly say that the beauty had to be *mathematical*:

> ‘It is possible to know when you’re right way ahead [...] of checking of all the consequences. You can recognize truth by a beauty and simplicity. It’s always easy when you’ve got the right guess and make two or three little calculations to make sure it isn’t obviously wrong, to know that it’s right.’[5]

Feynman said that it might not be possible to see beauty in particular experimental results, but that it was possible to see beauty in the equations that describe physical laws. For example, in the three-dimensional wave equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0, \qquad \text{(W)}$$

the regularity in the appearance of x, y, z, t ‘suggests to the mind still a greater beauty’, namely that space has a four-dimensional symmetry; there was thus beauty associated to the equations.[6]

[1] Jeans, *The New Background of Science*, p. 202; Feynman, ‘Lectures on Physics’, lect. S2, 23:47 sqq.; Feynman, Leighton & Sands, *Lectures on Physics* (book), vol. 2, § 2-4.

[2] Feynman, ‘Lectures on Physics’, lect. 23, 02:10 sqq.; Feynman, Leighton & Sands, *Lectures on Physics* (book), vol. 1, § 22-1.

[3] ‘the Great Architect of the Universe now begins to appear as a pure mathematician’ (Jeans, *The Mysterious Universe*, p. 122).

[4] Feynman, ‘The Character of Physical Law’, lect. 2, 11:52 sqq., 12:28 sqq., 51:47 sqq.; with minor emendations in Feynman, *The Character of Physical Law* (book), pp. 39–40, 58.

[5] Feynman, ‘The Character of Physical Law’, lect. 7, 50:17 sqq.; with minor emendations in Feynman, *The Character of Physical Law* (book), p. 171.

[6] Feynman, ‘Lectures on Physics’, lect. S20, 50:37 sqq.; Feynman, Leighton & Sands, *Lectures on Physics* (book), vol. 2, § 20-3.

Thus, for Feynman, the mathematical beauty of a physical law was connected to what it said about nature. The same is implied by the following equations, which can be deduced from Maxwell's:[*]

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{\vec{j}}{\varepsilon_0 c^2}; \tag{A}$$

$$\nabla^2 \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}. \tag{B}$$

These equations were 'beautiful'.[1] But this beauty depended on their physical meaning: 'they are nicely separated — with the charge density, goes φ; with the current, goes $\vec{A}$';[2] furthermore, expanding the left-hand side of (B) gives

$$\frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0},$$

which, like (W), has a regularity or symmetry in x, y, z, t, with the $-1/c^2$ being necessary because time and space have different units.[3]

Maxwell's equations can be combined into the following form:

$$\Box^2 A_\mu = \frac{1}{\varepsilon_0} j_\mu, \qquad \nabla_\mu j_\mu = 0; \tag{M}$$

Feynman found this form beautiful and simple. But the syntactic form of an equation was not enough for beauty or simplicity. The physical meaning was important. To illustrate this point, Feynman said that jokingly that '*All of the laws of physics can be contained in one equation*':[4]

$$\mathsf{U} = 0; \tag{P}$$

all that was necessary was to define the quantity U — 'unworldliness' — to be the sum of the squares of the deviations from the laws of physics. For example, one term in the sum would be $(\vec{F} - m\vec{a})^2$, being the 'unworldliness of mechanical effects'. The syntactically simple and superficially beautiful equation (P) was thus a trick; there was no actual simplicity.[5]

But there was more to (M); it was not a trick. The fact that Maxwell's equations can be written using a notation that was created for the four-dimensional geometry of the Lorentz transformation means that they are Lorentz-invariant. The beauty of (M) was a consequence and a symbol of the Lorentz invariance of Maxwell's equations.[6]

Thus, for Feynman, the beauty of an equation describing a physical law was bound up with the physical meaning and the properties of nature of which the equation is a consequence. Conversely, the consequences of the equation seem to have had little bearing on its beauty: Feynman thought that it was possible for an equation to be beautiful and yet for its consequences to be difficult to understand directly; the Dirac equation was an example.[7][†]

[*] ▸ pp. 884–5

[1] Feynman, 'Lectures on Physics', lect. S18, 51:42 sqq.; Feynman, Leighton & Sands, *Lectures on Physics* (book), vol. 2, § 18-6.

[2] Feynman, 'Lectures on Physics', lect. S18, 51:44 sqq.; Feynman, Leighton & Sands, *Lectures on Physics* (book), vol. 2, § 18-6.

[3] Feynman, 'Lectures on Physics', lect. S18, 51:50; Feynman, Leighton & Sands, *Lectures on Physics* (book), vol. 2, § 18-6.

[4] Feynman, 'Lectures on Physics', lect. S26, 02:36 sqq.; Feynman, Leighton & Sands, *Lectures on Physics* (book), vol. 2, § 25-6, emphasis in orig.

[5] Feynman, 'Lectures on Physics', lect. S26, 02:46 sqq., 04:26 sqq.; Feynman, Leighton & Sands, *Lectures on Physics* (book), vol. 2, § 25-6.

[6] Feynman, 'Lectures on Physics', lect. S26, 04:46 sqq.; Feynman, Leighton & Sands, *Lectures on Physics* (book), vol. 2, § 25-6.

[7] Feynman, 'Lectures on Physics', lect. 20, 41:41 sqq.; Feynman, Leighton & Sands, *Lectures on Physics* (book), vol. 1, § 20-3.

[†] ▸ pp. 903

Elegance in method

Elegance, for Feynman, seemed to denote partly notational efficiency or a lack of ornamentation or sophisticated technique.[1] In a lecture, he used elementary plane geometry to deduce Kepler's first law, which states that the path of each planet around the Sun is an ellipse with the Sun at one focus. When he explained his motivation for making this effort, he made an implicit contrast between elegance and fanciness:

> 'On the other hand, in the beginning of our science — that is, in the time of Newton — the geometrical method of analysis in the historical tradition of Euclid was very much the way to do things. [...] We do it now by writing analytic symbols on the blackboard, but for your entertainment and interest I want you to ride in a buggy for its elegance, instead of in a fancy automobile.'[2]

The geometrical method lent itself to elegance in its demonstrations, but not to ease of discovery. On the other hand, the analytic method made it easier to discover things, but not elegantly: it led to 'a lot of dirty paper, with *x*'s and *y*'s and crossed out, cancellations and so on'.[3] Thus elegance could also mean a kind of notational efficiency. He suggested that there could seem to be a sense of defeat at the inelegance involved if one wrote out components rather than using vector operators, although he said that it could be better to write everything out when trying to understand a problem, and then package it up in vector form for publication.[4]

Numerical preferences

Finally, there also seems to have been something aesthetic in Feynman's declared preference for theories that involved small integers and purely mathematical constants. For instance, he asked where numbers like the fine structure constant[*] come from, which for him was a 'beautiful problem'.[5] A 'good' theory would explain each such constant by a calculation using small integers and π, (deliberately inaccurate examples being '$\frac{1}{2\pi}\sqrt[3]{3}\cdot 6$' or 'the square root of 3 over 2 pi squared').[6]

[A minor intellectual industry has produced many such expressions for the fine structure constant α,[7] with proposals including[8]

$$\alpha^{-1} = \frac{8\pi^4}{9}\sqrt[4]{2^4 5!/\pi^5}, \qquad 108\pi\sqrt[6]{8/1843}, \qquad \sqrt{137+\pi^2};$$

the author of the last suggestion said that '[s]ince this expression is very simple, it must be the true solution'.[9]]

[1] Feynman, 'Lectures on Physics', lect. 36, 54:10 sqq.; Feynman, Leighton & Sands, *Lectures on Physics (book)*, vol. 1, § 34-9.

[2] Feynman, 'The Motion of Planets Around the Sun', p. 149 (in the audio recording of the lecture, 08:47 sqq.)

[3] *Ibid*., p. 164 (35:40 sqq.)

[4] Feynman, 'Lectures on Physics', lect. S6, 10:42; Feynman, Leighton & Sands, *Lectures on Physics (book)*, vol. 2, § 6-3.

[*] ▸ pp. 893–5

[5] Feynman, 'Quantum Electrodynamics', 19:30 sqq.; a 'beautiful question' in Feynman, *QED*, p. 129.

[6] First example from Feynman, 'Quantum Electrodynamics', 17:03 sqq.; second from Feynman, *QED*, pp. 129–30.

[7] Kragh, 'Magic Number', pp. 419–23.

[8] Wyler, 'L'espace symétrique', p. 744; Aspden & Eagles, 'Aether Theory and the Fine Structure Constant'; Burger, 'Calculating the fine structure constant'.

[9] Burger, 'Calculating the fine structure constant'.

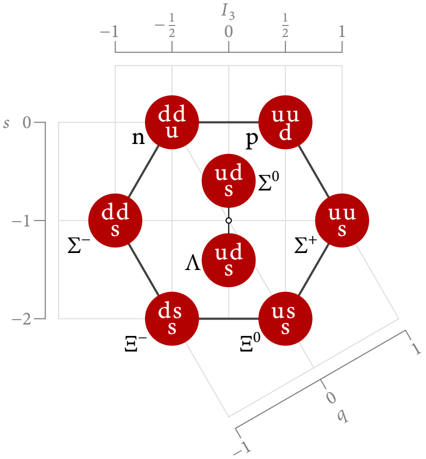

FIGURE 23.3
Spin-1/2 baryon octet.

s: Strangeness
I_3: Isospin component 3
q: Charge
p: Proton
n: Neutron
Λ: Lambda baryon
$\Xi^{\blacksquare}$: Xi baryon
$\Sigma^{\blacksquare}$: Sigma baryon
u: Up quark
d: Down quark
s: Strange quark

SU(3) AND PARTICLE PREDICTION

Murray Gell-Mann (1929–2019), who had a lively appreciation of the aesthetic value of theories, including their mathematical formulation,[*] sought a symmetry principle that would allow a unified classification of mesons, baryons, and hadrons. In his search, he went some way towards re-inventing part of Sophus Lie's (1842–99) work on group theory, before being pointed to the existing literature by a colleague.[1] Once equipped with the necessary theory, he fastened on SU(3) as a candidate for the required symmetry group. He formulated the correspondence between the eight known baryons and the 8-dimensional representation of SU(3) (see Figure 23.3). This led him to predict the existence of the thitherto undiscovered eta meson η to complete the analogous meson multiplet (see Figure 23.4); the existence η was confirmed experimentally a few months later.[2] [Note that Figures 23.3 and 23.4, like Figure 23.5, show the quark components of the baryons and mesons. The quark components were postulated and confirmed later as a consequence of, and in order to explain, the SU(3) symmetry.[†]]

The SU(3) symmetry's crowning success was in the completion of the spin-3/2 baryon decuplet (see Figure 23.5). Four such baryons had been known: Δ^-, Δ^0, Δ^+, Δ^{++}, all with strangeness 0. At a conference in July 1962, evidence was announced for more spin-3/2 baryons: a triplet of Σ^* baryons with strangeness −1 and a doublet of Ξ^* baryons with strangeness −2. Gell-Mann and Yuval Ne'eman (1925–2006) were both in the audience and independently inferred from the 10-dimensional representation of SU(3) that there must be another spin-3/2 baryon with strangeness −3; this Ω^- baryon's existence was detected in 1964.

The prediction of the Ω^- baryon is of an entirely different kind from the prediction by Pauli in 1930 of the neutrino, to account for observed discrepancies in the conservation of energy and momentum

[*] ▸ pp. 955–7

[1] Johnson, *Strange Beauty*, ch. 9.

[2] Riordan, *The Hunting of the Quark*, p. 89.

[†] ▸ p. 949

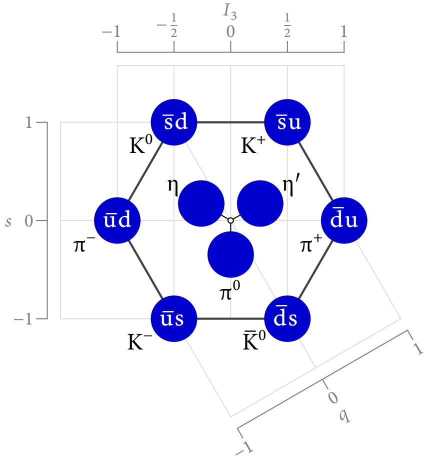

FIGURE 23.4
Meson nonet.

s: Strangeness
I_3: Isospin component 3
q: Charge
K▨: Kaon
π▨: Pion
η: Eta meson
u: Up quark
d: Down quark
s: Strange quark
▨̄: Antiparticle

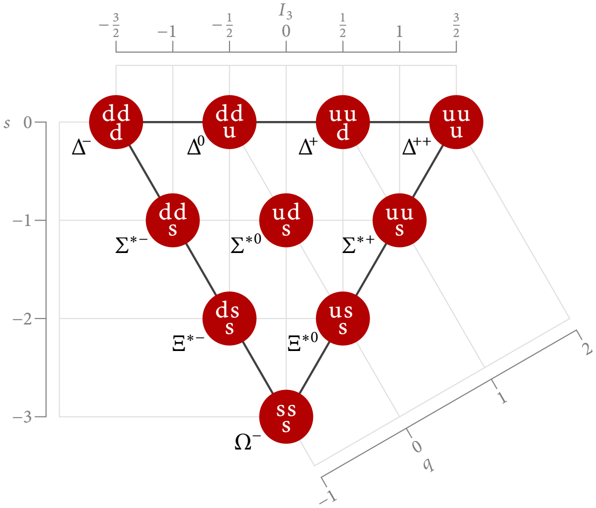

FIGURE 23.5
Spin-3/2 baryon decuplet.

s: Strangeness
I_3: Isospin component 3
q: Charge
Δ▨: Delta baryon
Ξ*▨: Xi baryon
Σ*▨: Sigma baryon
Ω⁻: Omega baryon
u: Up quark
d: Down quark
s: Strange quark

in beta decay,[1] or the gradually emerging prediction by various astronomers of an eighth planet, to account for observed discrepancies in the predicted motion of Uranus.[2] The predictions of the existences of the particle later named the neutrino and of the planet later named Neptune explained the observations in terms of the accepted physical laws. That is, these predictions were put forth to fulfil a requirement to comply with those established laws. But the prediction of Ω^- was based upon a desire to comply with a mathematical, non-physical, consideration of symmetry.[3] The non-existence of the neutrino or Neptune would violate physical law; the non-existence of Ω^- would violate only a putative mathematical symmetry.[4]

There is also a distinction to be made between the prediction of Ω^- and Dirac's prediction of the positron. Dirac's equation accurately described the electron's behaviour; it naturally extended to providing a theoretical description of an anti-electron as well. The positron

[1] Brown, 'The idea of the neutrino'.

[2] Baum & Sheehan, *In Search of Planet Vulcan*, ch. 7.

[3] Bangu, 'On the Pythagoreanism of Modern Physics', pp. 120–2.

[4] *Ibid.*, p. 124.

was predicted as the realization of this description. The prediction of Ω^- was not based upon a theoretical description that called out for realization, but upon a gap in a symmetrical and mathematical classification system.[1]

To find the closest historical parallel for the prediction of Ω^-, one must look back to the postulation of the counter-earth attributed by Aristotle to Philolaus, in order to bring the number of heavenly bodies in his cosmology to 10, matching the pythagorean tetractys.[*]

There has been a slight historiographical tendency to oversimplify symmetry arguments such as the one that predicted the Ω^- baryon, at least in popular science books. For example, Rudy Rucker's (1946–) explanation rather seemed to reduce the symmetry argument to the aesthetics of the geometric arrangement of the multiplet:

> 'a three-dimensional chart of all the known elementary particles if drawn up. The chart looks, let us say, like a regular dodecahedron with one corner missing. It would look prettier, more symmetric, if there were an additional particle with such and such characteristics to fill the missing corner, so the physicists postulate the existence of such a particle.'[2]

QUASI-CRYSTALS

A suitable coda for this chapter, and one of the few instances of aesthetic value in mathematics stimulating scientific research outside of fundamental physics, is the discovery of quasi-crystals: first theoretical, then synthetic, then natural.

Based on computer simulations that showed rapid cooling and solidification of liquid could produce arrangements of atoms with an icosahedral symmetry, in apparent defiance of the thitherto accepted laws of crystallography,[3] Dov Levine (1958–) & Paul Steinhardt (1952–) attempted to devise an atomic structure that exhibited such a symmetry. They were initially unsuccessful, but then heard of Penrose's aperiodic tiling[†] and immediately thought that they might be able to construct a three-dimensional analogue.[4]

The amateur mathematician Robert Ammann (1946–94) had independently discovered one of Penrose's aperiodic tilings in 1976,[5] and found that the tiles form patterns determined by what are now known as *Ammann bars*. These bars are straight lines on each tile, which serve as matching rules: tiles must be placed so that the Ammann bars on adjacent tiles form straight lines. The bars then form *Ammann lines* that cross the plane in five different directions, intersecting an

[1] Bangu, 'On the Pythagoreanism of Modern Physics', pp. 132–3.

[*] ▸ p. 48

[2] Rucker, *Infinity and the Mind*, p. 55.

[3] Steinhardt, *The Second Kind of Impossible*, ch. 2.

[†] ▸ p. 689

[4] *Loc. cit.*

[5] Gardner, *Penrose Tiles to Trapdoor Ciphers*, p. 19.

angle of 2π/5. Thus the angles between the Amman lines are the same as the angles between edges of a regular pentagon.

Levine and Steinhardt were confident that the Ammann lines were the key to showing that matter with pentagonal symmetry was theoretically realizable, despite the apparent conflict with the laws of crystallography.[1] This idea led them to construct a three-dimensional analogue of Penrose's tiling: they found two polyhedra that could be packed together to fill space only in non-periodic ways.[2] The resulting *quasi-crystal* had icosahedral symmetry,[3] and they computed the theoretically predicted diffraction patterns, which could show decagonal symmetry.[4]

When they advanced this idea in 1984, quasi-crystals were only a theoretical possibility, although there had been one empirical observation of icosahedral symmetry arising from rapid cooling, which had been announced as an experimental result without a theoretical explanation.[5] The idea of quasi-crystals remained controversial; notably, Linus Pauling (1901–94), who was Nobel laureate in chemistry in 1954 and in peace in 1962, repeatedly rubbished the concept and devised other hypotheses.[6] The first stable quasi-crystal was synthesized in 1987 by An-Pang Tsai (1958–2019), Akihisa Inoue (1947–) & Tsuyoshi Masumoto (1932–); it was an aluminium-copper-iron alloy and exhibited icosahedral symmetry (see Figure 23.6).[7] In 2009, the discovery of first natural quasi-crystal was announced.[8] It was subsequently named 'icosahedrite' for its icosahedral symmetry.[9]

Steinhardt's account of these discoveries contains repeated references to the beauty of the mathematics and the visual patterns.[10] In particular, the beauty of aperiodic tilings was a motivation for Peter J. Lu's (1978–) continued search for pre-modern examples of such tilings.[11]

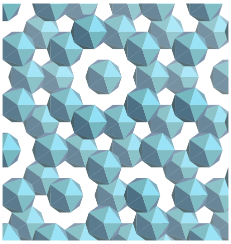
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FIGURE 23.6
Atomic structure of a Tsai-type quasi-crystal (Tamura et al., 'Experimental Observation', Fig. 1).

'EXPERIMENTAL LOGIC'

The physicist Frank Wilczek (1951–), one of the 2004 Nobel laureates in physics, gave an interesting characterization of the investigative processes that he thought Maxwell, Einstein, and Dirac had followed to reach, respectively, the theory of electromagnetism, the special and general theories of relativity, and the quantum theory of the electron. For Wilczek, their method was *experimental logic*. In experimental logic, the experiments are upon the mathematical hypotheses. The physicist attempted to improve the beauty and consistency of the equations and then to check whether the new (presumably more beautiful) equations explained some feature of the empirical evidence.[12] The creativity aspect of improving the equations was dependent upon the physicist's ability to manipulate patterns and to recognize beauty.[13]

[1] Steinhardt, *The Second Kind of Impossible*, ch. 3.

[2] Levine & Steinhardt, 'Quasicrystals', p. 2478.

[3] *Loc. cit.*

[4] *Ibid.*, Fig. 1(a).

[5] Shechtman et al., 'Metallic Phase with Long-Range Orientational Order and No Translational Symmetry'.

[6] Bindi, *Natural Quasicrystals*, pp. 4–6.

[7] Tsai, Inoue & Masumoto, 'A Stable Quasicrystal'.

[8] Bindi, Steinhardt et al., 'Natural quasicrystals'.

[9] Bindi, Steinhardt et al., 'Icosahedrite'.

[10] Steinhardt, *The Second Kind of Impossible*, ch. 4, 5.

[11] *Ibid.*, ch. 7.

[12] Wilczek, 'A Piece of Magic', p. 129.

[13] *Loc. cit.*

‘Experimental logic’

An important difference between experimental logic and rigorous mathematics was that the experimental logician was granted some leeway with regard to consistency: if a line of investigation seemed potentially fruitful in terms of its explanatory power, it should not be abandoned if a contradiction arose; rather, successful but contradictory mathematics should be tolerated temporarily while a resolution of the contradiction was sought.[1] This toleration was distinct from Dirac’s view that one should not immediately abandon a mathematically beautiful theory because of a contrary piece of experimental evidence,* for such problematical evidence was external to the theory. Wilczek’s argument was rather that one should be ready to ignore temporarily *internal* problems of consistency in the mathematical formulation of the theory.

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[1] Wilczek, ‘A Piece of Magic’, p. 129.

* ▸ pp. 906–7

24

Unreasonable Effectiveness

Beauty and the problem of the applicability of mathematics to nature

‘the search for effective precepts,
like the search for empirically adequate scientific theories,
is a search for regularities in the world.’
— James W. McAllister, *Beauty & Revolution in Science*, p. 101.

‘The whole Universe could have been fashioned in this way,
with no mathematics built into it, only aesthetics.
Then along we come with our mathematics
and proclaim the Universe to be intrinsically mathematical.’
— John D. Barrow, *The World within the World*, p. 248.

On the occasion of Richard Courant’s (1888–1972) seventieth birthday, some of his friends endowed a fund to provide for a series of two-yearly lectures at New York University named after him.[1] Presenting the gift, Kurt Friedrichs (1901–82) said that

> ‘within mathematics proper Courant has always fought against overemphasis of the rational, logical, legalistic aspects of this science and emphasized the inventive and constructive, esthetic and even playful on the one hand, and on the other hand those pertaining to reality. How mathematics can retain these qualities when it invades other sciences is an interesting and somewhat puzzling question. Here we hope our gift will help.’[2]

On 11 May 1959, the theoretical physicist Eugene Wigner (1902–95), who would go on to be one of the 1963 Nobel laureates in physics, delivered the first Richard Courant Lecture. This celebrated and influential address, entitled ‘The Unreasonable Effectiveness of Mathematics in the Natural Sciences’, fitted neatly with the characterization of Courant that Friedrichs had made, the question he had posed, and the hope he had expressed.

As his title suggests, Wigner argued that the very notion that natural phenomena should follow any kind of mathematical description was in itself remarkable. His starting-point was the observation that mathematical concepts often appeared in unexpected contexts in science, and nevertheless allowed for an accurate model of the phenomena. Wigner thought mysterious this usefulness of mathematics in the natural sciences, and maintained that there was no rational explanation for it.[1] Although the most fundamental concepts of mathematics, such as elementary arithmetic and geometry, might have been formulated though direct abstraction from the observed world, mathematics progressed by the formulation of new concepts, each still more advanced and seemingly more detached from that original observation. And yet such new concepts had continued to find application in physics.[2] Wigner was not the first to draw attention to the uncanny way in which the mathematics that had become ever more abstract was nevertheless drawn back into application in science. In a 1921 lecture, Albert Einstein[*] (1879–1955) called it an 'enigma [...] which in all ages has agitated inquiring minds':

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[1] Reid, *Courant*, p. 520.

[2] Friedrichs, quot. in *loc. cit.*

> 'How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality?'[3]

The answer he offered is a now-famous dictum:

> 'as far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality'.[4]

In 1925, Alfred North Whitehead (1861–1947) also commented that it was 'impressive' and a 'paradox' how the most extreme mathematical abstractions had become tools with which to understand concrete reality.[5]

But Wigner seems to have been the first to note the aesthetic aspect of the puzzle. He had been aware of the beauty of mathematics all his life, crediting his *gimnázium* (secondary school) mathematics teacher László Rátz (1864–1930) with introducing him to the beauty of the subject.[6] He thought that advanced mathematical concepts were formulated for aesthetic reasons: they were 'so devised that they are apt subjects on which the mathematician can demonstrate his ingenuity and sense of formal beauty'[7] and were 'defined with a view of permitting ingenious logical operations which appeal to our aesthetic sense'.[8] Wigner suggested the particular example of the complex numbers, which were not based on anything in our immediate experience. The value of complex numbers lay instead in the beautiful theorems in the vast area of mathematics that in some way depended upon them.[9] Wigner here seems to have indicate that he viewed mathematics as centred on its concept-formulation rather than on its deductive aspect.[10] This part of Wigner's argument may have been partly inspired by G. H. Hardy's (1877–1947) *Apology*:[†] although Wigner did not explicitly cite Hardy, he emphasized the role of aesthetic concerns in the motivation for mathematics and in the choice of direction in which it develops.

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[1] Wigner, 'The Unreasonable Effectiveness of Mathematics', p. 2.

[2] *Loc. cit.*

[*] ▸ pp. 886–92

[3] Einstein, 'Geometrie und Erfahrung', p. 385; trans. Einstein, 'Geometry and Experience', p. 233.

[4] Einstein, 'Geometrie und Erfahrung', pp. 385–6; trans. Einstein, 'Geometry and Experience', p. 233.

[5] Whitehead, *Science and the Modern World*, p. 41.

[6] Wigner, *The Recollections*, pp. 49–53.

[7] Wigner, 'The Unreasonable Effectiveness of Mathematics', p. 3.

[8] *Loc. cit.*

[9] *Loc. cit.*

[10] Islami, 'A match not made in heaven', p. 4845.

[†] ▸ pp. 498–509

Viewed from an aesthetic perspective, the problem of the applicability of mathematics to science takes on a special dimension. Physical laws are stated in terms of advanced mathematical concepts,[1] which, Wigner maintained, were developed by mathematicians for aesthetic reasons. Only a fraction of these concepts were used in physics, and usually the physicist did not choose from a menu supplied by the mathematician, but instead, guided by the demand of explaining empirical observations, developed a concept independently and then found that the mathematician, guided by an aesthetic sense, had been there first.[2] This kind of coincidence struck Wigner as a 'miracle', and the closest he had come to an explanation was the idea, which he attributed to Einstein, that we were only willing to accept beautiful physical theories, and so it was natural that mathematics developed for its own beauty should find a role in those theories.[3] He was ultimately unable to account for the applicability of mathematics to the natural sciences:

> 'The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.'[4]

Wigner himself played an important role in the introduction into physics of areas of mathematics such as group theory in order to describe quantum phenomena. Drawing on such theories, developed for purely mathematical ends and thitherto unexploited by science, naturally led to — or at least renewed — the question of the applicability of these new methods to physics; perhaps it was unsurprising that Wigner considered and spoke about it.[5]

The aesthetic dimension of Wigner's problem comprises two distinct but related mysteries, one metaphysical, one epistemological. First, why does mathematics developed for motivations that were at least partly aesthetic and certainly did not include potential applicability nevertheless fit the physical world? Second, why do humans, with an aesthetic sense that is *prima facie* not shaped by fundamental physics, develop mathematics that turns out to be applicable there?

Suppose that the simple applicability of mathematics to model the physical world is indeed a miracle. Then the role of aesthetic value in the development of mathematics causes the magic to redouble: it is what Jerry King (1935–2020) called 'a miracle of second order of magnitude', namely '*paradox of the utility of beauty*'.[6]

APPLICABILITY WITHOUT AESTHETICS

While the responses to Wigner discussed elsewhere in this chapter concentrate on the role of aesthetic concerns, it must be noted that Wigner's essay is the origin, or at least the focal point,

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[1] Wigner, 'The Unreasonable Effectiveness of Mathematics', pp. 6–7.

[2] *Ibid.*, p. 7.

[3] *Loc. cit.*

[4] *Ibid.*, p. 14.

[5] Ferreirós, 'Wigner's "Unreasonable Effectiveness" in Context', p. 65.

[6] King, *The Art of Mathematics*, p. 121, emphasis in orig.

Applicability without aesthetics

of a long philosophical debate, and it is thus useful to recapitulate briefly some of the non-aesthetic solutions to Wigner's problem.

Ineffectiveness of mathematics

Has mathematics actually been *unreasonably* effective in the sciences? Perhaps there has just been a filter of selective memory in place, which has occluded those occasions when mathematics has failed. The failures to apply mathematics to nature may vastly outnumber the successes. The philosopher Mark Steiner made this point,[1] and though he did not cite or present a historical survey to back it up, it is surely correct if one includes the whole field of the natural sciences. Loosely speaking, the further a field is from fundamental physics and the more concerned it is with complex (or chaotic) systems — for example, meteorology — the less accurate its mathematical models become.

Expanding the field of application to include engineering, one encounters other examples. The behaviour of early metal–oxide–semiconductor field-effect transistors, whose scale was in the micrometre range, could be modelled by 'elegant' equations derived from first principles; these equations allowed circuits to be designed correctly. But as transistor sizes diminished further, the equations were complicated with too many higher-order effects to be useful, and were displaced by empirical models.[2] Similarly, when it comes to developing integrated electromagnetic devices, analytical solutions to Maxwell's equations* have given way to numerical methods.[3]

Circumscription of science

In 1980, the mathematician Richard Hamming (1915–98) pointed out that science did not actually answer many problems. Very few human experiences in the world were addressed by science; its domain was tightly circumscribed. Thus the applicability of mathematics in science lost some of its impressiveness, for science itself had only narrow applicability.[4]

David Lindley (1956–), a theoretical physicist, put forward the rather stronger suggestion that the applicability of mathematics in science actually collapsed into a tautology: 'mathematics is the language of science because we reserve the name "science" for anything that mathematics can handle'.[5] He additionally observed that while physicists had successfully drawn concepts and tools from pure mathematics in order to build their theories, they had used only a small portion of the corpus. The part of the higher arithmetic that dealt with the distribution of the prime numbers had not found application in physics. The concepts of group theory had been used successfully to organize the elementary particles and their interactions, but physics

[1] Steiner, *The Applicability of Mathematics*, pp. 10, 46.

[2] Abbott, 'The Reasonable Ineffectiveness of Mathematics', p. 2149.

* ▸ pp. 884–5

[3] *Loc. cit.*

[4] Hamming, 'The unreasonable effectiveness of mathematics', p. 89.

[5] Lindley, *The End of Physics*, pp. 4–5.

made use of a minuscule and rather elementary fragment of a vast and sophisticated subject.[1]

Pythagoras unbound

The physicist and cosmologist Max Tegmark (1967–) offered the intriguing explanation that the physical world was purely mathematical, in the sense that the only intrinsic properties of anything in it were mathematical: the intrinsic properties of space were mathematical properties like dimensionality, curvature, topology; those of particles were also mathematical or numerical properties like charge and spin. Still more fundamental was the wavefunction and the Hilbert space it moved through in accordance with the Schrödinger equation.[2] With such a purely mathematical ontology, it would be unsurprising that mathematics described the universe.

Faust and Mephistopheles

During a 1986 symposium, Steven Weinberg (1933–2021) said that it was ‘positively spooky’ that the physicist always found themself entering a new vista only to come across the tracks of the mathematician, and offered the humorous explanation that at least some mathematicians must have bargained with the devil for advance knowledge of the kinds of mathematics the sciences would need.[3]

NICOLAS GOODMAN

In a 1990 article, the philosopher of mathematics Nicolas D. Goodman suggested that Wigner’s article might be viewed as ‘an over-reaction to the sort of arrogant aestheticism represented by perfectly “pure” mathematicians like G. H. Hardy’,[4] but Goodman’s point here was at least partly due to his falling prey to the mischaracterization of Hardy as taking pride in the lack of utility of pure mathematics.[*] He took issue with Wigner’s example of the complex numbers as having been introduced for aesthetic reasons, pointing out that they were introduced to fulfil a conceptual need.[5] While Wigner thought it surprising that mathematical concepts like complex numbers should be applicable to particles such as electrons and photons, Goodman argued that much of his puzzlement could be alleviated if mathematics was viewed as closer to being an empirical science:[6] from this perspective, such particles and complex numbers were alike *theoretical entities*, where the support for the latter was at least as good as for the former.[7] And if mathematics was akin to an empirical science, the applicability to physics of mathematics, with

[1] Lindley, *The End of Physics*, p. 5.

[2] Tegmark, *Our Mathematical Universe*, chs 2–4, 7, 8, summarized in ch. 10.

[3] Griffiths et al., ‘Mathematics’, pp. 725, 728.

[4] Goodman, ‘Mathematics as natural science’, p. 187.

[*] ▸ pp. 512–16

[5] *Ibid.*, p. 186.

[6] *Loc. cit.*

[7] *Loc. cit.*

all its abstractions, became analogous to the application to electrical engineering of physics, with its postulated perfect insulators and continuous currents.[1]

HAROLD OSBORNE

In 1984, the philosopher Harold Osborne (1905–87) pointed out that 'reliance on the heuristic value of mathematical beauty in scientific theory has become something of a commonplace'[2] and that 'it is taken for granted that anyone with a talent for scientific matters will recognize a beautiful theory',[3] but that the role of beauty in the natural sciences had not been subjected to systematic philosophical scrutiny.[4]

Osborne noted that beauty in the sciences was only a means to an end, and drew an important contrast with art. An artist would think meaningless Dirac's contention that 'if one is working from the point of view of getting beauty in one's equations, and if one has really a sound insight, one is on a sure line of progress'[5].[*] The artist who had created beauty had achieved their aim; they were not on a path to some other goal.[6] One could quibble with Osborne's argument, since not all artists attempt to create beauty, and an artist who does may have additional goals in mind, but he was correct that beauty in art is not a means towards truth (in the scientific sense of empirical adequacy).

Osborne warned that any attempt to determine whether mathematically beautiful theories were more likely to accord with observation would depend on a better understanding of what was meant by mathematical beauty.[7] Indeed, the question might not be meaningful at all.[8] Osborne did not explore his caveat about meaning, but he may have referred to the (*prima facie*) possibly higher degree of subjectivity in judgements of beauty and the higher degree of objectivity in judgements of empirical adequacy.

In an attempt to clarify what was actually meant by beauty in the sciences, Osborne turned to mathematics. He first reached for Aristotle's remarks,[†] then to G. H. Hardy's account,[‡] and to the qualities they saw in beautiful mathematics.[9]

The only hypothesis he discussed to explain the supposed connection between the mathematical beauty and the truth of a scientific theory was based upon *order*, which was one of Aristotle's forms of beauty and which was implicit in Hardy's discussion of pattern. The universe appeared to have an order that allowed intelligent life to arise; humans evolved in such a universe, and so human thought — which at minimum was closely *related* to the human brain — was able to grasp the order in the universe.[10] Osborne's unwritten conclusion

[1] Goodman, 'Mathematics as natural science', p. 188.

[2] Osborne, 'Mathematical beauty and physical science', p. 291.

[3] *Ibid.*, p. 292.

[4] *Ibid.*, pp. 291–2.

[5] Dirac, 'The Evolution of the Physicist's Picture of Nature', p. 47.

[*] ▸ pp. 906–7

[6] Osborne, 'Mathematical beauty and physical science', p. 292.

[7] *Ibid.*, p. 297.

[8] *Loc. cit.*

[†] ▸ pp. 89–91

[‡] ▸ pp. 498–509

[9] *Ibid.*, pp. 293–5.

[10] *Ibid.*, pp. 297–8.

seems to have been that humans thus had an aesthetic preference for the intrinsic order of the universe.

MARK STEINER

The philosopher Mark Steiner (1942–2020) argued in his 1998 book *The Applicability of Mathematics as a Philosophical Problem* that the solution to Wigner's problem was anthropocentrism, and the aesthetic component of the question was of particular importance in his argument.

Steiner reconstructed Wigner's argument schematically as follows:

(P1) 'Mathematical concepts arise from the aesthetic impulse in humans.'

(P2) 'It is unreasonable to expect that what arises from the aesthetic impulse in humans should be significantly effective in physics.'

(P3) 'Nevertheless, a significant number of these concepts are significantly effective in physics.'

(C) 'Hence, mathematical concepts are unreasonably effective in physics.'[1]

The conclusion (C) follows from the premisses (P1–P3). Therefore, in order to avoid the unreasonableness in the conclusion, Steiner had to reject at least one of the premisses.

Steiner maintained that mathematicians decided what was mathematical using internal criteria such as beauty and convenience.[2] For instance, a concept might be accepted into or excluded from mathematics depending on whether it was *convenient*, in the sense of easing deduction or calculation.[3] More important here is beauty: 'The *beauty* of the theory of a structure is a powerful reason to call it mathematical.'[4] Steiner quoted Hardy's dictum about there being no permanent place in the world for ugly mathematics.[5][*] Hardy pointed to *seriousness* as one of the criteria that delimited mathematics, at least as practised by those called mathematicians, but also indicated a connection between beauty and seriousness.[†] Thus, Steiner concluded, Hardy held the same position as he did vis-a-vis the demarcatory role of beauty.

Concepts such as convenience were anthropocentric: if human mental powers were much greater, the need for tools to ease deduction or calculation would be lesser. Beauty — at least as humans judge it — was anthropocentric. In Steiner's premisses (P1) and (P2), he implicitly characterized mathematical beauty in terms of the *human* aesthetic impulse. (Steiner here seems to have leaned towards the Diracian view that mathematical beauty, like other kinds of beauty, was indefinable.[6][‡]) Hardian judgements of seriousness were similarly anthropocentric.[7]

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[1] Steiner, *The Applicability of Mathematics*, p. 46.

[2] *Ibid.*, pp. 7, 64–6.

[3] *Ibid.*, pp. 7, 66–7.

[4] *Ibid.*, p. 7.

[5] *Ibid.*, p. 65.

[*] ▸ p. 508

[†] ▸ pp. 499–501

[6] Islami, 'On the Applicability of Mathematics in Physics', p. 65.

[‡] ▸ p. 904

[7] Steiner, *The Applicability of Mathematics*, pp. 7, 66.

Thus mathematics, as a collection of concepts and methods, was anthropocentric. It did not follow that any individual concept — for example, the notion of a group — was anthropocentric, but its inclusion within the scope of mathematics *was*.[1]

Steiner's examination led him to reject (P2) in favour of *anthropocentrism*, which, as he put it, was 'the teaching that the human race is in some way privileged, central to the scheme of things'.[2] With the assumption of anthropocentrism, there was no longer anything unreasonable about mathematics directed by 'the aesthetic impulse in humans'[3] being effective in physics.

What led Steiner to decide in favour of anthropocentrism? He argued that physicists formulated the laws of nature by mathematical analogy. Mathematical analogies had of course been used in past scientific inquiry, but Steiner noted an important distinction: modern scientists were forced to reach for mathematical analogies because there was no alternative. In particle physics especially, the scientist seemed to have no choice but to proceed solely by seeking mathematics that somehow resembled the laws they were trying to refine or replace.[4] Furthermore, the analogies used were what Steiner called *pythagorean*, meaning that they could only be expressed mathematically.[5]

Physicists used this pythagorean method to progress; mathematics, as Steiner argued, was shaped by anthropocentric judgements, including judgements of beauty. Thus in using mathematics the way they do, physicists were following an anthropocentric strategy. They were acting as though naturalism — which stood in opposition to anthropocentrism[6] — were false. By following this strategy, mathematics, and thus anthropocentric judgements of beauty and convenience and seriousness, had become embedded in the scientific enterprise. And yet science had continued to work, and to work well: the pythagorean strategy — this anthropocentric strategy — had proven to be successful.[7] And, without acceptance of anthropocentrism, this success was startling:

> 'An anthropocentric strategy for making a discovery is one which makes no sense unless the strategist believes, if only implicitly or unconsciously, that the human species has a special place in the scheme of things.'[8]

Steiner's claim was an empirical one: that anthropocentric concepts like human categories of beauty and convenience were necessary in the discovery of the fundamental theories of physics because of their role in shaping mathematics. Thus the universe appeared to be *user-friendly* (a term he ascribed to George N. Schlesinger; 1925–2013),[9] in the sense that these anthropocentric concepts were found in 'the "real essences" of things'.[10] Steiner's goal was a refutation of non-anthropocentrism, and, as he pointed out, his argument posed a challenge to philosophical naturalism.[11]

[1] Steiner, *The Applicability of Mathematics*, p. 7.

[2] *Ibid.*, p. 55.

[3] *Ibid.*, p. 46.

[4] *Ibid.*, p. 3.

[5] *Loc. cit.*

[6] *Ibid.*, p. 55.

[7] *Ibid.*, pp. 5, 72, 75.

[8] *Ibid.*, p. 5.

[9] *Ibid.*, pp. 8, 176.

[10] *Ibid.*, p. 8.

[11] *Ibid.*, p. 176.

JOHN POLKINGHORNE

John Polkinghorne (1930–2021) was a professor of mathematical physics until 1977, when he resigned in order to study for the priesthood. After being ordained in 1982, he wrote a series of books on the relationship between science and religion.

Polkinghorne thought that subjective experiences of beauty and goodness were just as significant as objective encounters with the physical world. Such experiences he saw as self-authenticating: while subjective and personal, they would be 'grotesquely diminished and distorted if treated as mere epiphenomena'.[1] He held that beauty in mathematics was easy for mathematicians to recognize and agree on. It was harder to describe, but he thought it associated with economy, elegance, and depth.[2] Depth he explained to be the existence of hidden consequences that were extensive, profound, and unexpectedly fruitful.[3]

Polkinghorne considered the intelligibility of the universe to be, *prima facie*, astonishing.[4] This intelligibility overlay an even deeper mystery, in that it was mathematics, and mathematics that was beautiful, that expressed fundamental physics. The order of the world was 'expressible in concise and elegant mathematical terms',[5] so much so that mathematical beauty had proved to be a successful guide in theory choice:[6]

> 'In fundamental physics it is an actual technique of discovery to look for equations that have about them the unmistakeable character of mathematical beauty. Time and again we have found that it is only equations possessing economy and elegance of this kind that will prove to be the basis for theories whose long-term fruitfulness [*sic*] convinces us that they are indeed verisimilitudinous descriptions of physical reality.'[7]

Mathematical beauty, therefore, was not just about the private satisfaction of the (perhaps subjective) aesthetic tastes of the mathematical elect, but was 'the opening of a window into the reality of the structure of the physical world'.[8] Some of the most beautiful structures in mathematics, studied primarily for aesthetic reasons, had later been found to correspond to fundamental physical reality.[9] Although Polkinghorne cited Wigner's lecture here, he seemed to place greater emphasis on the aesthetic component of the problem than Wigner did.

In brief, the universe was intelligible and the equations that govern it were beautiful. But this being the case was not an *a priori* truth.[10] What explanation might there be?

Polkinghorne rejected the idea that evolution offered an explanation, which Osborne had suggested.* Evolution had led to brains that could deal with the everyday experiences that our progenitors had faced. And, arguably, the same selection pressures that had led to hominid brains capable of surviving as groups of hunter-gatherers might

[1] Polkinghorne, *Belief in God*, pp. 125–6; cf. Polkinghorne, 'Mathematical Reality', p. 33.

[2] Polkinghorne, *Science and Theology*, p. 72; Polkinghorne, 'Mathematical Reality', pp. 32–3.

[3] Polkinghorne, *Science and Theology*, p. 72; Polkinghorne, 'Mathematical Reality', p. 33.

[4] Polkinghorne, *Science and Theology*, p. 72.

[5] Polkinghorne, *Belief in God*, p. 2.

[6] Polkinghorne, *Science and Theology*, pp. 72–3.

[7] Polkinghorne, *Science and the Trinity*, p. 63.

[8] Polkinghorne, *Belief in God*, p. 16.

[9] Polkinghorne, *Science and Theology*, p. 73.

[10] Polkinghorne, *Belief in God*, p. 4.

* ▸ pp. 934–5

have favoured brains capable of basic arithmetical and geometrical reasoning, and of an enormous degree of flexibility that could be harnessed for purposes not related to survival, such as the development of mathematics or science.[1] But human brains could fathom quantum mechanics and general relativity, which concerned concepts that were unimaginably far removed from the experiences that drove evolution, so an explanation that treated this ability as a happy side-effect was hardly adequate.[2] Likewise, Polkinghorne rejected the position that humans simply *like* to use mathematics and thus constructed or shaped their accounts of nature to fit a mathematical mould.[3]

For Polkinghorne, such evolutionary or sociological views were simply attempts to explain away, rather than to explain, the correspondence between beautiful mathematics and reality.[4] Naturalistic thinking could do no more than accept this correspondence as 'just the way it is', but Polkinghorne felt that it indicated something profound.[5] He suggested that a religious view could offer an answer where a purely naturalistic perspective could not: if we and the universe were both created by God, then there was a common origin of our minds and the world we inhabit. This account would explain the correlation between human reason and aesthetic taste and the rational structure of the universe.[6] In terms of Christian doctrine,

> 'our scientific ability to explore the rational beauty of the universe is seen to be part of the Father's gift of the *imago Dei* to humankind, and the beautiful rational order of the universe is the imprint of the divine Logos'.[7]

RUSSELL HOWELL

In a 2006 essay, the mathematician Russell W. Howell (1947–) listed three kinds of naturalistic explanation for the role of mathematical beauty in physical science. First, there might be some mechanism that influenced evolution so that human perceptions of beauty and related values were compatible with the laws of the physical world. Second, it just so happened that the universe was structured so that our perceptions are so compatible. Third, that science, formulated using mathematics that humans found beautiful, had been successful in explaining the world because of the effort that had been invested therein over the preceding centuries; if this effort had been directed elsewhere, there would have emerged a different but perhaps equally successful paradigm for understanding the universe.[8]

Howell thought none of the three kinds of explanation was satisfactory. Polkinghorne had rejected the evolutionary explanation on different grounds,* but Howell thought it was inadequate because there was no evidence for such a mechanism influencing evolution in

[1] Polkinghorne, 'Mathematical Reality', p. 32.

[2] Polkinghorne, *Science and Theology*, p. 72; cf. Polkinghorne, *Belief in God*, pp. 2–3.

[3] Polkinghorne, *Belief in God*, p. 3.

[4] *Ibid.*, p. 2.

[5] Polkinghorne, *Science and the Trinity*, pp. 64–5.

[6] *Ibid.*, p. 65.

[7] *Loc. cit.*

[8] Howell, 'Does Mathematical Beauty Pose Problems for Naturalism?', p. 10.

* ▸ pp. 937–8

that way, and even if there were, it led to the question of why such a mechanism would favour humans. The second 'just-so' explanation Howell considered to be a rabbit-from-a-hat explanation that in effect dismissed all argument. The third explanation also had the flaw of leaving unspecified the alternative way of understanding, and also did not take into account the parallel mathematical development in different cultures such as ancient China and Greece.[1]

For a theist, the correlation between human notions of beauty and the physical theories of the world reinforced 'the belief that human creation is the result of the purposive activity of an intelligent being'.[2] Specifically, it fitted with the Christian view that humans were created in God's image:[3] humans perceived beauty in the same things that God did.

Howell admitted that his conclusions were tentative and made no pretence at a complete analysis.[4] In particular, his argument did not take into account the influence of the physical sciences on the development of mathematics, nor on the vast areas of beautiful mathematics that have *not* found application in the natural sciences.

There remain further problems: why would God perceive as beautiful mathematics that happened to be describe a universe in which intelligent life could develop? Why should an omnicogniscent (or at least vastly superior) intellect make the same judgements of beauty as humans?

FRANK WILCZEK

Frank Wilczek, who shared in the 2004 Nobel Prize in physics, published two essays in November 2006 and May 2007 commenting on Wigner's problem. Only the first addressed aesthetic concerns and so the second is not discussed here.

Wilczek pointed out that the question of why nature seemed to be governed by mathematically beautiful laws could be illuminated by asking whether the opposite could hold.[5] That is, were there conceivable worlds that were *not* based on such laws? Yes: aristotelian physics or the physics models of various computer games supplied examples. There thus seemed to be something special about the actual world.[6] Wilczek thought that the most important special qualities of the laws governing the real world were *symmetry* and *locality*. It was symmetry that was responsible for the invariance of physical law under changes in temporal and spatial position. Locality, on the other hand, allowed a mathematical description of nature to be assembled from simple interactions among individual components. Together these qualities implied physical laws that had a form that could be expressed naturally using certain kinds of differential equations.[7] Wilczek accepted that there was a near-circularity in his argument:

[1] Howell, 'Does Mathematical Beauty Pose Problems for Naturalism?', p. 10.

[2] *Loc. cit.*

[3] *Ibid.*, p. 11.

[4] *Ibid.*, p. 10.

[5] Wilczek, 'Reasonably effective I', p. 9.

[6] *Loc. cit.*

[7] *Loc. cit.*

nature was describable in beautiful mathematics because it obeyed the mathematically beautiful properties of symmetry and locality.[1] Asking why the world should have these properties, Wilczek admitting that 'we pass, necessarily, from reason to faith': experience seemed to show that they hold.[2]

Wilczek also thought that the areas that physicists choose to research, and which are thus foci of scientific success, are those that are amenable to mathematical treatment, and hinted at an aesthetic element in this choice. He gave an explicit example of such a topic: the behaviour of very pure semiconductors in very strong magnetic fields at very low temperature, the impulse for which is that 'a rich and beautiful mathematical theory of the quantum Hall effect'.[3]

SORIN BANGU

The philosopher of science Sorin Bangu used the term *pythagoreanism* (which he capitalized) to denote the doctrine that 'mathematics enjoys a privileged relation with respect to the physical world',[4] and distinguished *ontological pythagoreanism*, which was the position that at a fundamental level the physical universe consists of mathematical objects, and *epistemological pythagoreanism*, which held that mathematics was not only the symbolic description of physical theories, but played a role in constructing, explaining, and extrapolating from those theories.[5] The former would of course explain the latter, but Bangu's focus was on epistemological pythagoreanism, which was accepted, tacitly and virtually without question, in the sciences.

Bangu gave historical examples that he thought counted against the thesis that mathematical beauty was a guide to truth in physics. His chronologically earliest example was the preference for circles in cosmological systems before Johannes Kepler's* (1571–1630) revolution.[6] Some of the more modern examples he gave are less convincing. His first modern example was Niels Bohr's (1885–1962) contribution to fundamental physics, because Bohr sought physical understanding first rather than seeking mathematical beauty.[7] But all this example shows is that the path of mathematical beauty is not the only route to the destination of physical truth; it does not indicate that that path leads elsewhere. Bangu's next example was based on Wigner's account[8] of Dirac's theoretical progress being impeded by his being a 'captive' of the Hamiltonian formalism.[9] Even assuming Wigner's diagnosis was correct, Dirac's preference for this formalism does not necessarily indicate that he worked solely with it for aesthetic reasons.

Bangu's most solid modern example was Dirac's supposed prediction of magnetic monopoles in a celebrated 1931 paper 'which ends with the prophecy that given the beauty of the theory predicting the existence of monopoles, “one would be surprised if Nature had made no use of it”[1]’.[2] Monopoles have never been observed in experiment. (The one possible observation on 14 February 1982 is now believed to have been unlikely to be an actual monopole.[3]) This episode is a better modern example than Bangu’s first two, but is not entirely unproblematic. Dirac seems never to have stated that mathematical beauty was involved in the reasoning that led to his prediction of the existence of magnetic monopoles. Indeed, Dirac’s explicitly stated aim in the 1931 paper was ‘to show that quantum mechanics does not really preclude the existence of isolated magnetic poles’;[4] this conclusion was as far as pure mathematics could take him.[5] That is, he did not have to follow his aesthetic sense in devising a new, mathematically beautiful, theory; he simply showed that the existing theory could accommodate monopoles. Predicting physically existing and experimentally detectable magnetic monopoles required an implicit additional *principle of plenitude*; loosely, that *possibility* — at least in terms of the mathematical formulation of quantum mechanics — implies *existence*.[6] Dirac’s monopole prediction therefore had a different foundation from his postulation of the positron,[*] which was a direct consequence of his own aesthetically-guided formulation of his wave equation.

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[1] Wilczek, 'Reasonably effective I', p. 9.

[2] *Loc. cit.*

[3] *Ibid.*, p. 8.

[4] Bangu, 'On the Pythagoreanism of Modern Physics', p. 34.

[5] *Ibid.*, pp. 34–5.

* ▸ pp. 279–94

[6] Bangu, 'Pythagorean Heuristic', pp. 404–5. The argument of this paper is also found in Bangu, 'On the Pythagoreanism of Modern Physics'.

[7] Bangu, 'Pythagorean Heuristic', p. 405.

[8] Kuhn, 'Interview Wigner, III'.

[9] Bangu, 'Pythagorean Heuristic', pp. 405–6.

[A very explicit failure of mathematical beauty as a guide would be Chen-Ning Yang (1922–2025) and Tsung-Dao Lee’s (1926–2024) failure to formulate a quaternion-based field theory. They openly stated that their attempt to do so was based on a faith that nature would involve the beautiful structure of the quaternion algebra.[†]]

Bangu made the observation that there was nothing problematic in the coincidence that *in the past* mathematics developed for aesthetic reasons — or at least independently of reference to the natural world — should happen to find application in the sciences. What was *prima facie* problematic was the *expectation* that this coincidence should continue to hold. This expectation seemed to endorse an anthropocentrism that most modern scientists and mathematicians would scorn. Bangu put the accepted anti-anthropocentric position of science bluntly: ‘The universe of the modern physicist couldn’t care less about what humans (in particular, mathematicians) like or dislike.’[7] But Bangu’s conclusion that mathematical beauty was at best of questionable helpfulness in formulating physical theories meant that he was able to undermine the premiss of the argument that authors such as Polkinghorne[‡] and Howell[§] had made: that the presence of anthropocentric beauty in fundamental physics led to a theistic conclusion. For Bangu, mathematical beauty instead simply turned out to be a worthwhile strategy in fundamental physics when no other approach presented itself as credible, such as when physics around the end of the nineteenth century had struggled to find a way to incorporate experimental results that clashed with the pre-relativistic world-view.[8]

[1] Dirac, ‘Quantised Singularities’, p. 71.

[2] Bangu, ‘Pythagorean Heuristic’, p. 406.

[3] Brumfiel, ‘The waiting game’.

[4] Dirac, ‘Quantised Singularities’, p. 71.

[5] Kragh, ‘The Concept of the Monopole’, p. 145.

[6] *Ibid.*, pp. 147–9.

[*] ▸ p. 903

[†] ▸ pp. 917–18

[7] Bangu, ‘Wigner’s Puzzle for Mathematical Naturalism’, p. 257.

[‡] ▸ pp. 937–8

[§] ▸ pp. 938–9

[8] Bangu, ‘Pythagorean Heuristic’, pp. 409–10.

JESPER LÜTZEN

Jesper Lützen

In 2011, the historian of science Jesper Lützen (1951–) published an essay that critiqued the very premiss of the aesthetic dimension of Wigner's problem.

Starting from Wigner's characterization of mathematics as 'the science of skillful operations with concepts and rules invented just for this purpose',[1] Lützen concluded that Wigner held to a formalist philosophy of mathematics, in which mathematics dealt with axioms and objects that were defined implicitly through the axioms.[2] Wigner's view would then be that the fundamental concepts are supplied by nature, but that the higher-level axiom systems derived from that source were inspired by 'formal beauty'.[3] Lützen pointed out that this account was not an accurate portrayal of how mathematics had in fact developed: axiomatization, when it happened, came late. For instance, when Euclid* (*fl.* c. 300 BCE–) axiomatized geometry (in an incomplete way, by anachronistically modern standards), he formulated the parallel postulate in order to prove already-known results like the pythagorean theorem within his systematized *Elements*. Similarly, David Hilbert (1862–1943) postulated completeness as an axiom of the real numbers after it had emerged from nineteenth-century efforts to clarify the concept of continuity. Thus the more logically fundamental notions — the axioms — were formulated after the less fundamental ones — the theorems that the axioms were intended to ground.[4] In a sense, as Lützen put it, 'axioms are often the results of theorems rather than the other way round'.[5]

Lützen agreed that some areas of mathematics, such as number theory, had developed conceptually in essentially complete independence from the world. But the areas of mathematics that had been most applicable to the world have been shaped by applications.[6] The development of non-euclidean geometry† was, at least for some of its pioneers, an investigation of the geometry of the physical universe: Nikolai Ivanovich Lobachevsky (1792–1856) examined the parallax of certain stars for evidence that the universe was non-euclidean.[7] Lobachevsky did not consider geometry to be an axiomatic subject, but an empirical one.[8] William Rowan Hamilton's (1805–65) discovery of the quaternions‡ was motivated by a desire to extend the complex number plane to three dimensions, to serve physics.[9] The calculus and subsequent developments in analysis walked hand-in-hand with mechanical investigations.[10] Such examples are not compatible with Wigner's idea that the conceptual development of higher mathematics was guided by formal beauty. Indeed, of Wigner's four explicit examples of concepts developed for their beauty — complex numbers, algebras, linear operators, and Borel sets[11] — all but complex numbers had been influenced by physical problems.[12] Lützen conceded that the complex numbers did have a purely mathematical origin, and that there might be something 'miraculous' about their applicability, but he located the miracle *within* mathematics, in the connection of the com-

[1] Wigner, 'The Unreasonable Effectiveness of Mathematics', p. 2.

[2] Lützen, 'The Physical Origin of Physically Useful Mathematics', pp. 230–1.

[3] *Ibid.*, pp. 233–4.

* ▸ pp. 95–103

[4] *Ibid.*, pp. 234–5.

[5] *Ibid.*, p. 235.

[6] *Ibid.*, p. 236.

† ▸ pp. 427–32

[7] Gray, *Ideas of Space*, pp. 122–3.

[8] Lützen, 'The Physical Origin of Physically Useful Mathematics', p. 238.

‡ ▸ pp. 419–20

[9] *Loc. cit.*

[10] *Ibid.*, p. 239.

[11] See Wigner, 'The Unreasonable Effectiveness of Mathematics', p. 3.

[12] Lützen, 'The Physical Origin of Physically Useful Mathematics', pp. 240–1.

plex exponential and trigonometric functions, not at the interface of mathematics and the world. For Lützen, the applicability of complex numbers to quantum mechanics added little to the miracle.[1] [Derek Abbott (1960–), by contrast, appears to have seen no such magic in the connection of the exponential and trigonometric; writing in 2013 that 'Euler's remarkable formula $e^{j\pi} = -1$ is somewhat demystified once one realizes it merely states that a rotation by π radians is simply a reflection or multiplication by −1'.[2]]

Lützen thus saw the 'miracle' of the applicability to physics of mathematical concepts developed for aesthetic reasons as a mirage: the concepts were not developed for their beauty (or at least not *only* for their beauty), but were influenced by physics. It was thus rather less mysterious that they should find application there.

AREZOO ISLAMI

The philosopher of science and mathematics Arezoo Islami argued that Wigner's lecture had been at least partly misunderstood, and that it contained an argument for 'the *limited* and *reasonable* effectiveness of mathematics in physics'.[3] Wigner, she pointed out, had argued that observation and experience led to the view that the laws of nature are subject to *invariance principles*. It was only through such invariance that the laws of nature could be expressed mathematically. But such invariance was a *contingent* aspect of the world. And thus Wigner's term for 'the appropriateness and accuracy of the mathematical formulation of the laws of nature in terms of concepts chosen for their manipulability':[4] the *empirical* law of epistemology.[5] The misconception came about when this law was construed as *a priori*: only from that perspective did it seem that nature had to be somehow mathematical.[6]

Islami made a number of important points about aesthetics arguments present in Wigner's lecture and in its later reception. There was an inconsistency in Wigner's use of the complex numbers as an illustration of how pure mathematics whose development was driven by aesthetic concerns found its place in physics. The professional mathematician might, as Wigner thought, defend the worth of complex numbers using the beauty of the theorems that concern them. But Islami pointed out that, as a matter of historiographical fact, the formulation and acceptance of the concept of complex numbers was an involved and protracted affair, driven by a variety of factors, including problems both internal to mathematics and originating in the sciences.[7] Wigner's citation of the aesthetic judgement of later mathematicians was not relevant to the motives that drove the development of complex numbers.

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[1] Lützen, 'The Physical Origin of Physically Useful Mathematics', p. 241.

[2] Abbott, 'The Reasonable Ineffectiveness of Mathematics', p. 2148.

[3] Islami, 'A match not made in heaven', p. 4841, emphases in orig. The argument of this paper is also found in Islami, 'On the Applicability of Mathematics in Physics'.

[4] Wigner, 'The Unreasonable Effectiveness of Mathematics', p. 10.

[5] *Ibid.*, pp. 4858–9.

[6] *Ibid.*, p. 4859.

[7] Islami, 'On the Applicability of Mathematics in Physics', ch. 5, *passim*.

Islami also thought that Steiner* had erred in his interpretation of Wigner by characterizing mathematical beauty in terms of the human aesthetic impulse. Wigner had rather understood beauty in mathematics with relation to values such as generality, simplicity, and manipulability, and it was conceivable that these notions might admit of rigorous definition. Islami thus held that beauty, for Wigner, was not an unanalysable concept emerging from the human aesthetic impulse.[1] Hence Steiner's premiss (P1) was stronger than Wigner would have admitted. Training was required to make accessible much of the intellectual beauty of the higher mathematics, and even if the development of a concept were motivated by an aesthetic impulse, its acceptance into the body of mathematics depended on how it fitted into existing mathematics. Did the concept enable simpler or more general proofs? Did it leads to beautiful results? (And the beauty of such results seemed to have *something* to do with the logical context in which they sat.) Thus, said Islami, mathematical beauty might be less subjective than Steiner had implied, which would undermine Steiner's conclusion that anthropocentrism was the solution to the applicability problem.[2]

JANE MCDONNELL

In a 2016 paper and in her 2017 book *The Pythagorean World*, the philosopher Jane McDonnell made a careful analysis and response to Wigner's problem.

She first dismissed the bare claim that mathematics was unreasonably effective in the natural sciences: first, mathematics was not effective in all physical situations, such as chaotic and non-linear systems; second, there was an abundance of mathematics, so it was not unreasonable that a sufficiently tenacious physicist could locate a mathematical tool that fitted their purpose; third, mathematics and physics had developed together as humans developed an understanding of the world that was sufficiently advanced to make precise predictions.[3]

Further, McDonnell thought that the narrower assertion that there was something mysterious about the applicability of pure mathematics — that is, mathematics that was developed for its aesthetic value and other concerns separate from empirical considerations — was questionable. She pointed out that close conceptual parallels between mathematics and physics had been maintained as both subjects developed. Furthermore, the fundamental concepts such as number, arithmetical operations, and symmetry were based upon empirical experience.[4]

There was a still narrower claim that the role of pure mathematics in formulating the laws of nature was mysterious. Physicists thought in mathematical terms, and often found that when they had developed what was to them a new mathematical concept, it turned out to have been formulated already in mathematics.[1] In particular, it seemed mysterious that *deep* mathematics, far divorced from the fundamental concepts derived from immediate experience, was needed to describe fundamental physics.[2] The putative mysteries in these claims — the role of pure mathematics, and in particular *deep* mathematics in formulating physical theories — were, McDonnell thought, 'genuinely puzzling'.[3]

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* ▸ pp. 935–6

[1] Islami, 'On the Applicability of Mathematics in Physics', p. 65.

[2] *Ibid.*, pp. 65–6.

[3] McDonnell, 'Wigner's puzzle and the Pythagorean heuristic', pp. 2933–4.

[4] *Ibid.*, pp. 2934–5.

McDonnell distinguished two main mathematical strategies for developing physical theories. The first she termed 'pragmatic':

> 'Use pure mathematical analogies to extrapolate from mathematically-characterised systems to uncharacterised systems; test the boundaries but keep close to experimentally verifiable phenomena; develop theoretical and experimental tools in tandem to discover new phenomena.'[4]

The second she called 'more Pythagorean':

> 'Broaden and deepen the conceptual basis of mathematically beautiful theories to eliminate the known ugly bits and to encompass more phenomena; focus on developing a unified basis for all of physics; resolve the theoretical contradictions whilst retaining beautiful features.'[5]

This distinction was very close to being an echo of the dichotomy Dirac described between experimental or mathematical concerns directing theoretical physics.[*] Just as Dirac thought, direction by mathematical concerns entailed the seeking of mathematically beautiful theories; McDonnell said that the 'more Pythagorean' approach indicated 'the belief that new physics will always be expressible in terms of deep, beautiful mathematics'.[6]

McDonnell disagreed with Bangu's conclusion that the pythagorean heuristic had only coincidental success;[†] the development of any heuristic would involve some failures, but, as with James W. McAllister's (1962–) aesthetic induction,[‡] those failures could be used to refine the heuristic.[7] For McDonnell, the pythagorean approach to fundamental physics required only that the beauty was revealed at high energies that reflected the conditions of the early universe.[8] There was a layered hierarchy of laws of nature, each higher-energy layer unifying some of the laws of lower-energy layers. At high enough energies, the theories were mathematically beautiful and exhibited a high degree of symmetry.[9] As the universe cooled, the symmetries became broken,[§] leaving only 'less perfect remnants in the everyday world'.[10] McDonnell called this view 'a version of Pythagorean metaphysics'.[11] Such a pythagorean metaphysics stopped short of a Tegmarkian mathematical universe,[❡] but still gave a complete answer to Wigner's problem. It explained both why deep and beautiful mathematics worked in physics, especially at the fundamental level, and

[1] McDonnell, 'Wigner's puzzle and the Pythagorean heuristic', pp. 2935–6.

[2] *Ibid.*, p. 2936.

[3] *Ibid.*, p. 2937.

[4] *Ibid.*, p. 2940.

[5] *Loc. cit.*

[*] ▸ p. 909

[6] *Loc. cit.*

[†] ▸ pp. 940–1

[‡] ▸ pp. 793–5

[7] *Ibid.*, p. 2942.

[8] *Loc. cit.*

[9] *Ibid.*, p. 2932.

[§] ▸ p. 951

[10] *Ibid.*, p. 2942.

[11] *Ibid.*, p. 2932.

[❡] ▸ p. 933

why human mathematicians, with an aesthetic sense that had been shaped by the everyday world, had developed mathematics that fitted a physical reality where everyday expectations broke down.[1]

◆

[1] McDonnell, 'Wigner's puzzle and the Pythagorean heuristic', p. 2932.

25

A Dangerous Concept

Mathematical beauty and the search for a final theory

> ‘ Beauty is a dangerous concept,
> because it can always mislead […].
> In your eyes a theory might be beautiful,
> but it might just be wrong.
> There’s nothing you can do about it. ’
>
> — Gerardus ’t Hooft,
> quot. in Hossenfelder, *Lost in Math*, ch. 1.

> ‘ his new procedure
> for calculating particle masses from first principles
> […] was beautiful, elegant, brilliant, and wrong. ’
>
> — John Cramer, *Einstein’s Bridge*, ch. 1.4.

Fundamental physics in the late twentieth and early twenty-first centuries openly and deliberately aimed at finding a final theory that would complete and replace both the standard model and the general theory of relativity. The standard model deals with the small scale: it classifies the known elementary particles and describes the fundamental interactions of nature except for gravitation. The general theory of relativity deals with the large scale, and describes gravitation. Both theories are empirically highly successful. But physics, and science more generally, reaches ever toward a unified theory from which the specialized theories of different domains can be deduced as particular cases.

There was confidence that there was a realistic chance of achieving this goal, with string theory giving at minimum an outline of the final theory of physics. But a countervailing sense of crisis emerged, produced in large part by a separation of theoretical and experimental physics. Until the late twentieth century, physics was driven and checked by empirical data. But the equipment needed to test theories grew ever more complex and expensive. Theoretical physicists ran far ahead of what could be empirically tested. The Higgs boson was theorized in 1964, but was only detected using the Large Hadron Collider in the early 2010s.

This separation increased to the degree that it started to be acknowledged that it might never be economically or technically feasible to test candidates for the final theory. The philosopher of science Richard Dawid (1966–) argued that fundamental physics had changed, not just in terms of the theories it formulated, but on the meta-theoretic level of how those theories were assessed. There had been a shift in the balance between the empirical and abstract aspects of theory evaluation.[1] This shift was acknowledged and sometimes heavily criticized. By 1987 the term 'postmodern physics' was in circulation for theoretical physics that could not be confronted with experiment. [The term has sometimes been ascribed to Yoichiro Nambu (1921–2015),[2] but it is not to be found in the alleged 1985 source.[3]] The science writer John Horgan (1953–) coined the term 'ironic science' for the practice of developing empirically untestable theories.[4]

This chapter studies the role of mathematical beauty in this age of physics. Theories that cannot be experimentally tested at present — and perhaps never will be testable — started to be evaluated largely in terms of the value of their mathematical formulation, including the elegance, simplicity, and beauty of that formulation. Some theoretical physicists extolled the role of the mathematical beauty of candidates for the final theory; others disparaged it and the theories in support of which it was cited. Others simply acknowledged the beauty of the candidate theories as exercises in pure mathematics, regardless of whether they reflected reality. As Freeman Dyson (1923–2020) said in a 1987 retrospection on mathematics connected with the work of Srinivasa Ramanujan (1887–1920):

> 'Whether or not superstring theory is a true image of nature, it is certainly a magnificent creation of pure mathematics. As pure mathematics, it is as beautiful as any of the other flowers that grew from the seeds that ripened in Ramanujan's garden.'[5]

CONCEPTS

To set in context the history examined in this chapter, it is necessary to recapitulate briefly some of the concepts that arise, such as the *standard model*, *symmetry breaking*, *supersymmetry*, and *string theory*. The discourse about beauty only very rarely engaged with the mathematical detail, so a very high-level discussion, eliding a great deal, will suffice.

Standard model

In the 1920s, the known elementary particles were the proton, the electron, and the photon. In the late 1920s and 1930s came the predictions and experimental detection of the positron and the neutron and the positing of the neutrino. As theory and detection

[1] Dawid, *String Theory and the Scientific Method*.

[2] Friedan, 'A New Formulation of String Theory', p. 78; Peat, *Superstrings*, p. 276.

[3] Nambu, 'Directions of Particle Physics'.

[4] Horgan, *The End of Science*, pp. 7–8.

[5] Dyson, 'A Walk Through Ramanujan's Garden', p. 19.

techniques developed, particles proliferated, until in 1961 Chen-Ning Yang (1922–) could list 26 experimentally known particles and another four predicted to exist.[1] While the particles could be classified into families of baryons, anti-baryons, leptons, anti-leptons, and 'bosons' (including mesons and thus not corresponding to today's notion of bosons), these divisions merely constituted a taxonomy: there was no underlying structure that explained the classification. According to Leon Lederman (1922–2018), Enrico Fermi (1901–54) said: 'If I could remember the names of these particles, I would have been a botanist.'[2]

Murray Gell-Mann* (1929–2019) and George Zweig (1937–) independently posited the entities that would supply the necessary system in 1963. Gell-Mann called them *quarks*; Zweig called them *aces*. Gell-Mann's term became the standard, as did his names for the three quarks of which the then-known baryons were composed: *up*, *down*, and *strange*. But, at the time, Gell-Mann viewed quarks as convenient mathematical abstractions, while Zweig thought of them as existing physical objects.[3] Only later did experimental evidence emerge for their reality.[4]

The SU(3) symmetries that Gell-Mann and Yuval Ne'eman (1925–2006) had found among baryons and mesons and had used to predict the Ω^- baryon† was thus explained with reference to more fundamental particles.[5] (See the quark composition of baryons and the quark/anti-quark composition of mesons shown in Figures 23.3, 23.4, and 23.5.) This episode marked the rise of what became known as the 'standard model'.[6]

To summarize, the *standard model* of particle physics classifies all the known elementary particles and describes three of the four fundamental interactions of nature. The elementary particles comprise the *fermions* and *anti-fermions*, which have spin 1/2, and the *bosons*, which have spin 1. The fermions and the anti-fermions interact by exchanging bosons. (The chart in Figure 25.1 lists the fermions and bosons.) The fermions divide into the *quarks* and *leptons*, each of which divides into three generations of successively more massive particles. Among the leptons is the electron, the earliest-discovered particle in the standard model. The proton comprises two up quarks and one down quark; the neutron comprises one up quark and two down quarks. Mesons are made up of a quark and an anti-quark. The bosons are divided into the gauge bosons and the Higgs boson, which is the unique scalar boson. The gauge bosons comprise the photon, which mediates the electromagnetic interaction; the gluon, which mediates the strong interaction, confining quarks into protons, neutrons, and other particles; and the W and Z bosons, which mediate the weak interaction, involved in radioactive decay, fission, and fusion. The Higgs boson gives mass to the gauge bosons and fermions. The standard model does not describe gravitation, which is vastly weaker than the other three interactions at the relevant scales. [Each quark

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[1] Yang, *Elementary Particles*, Fig. 18.

[2] Fermi, quot. in Lederman, 'Neutrino Physics', p. 1.

* ▸ pp. 955–7

[3] Riordan, *The Hunting of the Quark*, pp. 105–8.

[4] *Ibid.*, chs 8, 13.

† ▸ p. 924

[5] *Ibid.*, p. 106.

[6] Brown, Riordan et al., 'The Rise of the Standard Model', p. 11.

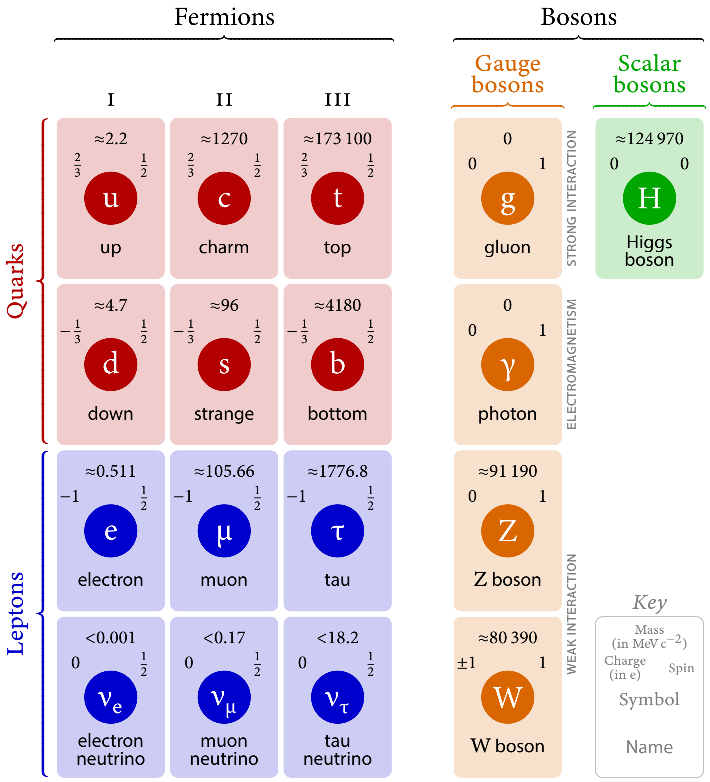
Fermions
Bosons
I
II
III
Gauge bosons
Scalar bosons
Quarks
Leptons
≈2.2
2/3
1/2
u
up
≈1270
c
charm
≈173 100
t
top
≈4.7
−1/3
d
down
≈96
s
strange
≈4180
b
bottom
≈0.511
−1
e
electron
≈105.66
μ
muon
≈1776.8
τ
tau
<0.001
0
νe
electron neutrino
<0.17
νμ
muon neutrino
<18.2
ντ
tau neutrino
0
1
g
gluon
STRONG INTERACTION
γ
photon
ELECTROMAGNETISM
≈91 190
Z
Z boson
≈80 390
±1
W
W boson
WEAK INTERACTION
≈124 970
H
Higgs boson
Key
Mass (in MeV c⁻²)
Charge (in e)
Spin
Symbol
Name

FIGURE 25.1 The particles of the standard model. The elementary fermions, made up of three generations of quarks and three generations of leptons, and the bosons, made up of the gauge and scalar bosons. Each fermion has a corresponding anti-particle with the same mass and spin and opposite charge (not shown).

can have one of three *colour charges*, which gives rise to eight different gluons.]

The gauge bosons are subject to gauge symmetry; loosely speaking, this refers to invariance under certain transformations. The overall gauge group is SU(3) × SU(2) × U(1). The first factor, SU(3), describes the gauge symmetry of the gluons. (Although this is the same group — up to isomorphism — as the one Gell-Mann and Ne'eman used and which led to the discovery of quarks, it describes here a different symmetry.) The second and third factors, SU(2) × U(1), describe the gauge symmetry of the unified electromagnetic and weak interactions.

The standard model has been highly empirically successful, predicting the existence and masses of the W and Z bosons and of the charm quark, and approximate masses for the top quark and Higgs boson masses. Together with the general theory of relativity, it accounts for most of the observed phenomena in nature.[1] The combined theories are certainly not the whole story: for instance, neither theory explains 'dark matter', whose existence has been posited to explain large-scale phenomena that suggest that the universe contains more matter than can be observed.[2]

But there is another class of objections that have little to do with the empirical adequacy of the combined standard model and general theory of relativity. Broadly, these objections concern the arbitrariness of the standard model. It contains too many unexplained numbers, such as particle masses: a true final theory should explain *why* the particles have their observed relative masses. It is unparsimonious: all matter that exists under ordinary conditions is built up out of the first generation of fermions (the up and down quarks, the electron, and the electron neutrino): a true final theory should explain why the other generations exist.[1] Such objections have an aesthetic component, and this has fed into the way in which a final theory has been sought.[2]

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[1] Langacker, *The Standard Model and Beyond*, p. 425; Dodd & Gripaios, *The Ideas of Particle Physics*, pt x.

[2] Langacker, *The Standard Model and Beyond*, p. 425; see also Dodd & Gripaios, *The Ideas of Particle Physics*, pp. 211–12.

Symmetry breaking

Above the so-called *unification energy*, the electromagnetic and weak interactions are known to merge into a unified *electroweak* interaction, which has the gauge symmetry SU(2) × U(1). Very early in the history of the universe, as temperatures fell, the electroweak interaction divided into the separate electromagnetic and weak interactions and this symmetry was thus 'broken'.

This idea that two fundamental interactions were once unified leads to the question of whether the gauge symmetry of the standard model, SU(3) × SU(2) × U(1), is in fact a broken version of some greater symmetry; perhaps this greater symmetry is a single group, not a product. Theories that posit such a greater symmetry are called *grand unified theories*.[3] The first such theory was put forward by Howard Georgi (1947–) & Sheldon Glashow (1932–) in 1974, who presented 'a series of hypotheses and speculations leading inescapably to the conclusion that SU(5) is the gauge group of the world'.[4] [In 1994, Georgi called SU(5) 'a beautiful little jewel'.[5]]

More generally, there recurs the idea that any partial symmetry or asymmetry observable in the world — or at least in particle physics — is the remnant of some broken higher symmetry.

Supersymmetry

The standard model groups the elementary particles into fermions, with half-integer spin, and bosons, with integer spin. Supersymmetry posits a symmetry between these classes of particles, in which each fermion has a corresponding bosonic *superpartner*, and each boson has a corresponding fermionic superpartner. None of the known fermions and bosons can form a superpartner pair, so supersymmetry would double the number of elementary particles. The symmetry between fermions and bosons may not be a perfect unbroken symmetry; if it were, each particle and its superpartner would have equal mass and other properties besides spin. But many supersymmetric theories posit a broken symmetry, where a particle and its superpartner can have different mass.

There were various motivations for the theoretical formulation of supersymmetric theories, but the concept could have been viewed as

[1] Langacker, *The Standard Model and Beyond*, pp. 425–6; Zee, *Fearful Symmetry*, p. 259.

[2] See Hossenfelder, *Lost in Math*, ch. 4.

[3] Dodd & Gripaios, *The Ideas of Particle Physics*, pp. 220–1.

[4] Georgi & Glashow, 'Unity of All Elementary-Particle Forces', p. 438.

[5] Georgi, 'Grand Unified Theories', p. 604.

'a solution in search of a problem' until the early 1980s, when it started to be realized that it could be used to resolve other problems.[1] One such problem was the naturalness — or lack thereof — of the Higgs boson.

The term *natural* has a precise interpretation in physics that was formulated by Gerardus 't Hooft (1946–) in what he termed a 'dogma'. Per this 'dogma', in a natural theory, a parameter of a theory being close to zero (relative to some mass scale) was not an accident; it was the result of a near symmetry. If, in a theory, setting such a parameter to zero would increase the symmetry of the theory, then the theory was natural.[2] The Higgs boson turned out to be unnatural in the sense that it failed to satisfy this dogma.[3] Around 1980 this problem was clear, and it was pointed out that supersymmetry, which was already in development, could offer a solution.[4] The general idea of introducing extra symmetries to make a theory more natural — either in 't Hooft's or in some other sense — is one that recurs. Note that the purported increase in naturalness comes at the expense of a less parsimonious ontology.[5]

Supersymmetry has consequences that have seemed to suggest its validity. In 1991, Ugo Amaldi (1934–), Wim De Boer (1948–2020) & Hermann Fürstenau examined the theoretical extension to higher energies of the measured electromagnetic, weak, and strong coupling constants. Grand unification predicts that these theoretical parameters will become equal. In the context of the non-supersymmetric standard model, there was no predicted single unification point; in the supersymmetric standard model, there was a predicted unification at $10^{16.0\pm0.3}$ GeV.[6]

Regardless of all the consequences of supersymmetry that might make one wish it to be true, and regardless of the almost universal praise for its beauty,[7] there is an important empirical caveat: *no superpartner has ever been experimentally observed*.

String theory

The entities of string theory are one-dimensional systems called *strings* that move in space-time and which give rise to the phenomena that previous theories had described in terms of point-like particles. The theory is remarkably parsimonious, in that it only posits strings of only one kind (in contrast to the multiple species of particle that inhabit the standard model), and thus there is only one free parameter, namely the coupling constant that describes the strength of the strings' interaction.[8]

Early versions of string theory predicted tachyons, particles that move faster than light. But certain supersymmetric string theories, now called *superstring* theories, did not have this problem, and thus became the focus. A consistent string theory requires that space-time have ten or twenty-six dimensions, rather than the observed four.

[1] Kane & Shifman, 'Foreword', p. vii.

[2] 't Hooft, 'Naturalness, Chiral Symmetry, and Spontaneous Chiral Symmetry Breaking', p. 136.

[3] Giudice, 'Naturally Speaking', pp. 163–4.

[4] *Ibid.*, p. 167.

[5] Cf. Orrell, *Truth or Beauty*, p. 180.

[6] Amaldi, De Boer & Fürstenau, 'Comparison of grand unified theories'.

[7] Hossenfelder, *Lost in Math*, ch. 7.

[8] Dodd & Gripaios, *The Ideas of Particle Physics*, pp. 272–3.

The extra dimensions, it seems, are compactified, or 'rolled-up', and the behaviour of the strings is dependent on the geometry of this rolled-up space.

There has been no experimental confirmation or refutation of string theory.[1] And it has been commonly complained that, at least as it has been formulated so far, string theory is immune to empirical testing.

GABRIELE VENEZIANO

In 1968 Gabriele Veneziano (1942–) formulated what became the germ of string theory to explain experimentally observed properties of the strong interaction. There were hints that the model he developed had a physical interpretation as strings; this idea was advanced by Yoichiro Nambu (1921–2015), Holger Nielsen (1941–), and Leonard Susskind (1940–) in 1970.[2] But it was soon found that this early hadronic string theory made predictions that clashed with experimental data, and what Veneziano saw as its '[l]ess revolutionary' competitor, quantum chromodynamics, supplanted it.[3]

In a 2012 collection of historical essays on the emergence of string theory, Veneziano wrote of hadronic string theory reaching an 'apotheosis in terms of mathematical beauty and physical consistency'[4] before its fall in the face of contrary experimental evidence. André Neveu (1946–) recalled that he and his colleagues 'were particularly attracted by the mathematical beauty'[5] of what grew out of Veneziano's work, in particular the way change of variables guaranteed a certain symmetry in the multiparticle generalization of the Veneziano formula.

JOHN SCHWARZ

John Schwarz (1941–) was one of the founders of string theory. When others left the field after quantum chromodynamics displaced it, he continued working on it, along with Joël Scherk (1946–80). One of their reasons for doing so was that a consequence of the theory was the existence of a massless spin-2 boson. Some saw this as a difficulty, but such a particle would be a candidate for a boson that mediated gravitation.[6] Later, this prediction came to be seen as a superiority of string theory over quantum field theories: it possessed — in the words of Edward Witten (1951–) — 'the remarkable property of *predicting gravity*'.[7]

But in 1996, Schwarz recalled that he and Scherk 'felt strongly that string theory was too beautiful a mathematical structure to be completely irrelevant to nature';[8] similarly, in 1997 he said that he had

[1] Dodd & Gripaios, *The Ideas of Particle Physics*, p. 282.

[2] Veneziano, 'Rise and fall of the hadronic string', pp. 29–30.

[3] *Ibid.*, pp. 31–3.

[4] *Ibid.*, p. 17.

[5] Neveu, 'Personal recollections', p. 373.

[6] Schwarz, quot. in Davies & Brown, *Superstrings*, pp. 70–1.

[7] Witten, 'Reflections on the Fate of Spacetime', p. 25, emphasis in orig.

[8] Schwarz, 'Superstrings', p. 698.

felt that 'the mathematical structure of string theory was so beautiful and had so many miraculous properties that it had to be pointing toward something deep';[1] again in 2001 he said that they were 'taken by the mathematical beauty of the subject' and that they 'that it should be useful for something'.[2]

Another potential problem was that the mathematical consistency of the theory necessitated space-time having more than four dimensions. Schwarz thought that attempts to force string theory into a four-dimensional mould

> 'always started with a very mathematically beautiful system and made it really ugly and unconvincing and inevitably led to mathematical inconsistencies.'[3]

Schwarz characterized the beauty of string theory by contrasting it with the beauty of other theories. Newtonian and Einsteinian physics had simple equations that explained a wide range of phenomena.[4] But string theory had no such equations, so the beauty had to be of a different kind. It seemed, rather, to lie in what caused the internal simplicity of the mathematics: complicated calculations yielded surprisingly simple answers, which pointed to some deeper, not yet understood, mathematical foundation:[5]

> 'In the case of string theory we have a certain set of equations, but we really don't understand the generalization of the equivalence principle that's responsible for those equations. But it's clear that there is a *very deep and beautiful mathematical structure* that underlies all of the startling results that we are finding, and that some *very elegant and profound principle* is there to be found.'[6]

SHING-TUNG YAU

A *Calabi–Yau manifold* or *space* is a particular kind of manifold satisfying certain properties that are responsible for its having found applications in physics. In the 1950s, Eugenio Calabi (1923–2023) conjectured the existence of manifolds having these properties; in 1978, Shing-Tung Yau (1949–) proved that they existed.

Yau later wrote that he investigated what became known as Calabi–Yau manifolds because they were beautiful. In particular, they fitted with the Einstein field equations 'in a particularly elegant way' that Yau 'found arresting'. This beauty and elegance made him think that they should find some application in physics, though he did not know what.[7]

Such an application arose in 1985, when Philip Candelas (1951–), Gary Horowitz (1955–), Andrew Strominger (1955–) & Edward Witten showed that the conditions needed for string theory to produce

[1] Schwarz, quot. in Greene, *The Elegant Universe*, p. 137.

[2] Schwarz, 'Strings and the Advent of Supersymmetry', p. 13.

[3] Schwarz, quot. in Davies & Brown, *Superstrings*, p. 73.

[4] Schwarz, 'Strings and the Advent of Supersymmetry', pp. 16–17.

[5] *Ibid.*, p. 17.

[6] Schwarz, quot. in Davies & Brown, *Superstrings*, p. 83, emphases added.

[7] Yau & Nadis, *The Shape of Inner Space*, pp. 123, 129.

Murray Gell-Mann

the supersymmetric version of the standard model are those that define a six-dimensional Calabi–Yau manifold.[1] Indeed, Calabi–Yau manifolds constitute the only solution that satisfies a variety of conditions: the Einstein field equations, supersymmetry, hidden extra dimensions.[2] To this set of constraints they were 'the beautiful mathematical answer'[3] as the physicist Raman Sundrum (1964–) called them.

Yau was greatly pleased that Calabi–Yau manifolds had found their role in string theory. But he did not think their beauty would be a reason to hold fast to them in the face of evidence:

> 'I personally think Calabi-Yau manifolds are the most elegant formulation, as well as the most beautiful manifolds constructed so far among all the string vacua. But if the science leads us to some other kind of geometry, I'll willingly follow.'[4]

MURRAY GELL-MANN

Murray Gell-Mann (1929–2019), who received the 1969 Nobel Prize in physics for his work on the classification of elementary particles, said that he had 'many times' favoured an aesthetically pleasing theory over experimental results that contradicted it, results that were only later shown to be wrong.[5] He recalled specifically that in 1957 he and colleagues put forward a partial theory of the weak interaction, which disagreed with what he variously recalled as either seven or nine experimental results, but which was so beautiful that they believed the theory over the experimental evidence. In the end, all the experimental evidence turned out to be incorrect.[6]

The beauty or elegance of a theory, for Gell-Mann, lay essentially in the brevity and simplicity (which he seemed to identify) of its mathematical formulation:

> 'when the mathematics is very simple — when in terms of some mathematical notation, you can write the theory in a very brief space, without a lot of complication — that's essentially what we mean by beauty or elegance.'[7]

Gell-Mann seemed to think that the shorter an equation, the more beautiful, and he connected brevity with symmetry considerations that allowed equations to be stated more concisely: 'symmetries, then, make the equations even shorter, and even prettier, therefore.'[8] An example was the formulation of Maxwell's equations* using the Lorentz symmetry of special relativity: in this form, the laws require only two equations, not four.[9]

Gell-Mann suggested in 1996 that the continuing usefulness of mathematical aesthetic value as a guide in science, despite its progress

[1] Candelas et al., 'Vacuum Configurations for Superstrings'.

[2] Yau & Nadis, *The Shape of Inner Space*, p. 131.

[3] Sundrum, quot. in *loc. cit.*

[4] *Ibid.*, p. 251.

[5] Gell-mann, quot. in Judson, *The Search for Solutions*, p. 22.

[6] Gell-Mann, 'Beauty, truth and ... physics?', 01:05 sqq.; Gell-mann, quot. in Judson, *The Search for Solutions*, p. 22.

[7] Gell-Mann, 'Beauty, truth and ... physics?', 03:46 sqq.

[8] *Ibid.*, 09:12 sqq.

* ▸ pp. 884–5

[9] Gell-Mann, 'Beauty, truth and ... physics?', 09:42 sqq.; cf. Gell-Mann, *The Quark and the Jaguar*, pp. 81–4.

Murray Gell-Mann

into realms that were alien to everyday experience, was due to a 'self-similarity' in nature: there was a resemblance between the laws that govern nature at different levels. Thus the known physical laws at one level would resemble those that were uncovered in the investigation of the next level deeper.

> 'As we peel the skins of the onion, penetrating to deeper and deeper levels of the structure of the elementary particle system, mathematics with which we become familiar because of its utility at one level suggests new mathematics, some of which may be applicable at the next level down — or to another phenomenon at the same level. Sometimes even the old mathematics is sufficient.'[1]

More formally, the investigation of fundamental physics involved working downwards from large distances and upwards from low energies. The laws that were manifested at a given distance and energy were formulated using a certain kind of mathematics. Stepping down to finer resolutions and up to higher energies involved a more general kind of mathematics, but each successive step was relatively simple.[2] When one compared the mathematical formulation of the laws, those of the established level and of the next one were 'beautifully equivalent'.[3]

What Gell-Mann seems to have meant was that the perception of this resemblance was manifested in the experience of elegance in the mathematical formulation, and, as a corollary, that the evaluation of such elegance was not grounded solely in human judgement:

> 'And that takes the human being, human judgment, out of it a little. You might object that after all we are the ones who say what elegance is. But I don't think that that's the point. One way to describe what's going on is to say that nature apparently *resembles itself* at different levels.'[4]

Thus if a formula describing the new area was elegant, it pointed to the resemblance between levels, and thus, by the postulated self-similarity of nature, suggested that the formula was empirically correct.

There does not seem to be any suggestion that Gell-Mann was familiar with James McAllister's work, but his argument would fit into McAllister's aesthetic induction.* Physicists' aesthetic preferences for mathematics were shaped by the theoretical investigation of the phenomena accessible to experiment by instruments that could measure at a certain resolution or achieve a certain energy. When the next generation of instruments uncovered a finer resolution or reached a higher energy level, the mathematics required to formulate the theory was close enough to the previous iteration to be judged positively by the standards of the previous aesthetic canon. Gell-Mann's reason for the success of the aesthetic induction — that is, the resemblance of the mathematics required for one level and for the next — was the

[1] Gell-Mann, 'Nature Conformable to Herself', p. 11.

[2] *Ibid.*, p. 12.

[3] Judson, *The Search for Solutions*, p. 22.

[4] Gell-Mann, quot. in *ibid.*, p. 22, emphasis in orig.

* ▸ pp. 793–5

ontological claim of the self-similarity of nature at different scales and energy levels.

STEVEN WEINBERG

Steven Weinberg (1933–2021) was a prolific researcher who made numerous contributions to theoretical physics.[1] He, Sheldon Glashow* (1932–), and Abdus Salam (1926–96) were awarded the 1979 Nobel Prize for physics for their work on the unification of electromagnetism and the weak interaction.

In several of his books and lectures, he discussed the beauty of physical theories and its relationship to mathematics. Late in life, Weinberg recognized that the expected mathematical beauty of a theory could sometimes drive research across science: 'Some research is done because we suspect the phenomena that we study (e.g., superconductivity again, turbulence, sex ratios in animal populations) will have explanations that are mathematically beautiful.'[2] But his views on the relationship between mathematics and beauty in science seem to have changed over time, although the apparent change may be partly explained by a varying level of precision in his exposition of them.

Early discussions

In a panel discussion during the Einstein Centennial Symposium at the Institute for Advanced Study in March 1979, Weinberg made the distinction between the beauty in a physical theory and in its mathematical expression:

> 'I have heard discussions of this that make me nervous, because they talk about the beauty of the mathematics inspiring the physics. Well, that may be true for some physicists, but it seems to me that there is a slight confusion there. What sometimes inspires the theorist is the beauty of a theoretical principle. The theoretical principles underlying thermodynamics are beautiful. I don't think the particular mathematical formulation in terms of $dS = dQ/T$ inspires any affection.'[3]

Weinberg suggested that the general theory of relativity was a good example. Most of the theory followed from the 'beautiful idea' of the equivalence of gravitation and inertia. Thus, even if the perihelion precession of Mercury had not been noticed as a phenomenon that the general theory of relativity could explain, a theorist would have been justified in accepting it as the theory of gravitation because 'the physical idea was so grand', and *not* because of any beauty in the equations.[4]

[1] Burgess & Quevedo, 'Steven Weinberg'.

* ▸ pp. 969–70

[2] Weinberg, 'Reductionism Redux', p. 121.

[3] Weinberg, quot. in Bethe et al., 'Einstein and the Physics of the Future', p. 503.

[4] Weinberg, quot. in *loc. cit.*

In his 1986 Dirac lecture, Weinberg recalled hearing Paul Dirac* (1902–84) once say, to an audience mainly comprising students, that physics students should not be too concerned about the meaning of physical equations, but only with their beauty.[1] Weinberg went on to say that he agreed with Dirac, but only in part: he thought that beauty as a guide in theoretical physics was not to be found in the equations, but in the principles and how they combined in an inevitable way: 'It *doesn't matter* whether the equations are more or less beautiful.'[2] Newton's and Einstein's theories of gravitation are both described by second-order differential equations; thus their equations were of equal beauty. But Newton's theory involved fewer equations and so in this sense was more beautiful. But Einstein's theory had greater inevitability, for the inverse square law emerges (in non-extreme situations) as part of a larger structure and cannot be altered without disrupting the theory. In Newton's theory, one could alter the inverse square law, and, although a large enough change would make it empirically unsound, it would not internally disrupt the theory.[3]

But, in a different sense, Weinberg did agree with Dirac. Researchers in theoretical physics were often uncertain which principles should be applied. Given this doubt, the best guide was often mathematical beauty.[4] Weinberg gave the example of the Dirac equation, which had originated in an attempt to unify quantum mechanics and special relativity by formulating a relativistic generalization of the Schrödinger wave equation. The principles had shifted since then: unification of special relativity and quantum mechanics was thought to require the context of quantum field theory. But Dirac's 'beautiful equation' remained.[5]

But Weinberg later pointed out such advice was only good for those 'whose sense of purely mathematical beauty is so keen that they can rely on it to see the way ahead', and added that Dirac might have been the only physicist whose sense was so acute.[6]

Beauty in physics vs beauty in mathematics

In his 1992 book *Dreams of a Final Theory*, Weinberg did not try to define beauty, instead calling it one of those experiences where 'you know them when you feel them'.[7] But he did offer some pointers.

Concerning *mathematical* beauty, he wrote that the five regular solids gave a 'a prime example of mathematical beauty', but the context suggests that he saw this beauty in the *classification*, not the objects themselves: he said that the same sort of beauty existed in Élie Cartan's (1869–1951) classification of simple Lie groups.[8] The theory of Riemannian geometry was beautiful, largely due to its inevitability, in the sense of fitting together in a way that made it seem that no component could be changed.[9] Note that, although Weinberg here applied ‘inevitability’ to mathematics, his use was quite different from the Hardian conception.[*] For Hardy, inevitability was a property of proofs and how they unfolded; for Weinberg, it was a property of a theory. For Hardy, inevitability was dynamic; for Weinberg, static.

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* ▸ pp. 900–13

[1] Weinberg, 'Toward the Final Laws of Physics', p. 107.

[2] *Ibid.*, p. 107, emphasis added.

[3] *Ibid.*, pp. 107–8.

[4] *Ibid.*, p. 108.

[5] *Ibid.*, pp. 108–10.

[6] Weinberg, quot. in Farmelo, *The Strangest Man*, ch. 31.

[7] Weinberg, *Dreams of a Final Theory*, p. 134.

[8] *Ibid.*, pp. 163–4.

[9] *Ibid.*, p. 154.

As regards beauty in physics, he acknowledged the difficulty of characterizing beauty in physical theories,[1] but said that it was limited to ‘the beauty of simplicity and inevitability — the beauty of perfect structure, the beauty of everything fitting together, of nothing being changeable, of logical rigidity’.[2] He thought that physical theories could have beauty comparable to works of art because they had inevitability,[3] and he compared their beauty to the ‘spare and classic’ beauty of Greek tragedy.[4] As with a mathematical theory, a physical theory was inevitable if it fitted together in a way that made it seem impossible that any part should be different.[5] But just like inevitability in visual art or music, the judgement of inevitability was ‘a matter of taste and experience’,[6] not something calculable from a formula. For Weinberg, both the general theory of relativity and the standard model had this inevitability.[7]

Simplicity too was a part of beauty; but only in the sense of *conceptual* simplicity. What Weinberg called ‘mechanical’ simplicity, which was measurable by counting equations or symbols, was no part of beauty.[8] The Newtonian theory of gravitation was simpler than Einstein’s general theory of relativity, for the former had fewer equations than the latter. But Einstein’s theory was conceptually simpler because it unified gravitation and inertia.[9] As discussed in the previous subsection, in his 1986 Dirac lecture, Weinberg had argued that having fewer equations made Newtonian physics more *beautiful*. When he came to write *Dreams of a Final Theory*, he seems to have made a more careful distinction of beauty, simplicity, and *types* of simplicity.

Weinberg distinguished beauty in the physicist’s sense from ‘the quality that mathematicians and physicists sometimes call elegance’.[10] A proof or calculation was elegant if its result was reached efficiently, with ‘a minimum of irrelevant complication’.[11] Weinberg’s choice of the term *irrelevant* is curious: strictly speaking, anything irrelevant to a calculation or proof could simply be removed. A proof via division into a large number of necessary cases would surely be judged inelegant, especially if there was a different proof that avoids enumeration of cases, even though all the cases are relevant. Perhaps Weinberg meant *unnecessary*, in the sense of an elegant calculation or proof being no more complicated than required to establish its conclusion.

But the key point Weinberg made was that the beauty of a theory was independent of the elegance of the solutions of its equations. He agreed that the general theory of relativity was a beautiful theory, but noted that its equations were ‘notoriously difficult to solve’, save in the simplest of cases.[12]

Despite — or perhaps because of — this separation of the beauty of physical theory from the aesthetic value of its mathematical expres-

[*] ▸ pp. 502

[1] Weinberg, *Dreams of a Final Theory*, p. 149.

[2] *Loc. cit.*

[3] *Ibid.*, p. 148.

[4] *Ibid.*, p. 149.

[5] *Ibid.*, p. 135.

[6] *Ibid.*, p. 148.

[7] *Loc. cit.*

[8] *Ibid.*, p. 134.

[9] *Ibid.*, p. 135.

[10] *Ibid.*, p. 134.

[11] *Loc. cit.*

[12] *Loc. cit.*

sion, Weinberg's account in *Dreams of a Final Theory* indicated that he continued to think that beautiful mathematics expressing a physical theory could survive the modification of the physical principles that underlay it:

> 'the mathematical structures that physicists develop in obedience to physical principles have an odd kind of portability. [...] We are led to these beautiful structures by physical principles, but the beauty sometimes survives when the principles themselves do not.'[1]

There is a difference between this statement and his view as expressed in his 1986 Dirac lecture, in which he had he suggested that mathematical beauty could be a guide when the principles were uncertain.[*] Here he seems to have thought that beautiful mathematics would emerge even from physical principles that were later considered incorrect.[2]

Effectiveness explained

Referencing Eugene Wigner's (1902–95) lecture 'The Unreasonable Effectiveness of Mathematics',[†] Weinberg noted that physicists had found valuable the concepts that pure mathematicians had formulated and developed in a search for beauty.[3] Notably, non-euclidean geometry emerged in the early nineteenth century,[‡] and then Bernhard Riemann (1826–66) developed a general theory of curved spaces. Riemannian geometry was thus ready when Albert Einstein[§] (1879–1955) needed it to formulate the general theory of relativity.[4] Wigner thought that the way in which aesthetically-guided mathematics ran in advance of physics was 'very strange'.[5]

There was then a twofold mystery: that judgements of beauty helped not only in discovering but also in assessing the validity of physical theories, and that mathematicians' pursuit of beauty produced structures and theories that were later found valuable by physicists.[6] Weinberg thought there were three plausible explanations.

The first explanation was that experience in studying and trying to understand the universe had shaped our aesthetic perception so that we had come to see nature as beautiful. As evidence, Weinberg pointed to the historical fact that symmetry — in a strictly mathematical sense — was in the 1930s not viewed as conducive to the beauty of a physical theory. But later, given the importance of symmetry in physical theories, it had become seen as a important contributor to beauty.[7] Weinberg wrote on this topic *before* McAllister published his account of aesthetic induction.[¶]

Weinberg's second explanation was that scientists had a tendency to choose problems that they believed would have beautiful solutions.[8] On its own, this explanation begs the question. It shifts the judgement from the solution to the problem, without shedding light on how scientists could judge which problems would have beautiful solutions. But, in further discussion, Weinberg suggested that one

[1] Weinberg, *Dreams of a Final Theory*, p. 152.

[*] ▸ p. 958

[2] *Ibid.*, p. 163.

[†] ▸ pp. 929–31

[3] *Ibid.*, p. 153.

[‡] ▸ pp. 427–32

[§] ▸ pp. 886–92

[4] *Ibid.*, p. 157.

[5] *Loc. cit.*

[6] *Loc. cit.*

[7] *Ibid.*, pp. 158–9.

[¶] ▸ pp. 793–5

[8] *Ibid.*, p. 160.

clue to a problem having a beautiful solution would be that it concerned a phenomenon that was widely observed in different situations. He did not say so explicitly, but there is a hint here that he thought the generality of the phenomenon might point to the simplicity and inevitability of the solution.[1]

The third explanation that Weinberg offered was in some sense just a special case of the second: he held that the truly fundamental problems were among those that would have beautiful solutions:

> 'if we ask why the world is the way it is and then ask why that answer is the way it is, at the end of this chain of explanations we shall find a few simple principles of compelling beauty. We think this in part because our historical experience teaches us that as we look beneath the surface of things, we find more and more beauty. […] For us, too, the beauty of present theories is an anticipation, a premonition, of the beauty of the final theory. And in any case, we would not accept any theory as final unless it were beautiful.'[2]

[The elided part of this quotation is a rather flawed analogy: 'Plato and the neo-Platonists taught that the beauty we see in nature is a reflection of the beauty of the ultimate, the *nous*.'[3] This statement is at odds with Plato's and Plotinus's view that beauty comes about through participation in the Form of the Beautiful.*]

Beauty and fundamentality

Weinberg had mentioned this faith in the beauty of the fundamental theories of nature in a lecture given in June 1987:

> 'We are beginning to suspect that […] there is simplicity, a beauty, that we are finding in the rules that govern matter that mirrors something that is built into the logical structure of the universe at a very deep level.'[4]

In another 1987 lecture he held that experience had shown that beauty was more important in theories describing the most fundamental phenomena. Planetary orbits need not be perfect circles because the orbits of planets were not fundamental. However,

> 'when we formulate the equations of quantum field theories or string theories we demand a great deal of mathematical elegance, because we believe that the mathematical elegance that must exist at the root of things in nature has to be mirrored at the level where we are working.'[5]

Note that here Weinberg had not made the careful distinction between the beauty of a theory and the elegance of its mathematical formulation that he would make in 1992's *Dreams of a Final Theory*. But he still thought that the fundamental theory would be beautiful, reflecting 'something that is built into the structure of the universe at a very deep level'.[1]

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[1] Weinberg, *Dreams of a Final Theory*, pp. 160–1.

[2] *Ibid.*, p. 165.

[3] *Loc. cit.*

* ▸ pp. 71–4

[4] Weinberg, 'Newtonianism, Reductionism', p. 24.

[5] Weinberg, 'Newton's Dream', p. 39.

This belief in the beauty of the fundamental theory could lead the scientist down the wrong path, but the error lay not in the expectation of beauty, but in the incorrect judgement that the sought theory was fundamental. Johannes Kepler* (1571–1630) thought that the geometric structure of the heliocentric cosmos was fundamental and had been deliberately shaped thus by the creator. Thus he applied his faith in the beauty of the fundamental theory at what can now been seen as the wrong level, and postulated the regular solids as an explanation for the sizes of the planetary orbs.[2]

If relying on aesthetic judgement to formulate a theory led to failure, the physical phenomena were not fundamental. And so, if such reliance led to success, then the phenomena and the theory that explained them should be fundamental, or close to it. Since reliance on beauty seemed to be successful in the physics of elementary particles, Weinberg thought that physics was close to its goal of a final theory.[3]

Symmetry

Weinberg thought that inevitability and simplicity contributed to the beauty of a physical theory. But there was a feature that contributed to this, at least in the cases of the general theory of relativity and the standard model: principles of symmetry, in the sense of an invariance of the laws of nature under certain changes in the perspective from which the phenomena were observed.[4]

For Weinberg, the concept of spontaneous symmetry breaking was important precisely because it pointed to a greater symmetry in the laws of nature than the observed properties of elementary particles suggested. Weinberg noted that this idea was thoroughly platonic: the observed reality, with its broken symmetry, 'is only an imperfect reflection of a deeper and more beautiful reality, the reality of the equations that display all the symmetries of the theory'.[5]

String theory

Weinberg knew that the standard model was not the final, most fundamental theory, because of its inherent aesthetic imperfections: it included various fields, but did not explain why *those* fields and not others; it had 'about eighteen' constants that were free parameters within the theory, in the sense that they could not be deduced from the theory itself but must rather be measured experimentally. Most importantly, the standard model does not account for gravitation.[6]

In a lecture delivered in February 1995, Weinberg said that, twenty years earlier, he had imagined that the next fundamental theory would be 'a better field theory that somehow tied things up more elegantly mathematically'.[1] By the time of his 1986 Dirac lecture, his views had shifted to support string theory as the fundamental theory, and this change was largely due to his faith in beauty. He wondered if Dirac would have thought string theory to be mathematically beautiful enough to be part of the final theory, but he was sure that Dirac would have approved of the aesthetically-guided search for it.[2]

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[1] Weinberg, *Dreams of a Final Theory*, p. 243.

* ▸pp. 279–94

[2] *Ibid.*, p. 163.

[3] *Ibid.*, p. 165.

[4] *Ibid.*, pp. 136–7.

[5] *Ibid.*, p. 195.

[6] Weinberg, 'Night Thoughts of a Quantum Physicist', p. 97.

In 1995, he admitted that the development of string theory had been 'mostly driven by a sense of mathematical beauty and by a search for consistency', and that, while it had pleasing features, such as the natural emergence of gravitation, it had made no empirically verified predictions.[3] Furthermore, it was still a collection of theories rather than a unified whole.[4] At least by July 2003, his aesthetically-motivated faith in string theory had become even more solid: 'I would find it hard to believe that that much elegance and mathematical beauty would simply be wasted.'[5]

By 2015, his faith seemed to be wavering somewhat, or at least he asserted it less firmly. He still considered string theory to have the 'beauty of a rigid art form — a sonnet or a sonata' as a result of the requirement of mathematical consistency, which it 'just barely' satisfied.[6] But because of the lack of empirically testable predictions, 'theorists (at least most of us) are keeping an open mind as to whether the theory actually applies to the real world'.[7]

ANTHONY ZEE

The physicist Anthony Zee (1945–) wrote that he thought of himself and his colleagues in fundamental physics as the 'intellectual descendants of Albert Einstein': just as Zee thought Einstein had before them, they sought beauty, and had a faith that beautiful equations governed the design of the universe.[8] His 1986 book *Fearful Symmetry* was specifically aimed at discussing the aesthetic motivation for physics.[9]

Zee held that physicists followed Dirac in thinking that beauty should be sought first, and that sooner or later the empirical evidence would fall into line with beauty:* '"Let us worry about beauty first, and truth will take care of itself!"'[10] But Zee thought that the aesthetics of modern physics admitted precise formulation:[11] for Zee, beauty equalled symmetry.[12] And he thought that physicists generally made the same connection:

> 'physicists developed the notion of symmetry as an objective criterion in judging Nature's design. Given two theories, physicists feel that the more symmetrical one, generally, is the more beautiful. When the beholder is a physicist, beauty *means* symmetry.'[13]

[1] Weinberg, 'Night Thoughts of a Quantum Physicist', p. 97.

[2] Weinberg, 'Toward the Final Laws of Physics', p. 110.

[3] Weinberg, 'Night Thoughts of a Quantum Physicist', p. 100.

[4] *Loc. cit.*

[5] Weinberg, quot. in McMaster, 'Viewpoints on String Theory'.

[6] Weinberg, *To Explain the World*, ch. 1.

[7] *Loc. cit.*

[8] Zee, *Fearful Symmetry*, p. 3.

[9] *Ibid.*, p. xvii.

* ▸ pp. 906–7

[10] *Ibid.*, p. 3.

[11] *Ibid.*, p. 4.

[12] *Ibid.*, p. 9.

[13] *Ibid.*, p. 13, emphasis in orig.

Thus it was part of the intellectual development of a physicist to train their aesthetic perception to see this beauty in the fundamental theories of nature.[1]

The development of Yang–Mills theory was, for Zee, an exemplar of this desire for symmetry:

> 'Yang and Mills invented a new exact symmetry of dazzling mathematical beauty. The symmetry was not motivated by any experimental observation [...]; rather, it was an intellectual construct based on aesthetics.'[2]

[The non-abelian gauge symmetry that Chen-Ning Yang* (1922–) & Robert Mills (1927–99) put forward explained precisely the strong interaction, but necessitated thitherto unknown particles; it thus lay in abeyance for some years.]

Zee admitted to being 'dazzled by the mathematical beauty of superstring theory', and to hoping that superstring theory was heading in the right direction.[3] But this aesthetic desire was tempered by a feeling that the theory was too conservative: he queried whether the fundamental framework of quantum physics might give way to something entirely new at the distances string theory involved, just as classical physics yielded to quantum physics on descending to atomic scales.[4] In so doing, he aligned himself with those doubters who 'want their beliefs anchored in hard facts, not in the soft elegance of higher mathematics'.[5]

DAVID BOHM AND F. DAVID PEAT

David Bohm (1917–92) started as a physicist and made important contributions to the understanding of plasmas and the behaviour of electrons in metals.[6] Over time, he became more interested in the philosophical implications of his research and of physics generally, and began to formulate a new metaphysics that could account both for known physics and for the relationship between mind and matter.[7] Partly as a consequence of this philosophical work, he had a nuanced view of the place of mathematics in the physical sciences and of the role of mathematical beauty.

Bohm recognized that both the scientist and the mathematician sought beauty. During his extensive nine-year correspondence[8] with the abstract artist Charles Biederman (1906–2004), he wrote:

> 'There is something of art in science [...]. Whenever a deep scientific or mathematical problem is involved, we have to do with a vision of totality, which inevitably has "beauty", "elegance" in it, in the sense that a deep truth cannot fail to be beautiful, as well as elegant in its pattern and structure.'[9]

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[1] Zee, *Fearful Symmetry*, p. 4.

[2] *Ibid.*, p. 187.

* ▸ pp. 916–18

[3] *Ibid.*, p. 291.

[4] *Ibid.*, pp. 291–2.

[5] *Ibid.*, p. 319.

[6] Hiley, 'Bohm', pp. 110–11.

[7] Pylkkänen, *Mind, Matter and the Implicate Order*, § 1.2.

[8] Pylkkänen, intro. to Bohm & Biederman, *Correspondence*, vol. 1, p. xiii.

[9] Bohm, letter to Biederman, dated 17 Mar. 1962, quot. in *ibid.*, vol. 1, pp. 183–4.

A 1968 essay indicates that Bohm suspected that judgements of beauty were not merely subjective, precisely because scientific theories that were perceived as having beauty also had order, coherence, harmony, and unity. And on these properties depended not only the beauty of a theory, but also its *truth*.[1] Thus the beauty and truth of a theory were connected. While the truth of a theory depended upon empirical adequacy, there were times when no experiment or observational evidence could distinguish between two rival theories, and the choice between them rested on the basis of beauty or how they fitted into the totality of scientific understanding.[2] Here, the beauty was not necessarily mathematical. But, writing with the physicist F. David Peat (1938–2017), he noted that, as a matter of practice, the choice between two mathematically equivalent theories, such as Schrödinger's and Heisenberg's formulations of quantum mechanics, was made upon the basis of aesthetic judgements *of the mathematics* or using principles such as Occam's razor.[3]

Bohm had once been suspicious of the role of 'elegant mathematics' in physics.[4] He seems to have moderated this view as his philosophical work progressed, but always maintained that mathematics should have a limited scope in science and that it was an error to confuse the mathematical formulation of a theory with what it described.[5] In a (non-extemporaneous) dialogue written with Peat, and published in their 1987 book *Science, Order, and Creativity*, Bohm lamented what he saw as an over-mathematization of physics, which had reached the point where the mathematics was being treated as the content, rather than as a description. He thought that this development could be traced back to the 1920s, and especially to quantum mechanics.[6] A key step had been when Heisenberg's work on quantum mechanics led to him to make mathematics prior to any visual intuition of physical reality: the mathematics came first and that *mathematics* might or might not be visualizable.[7]

In the dialogue, Peat responded to Bohm's complaint about this overmastering role for mathematics by pointing to how beauty in mathematics served as a guide in physical science and by asking Bohm about its role in his own work:

> 'DAVID PEAT: [...] many of the deepest scientific thinkers have used criteria of mathematical beauty in the development of their theories. They believed that the deepest scientific explanations must also be mathematically beautiful. Without the requirement of mathematical aesthetics a great many discoveries would not have been made. Surely in your own work the criteria of mathematical elegance must have acted as a signpost that you were on the right track?'[8]

Peat here asserted that without the guide of mathematical beauty or mathematical elegance (he seemed to treat the terms as synonymous), certain discoveries would not (and perhaps would never) have been made. He made a powerful statement of belief that mathematical beauty or elegance was not simply an aid, but was in fact an *essential* input into the scientific process.

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[1] Bohm, 'On the relationships of science and art', p. 42.

[2] *Ibid.*, p. 39.

[3] Bohm & Peat, *Science, Order, and Creativity*, pp. 43–4.

[4] Peat, *Infinite Potential*, p. 181.

[5] *Ibid.*, p. 183.

[6] Bohm & Peat, *Science, Order, and Creativity*, pp. xiv–xv.

[7] Miller, *Imagery In Scientific Thought*, pp. 253–5.

[8] Bohm & Peat, *Science, Order, and Creativity*, p. xv.

Bohm was more cautious. He accepted that mathematics could spur insights and that mathematical beauty could help, as the work of Kepler, Heisenberg, and Dirac has shown. But these scientists never evaluated their discoveries solely through their mathematical formulations, and others made use of verbal concepts, visualization, or philosophical thinking.[1] While Einstein appreciated mathematical beauty, it was not by seeking it that he came upon his great discoveries: rather, he started from what Bohm described as 'unspecifiable feelings and a succession of images'.[2] And Bohm deplored basing scientific theories, such as superstrings, on nothing more than 'elegant mathematics', with no appeal to physical or philosophical motives.[3]

For Bohm, aesthetic value, whether mathematical or not, did play a role in distinguishing minor acts of insight from truly creative discoveries. Bohm discussed the difference between them in an essay first published in 1968. The 'really creative' discoveries were characterized by the perception of a new order that led to the development of new structures that possessed harmony and totality and thus evoked the feeling of beauty.[4] That is, a 'really creative' act in science was characterized by the aesthetic aspects of its consequences: 'the harmony, beauty, and totality that is characteristic of real creation'.[5] The difference was not necessarily in any role the aesthetic value played as a mechanism or as a guide for creativity; rather, aesthetic perception served an evaluative and classificatory role not during but *after* the act of creation.

In 1988, the year after the appearance of *Science, Order, and Creativity*, Peat published *Superstrings and the Search for the Theory of Everything*. In this book, he wrote that physics had detached from experiment, and that theories had come to emerge, not out of experiment, but out of other theories. The validity of such theories was not empirical adequacy, but 'aesthetics, mathematical consistency, and their interrelationship to yet other theories'.[6] He seemed not to regret this development, but thought rather that this new mode of doing physics was successful. His example was superstring theory, which was not designed to accommodate new experimental evidence. Rather, it was intended to have 'internal consistency, freedom from infinities and anomalies, logical inevitability of the arguments, and mathematical aesthetics'.[7]

Unlike many authors who simply acquiesced in, or even celebrated, the separation of theoretical physics from experiment, Peat offered a defence of this change by analogy with language. Peat considered physics to be a social dialogue about reality. He thought that physics as a practice had advanced to a new level just as human use of language advances. The use of language *only* to refer to things in our experience was limited to an early stage of cognitive development. But language had vastly greater scope; it became a means for thinking.

[1] Bohm & Peat, *Science, Order, and Creativity*, p. xv.

[2] *Loc. cit.*

[3] Peat, *Infinite Potential*, p. 183.

[4] Bohm, 'On Creativity', p. 8.

[5] *Ibid.*, p. 21.

[6] Peat, *Superstrings*, p. 276.

[7] *Ibid.*, p. 277.

Beyond an elementary level, the meaning of language was not found in the external world, but within the social use of language. While language might have started from the world of experience, it reached far further, and there was freedom to extend it in new ways. Peat thought that the practice of physics had attained an analogous level of freedom, where it was no longer concerned with describing the physical world:[1]

> 'Modern physics is about theories [...]. Its main concern is for coherence and internal consistency, for freedom of expression, and for aesthetic value. Theories are increasingly about other theories, just as language is about language — in the sense that the structure and meaning of language can only be discussed using language itself.'[2]

For Peat, the separation of theory and experiment did not imply that the theoretical physicist had complete freedom: while it might not be possible to test a new theory, that theory had to, at however distant a remove, say something about — and be in accord with — the phenomena in the world. But there could be no recourse to experiment in order to choose between alternative such theories. Thus other criteria — among them the beauty of the mathematical formulation — were used to make the selection.[3] In time, Bohm's desire for new physical and philosophical foundations might be fulfilled, but presently the mathematical road was the only one open.[4]

Thus, in Peat's view, physics had simply advanced into a new kind of practice. Others, of course, might argue that in so doing, it had ceased to be physics, for, as Peat admitted,

> 'Clearly a physics of this nature can have no end but will be involved in a ceaseless process of transformation, clarification, and unfolding. Its goals may well turn out to be aesthetic rather than practical.'[5]

DAVID GROSS

David Gross (1941–), who shared the 2004 Nobel Prize in physics with Frank Wilczek* (1951–) and H. David Politzer (1949–,) declared strong support for string theory, and seems to have considered the mathematical beauty of the theory as a reason to think it true: Leonard Susskind wrote in a volume to which Gross also contributed that

> 'David Gross told me that string theory could not be wrong because its beautiful mathematics could not be accidental.'[6]

Yet Gross also acknowledged that there was something odd about how the mathematics that governed fundamental physics was beautiful:

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[1] Peat, *Superstrings*, pp. 277–9.

[2] *Ibid.*, p. 279.

[3] *Ibid.*, pp. 279–80.

[4] *Ibid.*, pp. 333 sqq.

[5] *Ibid.*, p. 280.

* ▸ pp. 985–7

[6] Susskind, 'Quark Confinement', p. 235.

> ‘The strangest thing is that for the fundamental laws of physics we still need deep mathematics and that as we probe deeper to reveal the ultimate microscopic simplicity, we require deeper and deeper mathematical structures. Even more, these mathematical structures are not just deep but they are also interesting, beautiful, and powerful.’[1]

But in what did this beauty consist? After quoting some of Dirac’s remarks on the beauty and interest and power of the mathematical expressions of fundamental physics, Gross pointed out that the terminology was ill-defined. In particular, what did it mean to say that an equation was beautiful? He presented the mathematical expression of the action of the standard model:[2]

$$\left.\begin{aligned} S = \int \mathrm{d}^4x\,\sqrt{g}\Bigg[& \frac{1}{4} g_{\alpha\beta} g_{\beta\delta} \{ A^{\alpha\gamma} A^{\beta\delta} \\ & \qquad\qquad + \mathrm{Tr} B^{\alpha\gamma} B^{\beta\delta} + \mathrm{Tr}\, C^{\alpha\gamma} C^{\beta\delta} \} \\ & + \frac{\vartheta_2}{32\pi^2} \mathrm{Tr} B^{\mu\nu} \widetilde{B_{\mu\nu}} + \frac{\vartheta_3}{32\pi^2} \mathrm{Tr} C^{\mu\nu} \widetilde{C_{\mu\nu}} \\ & + \sum_{i=1}^{3} g_{\mu\nu} (\bar{Q}_i \gamma^\mu D^\nu_Q Q_i + \bar{L}_i \gamma^\mu D^\nu_L L_i) \\ & + g_{\mu\nu} \mathrm{Tr} (D^\mu_\varphi \Phi)^\dagger (D^\mu_\varphi \Phi) - V(\Phi) \\ & + \sum_{ija} (\bar{Q}_i \Gamma^{ij}_{Qa} Q_j \Phi^a + \bar{L}_i \Gamma^{ij}_{La} L_j \Phi^a) + R \Bigg]. \end{aligned}\right\} \quad \text{(A)}$$

Gross thought that an equation such as (A) could only be beautiful ‘in the eye of an informed beholder’.[3]

For Gross, beauty in mathematics was ‘an acquired taste’ whose appreciation required training and education and which remained subjective, although there was a large consensus about beauty amongst mathematicians and physicists.[4] But perhaps mathematicians collectively differed from physicists collectively in the reasons they had for seeing beauty in a theory. In the standard model, the most beautiful parts were those that explained the forces of nature using the symmetry principles that were fundamental to gauge theories. Physicists found those parts beautiful because one could deduce from the simple symmetry principles the forces of nature and the bosons that mediate those interactions. Mathematicians found them beautiful because they provided interesting mathematical structures. That is, the physicists’ beauty came from the explanatory power of the theories with reference to the physical world; the mathematicians’ beauty was internal to mathematics.[5]

Gross thought that there was a close connection between the symmetry and beauty of a theory, but the symmetries that had continually emerged in the development of new theories were not just a source of aesthetic pleasure:

[1] Gross, ‘Physics and mathematics at the frontier’, p. 8372.

[2] *Ibid.*, p. 8373.

[3] *Loc. cit.*

[4] *Loc. cit.*

[5] *Loc. cit.*

> ‘That is the lesson that we’ve learned during the evolution of physics [...][.] All our advances introduce bigger and bigger symmetries. I think it’s beautiful and I like it, but most importantly, it seems to be a fact about the world, about nature.’[1]

Conversely, the ugly aspects of the standard model were those that did not adhere to any symmetry principle. Together with the nineteen parameters that did not fall out of the mathematics but had to be experimentally determined, this lack of beauty indicated that the standard model was not the true fundamental theory. For Gross, ‘it is simply not pretty enough’.[2]

Referring to Shiing-Shen Chern’s (1911–2004) response — that the objects of mathematics *exist* — to Yang’s musing on the coincidence between ideas in the physical theory of non-abelian gauge fields and the mathematical theory of fibre bundles,[*] Gross suggested that Chern’s answer offered an explanation for both the appearance of ideas from pure mathematics in physics and the beauty of those ideas. If mathematics studied ‘structures that are a real part of the natural world, as real as the concepts of theoretical physics’, then the effectiveness of mathematical ideas in physics was not so surprising. Further, one might expect that in both mathematics and physics, beauty would be seen in the parallel concepts. None of this was contradicted by the fact that for physicists the ultimate test of a theory was successful prediction of experimental outcomes, whereas mathematicians were free from such constraints.[3] If a mathematical reality underpinned the natural world, and especially if our aesthetic judgement was a valid guide to mathematical or physical truth, then mathematicians and physicists should seek cross-fertilizations between the two fields.[4]

Gross suggested that the beauty perceived in mathematics and in those parts that were applied to physics was due to our minds having evolved to take pleasure in natural patterns.[5] He did not explore this point further, but it closely resembles Harold Osborne’s (1905–87) explanation,[†] except that Osborne referred to order rather than natural patterns. (Almost two decades after Gross suggested this explanation, John Polkinghorne[‡] (1930–2021) and Russell Howell[§] (1947–) both argued that such evolutionary explanations were unsatisfactory.)

SHELDON GLASHOW

Sheldon Glashow (1932–), who shared the 1979 Nobel Prize in physics with Steven Weinberg[¶] and Abdus Salam, certainly had a lively appreciation for mathematical beauty, writing that in the theory of Lie groups ‘three great branches of classical mathematics — analysis, algebra and topology — converged to reveal the fundamental unity and beauty of mathematics’.[6] He also wrote that Maxwell’s equa-

[1] Gross, quot. in Cao, *Conceptual Foundations of Quantum Field Theory*, p. 72.

[2] Gross, ‘Physics and mathematics at the frontier’, p. 8373.

[*] ▸ p. 917

[3] *Loc. cit.*

[4] *Ibid.*, pp. 8373–4.

[5] *Ibid.*, p. 8373.

[†] ▸ pp. 934–5

[‡] ▸ pp. 937–8

[§] ▸ pp. 938–9

[¶] ▸ pp. 957

[6] Glashow & Bova, *Interactions*, pp. 155–6.

tions* had 'the logical consistency and the uniqueness that appear to the physicist as the elegance and beauty of revealed truth'.[1]

In his own work, Glashow admired the 'mathematical elegance' of SU(5) grand unification. But there seems to have been some conceptual overlap between the mathematical elegance of the theory and the physical unity it implied: his admiration for the mathematics was juxtaposed to the observation that it treated leptons and quarks together.[2]

Glashow was sceptical of the idea that aesthetic value could be a criterion for truth, but did recognize that it had an attraction. Writing with Paul Ginsparg (1955–) in 1986, he accepted that the likely correctness of quantum chromodynamics had reaffirmed the belief in aesthetic value as a criterion for truth.[3]

But the aesthetic value of the mathematical formulation of a theory was not sufficient: for Glashow, a theory of physics that was not confronted with experimental verification was akin to 'mediaeval theology'.[4] Because he thought that string theory invoked magnitudes so small and energies so high as to render it forever inaccessible to experimental testing, he asked rhetorically whether it was 'a theory of physics or a philosophy'.[5] With Ginsparg, he wrote:

> 'In lieu of the traditional confrontation between theory and experiment, superstring theorists pursue an inner harmony where elegance, uniqueness and beauty define truth. The theory depends for its existence upon magical coincidences, miraculous cancellations and relations among seemingly unrelated (and possibly undiscovered) fields of mathematics. Are these properties reasons to accept the reality of superstrings? Do mathematics and aesthetics supplant and transcend mere experiment?'[6]

Glashow said that he was 'particularly annoyed' by string theorists who thought that the uniqueness and beauty of the theory was sufficient reason for thinking it true, and who were then motivated to criticize the value of experimental testing.[7]

DAVID LINDLEY AND *THE END OF PHYSICS*

The physicist David Lindley (1956–) published in 1993 the provocatively-titled book *The End of Physics: The Myth of a Unified Theory*, which seems to have been the first extended account and analysis of the direction of physics and the apparently ever-strengthening role of aesthetic judgement of theories and their mathematical formulations.

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* ▸ pp. 884–5

[1] Glashow & Bova, *Interactions*, p. 43.

[2] *Ibid.*, p. 302.

[3] Ginsparg & Glashow, 'Desperately seeking superstrings?', p. 7.

[4] Glashow & Bova, *Interactions*, p. 77.

[5] McMaster, 'Viewpoints on String Theory', Glashow, quot. in.

[6] Ginsparg & Glashow, 'Desperately seeking superstrings?', p. 7.

[7] Glashow, quot. in Davies & Brown, *Superstrings*, p. 182.

Lindley noted that the process of scientific research occupied a range between one extreme in which data are accumulated with the hope that there would be perceived a mathematical relationship between them, and another in which theorists seek, without recourse to data, ‘mathematical laws whose beauty and simplicity have some particular appeal’ and only subsequently see if they fit the observed phenomena.[1] Lindley seemed not to object to the latter approach: aesthetic concerns were a valid part of the scientific process, and some of the triumphs of modern physics had emerged from theorists following their aesthetic and mathematical instincts. But Lindley insisted on the proviso of an ultimate confrontation of theory and observation.[2] Einstein, for instance, had steered a purely mathematical course in formulating the general theory of relativity, but it has been found to explain observation, starting with the precession of the perihelion of Mercury and the gravitational deflection of light.[3] But Lindley thought that Einstein’s mathematical instincts had started to fail, and that in his later research he had been drifting. Such aimlessness was the danger of this style of scientific research: it could become an exercise in seeking interesting mathematics, never being subject to empirical justification.[4]

Indeed, history showed there were times when seeking only to maximize accuracy could impede progress: the geocentric Ptolemaic cosmology agreed better with the observation than the heliocentric Copernican one. Only when Kepler presented his elliptical orbits was the Ptolemaic system dethroned as the most empirically accurate. Science as a process allowed theorists to hare off, ‘entranced by some new vision of mathematical beauty’, because they were in the end restrained by the practical test of experiment and observation.[5] Lindley pointed to the Titius–Bode law* as an example of science misled by ‘an excessive devotion to the search for mathematical simplicity’.[6] It was a kind of numerology. But numerology was not to be dismissed entirely: Balmer’s formula† was ‘a pure exercise in numerology’, but was explained by the Bohr quantum theory of the atom.[7] A scientist could judiciously choose to ignore certain contrary evidence, in the expectation that the problems would be resolved in time,[8] but ultimately ‘if science is to work properly the idea must be tested, and thrown away if it fails’.[9]

In modern theoretical physics, there was a belief that ‘a sense of mathematical rigor and elegance’ was vital in the search for truth.[10] Theoretical physicists could be lured by ‘the pure aesthetic appeal of mathematical ideas and systems in themselves, regardless of any applicability to the real world’, and following this path had proved successful many times.[11]

The mathematics of unified theories, such as grand unification, was a ‘mathematical tour de force’, possessed of a ‘certain elegance and power’. This judgement was not in dispute. But there were two related fundamental questions: Did the elegant and powerful mathematics yield empirically testable predictions? Did it actually represent the true fundamental laws of nature?[1] If the answer to the first question was 'no', the second seemed to become unanswerable, at least in the traditional conception of the scientific method. 'Does physics then become a branch of aesthetics?' That is, if a theory could not be evaluated empirically, could its success be judged in any way except in its conceptual, mathematical, and other aesthetical value?[2] Lindley pointed to Richard Feynman* (1918–88), Paul Ginsparg, and Sheldon Glashow† as exemplars of physicists who worried that their science was drifting into mathematical invention that would remain forever experimentally unverifiable.[3]

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[1] Lindley, *The End of Physics*, p. 7.

[2] *Ibid.*, p. 10.

[3] *Ibid.*, pp. 10–12.

[4] *Ibid.*, pp. 12–13.

[5] *Ibid.*, p. 9.

* ▸ pp. 882–3

[6] *Ibid.*, p. 14.

† ▸ p. 884

[7] *Ibid.*, p. 15.

[8] *Ibid.*, p. 8.

[9] *Ibid.*, p. 13.

[10] *Loc. cit.*

[11] *Loc. cit.*

In the case of string theory, even if it was admitted to be based on 'compellingly unique and beautiful mathematical notions',[4] one must descend from that almost platonic heaven to the observed reality of broken symmetries and measurable masses. And Lindley thought that in the course of that descent the theory ceased to have that compelling uniqueness and beauty.[5] This fall seemed to be common to all proposed final theories: their aesthetic perfection corresponded only to physical reality in the primordial universe. As the temperatures fell from the extremes of the first moments after the Big Bang, the symmetrical perfection of the theory ceased to hold.[6]

Even the most powerful feasible particle accelerators would never reach the energies where the symmetries would become observable again. In general, gathering empirical data with which to evaluate the correctness of fundamental physical theories required experimental apparatus that was already pushing the limits of what was practical or economical to construct and run. It might not ever be possible to test new theories.[7] Thus the importance of aesthetic judgements seemed to be gaining in importance, at the expense of empirical evaluation.[8] Lindley suggested the natural end to this process:

> 'Perhaps physicists will one day find a theory of such compelling beauty that its truth cannot be denied; truth will be beauty and beauty will be truth — because, in the absence of any means to make practical tests, what is beautiful is declared ipso facto to be the truth.'[9]

For Lindley, such a final theory would be a myth: accepted not because its truth could be demonstrated but because it would be convenient to accept it. Physics would have reached its end, because it would no longer have any power to explain.[10]

BRIAN GREENE

The theoretical physicist Brian Greene (1963–) wrote several popular science books in which he touched on the aesthetics

[1] Lindley, *The End of Physics*, p. 18.

[2] *Ibid.*, p. 20.

* ▸ pp. 920–3

† ▸ pp. 969–70

[3] *Ibid.*, p. 19.

[4] *Ibid.*, p. 244.

[5] *Loc. cit.*

[6] *Ibid.*, p. 253.

[7] *Ibid.*, pp. 11, 209, 255.

[8] *Ibid.*, p. 11.

[9] *Ibid.*, p. 255.

[10] *Loc. cit.*

of physical theories. Notably, his 1999 book *The Elegant Universe* expounded on string theory as a solution to the incompatibility of the general theory of relativity and standard model.

For Greene, the mathematical beauty of a theory could have a psychological impact that suggested that it described the world:

> 'a set of mathematical equations, through their sleek internal logic, their intrinsic beauty, and their potential for wide-ranging applicability, can seemingly radiate reality'.[1]

Experience, he thought, had shown that physicists' propensity for using aesthetic value as a guide had proven successful.[2] In particular, symmetry was a component of aesthetic value, and symmetry in physics had a precise mathematical meaning, and it was following this mathematical meaning to its conclusion that had led to supersymmetry.[3] Again, for aesthetic reasons, physicists tended to reject the idea that nature is symmetrical in only *almost* all mathematically possible ways.[4]

But he admitted that the apparent usefulness of aesthetic value as a guide to the truth of physical theories was a contingent fact; there was no necessary reason for it. He observed that the currently-held aesthetic criteria might need to be 'refined' as physics probed ever more alien phenomena; he did not expand on what form this refinement might take.[5]

It is unclear whether Greene distinguished between aesthetic value in the mathematical formulation of a theory and in its other aspects. For instance, consider how Greene described his reaction to reading the paper[6] by Philip Candelas, Gary Horowitz, Andrew Strominger & Edward Witten that demonstrated how Calabi–Yau compactifications could yield a supersymmetric theory with many realistic features. For Greene, this was 'one of those rare mathematical forays bordering on spiritual enlightenment',[7] and that the idea that the hidden spatial dimensions might govern fundamental physics as 'one of the most beautiful ideas'[8] he had ever come across. From his description, the mathematics seemed to have had a profound effect on him, but he located the beauty in the concept of the hidden dimensions.

LEONARD SUSSKIND

The theoretical physicist Leonard Susskind (1940–) maintained that physicists have a keen sense of 'beauty, elegance, and uniqueness'.[9] For Susskind, these values seem to have been closely related and were sometimes not clearly distinguished. In particular, he suggested that an absence of elegance and neatness would render the laws of physics ugly.[10]

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[1] Greene, *The Hidden Reality*, p. 320.

[2] Greene, *The Elegant Universe*, p. 167.

[3] *Loc. cit.*

[4] *Ibid.*, p. 174.

[5] *Ibid.*, p. 167.

[6] See Candelas et al., 'Vacuum Configurations for Superstrings'.

[7] Greene, 'Why String Theory Still Offers Hope We Can Unify Physics'.

[8] *Ibid.*

[9] Susskind, *The Cosmic Landscape*, p. 12.

[10] *Loc. cit.*

Physicists have such a strong belief that 'the laws of nature are the unique inevitable consequence of some elegant mathematical principle' that many would be disappointed if the laws of physics lacked uniqueness and elegance.[1] But cosmologists had perhaps been less faithful to the idea that fundamental laws of physics should be simple and elegant, because they spent more time observing the universe and what they saw there indicated arbitrariness, coincidence, fine-tuning, and tailoring for our existence.[2]

Susskind spent his whole career doing theoretical physics, and thought that is was 'the most beautiful and elegant of all the sciences', an assessment in which he expected other physicists to concur.[3] But he admitted that he and most other physicists he knew were unclear about what was meant by beauty in physics. When he questioned colleagues on this point, most said that it meant that the equations were elegant; a few that the actual phenomena were beautiful.[4] For Susskind, the general theory of relativity was 'the most beautiful physical theory yet devised' because it was expressed in elegant equations, had 'an element of uniqueness', and was able to describe many observed phenomena.[5]

Physicists certainly made aesthetic judgements of theories, using terms like *beautiful*, *elegant*, *powerful*, *simple*, *unique*. But Susskind doubted that there was precise agreement on the meaning of any of these words, but gave some characteristics on which he thought there would be general agreement.[6] For Susskind, there was no real difference between elegance and simplicity, and he thought that physicists, mathematicians, and engineers all used the terms interchangeably.[7] But he went on to suggest that in mathematics, theories were elegant if they were based on a minimal number of axioms, and demonstrations were elegant if they contained as few steps as possible, pointing out that '[m]athematicians love to streamline their arguments, sometimes to the point of incomprehensibility'.[8] There was an apparent paradox here, in that an elegant, and therefore simple, argument might be incomprehensible, or at least barely so. A resolution might lie in the multifaceted concept of simplicity: verificational simplicity, which concerns the ease of checking an argument; inventional simplicity, involved in the ease of discovery of the argument;[9] explanatory simplicity, which concerns understanding how the argument supports its conclusion;[10] conceptual simplicity, relating to the intellectual resources upon which the argument draws. (This list could be continued.) Susskind's 'simplicity' seems to have related especially to conceptual simplicity, and perhaps not at all to verificational or explanatory simplicity.

Thus, for Susskind, the general theory of relativity was elegant because it made so few assumptions and so much followed from them,[11] and because, like any elegant theory, it could be expressed in a small number of equations, each of which was simple. A simple equation

[1] Susskind, *The Cosmic Landscape*, p. 12.

[2] *Ibid.*, pp. 78, 81, 129–30.

[3] *Ibid.*, p. 112.

[4] *Loc. cit.*

[5] *Ibid.*, pp. 114–16.

[6] *Ibid.*, p. 112.

[7] *Ibid.*, pp. 112–3.

[8] *Ibid.*, p. 113.

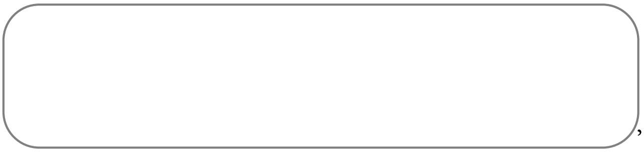
[9] Detlefsen, 'Philosophy of mathematics in the twentieth century', p. 87.

[10] Cain, 'Visual thinking and simplicity of proof', p. 3.

[11] Susskind, *The Cosmic Landscape*, p. 113.

was one that was short, without too many symbols.[1] Susskind gave what is possibly the bluntest ever characterization of simple equations:

> ‘Simplicity, roughly speaking, means it should be possible to write the equations in a box about this big:



> ’[2]

Susskind was not quite consistent in his application of aesthetic terms, also describing as ‘beautiful’ the equations of the general theory of relativity.[3]

The anthropic principle

In his 2006 book *The Cosmic Landscape*, Susskind introduced his discussion of the anthropic principle as follows:

> ‘the *Anthropic Principle* — a hypothetical principle that says that the world is fine-tuned so that we can be here to observe it!’[4]

While Susskind found the principle to be ‘a silly, half-baked notion’, he thought that it was a label for a deeper idea. But precisely because of its history as an unscientific idea, hinting at some flavour of creationism,[5] physicists had been reluctant to engage with it: it was ‘the bête noire of theoretical physics’.[6]

Susskind thought that physicists had seemed to ignore, or at least to *want* to ignore, the laws of the universe seeming to be such as to allow life as we know it to develop.[7] Physicists, Susskind included, had disdained this idea, preferring to believe that the laws follow from some underlying ‘elegant mathematical principle’[8] or ‘the beautiful symmetry of some mathematical theory’.[9]

Susskind himself pointed out that if we inhabited a precisely supersymmetric universe, every fermion would have a boson superpartner with the same mass, and this would disrupt the chemistry that formed our kind of life. Each electron orbiting in a carbon atom could spontaneously emit a photino and become a selectron. Selectrons would be bosons and thus not subject to the Pauli exclusion principle: multiple selectrons could occupy the same state. Thus all selectrons could fall to the lowest energy orbits in the atom. What resulted would no longer be *chemically* carbon and thus could not participate in the kinds of reactions upon which all known biochemistry depends.[10] In short, if the universe possessed that kind of mathematical beauty, we would not be here to observe it.

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[1] Susskind, *The Cosmic Landscape*, p. 113.

[2] *Ibid.*, p. 118.

[3] *Ibid.*, p. 71.

[4] *Ibid.*, p. 7.

[5] *Ibid.*, pp. 83–4.

[6] *Ibid.*, p. 187.

[7] *Ibid.*, p. x.

[8] *Ibid.*, p. xi.

[9] Susskind, ‘The Landscape’.

[10] Susskind, *The Cosmic Landscape*, pp. 250–1.

String theory

In the conclusion of *The Cosmic Landscape*, Susskind said that he had tried to dismiss beauty, uniqueness, and elegance as what he rather redundantly termed 'false mirages'.[1] He maintained that the laws of physics were neither unique nor elegant. But he *hoped* — like, he believed, many other physicists[2] — that underlying everything was a fundamental theory that was 'unique, elegant, and wonderfully simple', and that the known laws were merely the inelegant consequences of that theory. He drew an analogy with quantum mechanics and the higher-level phenomena that emerge from it. Quantum mechanics was an elegant description of the behaviour of atoms, but the things made from atoms — complex molecules and structures — lacked such an elegant description.[3] An elegant formulation of the emergent behaviour of atoms was lacking even in nuclear physics: Susskind thought that '[m]ost theoretical physicists agree that the equations of nuclear physics are not in the least elegant'.[4]

Without quite stating it, Susskind seems to have held to the same principle as Weinberg: that aesthetic value in mathematics could be a guide at the most fundamental level of physics. Just as Weinberg did,[*] Susskind presented Kepler's geometric cosmos as a cautionary tale of how reliance on aesthetic value could mislead.[5]

Susskind thought that the known laws of particle physics were not elegant: 'There are too many particles, too many vertex diagrams, and too many coupling constants. I haven't even told you yet about the random collection of masses that characterize the particles.'[6] That is, it seemed too arbitrary: Why *those* particles? Why *those* masses? Why *those* constants? The only reason the known laws had any appeal was their empirical success; the mess of particles and constants *worked*.[7] This seems to have been a reason for him to seek a deeper foundation that would be elegant.

For Susskind, string theory was potentially 'a very elegant mathematical structure', even if its foundations had not yet been found,[8] and he hoped that it would be elegant, simple, and beautiful. But he warned that, like the general theory of relativity, it might happen that the equations would 'satisfy every esthetic criterion that a physicist could hope for', while the solutions of those equations would be neither simple nor elegant.[9]

Susskind also acknowledged that, if paucity of defining equations was a measure of elegance, then string theory was in a sense the *ne plus ultra* of elegant theories, for string theory had precisely *zero* known defining equations.[10]

String theory allowed for a possibly huge number of different vacua and different laws of physics;[11] In February 2003, Susskind coined the term 'the Landscape' for this space of possibilities.[12] The uniqueness of the vacuum in the standard model was not due to any mathematical elegance, but simply because the theory was built to explain the observed phenomena and thus the vacuum of the particular world we inhabit.[1]

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[1] Susskind, *The Cosmic Landscape*, p. 377.

[2] *Ibid.*, p. 118.

[3] *Ibid.*, p. 377.

[4] *Ibid.*, p. 117.

[*] ▸ p. 962

[5] *Ibid.*, p. 120.

[6] *Ibid.*, p. 59.

[7] *Loc. cit.*

[8] *Ibid.*, p. 378.

[9] *Ibid.*, p. 379.

[10] *Ibid.*, p. 124.

[11] *Ibid.*, pp. 290–4.

[12] Susskind, *The Cosmic Landscape*, pp. ix, 90; Susskind, 'The Anthropic Landscape of String Theory', p. 2.

There were various hypothesized mechanisms for each possibility in the Landscape to have a physical instantiation, which Susskind explicitly stated to be 'obviously' different from mathematical existence.[2] One such mechanism, which Susskind discussed, was 'eternal inflation', in which space constantly expands and produces separate 'bubbles' that expand into universes in which the laws of physics have crystallized into one of the points in the Landscape.[3]

From Susskind's perspective, postulating the Landscape would have had a twofold benefit. First, the resolution of the anthropic principle in a way that does not require a supernatural entity to carry out the fine-tuning: instead, we simply inhabit a region in which the laws of physics have settled into a configuration suitable for our kind of life. This explanation is an analogue of life having evolved on a planet with suitable conditions, rather than in one without. Second, the laws of physics, with their apparent lack of mathematical elegance, are grounded upon deeper laws, and so is restored the mathematical elegance, and thus the beauty, of the fundamental laws of physics.

LEE SMOLIN AND *THE TROUBLE WITH PHYSICS*

> ‘ String Theory has no lack of enemies
> who will tell you that it is a monstrous perversion. ’
>
> — Leonard Susskind, *The Cosmic Landscape*, p. 126.

The physicist Lee Smolin (1955–) published in 2006 *The Trouble With Physics*, in which he argued that string theory could not be considered to be scientific.

Smolin diagnosed the root of the problem as a particular style of doing science that had dominated mid-twentieth-century physics. The revolutions in physics were shaped by the work of theorists like Einstein, Niels Bohr (1885–1962), Werner Heisenberg (1901–76), and Erwin Schrödinger (1887–1961), work that arose from deep reflection on the fundamental concepts of and questions about space, time, and matter. But their mode of theoretical physics gave way to a style that favoured virtuosity in calculation,[4] which had evolved into a way of doing fundamental physics that was entirely mathematical, and made no attempt to ground itself in experiment. Smolin wrote disparagingly that this abandoning of reliance on experiment in favour of a purely mathematical exploration of the laws of nature meant that the era of 'postmodern physics' had arrived.[5]

An 'embarrassment' of the standard model was that it had 'about twenty' constants that had to be empirically determined.[6] String theory had exactly one such constant.[7] If string theory was correct, all

[1] Susskind, *The Cosmic Landscape*, p. 273.

[2] *Ibid.*, p. 293.

[3] *Ibid.*, pp. 304–10.

[4] Smolin, *The Trouble With Physics*, pp. xii–xiii.

[5] *Ibid.*, pp. 116–17.

[6] *Ibid.*, p. 13.

[7] *Ibid.*, pp. 112–13, 117.

the *prima facie* independent constants of the standard model would follow from this one more fundamental constant.[1] But different background geometries gave different theories, and these geometries were specified by a list of constants, which was in fact vastly longer than the list of constants of the standard model,[2] and the problem became to specify the geometry that would give rise to the standard model.[3] Candelas, Horowitz, Strominger & Witten found that six-dimensional Calabi–Yau manifolds, which Smolin called 'particularly beautiful', provided the correct geometry.[4][*] But Strominger then went on to show how to construct large numbers of supersymmetric string theories.[5]

There seems to have been little disagreement, even from Smolin, that string theory was beautiful or elegant;[6] certainly, supersymmetric string theories were more elegant than the 'ugly and complicated' supersymmetric versions of the standard model.[7]

But the problem with so many string theories was that it becomes difficult to make a prediction. All of the evidence could be made to fit a huge number of theories.[8] It seemed unlikely that an experiment could be devised whose outcome would not fit at least some of them. Thus, string theory could not be experimentally falsified; neither could it be verified.[9] Even leaving this worry aside, there was another objection: as Smolin put it, string theory began as an attempt to unify. '[i]f an attempt to construct a unique theory of nature leads instead to 10^{500} theories, that approach has been reduced to absurdity'.[10]

Smolin came to the conclusion that beautiful mathematics should not be a guide in physics. For his main historical example, he looked to the same cautionary tale as Weinberg and Lindley and Susskind had: Kepler's model of the cosmos as based on the regular solids,[†] which was indeed 'mathematically beautiful'.[11] Smolin's account of Kepler's thought is a moderate historiographical simplification, eliding for instance Kepler's attempt to fit rhombic solids into the orbits of the four Galilean moons of Jupiter.[‡] But Smolin's point was that, although Kepler's scheme gave a unified explanation for the orbits of the then-known planets, it did not lead anywhere.[12] For Smolin, Kepler's later three laws[§] had 'none of the beauty of his other proposals', but successfully described the motions of the planets and the Jovian satellites.[13] The moral Smolin drew was that mathematical beauty could be a treacherous guide, and that observation could be more important.[14]

A modern example was the posited SU(5) grand unification,[¶] which was a 'beautiful idea', the 'most elegant way imaginable of unifying quarks with leptons'. However, this theory predicted that protons should decay. Despite ongoing large-scale experiments aimed at detecting proton decay, it has never been observed. This, for Smolin, was enough to dismiss SU(5) grand unification.[15] Importantly, Smolin admitted that he was 'stunned' by the failure of SU(5) unification, though

[1] Smolin, *The Trouble With Physics*, p. 117.

[2] *Ibid.*, pp. 119–21.

[3] *Ibid.*, pp. 121–2.

[4] *Ibid.*, pp. 122–3.

[*] ▸ pp. 954–5

[5] *Ibid.*, p. 124.

[6] *Ibid.*, p. 194.

[7] *Ibid.*, p. 112.

[8] *Ibid.*, p. 124.

[9] *Ibid.*, p. xiv.

[10] *Ibid.*, p. 159.

[†] ▸ pp. 281–5

[11] *Ibid.*, p. 29.

[‡] ▸ pp. 293–4

[12] *Loc. cit.*

[§] ▸ pp. 285–7

[13] *Loc. cit.*

[14] *Ibid.*, p. 30.

[¶] ▸ p. 951

[15] *Ibid.*, pp. 64–5.

he did not make explicit whether his expectation of its correctness had been grounded in its mathematical beauty.[1]

For Smolin, a more ambiguous example of the role of mathematical beauty was Weyl's attempt to devise a unified theory of electromagnetism and gravitation. Weyl's suggested theory contained the first formulation of what would become gauge invariance, which Smolin called 'a beautiful mathematical idea', but it failed because it had major consequences that were at variance with observation. Nevertheless, gauge invariance would go on become fundamental to the standard model.[2]

Leaving aside the mathematical beauty of the theory itself, Smolin pointed out that while string theory had led to valuable and beautiful developments in pure mathematics, such a consequence 'cannot save a theory that has no clearly articulated core principles and makes no physical predictions'.[3]

PETER WOIT AND *NOT EVEN WRONG*

According to his obituarist Rudolf Peierls (1907–95), who either heard the story himself or had it from a source he considered reliable, Wolfgang Pauli (1900–58) once described a physics paper as 'not even wrong'.[4] The phrase has entered the lexicon of science and philosophy to disparage works that do not present any thesis that can be confronted critically: a paper that makes a claim with which one can engage may be *wrong*, but a paper that makes no such claim is *not even wrong*.

Thence came the title of Peter Woit's (1957–) book *Not Even Wrong*, a critique of string theory that appeared in 2006 after having been rejected by a number of academic publishers, seemingly for its supposedly controversial content. [The putative pusillanimity of publishers in preventing vigorous debate is beyond the scope of this history.]

Woit had first published an evaluation of string theory in February 2001, in the form of a five-page preprint.[5] He noted that string theory had gained popular attention.[6] He also pointed out that its development had yielded concepts that had been useful in pure mathematics.[7] But he was strongly critical of the lack of predictions that could be empirically tested.[8]

Woit thought that it was difficult to engage with those who extolled the elegance or beauty of string theory, since there was no well-defined theory.[9] Rather, he thought that the standard model was more aesthetically attractive:

> 'The Standard Model is dramatically more "elegant" and "beautiful" than string theory in that its crucial concepts are among the deepest and most powerful in modern mathematics.'[10]

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[1] Smolin, *The Trouble With Physics*, p. 65.

[2] Smolin, *The Trouble With Physics*, p. 45; see also Goenner, 'On the History of Unified Field Theories', § 4.1.

[3] Smolin, *The Trouble With Physics*, p. 196.

[4] Peierls, 'Wolfgang Ernst Pauli', p. 186.

[5] Woit, 'String Theory'.

[6] *Ibid.*, p. 1.

[7] *Ibid.*, p. 2.

[8] *Loc. cit.*

[9] *Loc. cit.*

[10] *Ibid.*, p. 3.

This statement was not quite an assertion of mathematical beauty in the standard model, but Woit returned to the comparison in *Not Even Wrong*, where he made it clear that he did not think of superstring theory as beautiful, but that there was mathematical beauty in the standard model:

> 'the quantum field theory of the standard model contains physical and mathematical ideas that are strikingly beautiful, the likes of which superstring theory cannot approach.'[1]

For instance, the Dirac equation and the Yang–Mills field corresponded exactly with the mathematical structures that were at the centre of the modern view of geometry. Superstring theory claimed that these phenomena were only low-energy approximations of something more fundamental, without specifying what that more fundamental thing was.[2]

Woit disagreed with David Lindley's[*] suggestion that mathematical beauty had somehow prevented superstring theorists from making empirically testable predictions.[3] Woit's diagnosis had little to do with the putative mathematical beauty of the theory. Rather, it was that string theory was *not well-defined enough* that one could extract a prediction whose failure would refute the theory. There are countless parameters that characterize Calabi–Yau manifolds, and no way to pin them down so as to make a definitive prediction.[4]

Woit admitted that the standard model failed to answer certain questions. For instance, the standard model is a gauge theory with symmetry group $U(1)\times SU(2)\times SU(3)$.[†] A hypothetical more fundamental theory would explain *why* this symmetry group arises. Quantum chromodynamics, which corresponds to the SU(3) direct factor, had the 'beautiful property of having no free parameters'; the other two factors introduce free parameters that have been determined experimentally; the fine structure constant[‡] is one.[5] There was at least a tacit admission here that having no free parameters — having the parameters emerge as consequences of the fundamental theory — would be a beautiful property.

CUMRUN VAFA

The theoretical physicist Cumrun Vafa (1960–) wrote in 2000 an essay on the history and possible future of the interaction of mathematics and physics, which in part discussed the role of beauty. Vafa thought that the long historical connection between mathematics and physics, which traces back to ancient Greece, had left physicists convinced that the principles that govern nature are 'associated with deep and beautiful mathematics'.[6] But he noted that beautiful theories need not involve deep mathematics. His example was the special theory of relativity, which uses only elementary mathematics. A theory could be beautiful but not mathematically beautiful.[1]

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[1] Woit, *Not Even Wrong*, p. 196.

[2] *Loc. cit.*

[*] ▸ pp. 970–2

[3] *Ibid.*, p. 261.

[4] *Ibid.*, pp. 179–87.

[†] ▸ p. 917

[‡] ▸ pp. 893–5

[5] *Ibid.*, p. 91.

[6] Vafa, 'On the Future of Mathematics/Physics Interaction', p. 322.

However, there seemed to be deep connections between concepts used in theoretical physics and in mathematics, notably between gauge theories in physics and vector bundles in mathematics.[2] But on a human level, the collaboration between mathematicians and physicists was distressingly shallow: there was an exchange of problems and results, but not in ways of tackling those problems. Vafa thought this limited collaboration was due to the differing notions of rigour in the two fields. Physicists would happily reason about objects and concepts whose existence had not been formally proved. In establishing their existence rigorously, mathematicians would pull them away from the original physical ideas. Vafa was careful to note that this diagnosis was not a criticism of either side, merely an observation about their different cultures: the mathematician insisted on rigour because it had been important in mathematics; the physicist worked to determine how nature functioned, regardless of any difficulties in making the mathematical description rigorous.[3] Further, ideas from mathematics rarely led to new physics; rather, mathematical ideas were adopted by physicists when they were applicable in some physical theory.[4]

Because physical theories were becoming less accessible to experiment, physicists found themselves short of ways to guide them in selecting and evaluating new ideas. Therefore, physicists' conviction that physical theories should have mathematical beauty had become a tool to guide them.[5] But Vafa thought that

> 'most physicists believe that this tool should be used with great caution, as one can be led astray to areas which indeed have beautiful mathematics but very little physical content. In fact there are many beautiful areas of mathematics with very little physical application. So in some sense *mathematical beauty is viewed as a check and an auxiliary tool*, rather than a primary tool in search of physical theories.'[6]

DANIEL FRIEDAN

In a 2003 paper that put forward a possible theory for the large distance physics of spacetime, Daniel Friedan (1948–) included one of the rare discussions of mathematical beauty in the formal academic physics literature, in what seems to have been a reaction to the confidence reposed in string theory on the basis of its mathematical beauty. While not denying that fundamental physics might continue to exhibit beauty, his discussion almost entirely urged caution.

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[1] Vafa, 'On the Future of Mathematics/Physics Interaction', p. 322.

[2] *Loc. cit.*

[3] *Ibid.*, p. 323.

[4] *Loc. cit.*

[5] *Ibid.*, p. 324.

[6] *Ibid.*, p. 324, emphasis in orig.

Friedan acknowledged the beauty of the mathematics of physical laws in the time from the formulation of the general theory of relativity to that of the standard model, but pointed out that it did not follow that the fundamentals of the physical world fitted any standard of mathematical beauty.[1] He made two important observations. First, the mathematical beauty of physical law applied on a range of scales that was wide by historical standards, and thus psychologically impressive; nevertheless, the range remained limited. Second, only a *very* small part of mathematics had both been found beautiful and applied to the formulation of physical laws.[2] It was inadvisable to extrapolate from past experience *any* property, aesthetic or not, of the mathematics that had been used successfully in physics.

Both the general theory of relativity and the Dirac equation, which Friedan considered '[t]he historical exemplars of mathematical beauty in physics', explained previously observed phenomena and more specific physical laws, and made empirically testable predictions.[3] Their proving useful, thought Friedan, came first; only secondarily could be their mathematical beauty be appreciated.[4] Previous attempts by theoretical physicists to seek a particular standard of mathematical beauty — Friedan did not specify which — had ended in failure.[5]

Later, he warned that even if there was a successful theory of everything that was expressed in beautiful mathematics, there was no way of knowing how far off its formulation might be, nor of knowing in what direction it lay, nor of knowing which parts of beautiful mathematics would play a role.[6] Thus it was practical to seek, not mathematically beautiful laws, but improvements in current applications of mathematics to physics, improvements that would be of benefit in describing reality.[7]

ROGER PENROSE

The mathematician and theoretical physicist Roger Penrose (1931–), who shared in the 2020 Nobel Prize in physics for his work on the formation of black holes, recognized and at various times discussed mathematical beauty in physical theories. He variously described Maxwell's equations* as having 'a special beauty'[8] and being 'exquisite'.[9] He admired the 'supreme mathematical beauty' of the general theory of relativity and of the theory of quantum mechanics.[10] He seems to have considered mathematical elegance a contributor to, or aspect of, mathematical beauty, pointing to 'the extraordinary mathematical elegance' of spin in quantum mechanics, of the Dirac equation, and of Feynman's approach to quantum field theory as exemplars of beauty.[11] In particular, he thought that a mathematically elegant and unexpected feature of quantum mechanics was the fundamental role of complex numbers.[12]

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[1] Friedan, 'A tentative theory of large distance physics', p. 11.

[2] *Loc. cit.*

[3] *Loc. cit.*

[4] *Loc. cit.*

[5] *Loc. cit.*

[6] Friedan, 'The shape of a more fundamental theory?', pp. 17–18.

[7] *Ibid.*, p. 18.

* ▸ pp. 884–5

[8] Penrose, 'The Rôle of Aesthetics', p. 268.

[9] Penrose, *The Road to Reality*, p. 1038.

[10] Penrose, *The Road to Reality*, p. 1038; cf. Penrose, 'Mathematical Physics of the 20th and 21st Centuries', p. 219.

[11] Penrose, *The Road to Reality*, p. 1038.

[12] Penrose, 'Mathematical Physics of the 20th and 21st Centuries', pp. 229–30.

Analysis of mathematical beauty in physical theories

Penrose made an important distinction between the levels on which beauty could be perceived. Minkowski geometry is completely symmetric. The pseudo-Riemannian geometry of the general theory of relativity is highly non-symmetrical. But the mathematical object that describes this irregular geometry, the Riemann curvature tensor, was symmetrical and beautiful. Thus in moving from the special to the general theory of relativity, the mathematical beauty became apparent at a higher level of abstraction. Indeed, only the development of new mathematics, by Einstein and others, elucidated this beauty.[1]

Penrose also thought that the beauty of the mathematical formulations of physical laws was in part dependent upon their describing nature. Detached from reality, as pure mathematics, they would be — at minimum — less beautiful; it was their status as physical theories that elevated them.[2] For instance, he described Einstein's gravitational field equations[3]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}.$$

as having 'elegance and geometric simplicity'.[4] But he noted that 'without the meaning that lies behind these symbols, there is neither beauty nor physical significance'.[5] The 'physical significance' part of this statement indicates that 'meaning' should be interpreted as both the mathematical referents of the symbols and the physical referents of the mathematical objects.

For Penrose, mathematical beauty was closely related to a quality he termed *mathematical coherence*.[6] The term *coherence* he used

> 'to convey something that is just a little stronger than *consistency*. The word is to suggest that there is a certain economy as well as consistency in a fully coherent mathematical structure, where different aspects of its formalism work together in perfect accord.'[7]

Mathematical consistency was vital for a physical theory to make overall sense, and allowed for purely mathematical theoretical developments that implied consequences in the physical world.[8] Penrose considered this criterion of mathematical consistency to be *aesthetic*, but it had the advantage of being 'fairly clearly' objective, in contrast to the general subjectivity of aesthetic judgements.[9] A judgement of mathematical coherence could vary depending on how familiar one was with a body of mathematics: its unity may be perceptible only to those who have experience working with it. Thus different judgement of mathematical coherence might be made by 'insiders' and 'outsiders'.[10] Penrose's position seems to have been that among the different judgements, that of the 'insiders' would be *correct*, or at least would most closely approach an objective correctness.

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[1] Penrose, 'Einstein's Vision', p. 7.

[2] Penrose, *The Road to Reality*, p. 1038.

[3] See Grøn & Hervik, *Einstein's General Theory of Relativity*, ch. 8.

[4] Penrose, 'The Rediscovery of Gravity', p. 49.

[5] *Loc. cit.*

[6] Penrose, *The Road to Reality*, p. 1014.

[7] *Ibid.*, p. 1045, n. 34.5, emphasis in orig.

[8] *Ibid.*, p. 889.

[9] *Ibid.*, p. 1014.

[10] *Ibid.*, p. 1015.

Mathematical beauty as a guide

Penrose thought that the mathematics involved in fundamental physics had greater sophistication, subtlety, and beauty than was seen in less fundamental physics.[1] The search for 'mathematical beauty, depth, and sophistication' was part of the motivation for seeking better physical theories;[2] it was, however, a mystery why the mathematics of empirically successful theories had these properties.[3] Possibly the aesthetic value of the mathematical formulation of theories, especially theories that are unifying, were 'telling us something deep about the mathematical underpinnings of our physical universe'.[4]

Penrose seems to have considered the aesthetic value of the mathematical formulation to be at least a factor in deciding whether to accept a theory. In 2000, he wrote that the 'loop-variable' approach to unifying quantum mechanics and the general theory of relativity had 'the kind of mathematical elegance which' he found 'makes a fully convincing case'.[5] In contrast, he was not convinced that supersymmetry described reality: there was no empirical evidence, and the aesthetic appeal of its mathematics and its unifying role did not compensate for this shortcoming.[6]

When theoretical physics moved beyond the guidance and check of experiment, subjective aesthetic values started to play a much greater role.[7] This development was a base fact about the pratice of physics. But Penrose seems to have indicated that there was no other choice: if we could not rely on experiment to guide formulation of deeper theories, it seems that we had to have recourse to mathematical aesthetic judgement.[8] Objective values like mathematical coherence were no longer sufficient, although, given the connection Penrose saw — but did not explore — between mathematical beauty and coherence, some of the subjective judgements may in fact be based upon a need for mathematical coherence.[9] In any case, as a guide, such judgements were not reliable. In particular, mutually contradictory physical theories could be obtained by following subjectively different mathematical and physical aesthetic criteria.[10]

As a converse example, an argument in favour of inflation was that it led to an overall flat space-time, and those who advanced this argument seemed to consider this result to be an aesthetic positive.[11] Leopold Infeld (1898–1968) thought that all mathematicians would prefer, on the grounds of mathematical beauty, a universe with overall positive curvature.* Penrose included himself among many mathematicians who would consider negative overall curvature, or hyperbolic geometry, as more beautiful:[12] he professed a liking for 'the beauties of hyperbolic geometry'.[13]

Nevertheless, Penrose thought that researchers should 'follow their own aesthetic drives',[14] but had to be ready for rejection of their conclusions on aesthetic grounds. The subjectivity of aesthetic judgements meant that one researcher might present an aesthetically-motivated theory to another, only to find that the other sees there little or no aesthetic value.[1] Penrose seems to have been pointing out that this subjectivity meant that there was no guarantee of using such judgements to persuade another theorist. And there was no recourse to experiment when the theorist had ventured far beyond its reach.

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[1] Penrose, *The Road to Reality*, p. 1026.

[2] *Ibid.*, p. xx.

[3] *Ibid.*, pp. 20–1.

[4] *Ibid.*, p. 471.

[5] Penrose, 'Mathematical Physics of the 20th and 21st Centuries', p. 229.

[6] *Loc. cit.*

[7] Penrose, *The Road to Reality*, pp. 1014–15.

[8] *Ibid.*, p. 1026.

[9] *Ibid.*, p. 1014.

[10] *Ibid.*, pp. 1025–6.

[11] *Ibid.*, p. 754.

* ▸ p. 914

[12] *Loc. cit.*

[13] *Ibid.*, p. 48.

[14] *Ibid.*, p. 1015.

Penrose understood that pure mathematicians might desire that beautiful mathematics should say something about the world.[2] But he pointed out that there were historical examples where a determined yet futile attempt had been made to apply beautiful mathematics to the study of the physical world. He gave the example of William Rowan Hamilton's* (1805–65) efforts to incorporate all of natural law into a quaternion-based framework;† the quaternions, Penrose thought, had the beautiful property of being a division algebra.[3] And there were beautiful parts of mathematics that seemed very distant from direct physical application, such as Georg Cantor's‡ (1845–1918) theory of the infinite, or the Riemann hypothesis.[4]

These historical considerations led Penrose to wonder whether previous successes had deceived scientists into believing in a closer relationship between mathematics and physics than truly existed.[5] One could argue that there was a *non sequitur* here: if the success of *beautiful* mathematics in physics had been misleading, then it does not follow that there was a looser connection between mathematics and physics than had previously been thought, just that there was a looser connection between *beautiful* mathematics and physics. But, for Penrose, aesthetic value was an intrinsic part of the connection, and so the loss of this component would weaken the bond between mathematics and physics.

FRANK WILCZEK

Frank Wilczek (1951–) shared the 2004 Nobel Prize in physics with David Gross§ and H. David Politzer for their work on the strong interaction. His 2015 book *A Beautiful Question* examined one fundamental question: 'Does the world embody beautiful ideas?'[6] Much of Wilczek's examination of this question, and its implications for the ultimate laws of nature, concerned mathematical beauty.

Wilczek thought that symmetry was a part of beauty, and seems to have thought that symmetry was bound up with an economy of thought: from part of a symmetric thing, the rest can be understood.[7] But there was also 'a more primitive concept of beauty', which one recognized when one saw it, and which involved getting more out of a theory than one put in: postulating a concept to explain one thing and finding it explained another. This phenomenon was an aspect of beauty, but it was also economy and simplicity.[8]

[1] Penrose, *The Road to Reality*, p. 1015.

[2] *Ibid.*, p. 1017.

* ▸ pp. 412–21

† ▸ pp. 418–21

[3] *Ibid.*, pp. 1015–16.

‡ ▸ pp. 456

[4] *Ibid.*, pp. 1038–9.

[5] *Ibid.*, p. 1039.

§ ▸ pp. 967–9

[6] Wilczek, *A Beautiful Question*, p. 1.

[7] *Ibid.*, p. 15.

[8] Wilczek, quot. in Hossenfelder, *Lost in Math*, ch. 7.

In order to illustrate mathematical beauty, Wilczek made two specific analogies. In a musical or literary work, beauty may consist in the development of tension among certain themes and the resolution of this tension in a satisfactory way. In a work of visual art, 'beauty is symmetry', where the term *symmetry* should be interpreted as 'balance of proportions, intricacy towards a purpose', which leaned more towards the ancient Greek *symmetria* [συμμετρία] than the modern technical sense of symmetry. For Wilczek, a work of mathematical beauty should have both of these features.[1] In the Dirac equation, for example, there was a resolution of tension between the competing demands of relativity and simplicity.[2]

Wilczek made a more detailed analysis of the beauty of Maxwell's equations.[*] Their beauty derived from three sources: beauty as a tool, beauty as an experience, and beauty and symmetry.[3] In explaining 'beauty as an experience', Wilczek wrote:

> 'The Maxwell equations can be written pictorially, in terms of flows. When so presented, they depict a sort of dance. I often visualize them that way, as a dance of concepts through space and time, which is a joy.'[4]

The symmetry of the equations he termed a 'more intellectually precise perspective' on their beauty.[5] His linking of symmetry and beauty was also evident in his description of quantum chromodynamics as 'a theory of compelling symmetry and mathematical beauty'.[6]

For Wilczek, symmetrical equations tended to be particularly beautiful. The symmetry of equations was a semantic as well as a syntactic notion. An equation or system of equations was symmetric if one could make changes among the variables without breaking the equality.[7] Thus, Wilczek thought that the aesthetic evaluation of the equation was dependent in some degree on the physical interpretation of the variables. For instance, the Dirac equation has a high degree of symmetry, because space, time, energy, and momentum 'appear on an equal footing'.[8]

Wilczek held that the equations that described the world tended to be symmetrical: 'Nature loves to use such equations'.[9] He suggested thought that there might be an evolutionary reason for our recognizing beauty in successful theories. Evolution, he thought, had guided our development so that we took pleasure in correctness, and we have learned to see as beautiful certain patterns in the ideas that work.[10] Thus the link between beauty and symmetry depended ultimately on a selective pressure that has conditioned us to find symmetry pleasing. But since Wilczek's notion of symmetry in equations was bound up with the physical interpretation of those equations, this selection could not have acted in favour of only visual symmetry.

The effectiveness of beautiful mathematics in physics was, for Wilczek, due to the world having properties such as *locality* and *symmetry*.[†] As he put it in a 2006 essay, 'coarse-grained versions of local and symmetric equations remain local and symmetric'. Thus approxi-

[1] Wilczek, 'A Piece of Magic', p. 128.

[2] *Loc. cit.*

[*] ▸ pp. 884–5

[3] Wilczek, *A Beautiful Question*, pp. 117–18.

[4] *Ibid.*, ch. 'Maxwell 1'.

[5] *Ibid.*, p. 118.

[6] Wilczek, 'A Piece of Magic', p. 122.

[7] Wilczek, *A Beautiful Question*, p. 238.

[8] Wilczek, 'A Piece of Magic', p. 128.

[9] Wilczek, *A Beautiful Question*, p. 118.

[10] Wilczek, quot. in Hossenfelder, *Lost in Math*, ch. 7.

[†] ▸ pp. 939–40

mate versions of the currently-accepted laws of nature, such as were known in previous generations, were still beautiful. And thus the beauty of the currently-accepted laws of nature also suggests that the future and final laws of nature will be beautiful.[1] There is an echo here of Gell-Mann's view that there was a self-similarity at various levels in nature[*] which pointed to future physical theories being beautiful.

Wilczek worried that if McAllister's aesthetic induction model[†] were accurate, the successes in physics so far and the shaping of our sense of beauty around them might impede the discovery of a more fundamental theory because, by current standards, it might not be beautiful enough.[2] That same worry might apply to his own view that properties of the world such as locality and symmetry suggest that future theories should be mathematically beautiful, but perhaps Wilczek considered these to be objective properties that stood apart from the aesthetic judgement of beauty.

For Wilczek, supersymmetry was mathematically beautiful.[3] But he accepted that there was a lack of empirical evidence for it, since predicted particles that have not been observed experimentally.[4] One way of salvaging the concept, of 'having our supersymmetric cake and eating it', as he put it, would be to fall back on spontaneous symmetry breaking, where the fundamental physics laws have supersymmetry, but the less beautiful reality we inhabit exhibits only a partial symmetry.[5]

GARRETT LISI AND E_8

In 2007, Garrett Lisi (1968–) proposed a theory of everything which was based on the Lie group E_8 and which had the declared aim of 'describing all fields of the standard model and gravity as parts of a uniquely beautiful mathematical structure'.[6] The paper in which Lisi expounded his theory was entitled 'An Exceptionally Simple Theory of Everything', which was a play on words: E_8 is one of the Lie groups that do not fit into the four infinite classes, and is thus termed *exceptional*. The group E_8 is certainly acknowledged for its aesthetic value: for instance, the physicist Hermann Nicolai (1952–) said that E_8 was 'perhaps the most beautiful structure in all of mathematics, but it's very complex'.[7] Lisi almost seemed to suggest that it was the mathematical beauty of E_8 that made it the natural choice for the fundamental structure of the universe. But Jacques Distler (1961–) & Skip Garibaldi (1972–) argued, on an almost entirely mathematical basis, that all theories in a class that includes Lisi's lack certain properties that physical reality necessitates.[8]

In a reminiscence and meditation published in 2009, the mathematician Bertram Kostant (1928–2017) asserted the aesthetic supremacy of E_8 and agreed that, whatever the current problems with the idea,

[1] Wilczek, 'Reasonably effective I', p. 9.

[*] ▸ pp. 955–7

[†] ▸ pp. 793–5

[2] Wilczek, quot. in Hossenfelder, *Lost in Math*, ch. 7.

[3] Wilczek, *A Beautiful Question*, p. 312.

[4] *Loc. cit.*

[5] *Ibid.*, pp. 312–13.

[6] Lisi, 'An Exceptionally Simple Theory of Everything', p. 29.

[7] Nicolai, quot. in Whitfield, 'Journey to the 248th dimension'.

[8] Distler & Garibaldi, 'There is No "Theory of Everything" Inside E_8'.

there was an intuition that E_8 should play some fundamental role in a final theory:

> ‘In my opinion, E_8 is the most magnificent “object” in all of mathematics. E_8 is like a diamond with thousands of facets where each facet offers a beautiful different view of its internal structure. [...] Attempts have been made to involve E_8 in our understanding of elementary particles. Judging from criticisms, it is likely that these first attempts are flawed. Still, one cannot help having the gut feeling that a real understanding of our universe must somehow involve E_8.’[1]

OCTONIONS

The octonions are an exotic algebraic structure, discovered by analogue to the quaternions by John Graves (1806–70), who was a friend of William Rowan Hamilton* (1805–65) and brother to the latter’s biographer Robert Perceval Graves (1810–93).

The *Cayley–Dickson construction* allows one to ascend from the real numbers to the octonions and beyond. This construction can be applied to any algebra A equipped with addition and multiplication operations and a unary conjugation operation $\blacksquare \mapsto \blacksquare^*$. It yields a new algebra whose underlying set is pairs of elements of A and whose operations are defined by

$$\begin{aligned}(a,b)+(c,d) &= (a+c, b+d),\\ (a,b)(c,d) &= (ac-db^*, a^*d+cb),\\ (a,b)^* &= (a^*, -b).\end{aligned}$$

The real numbers $\mathbb{R}$ form such an algebra A, with the usual addition and multiplication operations and a conjugation operation that leaves a number unchanged. Starting with $\mathbb{R}$ and iteratively applying the Cayley–Dickson construction yields in turn the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, the octonions $\mathbb{O}$, and the sedenions $\mathbb{S}$. The dimension of each algebra is double the previous one, and in a sense properties are gradually lost: $\mathbb{R}$ is an ordered field; $\mathbb{C}$ is a (non-ordered) field; $\mathbb{H}$ is a division ring (or skew field); $\mathbb{O}$ is a division algebra;[2] $\mathbb{S}$ contains zero divisors and so is *not* a division algebra.[3] Just as one loses commutativity in passing from the complex numbers to the quaternions, so one loses associativity in passing from the quaternions to the octonions. A ghost of associativity remains in the *alternativity* of the octonions: the subalgebra generated by any two elements is associative.[4]

The division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ have dimensions 1, 2, 4, and 8, which are the only possible dimensions for a division algebra.

[1] Kostant, quot. in Cook, *Mathematicians*, p. 78.

* ▸ pp. 412–21

[2] Baez, ‘The Octonions’, § 2.2.

[3] *Ibid.*, p. 155.

[4] *Loc. cit.*

These four are not the only division algebras, but they are the only normed and the only alternative division algebras.[1] Among them, the algebra of octonions $\mathbb{O}$ is clearly the strangest beast. Ian Stewart (1945–) described the octonions as having 'a haunting, surreal mathematical beauty'.[2] They have fascinating and beautiful connections to other areas of mathematics. For instance, for any division ring $\mathbb{K}$, one can construct an n-dimensional projective space $\mathbb{KP}^n$.[3] In 1933 Ruth Moufang (1905–77) used the octonions to construct a non-Desarguesian projective plane $\mathbb{OP}^2$. This connection to projective geometry points to a deep relationship between the four normed and alternative division algebras and the theory of Lie groups. The groups of isometries of $\mathbb{RP}^n$, $\mathbb{CP}^n$, and $\mathbb{HP}^n$ are (eliding various technicalities) the classical simple Lie groups. The group of isometries of $\mathbb{OP}^2$ is the exceptional Lie group F_4, and those of $(\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2$, $(\mathbb{H} \otimes \mathbb{O})\mathbb{P}^2$, $(\mathbb{O} \otimes \mathbb{O})\mathbb{P}^2$ are (in some sense) the exceptional Lie groups E_6, E_7, E_8. The last exceptional Lie group, G_2, is the automorphism group of the octonions.[4]

In 2011, John Baez (1961–) & John Huerta suggested that octonions might play a role in string theory. It has long been held that string theory is only consistent in 10 dimensions. The octonions would account for 8 dimensions of space, the extension of the string itself for the ninth, and time the tenth. So if string theory is correct, the octonions would explain why the universe must have 10 dimensions.[5] Baez & Huerta noted that the lack of experimental evidence meant that science was far from deciding whether the octonions played such a fundamental role in physical reality or were 'merely a piece of beautiful mathematics'.[6] But, in a perhaps unconscious echo of Chen-Ning Yang (1922–2025) and Tsung-Dao Lee's (1926–2024) attempt to devise a quaternion-based field theory,* they admitted that it would be desirable if the universe turned out to have such a mathematically beautiful foundation:

> 'Of course, mathematical beauty is a worthy end in itself, but it would be even more delightful if the octonions turned out to be built into the fabric of nature.'[7]

SABINE HOSSENFELDER

In her 2018 book *Lost in Math*, Sabine Hossenfelder (1976–) wrote that physicists drew upon hidden rules to evaluate an untested theory; these rules were concepts of 'naturalness, simplicity or elegance, and beauty'. [One might contest whether these rules were actually hidden, or just *unarticulated* in the formal scientific discourse.] Hossenfelder thought them 'in utter conflict with the scientific mandate of objectivity'. Further, she was one of a seemingly

[1] Baez, 'The Octonions', p. 150, Thms 1, 2.

[2] Stewart, *Why Beauty is Truth*, p. 263.

[3] Baez, 'The Octonions', p. 165.

[4] *Ibid.*, § 4.

[5] Baez & Huerta, 'The Strangest Numbers in String Theory', p. 65.

[6] *Loc. cit.*

* ▸ pp. 917–18

[7] *Loc. cit.*

small number of physicists who thought that these hidden rules were unhelpful to science.[1]

Hossenfelder admitted that she was not immune to beauty in scientific theories. When the particle physicist Gian Francesco Giudice (1961–) said that encountering a beautiful theory and contemplating a piece of art produced the same emotional reaction, Hossenfelder recognized the experience.[2] But she wrote bluntly: 'I don't know why it matters.' She doubted that her sense of beauty could be a guide in formulating correct theories.[3]

In response to Schwarz's belief that the beauty of the mathematics of string theory meant it had to describe something physical,[*] Hossenfelder pointed out that mathematics was replete with beautiful concepts that had not found any role in the study of the physical world:

> 'I could belabor until the end of eternal inflation how unfortunate it is that we don't live in a complex manifold of dimension six because calculus in such spaces is considerably more beautiful than in the real space we have to deal with.'[4]

[There is an ironic contrast between Hossenfelder's point and Baez & Huerta's suggestion of an octonion-based physical reality.[†]]

Hossenfelder considered beauty and other aesthetic properties, such as simplicity, naturalness, and elegance. Each of these was subjective and anthropocentric.[5] In particular, she rejected the suggestion that the human brain might have an in-built sense for beauty in physical theories, since there was no apparent selection pressure that might have driven evolution to shape us thus.[6] From the subjectivity of these properties, she thus concluded that attempting to understand physicists' reliance on beauty led to it making even less sense.[7]

A particularly important point that Hossenfelder made was that naturalness had started off as an aesthetic value, then become formalized into the technical naturalness that 't Hooft espoused;[‡] this mathematical definition of naturalness seems to have had an effect of obscuring its aesthetic origin.[8]

CONCLUSIONS

> ʻ Whenever a theory appears to you as the only possible one, take this as a sign that you have neither understood the theory nor the problem which it was intended to solve. ʼ
>
> — Karl Popper, *Objective Knowledge*, p. 266.

Edward Witten (1951–), a 1990 Fields medallist and one of the pioneers of string theory, thought that a lesson of the development of twentieth-century physics was that the equations that describe nature had beauty — 'just as real to anyone who's experienced it as the beauty of music' — and inner harmony.[1] The upshot of this lesson was that the equations of string theory *should* apply to nature:

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[1] Hossenfelder, *Lost in Math*, ch. 1.

[2] *Loc. cit.*

[3] *Loc. cit.*

[*] ▸ pp. 953–4

[4] *Ibid.*, ch. 8.

[†] ▸ p. 989

[5] *Ibid.*, ch. 5.

[6] *Ibid.*, ch. 1.

[7] *Ibid.*, ch. 5.

[‡] ▸ p. 952

[8] *Ibid.*, ch. 4.

> 'there must be skeptics out there who will tell you that these beautiful equations might have nothing to do with nature. That's possible, but it's uncanny that they are so graceful and that they capture so much of what we already know about physics while shedding so much light on theories that we already have.'[2]

Witten here looked to the twentieth century, and this focus suggests a broader point. Mathematical beauty as a guide to truth in natural science has involved only a small and relatively recent part of the histories of science and mathematics. There is the possibility that the success of beautiful mathematics in science will prove to have been but a brief interlude. Perhaps, as it looks further and deeper, physics will pass beyond the realm governed by beautiful mathematics — or even by *any* mathematics. The descriptions of nature in terms of beautiful mathematics; the triumphs of Einstein[*] and Dirac:[†] perhaps these achievements only apply to a set of patterns that humans can see at a particular level of reality.

Many physicists have indicated that symmetry contributes to beauty: for example, Steven Weinberg[‡] and Marcelo Gleiser (1959–).[3] Some even equate beauty and symmetry: for example, Anthony Zee[§] and Michio Kaku (1947–).[4] The search for a mathematically beautiful fundamental theory has, broadly speaking, involved seeking higher symmetries. But what if the desire for symmetry — or at least the *kind* of symmetry that has hitherto been sought — represents an aesthetic canon that has run its course? There is, after all, much mathematical beauty in chaotic systems (see Chapter 17). As Hossenfelder pointed out, perhaps it is there that physicists should look for the mathematics of the final theory: 'What if I told you that a truly beautiful fundamental theory would be highly chaotic, not symmetric?'[5]

Indeed, as Gleiser acutely observed, the very idea of a final theory is cultural: it may be *suggested* by the experience of previous unifications, but there is no scientific evidence that demands its existence.[6] In the end, perhaps the momentary pessimism that Eugene Wigner (1902–95) expressed at an Einstein centennial symposium in 1979 will turn out to have been justified:

> 'It is, of course, possible that the laws of nature cannot be given a joint, simple, and mathematically beautiful formulation, or at least that man's mind is not powerful enough to discover such a formulation.'[7]

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[1] Witten, quot. in McMaster, 'Viewpoints on String Theory'.

[2] Witten, quot. in *ibid.*

[*] ▸ pp. 886–92

[†] ▸ pp. 900–13

[‡] ▸ p. 962

[3] Gleiser, *A Tear at the Edge of Creation*, ch. 26.

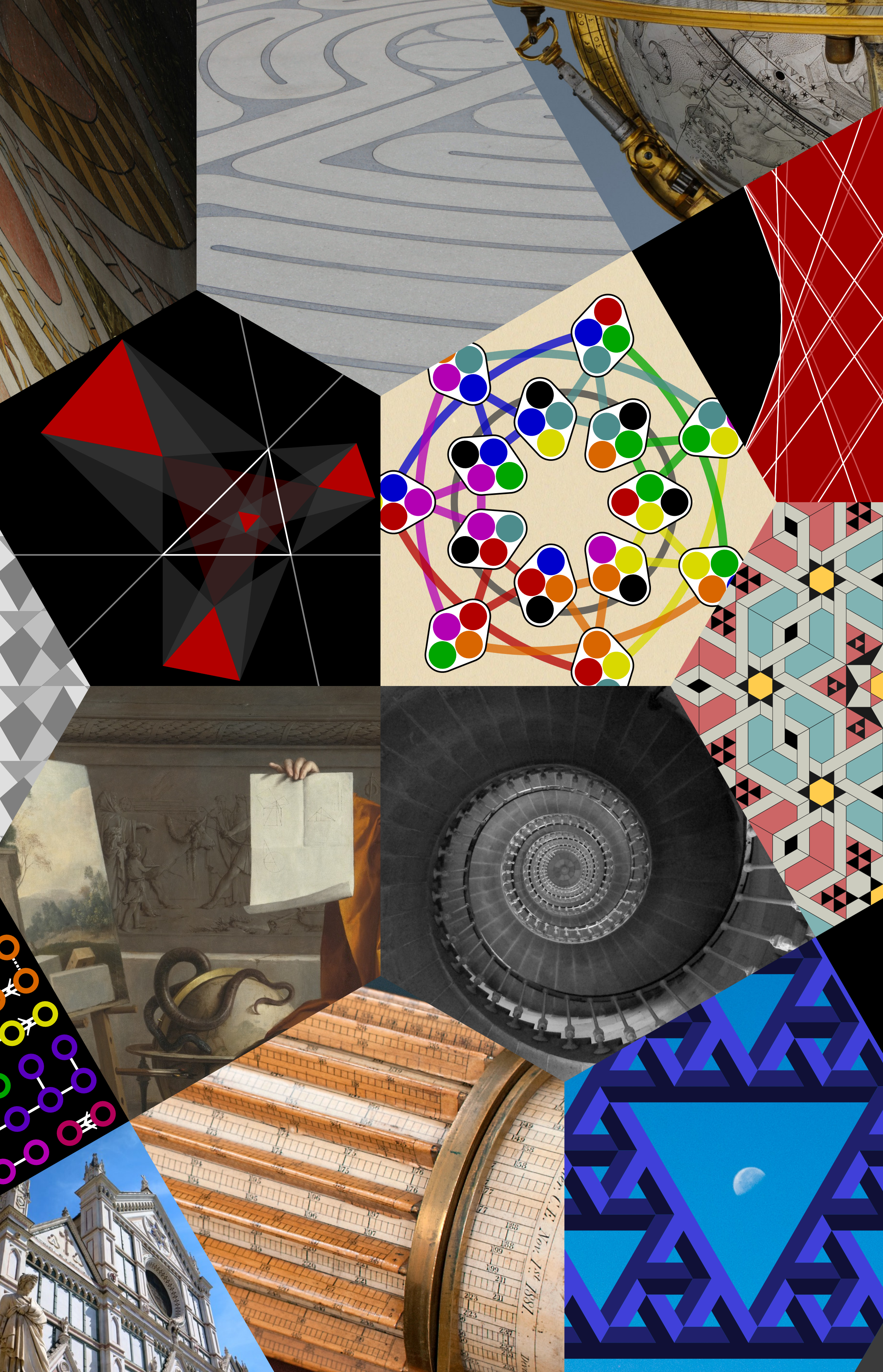
[§] ▸ pp. 963–4

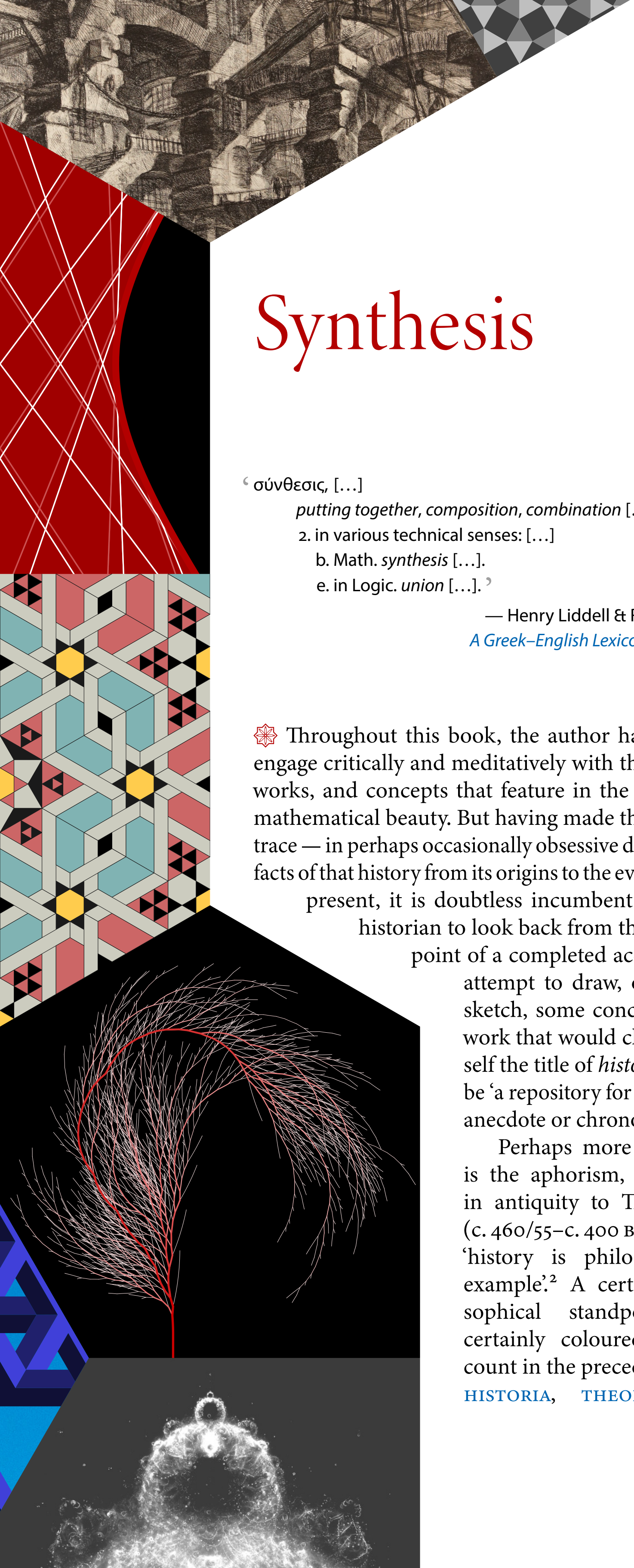
[4] Kaku, *The God Equation*, ch. 2.

[5] Hossenfelder, *Lost in Math*, ch. 7.

[6] Gleiser, *A Tear at the Edge of Creation*, ch. 31.

[7] Wigner, 'Thirty Years of Knowing Einstein', p. 466.





IV

Synthesis

> ‘σύνθεσις, […]
> *putting together*, *composition*, *combination* […].
> 2. in various technical senses: […]
> b. Math. *synthesis* […].
> e. in Logic. *union* […].’
>
> — Henry Liddell & Robert Scott,
> *A Greek–English Lexicon*, ‘σύνθεσις’.

Throughout this book, the author has tried to engage critically and meditatively with the authors, works, and concepts that feature in the history of mathematical beauty. But having made the effort to trace — in perhaps occasionally obsessive detail — the facts of that history from its origins to the ever-moving present, it is doubtless incumbent upon the historian to look back from the vantage-point of a completed account and attempt to draw, or at least sketch, some conclusions. A work that would claim for itself the title of *history* should be ‘a repository for more than anecdote or chronology’.[1]

Perhaps more pertinent is the aphorism, attributed in antiquity to Thucydides (c. 460/55–c. 400 BCE), that ‘history is philosophy by example’.[2] A certain philosophical standpoint has certainly coloured the account in the preceding parts, HISTORIA, THEORIA, and

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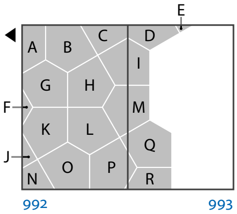
[1] Kuhn, *The Structure of Scientific Revolutions*, p. 1.

[2] [Unknown], *Ars Rhetorica*, § XI.2 ⟦p. 376⟧.

PHYSIKOS, but in the two chapters that comprise this short final part it becomes more evident. Chapter ω is a reflection and a meditation, and contains a series of questions about the aesthetics of mathematics that the author believes philosophy should address. The final Chapter Ω is less analytical and contains some more personal ruminations on mathematical beauty.

> ‘The history of the sciences and arts is a tale
> of recurrent crises, of traumatic challenges […];
> followed by the liberation from restraint of creative potentials,
> and their reintegration in a new synthesis.’
>
> — Arthur Koestler, *The Act of Creation*, p. 461.

◆



A Perspective-transformed detail from Hans Holbein the Younger, *The Ambassadors*. B Detail of a meditational labyrinth. C Detail from a 16th-century celestial globe. D Detail from Giovanni Battista Piranesi, *Carceri d'invenzione* (2nd ed., 1761) XIV: *l'Arco Gotico*. E Snub square tiling. F Egan's solution to the dissection of a triangle and a square with equal areas into four congruent pieces. G Morley's trisector theorem for internal and external angle trisectors. H Steiner quadruple system of order 8, showing symmetries and the derived Steiner triple system of order 7. I Hyperboloid. J Dynkin diagrams characterizing the simple Lie algebras. K Detail from Laurent de La Hyre, *Allegory of Geometry*. L Helical staircase. M Tiling from Prisse d'Avennes, *L'Art Arabe*, vol. 1, pl. LI. N Façade of Basilica di Santa Croce, Florence. O Thacher's Calculating Instrument, a cylindrical slide-rule. P Penrose triangles compounded into an approximation of a triangular Sierpiński gasket. Q A tree generated from Collatz sequences. R The Buddhabrot.

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Reflections

ω

Lessons and questions from the history of mathematical beauty

> ‘Beauty is eternity gazing at itself in a mirror.
> But you are eternity and you are the mirror.’
>
> — Kahlil Gibran, *The Prophet*, p. 86.

> ‘This is reflection, or meditation, or rumination;
> it is also celebration.
> In the loose sense of the word, it is “philosophical.”’
>
> — Robert Bringhurst, ‘The Philosophy of Poetry’, p. 98.

Edward Gibbon (1737–94) could sum up his work as an account of ‘the triumph of Barbarism and religion,’[1] but the historical development of mathematical beauty resists encapsulation in any such pithy phrase. Yet, if the account in the main parts of this book, HISTORIA, THEORIA, and PHYSIKOS, is even remotely coherent, there must be some conclusions that can be made from it. Moreover, surveying this history should make evident the lacunae in the scrutiny — philosophical, psychological, and practical — of mathematical beauty. Judgements and questions should both emerge.

A RECAPITULATION

Possibly the simplest place to begin is by listing the earliest extant attestations of beauty in some of the loci discussed in Chapter 0:[*]

Objects. According to one of the *akousmata*, Pythagoras (c. 570–c. 490 BCE) described the sphere as the most beautiful solid figure and the circle as the most beautiful plane figure.[†] The *akousmata* are sayings attributed to Pythagoras; the source for this one is the biographer Diogenes Laertius' (*fl.* 3rd century CE) life of Pythagoras,[2] but it is held to be authentic.[3]

If this *akousma* is not accepted, then Plato's (c. 427–348/347 BCE) *Timaeus* supplies multiple assertions of the beauty of mathematical objects:[‡] circles and spheres; the right-angled isosceles triangle; the scalene triangle with sides in the ratio $1 : \sqrt{3} : 2$; and the five regular solids. Depending on the demarcation between *object* and *concept*, the geometric mean might be included too.

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[1] Gibbon, *The Decline and Fall of the Roman Empire*, ch. LXXI.

[*] ▸ pp. 6–13

[†] ▸ p. 44

[2] Diogenes Laertius, ‘Pythagoras’, MEI 8.35.

[3] See Burkert, *Lore and Science*, p. 168, n. 18.

[‡] ▸ pp. 61–70

Results. The earliest known description of results as beautiful comes in Apollonius' (c. 260–c. 190 BCE) preface to the third book of his *Conics*, where he wrote that among the theorems on a particular topic, 'the greatest part and the most beautiful are new';[1] this phrase is also the only extant instance of an ancient Greek mathematician describing a result as 'beautiful'.[2][∗]

Proofs. The earliest extant judgement of a proof as beautiful seems to have been by al-Nasawī (*fl.* 1029–44), who so described a proof by Abū Sahl al-Kūhī (*fl.* c. 970–c. 1000) of a pseudo-Archimedean geometrical theorem.[†] In the European tradition, the first description of a proof as beautiful was Leonardo Pisano's (c. 1170–after 1240) described Archimedes' (c. 287–212 BCE) proof that $3\frac{10}{71} < \pi < 3\frac{1}{7}$ as beautiful in his *De Practica Geometrie*.[3][‡] Within a few years, Roger Bacon (c. 1219–c. 1292) also described geometric demonstrations in the *Book of Burning Mirrors* as beautiful.[§]

Theories. It is difficult to decide where to place the earliest attestation of beauty in a theory. Regiomontanus (1436–76) called algebra 'that most beautiful art',[4][¶] but he was referring to a disparate collection of results, not fully separate from arithmetic, established by Diophantus (*fl.* c. 250 CE) and mediaeval Islamicate authors. Luca Pacioli (c. 1445–1517) wrote of 'beautiful and gentle geometry',[5][♠] but he may have referred to its objects rather than have conceived it as a theory.

The earliest indisputable descriptions of a theory as beautiful seem to be due to Pierre de Fermat (1607–65), who described plane loci as being 'the most beautiful theory in all geometry'[6][♥] and the theory of numbers as 'very beautiful and very subtle'.[7][♦]

Definitions. The idea that a definition can be beautiful seems to have emerged very late; the earliest attestation seems to have been Gian-Carlo Rota's (1932–99) assessment of the definitions of category theory as beautiful.[8][♣]

Broadly, there seems to have been a shift from beauty being seen solely in mathematical objects, to its being seen in results, then in proofs, and later in theories and definitions. But a modicum of caution is required here. The attestations to the objects of mathematics being beautiful were not made by mathematicians: Pythagoras was a religious leader who may have contributed to philosophy; Plato was a philosopher with an admiration for mathematics. Neither is known to have been a mathematician, in the sense of having contributed to mathematics. One should not dismiss the aesthetic judgements of non-mathematicians as somehow invalid, but it may be significant that the later attestations to beauty in results, proofs, theories, and definitions were made by mathematicians.

[1] Apollonius, *Coniques*, vol. 1.2, p. 4, ll. 5–11; bk 1, preface; trans. Apollonius, *Conics* (Ed. Taliaferro), p. 603.

[2] Netz, 'What did Greek Mathematicians Find Beautiful?', p. 428.

[∗] ▸ pp. 113–14

[†] ▸ p. 212

[3] Fibonacci, *De Practica Geometrie*, ch. 3, § 194, p. 154.

[‡] ▸ pp. 179–80

[§] ▸ pp. 180–1

[4] Regiomontanus, 'Oratio', p. 46; trans. Byrne, 'A Humanist History of Mathematics?', p. 55.

[¶] ▸ p. 246

[5] Pacioli, *Summa*, pt II (new pagination), fol. 1r.

[♠] ▸ p. 247

[6] Fermat, 'Apollonii Pergæi Libri Duo', p. 3; Fermat, *Œuvres*, vol. III, p. 3.

[♥] ▸ p. 298

[7] Fermat, 'Observations', pp. 333–4; Fermat, *Œuvres*, vol. III, p. 269; trans. C., emphasis added.

[♦] ▸ p. 300

[8] Rota, 'Phenomenology of Mathematical Beauty', p. 173.

[♣] ▸ p. 743

REPRESENTATION AND AESTHETIC JUDGEMENT

Representation and aesthetic judgement

The study of mathematical beauty, like that of any intellectual beauty, must confront the question of the representation of the aesthetic object.* The study of other forms of beauty are also concerned with this question, but to a lesser degree: a painting is always considered as seen, a symphony as being heard. The painting may be seen in a photograph or other reproduction, a symphony may be heard in a recording. But visual and auditory beauty are always perceived through sight and hearing. A beautiful poem may be read or heard recited, but its beauty is always perceived through language and its rhythm.

* ▸ p. 764

But a beautiful mathematical proof may be read in a book, explained on a blackboard, described by a colleague using hand speech and hand-gestures over tea, or simply summoned from memory. A beautiful chess problem may be set up on a chessboard, shown on a diagram, or described by the notated moves that lead up to it, for a strong player can maintain an unaided mental image of the position. In some sense, the 'same' proof or chess problem is communicated. But a question arises:

QUESTION 1. Is there a difference in the aesthetic response, depending on how a piece of mathematics (or, more generally, other intellectual object) is communicated?

While there have been studies of the aesthetics of mathematical texts, notably in Reviel Netz's (1968–) book *Ludic Proof*, there seems to have been little examination, philosophical or empirical, of the degree to which the representation of a piece of mathematics influences aesthetic judgements of it. [In 1995, David Henley pointed out, apropos of G. H. Hardy's (1877–1947) analysis of seriousness in mathematics, that the interest or seriousness of a piece of mathematical may depend on the properties of mathematical language rather than of the mathematical objects; that is, that it may be *syntactic*.[1] His observation does not address beauty *per se*, but it suggests that the value of mathematics may depend on how it is represented.]

The scarcity of attention paid to representation is perhaps symptomatic of much of the literature on mathematical beauty being written by mathematicians, who are in general accustomed to — and even trained to — separate the mathematical content of a statement or proof and its form of expression.[2] This separation is related to an issue in the history of mathematics: the recasting of ancient mathematics in modern notation. For instance, T. L. Heath's (1861–1940) editions of the works of Archimedes and Apollonius proclaim on their title pages that they are 'edited in modern notation'; they are less translations than expositions for the modern reader.[3] Such translations are valuable insofar as they make more accessible the results and arguments of the ancients. But they obscure the *way* in which the results were

[1] Henley, 'Syntax-directed discovery', p. 247.

[2] See, for example, Artmann, *Euclid*, p. 147.

[3] Heath, *Aristarchus*; Archimedes, *Works*.

presented and proved, and the form of mathematical thinking generally, even if one does not go quite so far as Netz in saying that '[s]tyle and mode of presentation are not incidental to a mathematical proof: they constitute its soul'.[1]

NOVELTY, FAMILIARITY, AND CHANGING AESTHETIC EVALUATION

Roger Penrose (1931–) wrote that 'when one first comes across something it may have a certain beauty which later disappears when it becomes too familiar — indeed it may become quite boring after a while'.[2] He made this remark in the context of a discussion of aesthetic value that a child might see in very elementary arithmetic, such as in the result that $2 + 2 = 2 \times 2$, which might have a certain appeal that vanishes as one grows in mathematical (or even just arithmetical) ability. François le Lionnais (1901–84) thought that the reason Euler's equation holds had become so clear that the equation itself could not be called beautiful, but 'if not insipid, at least wholly natural'.[3]*

This raises a broader question of whether, or to what degree, aesthetic responses to the same piece of mathematics might change over time. Certainly one would expect that any judgement of novelty would change, but what about judgements of beauty or elegance? Gerhard Domanski, one of the respondents to David Wells's (1940–) questionnaire on the beauty of theorems,† said that he scored each result by trying to recall his feelings when he first encountered it.[4]

The judgement of the beauty of a piece of mathematics has been held to be connected to its context and the ideas it involves. Thus it is natural that as one learns (or indeed forgets) about related ideas, one's judgement of the beauty of a theorem or proof may evolve, which fits with the change Le Lionnais saw in the beauty of Euler's equation. But there seems to have been no systematic examination of how changes of knowledge influences judgements of beauty.

QUESTION II. Does a judgement of beauty of a piece of mathematics remain unchanged across successive contemplations? Or, for instance, does a sufficiently large number of encounters with the same piece of mathematics breed *ennui* and hence a diminution of beauty? If there is a change, is it connected with increased or diminished knowledge?

[1] Netz, intro. to Archimedes, *Sphere and Cylinder*, p. 3.

[2] Penrose, 'The Rôle of Aesthetics', p. 268.

[3] Le Lionnais, 'La Beauté en Mathématiques', p. 444; trans. C.

* ▸ pp. 840–1

† ▸ pp. 834–7

[4] Wells, 'Are these the most beautiful?', p. 39.

RECURRING AESTHETIC CRITERIA

Certain ideas and conceptions regarding aesthetic value in mathematics do find echoes over time, without there being any apparent historical connection. For instance, Dugald Stewart's (1753–1828) condition for an 'arrangement' of mathematical concepts, results, and proofs being beautiful was that it should be based on few fundamental ideas; a large variety of consequences should be shown to follow from a small number of premisses.[*] A similar thought was articulated in the twentieth century by Bertrand Russell (1872–1970) and Alfred North Whitehead (1861–1947): a mathematical theory should not contain superfluous premisses.[†]

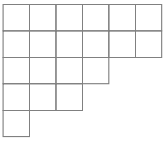
CORRECTNESS, RIGOUR, AND BEAUTY

QUESTION III. Is it possible for a piece of mathematics to be beautiful but incorrect or invalid?

The mathematician Reuben Hersh (1927–2020) thought so: 'Mathematicians prefer a beautiful proof, even if it contains a serious gap, over a dull, boring, correct one.'[1] And the various editors of Ramanujan's notebooks knew certain of his claims to be false but still thought them beautiful.[‡]

A colleague of the present author once said that if a beautiful piece of mathematics was found to be invalid, it did not cease to be beautiful; it simply ceased to be mathematics. But perhaps an invalid proof can survive as a heuristic or a plausibility argument.

Take a concrete example: an incorrect proof of the *hook length formula*, which concerns Young diagrams and Young tableaux. A *Young diagram* is an array of cells, left-justified, with each row at least as long as the one below it, such as the following:



(D)

A *standard Young tableau* is a Young diagram with n cells in which each number in $\{1, 2, \dots, n\}$ is placed in exactly one cell so that the entries in each row are increasing from left to right and the entries in each column are increasing from top to bottom; such as:

1	2	4	11	16	17
3	6	9	13	19	20
5	7	10	14		
8	15	18			
12					

---

[*] ▸ p. 384

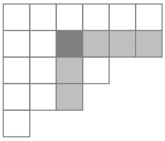
[†] ▸ pp. 481–2

[1] Hersh, 'Proving is convincing and explaining', p. 394; cf. Hersh, *What is Mathematics, Really?*, p. 52.

[‡] ▸ pp. 671–2

The *shape* of a standard Young tableaux is its underlying Young diagram. [Young tableaux play an important role in algebra, combinatorics, and representation theory.[1]]

There is a neat formula for counting the number of standard Young tableaux of a given shape. The *hook* of a cell in a Young diagram consists of the cell itself, the cells to its right in the same row, and the cells below it in the same column; the hook of the $(2, 3)$-th cell of the Young diagram (D) is



The *hook length* of a cell is the number of cells in its hook.

HOOK LENGTH FORMULA. *The number of standard Young tableaux with entries from* $\{1, 2, \dots, n\}$ *with a given shape is* $n!$ *divided by the product of the hook lengths of that shape.*

This result was discovered in 1954 by J. S. Frame (1907–97), G. de B. Robinson (1906–92) & R. M. Thrall (1914–2006).[2] The computer scientist and doyen of algorithmics Donald Knuth (1938–) remarked that such a simple rule deserved an equally simple proof and noted that there was a fallacious proof that functioned as a heuristic argument:[3]

Fallacious proof. A Young diagram with every cell filled forms a standard Young tableaux if and only if the entry in each cell is the smallest in its hook. The number of ways of filling a Young diagram with n cells using the numbers 1 to n is $n!$. Such a filling results in a Young tableau if and only if each cell's entry is the smallest in its hook; the probability of this is the reciprocal of the hook length. So multiplying $n!$ by the product of these probabilities gives the number of Young tableaux with the given shape. ☐

The argument fails because the probabilities are not independent. It therefore does not constitute a proof. But it continues to be a beautiful, or at least elegant, heuristic argument.

Replacing the terms 'Young diagram', 'Young tableau', 'hook', and 'hook length' in the incorrect proof of hook length formula above with 'binary tree', 'binary search tree', 'subtree', and 'subtree sizes' gives a valid proof of the corresponding result for binary search trees; the proof works because in this situation the probabilities *are* independent.[4] So there is a beautiful proof, concerning different objects, that has precisely the same structure as the heuristic argument.

[1] See, for example, Fulton, *Young Tableaux*.

[2] Frame, Robinson & Thrall, 'The Hook Graphs of the Symmetric Group', Thm 1.

[3] Knuth, *The Art of Computer Programming*, vol. 3, p. 60.

[4] *Ibid.*, vol. 3, Exer. 5.1.4.20.

PLATONISM AND BEAUTY

The chemist and philosopher of science Michael Polanyi (1891–1976) wrote in *Personal Knowledge* that

> ‘we dwell on mathematics and affirm its statements for the sake of its intellectual beauty, which *betokens the reality of its conceptions and the truth of its assertions*. For if this passion were extinct, we would cease to understand mathematics; its conceptions would dissolve and its proofs carry no conviction. Mathematics would become pointless and would lose itself in a welter of insignificant tautologies’.[1]

The part of this quotation after the emphasized phrase seems to indicate that Polanyi thought that, without the guidance and the sieve of aesthetic judgement, mathematics would dissolve into a logical morass which could only be grasped on a shallow level, in which proofs and results would have no value beyond validity, and that we would simply not *care*. But the emphasized phrase itself shows that mathematical beauty was even more important. For Polanyi, it was the beauty of mathematics that offered assurance that mathematics was *about something*; theorems and demonstrations were not merely *valid*, but *true*. Polanyi here made an inference from mathematical beauty to mathematical platonism.

Although Polanyi was a man of wide intellectual horizons, he was not a mathematician.[2] But Ivar Ekeland (1944–), who *was*, made a similar point even more bluntly in his book *The Broken Dice*:

> ‘Most mathematicians are Platonists. This is *because* of the transcendent beauty of mathematical structures, which don’t seem to have been created by the hand of man.’[3]

Polanyi’s implicit and Ekeland’s explicit statements are *prima facie* curious claims: why should an aesthetic response be an impulse towards mathematical platonism?

Perhaps a point made by the philosopher Jan Zwicky (1955–) can shed light on this problem. Reflecting on her own experience of understanding a proof and truly *knowing* the result thus established, Zwicky wrote that

> ‘the perception of necessary truth involves a kind of intellectual *phenomenology* — that necessary truth has a distinct *feel*’.[4]

This *feeling*, thought Zwicky, was evoked especially when a mathematical truth was beautiful or elegant.[5]

Other testimonies resonate with Zwicky’s: think of Russell’s description of ‘[t]he true spirit of delight, the exaltation, the sense of being more than man’.[6]* Or, further back, consider John Aubrey’s (1626–97) account of the philosopher Thomas Hobbes (1588–1679) encountering geometry for the first time at the age of forty. Hobbes,

[1] Polanyi, *Personal Knowledge*, ch. 6, § 11, emphasis added.

[2] Szabadváry, ‘Polányi, Mihály’.

[3] Ekeland, *The Broken Dice*, p. 54–5, emphasis added.

[4] Zwicky, *Plato as Artist*, p. 47, emphasis in orig.

[5] Zwicky, *Plato as Artist*, pp. 47, 50, 52; cf. Zwicky, *Wisdom & Metaphor*, pp. 37, 64, 66.

[6] Russell, ‘The Study of Mathematics’, p. 60.

* ▸ p. 479

wrote Aubrey, happened upon a copy of Euclid's *Elements* — not his own — open at the pythagorean theorem:

> '*By G—*, sayd he [...] *this is impossible!* So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read. [...] at last he was demonstratively convinced of that trueth. *This made him in love with Geometry.*'[1]

In the light of this *feeling* of necessary truth, Polanyi's and Ekeland's statements about mathematicians' platonism seem less mysterious. The moment of understanding in mathematics is accompanied by a special *feeling* to which the apprehension of mathematical beauty is somehow related, and the feeling leaves behind it a particular impression: 'we finally see things *as they are*, the mystery has been solved'.[2] The phenomenology of mathematical beauty, of necessary truth, inculcates the idea that it is about something *real*.

QUESTION IV. Do those who experience mathematical beauty tend to hold to mathematical platonism? Is platonism associated more to the experience of beauty in specific loci, such as proofs?

OTHER AESTHETIC CATEGORIES IN MATHEMATICS

Sublimity

The aesthetic term *sublime* dates back to the first century CE treatise *On the Sublime* [*Περὶ ῞Υψους Peri Hypsous*], whose author is conventionally called Longinus, for whom *sublime* denoted 'a consummate excellence and distinction of language' that 'alone gave to the greatest poets and prose writers their preeminence'.[3] With the emergence of aesthetics as a philosophical discipline in the eighteenth century, *sublime* ceased to denote simply a property of rhetoric, and came to mean a quality related to grandeur or greatness wherever found. The posthumously-published *Essay on the Sublime* (1747) by John Baillie (d. 1743), an otherwise obscure figure, seems to have been the earliest work to use the term in this way.[4] Baillie sought to demonstrate what the sublime was, since *On the Sublime* deliberately avoided this fundamental task.[5] He saw sublimity in language as reflecting sublimity in nature,[6] where a sublime thing, such as a great river or ocean, had immense magnitude.[7] For Edmund Burke (1729–97), the sublime was found in objects that produced feelings such as danger, horror, or reverence.[8] The greatest difference between the sublime and the beautiful, at least in eighteenth-century theories, was the role of emotion in the sublime. The beautiful might be subject to calm intellectual contemplation, and might be associated to Platonic–Aristotelian qualities

[1] Aubrey, *Brief Lives*, 'Thomas Hobbes'; last emphasis added, others in orig.

[2] Ekeland, *The Broken Dice*, p. 55, emphasis added.

[3] Longinus, *On the Sublime*, § 1, p. 163.

[4] Brady, *The Sublime in Modern Philosophy*, pp. 17–18.

[5] Longinus, *On the Sublime*, § 1, p. 163.

[6] Baillie, *Essay on the Sublime*, p. 3.

[7] *Ibid.*, p. 5.

[8] Burke, *Philosophical Enquiry*, §§ I.7, II.1.

such as order and harmony; the sublime provoked emotional and even physical response.[1] The philosopher and historian of philosophy Emily Brady suggested that the sublime was best exemplified by

> 'natural objects or phenomena having qualities of great height or vastness or tremendous power which cause an intense emotional response characterized by feelings of being overwhelmed and somewhat anxious, though ultimately an experience that feels exciting and pleasurable.'[2]

QUESTION V. Does the sublime, in a sense that fits the preceding description, exist in mathematics?

The term *mathematical sublime* is well-established in aesthetics, but primarily signifies a response to vast dimensions or numbers of things. Immanuel Kant* (1724–1804) distinguished it from the dynamical sublime. For Kant, both the mathematical and dynamical sublime involved a relationship between imagination and reason: theoretical reason in the case of the mathematical sublime, and practical reason in the case of the dynamical. The mathematical sublime was bound up with, on the one hand, the failure to grasp a vast thing in a single act of imagination, and, on the other, the intellectual ability to grasp the *idea* of the infinite. The dynamical sublime was connected with the fear that a vast and powerful thing might overwhelm and harm us, and the intellectual ability not to yield to that fear.[3]

Some of the eighteenth-century pioneers of philosophical aesthetics mentioned, without delving deeply into, the sublime *in* mathematics. Henry Needler (*fl.* 1690–1718), who died young but left a volume of poetry and essays successful enough to go through three editions in eleven years,[4] wrote 'On the Excellency of Divine Contemplation', which at least foreshadowed later discussions of the sublime. His remarks on mathematics, through phrased in terms of 'grandeur and magnificence', seem to point to sublimity, and are noteworthy for using the example of a specific mathematical object, namely Torricelli's solid:†

> 'The Grandeur and Magnificence of an Infinite Object affect it with an ineffable Complacency; Somewhat like that we take in beholding the vast Expanse of the Firmament, or the spacious Surface of the Ocean. Thus we find that those Theorems in Geometry, which have a relation to Infinite, afford the greatest Delight and Satisfaction to curious Minds: such, for example, as that of *Torricellius*, that a Solid of Infinite Length, may be equal to a Finite Quantity'.[5]

At the end of his *Essay on the Sublime*, Baillie regretted not having considered the sublime in the sciences, and only noted, in his penultimate paragraph, that 'in mathematics it is the universal theorems, and the vast similarity in the properties of infinite curves'.[6] Baillie had

[1] Brady, *The Sublime in Modern Philosophy*, p. 40.

[2] *Ibid.*, p. 6.

* ▸ pp. 374–82

[3] Kant, *Critique of the Power of Judgment*, §§ 25–8; KGS vol. 5, pp. 248–62.

[4] Drennon, 'Henry Needler and Shaftesbury', p. 1096.

† ▸ p. 305

[5] Needler, 'On the Excellency of Divine Contemplation', p. 80.

[6] Baillie, *Essay on the Sublime*, p. 41.

a recognized influence on Alexander Gerard (1728–95),[1] who made a narrower point when he mentioned 'the sublime of science',

> 'which lies in universal principles and general theorems, from which, as from an inexhaustible source, flow multitudes of corollaries and subordinate truths.'[2]

Needler and Baillie had allowed infinity *in* mathematical objects as a basis for sublimity, and Baillie also pointed to the universal range of theorems. Gerard said nothing of objects, but looked only to the scope and innumerable consequences of theorems. Joseph Priestley (1733–1804), in his *Lectures on Oratory and Criticism*, seemed to start by following Gerard's narrower conception, locating sublimity in the theorems and their consequences:

> 'The sublime of science consists in general and comprehensive theorems, which, by means of very great and extensive consequences, present the idea of *vastness* to the mind. A person of true taste may perceive many instances of genuine sublime in geometry, and even in algebra.'[3]

But, almost as an afterthought, he mentioned objects: 'Theorems may also be sublime by their relating to great objects.'[4] The context suggests that this sentence applied to statements in the sciences generally and the relation of these statements to physical things. Even if Priestley's intended meaning included mathematical statements and objects, the sublimity was seen in the *theorem*, not, as Needler or Baillie had suggested, the *object*.

Some modern writers have discerned sublimity in the physical sciences,[5] A modern suggestion of the sublime in mathematics was made by Roald Hoffmann (1937–), one of the 1981 Nobel laureates in chemistry, and who wrote that '[t]he sublime may be embodied in a paper that is unique, that solves a problem of long standing',[6] such as in Andrew Wiles's (1953–) proof of Fermat's last theorem. Since a difficult problem is *ceteris paribus* likely to take longer to solve, Hoffmann's suggestion points to aesthetic judgements of sublimity being bound up with (non-aesthetic) judgements of the difficulty of a problem.

But there seems to have been little scholarly attention paid to whether there exist mathematical results, objects, proofs, definitions, or theories that are sublime in the sense of producing a response that includes a sense of being overwhelmed or anxious. Ernst Kummer (1810–93) perhaps came close when he interpreted G. W. Leibniz's (1646–1716) reaction* to the discovery of his alternating series for $\pi/4$ as astonishment at the infinite summation and compared it to the experience of seeing the limitless sea or a towering mountain. This analogy suggests the aesthetic response of sublimity, but Kummer himself thought it pointed to beauty:

> 'Every mathematician will be aware of such impressions, for in the realm of mathematics there is a peculiar beauty which does

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[1] Monk, *The Sublime*, p. 113.

[2] Gerard, *An Essay on Taste*, p. 14.

[3] Priestley, *Lectures on Oratory and Criticism*, p. 157.

[4] *Loc. cit.*

[5] Greig, 'Quantum Romanticism', See, for example; Hoffmann, 'On the Sublime in Science'.

[6] Hoffmann, 'On the Sublime in Science', p. 160.

* ▸ p. 316

> not resemble the beauty of artworks, but rather the beauty of nature, and which affects a sensible person who has gained an appreciation of it, in a very similar way to the latter.'[1]

The present author cannot ever recall feeling overwhelmed by the notion of infinity *per se*. There is perhaps a hint of the sublime in the Cantorian *hierarchy* of infinite ordinals and cardinals, but it seems more evident in the very large rather than the infinite. For instance, Knuth's up-arrow notation can be used to denote numbers that are too large for the usual system of numeration.[2] In this system, x^n is denoted $x{\uparrow}n$, and, in general,

$$x\overbrace{\uparrow\uparrow\ldots\uparrow}^{k\text{ arrows}} n = \underbrace{x\overbrace{\uparrow\ldots\uparrow}^{\substack{k-1\\ \text{arrows}}}\big(x\overbrace{\uparrow\ldots\uparrow}^{\substack{k-1\\ \text{arrows}}}\big(\cdots\overbrace{\uparrow\ldots\uparrow}^{\substack{k-1\\ \text{arrows}}}x\big)\ldots\big)}_{n\text{ appearances of }x}.$$

For example,

$$\begin{aligned} 3{\uparrow}3 &= 3^3 = 27;\\ 3{\uparrow\uparrow}3 = 3{\uparrow}(3{\uparrow}3) &= 3^{3^3} = 3^{27} = 7\,625\,597\,484\,987;\\ 3{\uparrow\uparrow\uparrow}3 = 3{\uparrow\uparrow}(3{\uparrow\uparrow}3) &= 3{\uparrow}3{\uparrow}\ldots{\uparrow}3 = 3^{3^{\cdot^{\cdot^{3}}}}, \end{aligned}$$

where the last staircase of exponents has 7 625 597 484 987 steps.

What is the minimum n such that any 2-colouring of the graph formed by connecting each pair of the 2^n vertices of an n-dimensional hypercube contains a 1-coloured complete subgraph on 4 coplanar vertices? As a consequence of the Graham–Rothschild theorem,[3] such a minimum n exists. Ron Graham (1935–2020) gave g_{64} as an upper bound for the minimum n, where for $k \in \mathbb{N}$,[4]

$$g_k = \begin{cases} 3{\uparrow\uparrow\uparrow\uparrow}3 & \text{if } k = 1,\\ 3\underbrace{\uparrow\ldots\uparrow}_{\substack{g_{k-1}\\ \text{arrows}}}3 & \text{if } k > 1. \end{cases}$$

The sheer vastness of the recursions that define g_{64} seems to elude any attempt to grasp it through imagination. One can intellectually encompass it, reasoning it to be finite and definable using a certain number of symbols, and the least significant digits of its decimal representation can be calculated: … 439 059 104 575 627 262 464 195 387.[5]

There is a historical precursor that one might speculate could have produced a similar response: Archimedes' *Sand-Reckoner*, which describes a system of numeration suitable for estimating the number of grains of sand that would fill the cosmos, a number that went far beyond what Greek numeration had hitherto reached. Archimedes' system collected numbers into classes based on 'a myriad myriad', or,

[1] Kummer, 'Öffentliche Sitzung zur Feier des Leibnizischen Jahrestages', p. 395; trans. C.

[2] Knuth, 'Mathematics and Computer Science: Coping with Finiteness'.

[3] Graham & Rothschild, 'Ramsey's Theorem for n-parameter Sets'.

[4] Gardner, *Penrose Tiles to Trapdoor Ciphers*, pp. 243–4.

[5] OEIS, A133613.

in modern terms, 10^8. The first order was from 1 to 10^8, the second was from 10^8 to 10^{16}, the third from 10^{16} to 10^{24}, ... The first 10^8 orders are gathered into the first period. The greatest number in the explicitly enunicated part of Archimedes' system was the end of the 10^8-th period; 'a myriad myriad units of myriad myriadth numbers of the myriad myriadth period': that is, $10^{8 \cdot 10^{16}}$.[1]

Kitsch

The term *kitsch* has acquired a number of associated meanings that perhaps make it even more difficult to define than *beautiful*. It is certainly a derogatory term, referring to art of questionable aesthetic value. It has connotations of banality, sentimentality (in a negative sense), vulgarity, insincerity, a formulaic character, and a want of artistic taste.[2] Whatever kitsch is, it would not be out of place among what Natalia Ginzburg (1916–91) called 'the little virtues': it can 'provide but meagre fare for human nature'.[3] No kitsch thing ever elevated the soul.

Clement Greenberg (1909–94), in his celebrated essay 'Avant-Garde and Kitsch', characterized kitsch as

> 'ersatz culture [...] destined for those who, insensible to the values of genuine culture, are hungry nevertheless for the diversion that only culture of some sort can provide.'[4]

Where previously the leisure to enjoy culture had generally been accompanied by a degree of cultivation, Greenberg thought that urbanization and universal literacy had produced a demand for culture among people whose education had not including shaping their tastes.[5] Kitsch filled the gap. Kitsch drew on mature culture, borrowing tricks and devices — discarding the rest, including the substance — and used them superficially in a system.[6] Kitsch provided a shortcut to the pleasure of art that avoided the difficulties; it was superfluous to reflect on kitsch, for the kitsch included the reflection and spared the viewer any effort.[7]

'Kitsch is not simply bad art, but bad art of a particular kind. Here "bad" is used not so much in an aesthetic as in a moral sense.'[8] A work of art might be bad for many reasons, including not being beautiful. Nevertheless, a work of art might be good and yet not beautiful. Kitsch can be and often is beautiful, but behind the façade there is nothing. For instance, William-Adolphe Bouguereau's (1825–1905) paintings are often beautiful and indicate his mastery of technique, but came to be reviled as kitsch.[9]

QUESTION VI. Does the category of kitsch apply to works of pure intellect, and in particular to mathematics?

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[1] Archimedes, *Arenarius*, pp. 237–41; Archimède, *L'Arénaire*, pp. 145–7.

[2] Higgins, 'kitsch', pp. 393–4.

[3] Ginzburg, 'The Little Virtues'.

[4] Greenberg, 'Avant-Garde and Kitsch', p. 10.

[5] *Ibid.*, pp. 9–10.

[6] *Ibid.*, p. 10.

[7] *Ibid.*, p. 15.

[8] Harries, *The Meaning of Modern Art*, p. 75.

[9] *Ibid.*, pp. 75–6, 81.

Rob Riemen's (1962–) characterization of kitsch as 'beauty without truth'[1] might be taken to imply that there can be no kitsch in mathematics, since without truth, a thing cannot be part of mathematics. Against this implication, one might point to a beautiful but flawed proof.[*] But Riemen's use of 'truth' was obviously not intended in a mathematical or even in a coldly epistemological sense; rather, he referred to some less well-defined genuineness that could be bound up with beauty.

Mathematical kitsch could be found in situations where there is a very superficial beauty that upon consideration evaporates. Calvin Clawson's (1941–2012) generally admirable book *Mathematical Mysteries* contains the following isolated blemish, which could qualify as kitsch. Clawson used the expression 'the beauty of this astounding relationship'[2] to refer to the following equation involving sums of Fibonacci numbers:

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1.$$

But although it might be *prima facie* beautiful and surprising, on reflection it is neither: a trivial application of the definition of the Fibonacci sequence ($F_1 = F_2 = 1$; $F_{i+1} = F_i + F_{i-1}$) reveals the superficiality:

$$\begin{aligned} F_{n+2} &= F_{n+1} + F_n && [\text{since } F_{n+2} = F_{n+1} + F_n] \\ &= 2F_n + F_{n-1} && [\text{since } F_{n+1} = F_n + F_{n-1}] \\ &= F_n + 2F_{n-1} + F_{n-2} && [\text{since } F_n = F_{n-1} + F_{n-2}] \\ &\vdots \\ &= F_n + F_{n-1} + \ldots + 2F_2 + F_1 && [\text{since } F_3 = F_2 + F_1] \\ &= F_n + F_{n-1} + \ldots + F_2 + F_1 + 1. && [\text{since } F_2 = 1] \end{aligned}$$

Mathematical kitsch — or at least the analogue of kitsch in mathematics — might perhaps best be characterized as *beauty without depth*. As in art, there is something superficially pleasing, but it fails to sustain attentive contemplation.

PHYSICAL REACTIONS TO MATHEMATICAL BEAUTY

The experience of music can trigger effects such as 'goosebump' formation or 'shivers down the spine'.[3] Such reactions seem to be connected to an emotional response to the music, with sadness seeming to be more closely correlated to these reactions.[4] The psychiatrist Anthony Storr (1920–2001) thought that the occurrence of such a physical reaction was precisely what a person means when they are 'moved' by a piece of music.[5]

[1] Riemen, *To Fight Against This Age*, ch. I.

[*] ▸ pp. 999–1000

[2] Clawson, *Mathematical Mysteries*, p. 127.

[3] Goldstein, 'Thrills in response to music and other stimuli'.

[4] Panksepp, 'The Emotional Sources of "Chills" Induced by Music'.

[5] Storr, *Music and the Mind*, p. 184.

Physical reactions to mathematical beauty

In spite of Storr's own avowed lack of enjoyment of mathematics — he wrote that 'G. H. Hardy appreciated music as little as I appreciate mathematics'[1] — he recognized that there was a close analogy between music and mathematics as creative practices that sought pattern.[2] But, he maintained, while there were parallels in the aesthetic appreciation of musical and mathematical works, they were distinguished by the physical aspect of the aesthetic response:

> 'Although aesthetic appreciation of each has much in common, mathematics does not affect the body in the ways that music does. […]
>
> Music is less abstract than mathematics because it causes physiological arousal.'[3]

Yet there are occasional references in the literature to just such physical reactions to mathematics, sometimes connected to the aesthetic response. Both Camille Jordan (1838–1922) and Paul Painlevé (1863–1922) recalled this reaction to the works of Charles Hermite (1822–1901) in their February 1901 obituaries of him:

> 'The impression produced on the geometers by all these works is summarized quite well in the picturesque words that we once heard from the mouth of M. Lamé: "In reading the memoirs of Hermite, *one gets goosebumps*."'[4]

> 'Those who have had the lucky pleasure of being the students of the great geometer, would never forget the almost religious tone of his teaching, *the shiver of beauty or mystery* that passed through his audience confronted with some admirable discovery or something unknown.'[5]

In response to being asked, at the end of a public dialogue, why he did mathematics, Serge Lang (1927–2005) responded:

> 'Why do you compose a symphony or a ballad? I already told you why. Because it gives me *chills in the spine* [frissons dans le dos]. That's why.'[6]

QUESTION VII. How common are such physical responses to mathematics? Are they connected to beauty? If so, do they correlate with particular loci of mathematical beauty?

A more intense physical response to beauty was described by Stendhal (Marie-Henry Beyle; 1783–1842), who wrote of how he was overcome by the beauty and culture of Florence when he visited in 1817:

> 'Absorbed in the contemplation of sublime beauty, I saw it up close, I touched it so to speak. I had reached that point of emotion where meet the *celestial sensations* given by the fine arts and the passionate sentiments. In leaving *Santa Croce*, I felt my heart beating […]; life was exhausted in me, I walked afraid of falling.'[7]

[1] Storr, *Music and the Mind*, p. 178.

[2] *Ibid.*, pp. 179–83.

[3] *Ibid.*, p. 183.

[4] Jordan, 'Charles Hermite', p. 130, emphasis added; trans. C.

[5] Painlevé, 'Ch. Hermite', p. 145, emphasis added; trans. C.

[6] Lang, *The Beauty of Doing Mathematics*, p. 17, emphasis added; cf. p. 49; orig. Lang, *fait des maths en public*, p. 26.

[7] Stendhal, *Rome, Naples et Florence*, vol. 1, p. 325; trans. C., emphases in orig.

His account led to the coining of the term *Stendhal syndrome* for a psychosomatic response to art, beauty, and culture, with various effects that could include disorientation or panic attacks, which seemed to affect some visitors to Florence. Whether there is a genuine specific psychiatric disorder is uncertain,[1] but at least one survey showed that some neurologists who were aware of the syndrome came to realize that they had a partial form of it.[2]

Assuming that Stendhal syndrome is some genuine psychological phenomenon, it seems to be at least partly connected to the aesthetic response to great beauty such as is found in the art and architecture of the Tuscan capital, which naturally prompts a question that seems never to have been considered:

QUESTION VIII. Can intellectual beauty, and specifically mathematical beauty, produce Stendhal syndrome, or at least a psychologically similar response?

PLEASURE IN NUMBERS

QUESTION IX. What is the nature of putative aesthetic judgments of individual numbers?

Arithmologists from antiquity and the middle ages, and some of their successors in the Renaissance, seem to have found aesthetic value in individual numbers, which sometimes seem to have been linked to mathematical notions such as perfect numbers. There are scattered later reports of similar aesthetic judgements of numbers, some of which hint at a link with their arithmetical properties.

For instance, in his psychological study *Inquiries into Human Faculty and Its Development*, Francis Galton (1822–1911) reported various divergent affective responses to numbers among his respondents. The only 'approach to unanimity' he found was the positive response to 12, which was in particular 'a more beautiful number than 10, from the many multiples that make it up'.[3]

Salo Finkelstein (1896/7–after 1942) who could mentally calculate with large numbers and rapidly memorize random sequences of digits, expressed aesthetic judgements of the numbers he manipulated. The abilities of Finkelstein were tested in the 1930s by the psychologist James D. Weinland (1894–1987?). Weinland said that Finkelstein had 'an æsthetic feeling in regard to numbers'[4] and quoted him as saying that '[n]umbers can have a beauty as of music or art'[5] and believing himself to be unique in having such a response.[6] His aesthetic feeling seems to have been stimulated by individual numbers: 214 was a 'beautiful' number.[7] While Weinland reported no other specific number to which Finkelstein had a specifically aesthetic response, there were other numbers to which he had an affective response: 375 he liked, 8337 was 'very nice', to 0 he had an aversion.[1]

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[1] Innocenti et al., 'La sindrome di Stendhal'.

[2] Guerrero, Barceló Rosselló & Ezpeleta, 'Stendhal syndrome'.

[3] Galton, *Inquiries into Human Faculty and Its Development*, § 'Number-Forms'.

[4] Weinland & Schlauch, 'Computing Ability of Salo Finkelstein', p. 383.

[5] Finkelstein, quot. in Weinland, 'The Memory of Salo Finkelstein', p. 255.

[6] *Loc. cit.*

[7] Finkelstein, quot. in *ibid.*, p. 253.

But Finkelstein said that simple numbers were 'nothing to him'; only 'in complex form' did he like them.[2] Thus his affective response to individual numbers was perhaps bound up with his mental manipulation of those numbers, which was in turn entangled in a web of arithmetic, cultural, and personal connotations. For instance, 41681 was for Finkelstein both a concatenation of three square numbers and the telephone number of an acquaintance.[3]

[It is unclear whether Finkelstein's aesthetic judgements of numbers were shared by those with similar abilities. Finkelstein seems to have been the only subject in S. B. Smith's survey *The Great Mental Calculators* to express any aesthetic views on numbers or mathematics.]

Finally, almost as an incidental aside in his book *The Aesthetic Brain*, the neurologist Anjan Chatterjee (1958–) described his own affective response to the number of participants in his experiments. His language, while not aesthetic *per se*, brings to mind Galton's report: numbers like 16, 18, 20 were 'good'; 17 and 19 produced a feeling of 'discomfort'. For Chatterjee, numbers with many divisors seemed 'more right for an experiment than prime numbers'.[4]

One other famous story of pleasure in numbers, related by the neurologist Oliver Sacks (1933–2015), must be mentioned, if only to question it. Sacks's tale concerned a set of twins who were given the then-accepted label of 'idiot savants' because their limited measured intelligence and conventional arithmetical abilities were complemented by their remarkable memories and their extraordinary capacity to perform calendrical calculations.[5] According to Sacks, in 1966[6] he witnessed them exchanging six-digit numbers with what he described as the air of 'two connoisseurs wine-tasting, sharing rare tastes, rare appreciations'.[7] He consulted a table of prime numbers and was able to verify that the numbers the twins exchanged were all primes. He joined their exchange by offering an eight-digit prime, at which, after deep concentration, they expressed joy. The twins responded with nine-digit primes; Sacks, after consulting his reference, offered a ten-digit prime. This provoked the twins into exchanging numbers with twelve and higher digits, whose primality Sacks could not check in that era before pocket calculators.[8] Sacks speculated that there was something aesthetic in the twins' uncanny ability to identify and produce large primes: 'they must have "sense" in their numbers — in the same way, perhaps, as a musician must have harmony'.[9]

It is perhaps to be regretted that Sacks's account must be approached with caution. The psychologist Makoto Yamaguchi pointed out that Sacks could hardly have had a book listing all 455 052 511 primes with at most ten digits, or even all 50 847 534 primes with at most nine digits.[1] According to Yamaguchi, Sacks said in correspondence that he had lost the book he used.[2] Yamaguchi initially did not doubt the gist of Sacks's account, only the detail,[3] but grew sceptical,[4] noting in particular that Sacks's claim that the twins could give the correct date of Easter over a wide range of years[5] contradicted contemporaneous studies showing that their ability in this regard was both error-prone and much narrower in scope.[6] Perhaps Sacks's memory of the episode had simply grown unreliable in the almost two decades before he set it down.[7]

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[1] Weinland, 'The Memory of Salo Finkelstein', p. 253.

[2] Weinland & Schlauch, 'Computing Ability of Salo Finkelstein', p. 286.

[3] Weinland, 'The Memory of Salo Finkelstein', p. 253.

[4] Chatterjee, *The Aesthetic Brain*, p. 54.

[5] Horwitz, Kestenbaum et al., 'Identical Twin—"Idiot Savants"—Calendar Calculators'.

[6] Sacks, *The Man Who Mistook His Wife for a Hat*, pp. 195, 203.

[7] *Ibid.*, p. 202.

[8] *Ibid.*, pp. 202–3.

[9] *Ibid.*, p. 204.

Even taking Sacks's account at face value, there is a glaring omission: Sacks did not report offering the twins any large non-prime numbers. He thus presented no evidence of how their responses differed to prime and composite numbers. It is conceivable, for example, that if a number 'seemed' prime to the twins, in the sense of passing some tests, then they might have reacted in the same way.

PRODIGY AND BEAUTY

The historian of mathematics Morris Kline (1908–92) observed that while it was natural to speak of a gift or genius for art or music or mathematics, no-one spoke of 'a gift for history or for economics or even for biology'.[8]

The philosopher and essayist George Steiner (1929–2020) made a similar point when he wrote that '[p]rodigies in mathematics and the sciences are far more frequent and identifiable than in, say, poetics or metaphysics'.[9]

Combine the examples of Kline and Steiner: there are prodigies or individuals who are gifted in art, mathematics, music, science, but not in biology, economics, history, poetics (*not* poetry) or metaphysics. There is at least one evident difference between these two classes of fields of endeavour. In art, mathematics, music, and in at least the more mathematical sciences — at least those that G. H. Hardy (1877–1947) would annex to 'real mathematics'[*] — a gift for the subject could be engaged after a certain minimum has been learned. In biology, economics, history, poetics or metaphysics, a large stock of existing knowledge must be assimilated before a new contribution can be made. This difference alone would make prodigies in the first class of subjects more common than in the second.

But there is also an aesthetic difference between the classes. Art, mathematics, and music are subjects where beauty can be created and contemplated. There is scant evidence of beauty being found in biology (*qua* scientific practice), economics, history, poetics (again, *not* poetry), or metaphysics.

These two differences may be entirely coincidental. But perhaps there is a connection. Kline and Steiner seemed to think so: Kline thought that ability in mathematics mirrored ability in art or music;[1] Steiner pointed to 'what are intuited to be subterranean links between mathematics, music, and chess'.[2]

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[1] Yamaguchi, 'Questionable Aspects of Sacks' Report'; see also OEIS, A006880.

[2] Yamaguchi, 'Questionable Aspects of Sacks' Report'.

[3] *Ibid.*

[4] Yamaguchi, 'On the Lack of Rigor in Research on Savant Syndrome', p. 91.

[5] Sacks, *The Man Who Mistook His Wife for a Hat*, p. 197.

[6] Horwitz, Deming & Winter, 'A Further Account of the Idiots Savants', p. 413.

[7] See Sacks, *The Man Who Mistook His Wife for a Hat*, pp. ix–x; Sacks, 'The Twins'.

[8] Kline, *Mathematics in Western Culture*, p. 470.

[9] Steiner, *Lessons of the Masters*, § 6.

[*] ▸ p. 507

Aesthetic sense is certainly a guide in music, and seems to be held to be a guide in mathematics (see Chapter 19). In particular, Henri Poincaré (1854–1912) and Jacques Hadamard (1865–1963) both thought aesthetic sense played a role in mathematical discovery. It is thus conceivable that an early development of such an aesthetic sense could play a role in the flowering of prodigy.

QUESTION X. Is there a connection between the aesthetics of a field and the emergence of prodigies?

RECOGNITION OF MATHEMATICAL BEAUTY

The scholar and critic Ernst Robert Curtius (1886–1956) remarked in a note in his *magnum opus* *European Literature and the Latin Middle Ages* that '"[m]odern" man immeasurably overvalues art because he has lost the sense of intelligible beauty that Neo-Platonism and the Middle Ages possessed',[3] and referred to Augustine's* (354–430 CE) conception of beauty as an attibute of God. The scholastic conception of beauty had nothing to do with art.

But, as this book has shown, the sense of intelligible beauty is still with us, in the guise of mathematical beauty; nor was it ever lost, as Morris Kline (1908–92) claimed,† or even buried, as Henry David Thoreau (1817–62) might have thought:

> 'We have heard much about the poetry of mathematics, but very little of it has yet been sung. The ancients had a juster notion of their poetic value than we. The most distinct and beautiful statement of any truth must take at last the mathematical form.'[4]

There are abundant discussions of beauty in popular mathematics books, websites, and videos; Escherian tessellations, fractals, and complex geometric designs have been used in virtually every medium of visual and plastic art. Anyone with even a modicum of curiosity and leisure and access must surely be aware of the *idea* of mathematical beauty, even if they have never perceived it themself.

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[1] Kline, *Mathematics in Western Culture*, p. 470.

[2] Steiner, *Lessons of the Masters*, § 6.

[3] Curtius, *European Literature and the Latin Middle Ages*, ch. 12, n. 20.

* ▸ pp. 161–6

† ▸ p. xxix

[4] Thoreau, *A Week on the Concord and Merrimack Rivers*, 'Friday'.

Last Thoughts

An envoi

Ω

‘ “ […] if you are determined to stick with mathematics,
it, too, will lead you in a spiral from perfection to perfection;
and you can take solace in the precepts of our eighteenth century,
which taught that man was originally good, happy, and perfect,
[…] and that by working critically to rebuild society
he shall become good, happy, and perfect again.” ’

— Thomas Mann, *The Magic Mountain*, ch. 6, § ‘Someone Else’
(trans. John E. Woods).

‘ History for us is the memory which is not only known to us,
but from which we live. ’

— Karl Jaspers, *The Origin and Goal of History*, p. 231
(trans. Michael Bullock).

And so here, at the end, what have *I* to say about mathematical beauty? For myself, I would prefer to claim nothing: the contributions of my antecessors — amongst whom number Plato, Aristotle, Apollonius, Proclus, Cardano, Fermat, Descartes, Leibniz, Shaftesbury, Hutcheson, André, Hilbert, Russell, Hardy, Dirac, … — are not to be aped by a lesser man. But there are certain topics on which I feel obliged to make a few remarks.

THE ETHICS OF BEAUTY

‘ Und es neigen die Weisen
Oft am Ende zu Schönen sich. ’

— Friedrich Hölderlin, ‘Sokrates und Alcibiades’, ll. 7–8.

The philosopher and historian Ananda K. Coomaraswamy (1877–1947) wrote what I consider to be the single most important response to G. H. Hardy’s* (1877–1947) *A Mathematician’s Apology*, in the closing paragraph of his 1941 review:

* ▸ pp. 498–523

> ‘As an “Apology,” Professor Hardy’s book is a defence of real or higher mathematics against those who raise objection to their uselessness (in the crude sense of the word). All he need have said is that mathematics as a whole serves needs both of the soul and of the body, like the arts of primitive man and those which Plato would have admitted to his Republic.

> That the higher mathematics have served his own soul well is shown by his concluding statement that, if he had a statue on a column in London, and were able to choose whether the column should be so high that the features of the statue would be invisible, or so low that they could be clearly seen, he would choose the first alternative;[1] and since it is man's first duty to work out his own salvation (from himself), no further defence is needed.'[2]

'[*T*]*he higher mathematics have served his own soul well*': Coomaraswamy thought that Hardy's dedication to mathematics had at least played a part in inculcating the value-system that would have him choose anonymity and immortality through his work over a transient, more personal, fame.

Regardless of whether one also agrees with Hardy's choice, one must concede that Coomaraswamy made a point that deserves examination: that it was Hardy's pursuit of mathematics that guided his ethical development in a fundamentally sound direction. But, as Hardy admitted in his *Apology*, what initially spurred him to pursue mathematics was 'far from noble': he sought to use his evident mathematical abilities to beat others in examinations and scholarships and, later, to gain a fellowship at Trinity College.[3] These motives suggest an early desire for personal recognition. If Hardy's account is accurate, and if Coomaraswamy's analysis is correct, it was Hardy's mathematical career that changed his desired monument from short to tall.

From antiquity descends the idea that intellectual endeavour can shape one's character. Cicero told us that 'Cultura [...] animi philosophia est':[4] philosophy is the cultivation of the soul. And consider what Jan Patočka (1907–77) identified as a, if not *the*, fundamental theme in western philosophy: CARE FOR THE SOUL.[5] This phrase does not indicate an education toward any immediately utilitarian goal, but an internal and fundamental and deliberate *forming of the self*.[6]

This thread can be discerned as far back as Democritus* (c. 460–c. 370 BCE), who wrote:

> 'It is fitting for people to take account of the soul rather than the body. For perfection of the soul puts right the bad state of the dwelling, but strength of the dwelling without thought does not make the soul any better.'[7]

The Platonic Socrates, in the *Apology* he delivered at his trial, claimed that he regularly exhorted those whom he met in the street and engaged in conversation:

> '"Most excellent of men, as an Athenian, a citizen of the greatest of cities and one most distinguished for wisdom and strength, aren't you ashamed to be spending your time acquiring as much money as you can, or gaining reputation and honor, but show no interest or concern for wisdom and truth and seeing to it that your soul will be in the best possible state?"'[8]

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[1] See Hardy, *Apology*, Note.

[2] Coomaraswamy, Review of *A Mathematician's Apology*, parenthetical reference deleted.

[3] Hardy, *Apology*, § 29.

[4] Cicero, *Tusculan Disputations*, § II.iv ⟦p. 158⟧.

[5] See Patočka, *Plato and Europe*, *passim*.

[6] *Ibid.*, pp. 86, 97.

* ▸ p. 52

[7] DK, 68B187; orig. Stobaeus, *Anthologium*, § III.i.27 ⟦vol. 3, p. 18⟧; trans. Taylor, *The Atomists*, p. 23 [§ D52].

[8] Plato, *Apology*, STE 29d–e.

Patočka held that Plato had distinguished three aspects to care for the soul: the understanding of ontocosmological matters, of the structure of being, and indeed of reality; development of the role of the individual spirit in the community; and the cultivation of the soul regarding its inner life and of the fundamental experience of human life, namely confronting mortality.[1] And care for the soul is *necessary*; one can and must make an effort to shape oneself for the best; 'it's certainly hard to be truly good — perfect in hands, feet, and mind.'[2] The soul is shaped by the influences to which it is exposed.

> 'It makes no small difference, then, whether we form habits of one kind or of another from our very youth; it makes a very great difference, or rather *all* the difference.'[3]

A child has little or no choice over the these influences, but in growing, an individual assumes the responsibility to mould their own spirit. 'Thinking men and women are not Pavlov's dogs.'[4]

But what has all this to do with beauty in mathematics? Jan Zwicky (1955–) thought that Plato was conscious that necessary truth, especially when aesthetically pleasing, evokes a distinct and special *feeling*,[*] and suspected that Plato saw the recognition of this feeling as what set apart philosophers and mathematicians from any other thinkers.[5] But she also suspected that Plato thought that witnessing moral beauty produced a similar phenomenological response, with a similar depth and purity and simplicity, such as was reflected in Alcibiades' description of how he saw the virtue of Socrates.[6] For Plato, she thought, the *feeling* of proof and the *feeling* of moral beauty were alike.[7] This parallel underlies the dialogue *Meno*, which begins with the aristocratic Meno asking Socrates whether virtue [*aretē* ἀρετή] is teachable, or is acquired through practice, or is innate, or something else.[8] Socrates declares his ignorance of the answer to the more fundamental question of what virtue is. Through the ensuing discussion, Socrates causes Meno to admit his own bafflement.[9] In response to Socrates' wish to continue their inquiry, Meno asks with apparent sarcasm how one can seek something one does not know.[10] This question leads to Socrates' famous demonstration that one can obtain knowledge not learned in this life, when he guides a servant boy to discover how to duplicate the square.[11][†] For herself, Zwicky declared: 'I am myself struck by the phenomenological resemblance between moral and mathematical beauty that I believe struck him.'[12]

Even putting aside this mirrored phenomenology, I hold that an appreciation of mathematical beauty can serve as a component of — or an instrument for — care for the soul. Mathematics is first a search for truth: mathematical discoverers seek truths previously unknown to all; mathematical students seek truths hitherto unknown to them. But more: the discoverer can create, or at least can uncover, beauty

[1] Patočka, *Plato and Europe*, esp. pp. 97, 180.

[2] Simonides, in Page, *Poetae Melici Graeci*, 542; orig. Plato, *Protagoras*, STE 339a–b; trans. Beresford, 'Nobody's Perfect', p. 243.

[3] Aristotle, *Nicomachean Ethics*, BEK II.1.1103b21–5.

[4] Steiner, *Lessons of the Masters*, § 4.

[*] ▸ p. 1001

[5] Zwicky, *Plato as Artist*, pp. 47–8, 50–2.

[6] Plato, *Symposium*, STE 215e–16b; Zwicky, *Plato as Artist*, pp. 51–2.

[7] Zwicky, *Plato as Artist*, p. 50.

[8] Plato, *Meno*, STE 70a.

[9] *Ibid.*, STE 80a–b.

[10] *Ibid.*, STE 80d.

[11] *Ibid.*, STE 82a–5b.

[†] ▸ p. 74

[12] Zwicky, *Plato as Artist*, p. 52.

that others can admire. And there is so formed an intellectual nexus, both positive and negative, linking truth and beauty. Positive: there is a greater appreciation for truth, when it is accompanied by beauty, and a greater appreciation for beauty, when it is accompanied by truth. Negative: that beauty does not always point to truth, nor truth to beauty. Both the positive and negative aspects of the nexus are *good*: truth and beauty are simple human goods, and a healthy scepticism for their always being seen together is an ethical good.

I follow G. E. Moore[*] (1873–1958) in believing that truth and beauty are good in themselves, but that their intrinsic value is less than that of the consciousness of them.[1] And we should recognize that humanism holds that we share certain universal capacities, and that the ability to perceive truth and beauty lies among them. As the chemist, philosopher, and educator Radim Palouš (1924–2015) said,

> 'our souls enable us to know the absolute, the eternal, that which is not transitory: truth, goodness, beauty, love, and justice. Ecce homo.
>
> The greatness of human beings is their ability to make these spiritual values, which are eternal, their own in time. That is also the aim of every great artist: to allow us to experience this imperishable world.'[2]

Mathematics, with its truth, its beauty, and — as Hardy so rightly emphasized[†] — its *permanence*, is thus a natural component of a humanist education. Just as training in mathematics involves developing a critical judgement of the correctness of an argument or a result, so too should it include cultivation of aesthetic taste. As in visual art, poetry, and music, historical sense is perhaps the best foundation of taste in mathematics. Developments in knowledge and rigour and exposition have made obsolete neither Euclid[‡] nor Euler.[§]

> 'Some one said: "The dead writers are remote from us because we *know* so much more than they did." Precisely, and they are that which we know.'[3]

The historical basis should not force the student's taste to evolve in line with their masters'. But the student should confront mathematics that has previously been considered to be beautiful, and should confront it critically. They should not blindly assimilate the past, but be shaped by the experience of interrogating it. *Then* the student has the intellectual foundation to transcend the previous canon. Such is progress in art; and such is the last triumph of mastership: to be superseded by the student.[4]

None of this is to deny that utility is a good in mathematics. But utility is not a *fundamental* good. What goodness derives from the usefulness of a thing depends *how* it is applied. Every scientist can see something good in the physical theory that governs nuclear fission and fusion, whether that good is aesthetic or intellectual or something else.

[*] ▸ p. 478

[1] Moore, *Principia Ethica*, § 113.

[2] Palouš, quot. in Riemen, *To Fight Against This Age*, ch. 11.

[†] ▸ p. 508

[‡] ▸ pp. 95–103

[§] ▸ pp. 334–41

[3] Eliot, 'Tradition and the Individual Talent', p. 46, emphasis in orig.

[4] Cf. Steiner, *Lessons of the Masters*, § 6.

That this theory has utility *may* be good, depending on our evaluations of the present and potential future benefits and costs of nuclear power generation, of the destruction at Hiroshima and Nagasaki, and of the mutually assured destruction that helped maintain for half a century a precarious peace — or at least divert conflict from global conflagration into local yet often monstrous proxy wars.

And regardless of what utility a scholar sees in their own mathematical work, a pragmatist might judge that it is worth emphasizing the utility of an education in mathematics rather than attempt to convince a philistine of its intellectual and aesthetic value.

The greatness of a mathematician should not be judged solely by the hardness of the problems they solve or the technical ability shown by the proofs they discover, any more than the greatness of an artist should be judged solely by the subjects they consider or their raw talent for painting or sculpting or composing. They should be judged by all forms of value that adhere to that which they create, including aesthetic value, whether beauty or elegance or otherwise.

◆

There will doubtless seem to be something elitist in my views on humanist education. But I hope that it is 'elitist' not in the negative sense of desiring to mark off a social group by a special ability and access to knowledge, but rather in the positive sense of reaching for, and assuming a care for, *what is best* in culture and science. And judgement of *what is best* in mathematics should be informed by a cultivated taste for its beauty.

THE LONELINESS OF BEAUTY

‘ when *Jupiters* daughters were each of them married to the Gods, the *Muses* alone were left solitary ’

— Robert Burton, *The Anatomy of Melancholy*, § 1.2.3.15 ⟦vol. I, p. 310⟧.

The cultural philosopher Rob Riemen (1962–) wrote in his essay *Nobility of Spirit* that

> ‘[a]rt derives its ethical value exclusively from its aesthetic value, from its status as art — *l'art pour l'art* — from its independence, which is tied to its only purpose: to express beauty and truth.’[1]

Yet a work of plastic art cannot, in our present society, fulfil this ideal. Such a work of art is *owned*; it has economic value in addition to its aesthetic value, and this economic value bears an influence on its ethical value: a private collector may hide it away, denying others a chance to contemplate it and grow from that contemplation. A public

[1] Riemen, *Nobility of Spirit*, pp. 6–7.

owner — a national or provincial or municipal government — has in it a stock of value that could be deployed elsewhere.

As long as we as a society cling to what Walter Benjamin (1892–1940) called the *aura*,[1] an artwork has an exclusivity that can be possessed. A work of literature or music or photography, with its relative ease of reproducibility, has exclusivity only in the faintest degree, because of the equipment required to reproduce it. But a work of mathematics — as a piece of intellectual art — is completely free of it; mathematics can be communicated on paper (or even lines in the sand), or verbally, and, once communicated, can in principle be *summoned to the mind at will.*

And so perhaps we can wonder, with Plutarch, how anyone might think that the ‘trappings of wealth’ could compare to the beauty of mathematics.[2]*

THE SANCTUARY OF BEAUTY

> ‘ “My brain has become my refuge,” writes Charles Bonnet.
> But higher than the brain is the object of its devotion,
> and that is a very much safer refuge. ’
>
> — A.-G. Sertillanges, *The Intellectual Life*, p. xvii,
> (trans. Mary Ryan).

After Niccolò Machiavelli (1469–1527) retired to his farm estate, study was a haven from the cares of daily life. In a famous letter to his friend Francesco Vettori (1474–1540), he described how, after the day’s work, he would ‘put on garments regal and courtly’ and go to his *scrittorio*:

> ‘I enter the ancient courts of ancient men, where, received by them with affection, I feed on that food which only is mine and which I was born for [...]; and for four hours of time I do not feel boredom, I forget every trouble, I do not dread poverty, I am not frightened by death’.[3]

Intellectual life of whatever kind requires that a time and a space be set aside for reflection. But this *demand* is also a *gift*. When one is able to lose oneself to study and contemplation — even if only for an hour each day, even while working physically — one has freedom. ‘Life and death, imprisonment and hunger, the cultivation of the soul despite the captivity of the body’.[4]

Mathematics can be such an escape: when Paul Erdős (1913–96) fell into a depression on the death of his mother, his friend Paul Turán (1910–76) reminded him that ‘[a] strong fortress is our mathematics’, prompting Erdős to throw himself into his work.[5] Turán was doubtless here alluding — perhaps ironically, given his own Jewish origin — to Martin Luther’s (1483–1546) hymn ‘Ein feste Burg ist unser Gott’. But Turán’s point was serious: he knew, from his own experience of being

[1] Benjamin, ‘The Work of Art’, p. 22.

[2] Plutarch, *On Love of Wealth*, STE 528a [§ 10].

* ▸ pp. 134–5

[3] Machiavelli, letter to Vettori, dated 10 Dec. 1513, repr. in Machiavelli, *Chief Works*, vol. II, p. 929.

[4] Solzhenitsyn, *Between Two Millstones*, vol. 1, p. 230.

[5] Hoffman, *The Man Who Loved Only Numbers*, p. 143.

drafted into a wartime labour camp, the benefit of having the rod and staff of mathematics to comfort him.[*] And mathematical beauty can be part of the effectiveness of the escape: with its aid, Arthur Koestler (1905–83) was able to forget his sentence of death.[†]

It is arguable that, unlike the creation of beauty, which can at least be seen as contributing something of value to the world, treating mathematics as a refuge from the world is an entirely individualistic motivation for pursuing it. But this judgement is not a criticism: if an individual is powerless to ameliorate the suffering of others, mental or physical, then having ability to the escape from their own suffering must be counted as a good. Albert Einstein (1879–1955) and Arthur Schopenhauer (1788–1860) seemed to have thought that aesthetic contemplation can help achieve this escape.[‡] The experience of beauty can be a kind of 'relief from time';[1] it can certainly be ecstasy, not only in the sense of pleasure, but as the Greeks understood it: *ek-stasis* [ἔκστασις]; displacement.[2] But in a prison cell or a labour-camp, what aesthetic contemplation is available? Except for those few who can recall precisely every visual detail of a painting or every note of a symphony, all that remains is the contemplation of intellectual aesthetic value.

Intellectual life can be characterized as 'a sort of asceticism, a turning away from things'.[3] But, as Aristotle knew, it is not asceticism in the sense of a self-denial of happiness or pleasure. That the highest happiness, the greatest *eudaimonia* [εὐδαιμονία], is best found in contemplation of intellectual things was the conclusion of the *Nicomachean Ethics*:[4]

> 'that which is proper to each thing is by nature best and most pleasant for each thing; for man, therefore, the life according to intellect is best and pleasantest, since intellect more than anything else *is* man. This life therefore is also the happiest.'[5]

And there is no lack of happiness or pleasure in the pursuit of mathematics and other fields where beauty can be found. There is no self-denial in aesthetic pleasure; even less so in the elation, exhilaration, exultation that is brought about by the discovery of new beauty.

THE IMMORTALITY OF BEAUTY

‘ Wer möchte im Ernst unsterblich sein? […]
Es ist der Tod, der dem Augenblick
seine Schönheit gibt und seinen Schrecken.
Nur durch den Tod ist die Zeit eine lebendige Zeit. ’

— Pascal Mercier, *Nachtzug nach Lissabon*, pp. 201–2.

In his *Apology*, Hardy said that 'the noblest ambition is that of leaving behind one something of permanent value'.[6] Ozymandias may have ordered a physical monument, but the desire to

[*] ▸ p. 548

[†] ▸ pp. 547–8

[‡] ▸ p. 473

[1] Zwicky, *Wisdom & Metaphor*, 71(L).

[2] LSJ, 'ἔκστασις'.

[3] Hitz, *Lost in Thought*, p. 85.

[4] Aristotle, *Nicomachean Ethics*, BEK X.6–8.1176a29–79a31.

[5] *Ibid.*, BEK X.7.1178a6–8, emphasis in orig.

[6] Hardy, *Apology*, § 7.

The immortality of beauty

create an enduring intellectual work and the hope of so preserving one's name is no less powerful as a motive and is higher as an aim. Material things are subject to the corroding rains, the tempestuous winds, the endless passage of the years, and the flight of time.[1] For works of the mind, think of how Dante (c. 1265–1321) addressed with respect his teacher Brunetto Latini (1220/30–1293/4), whom he encountered in the seventh circle of the *Inferno*:

> 'You taught me how man makes himself immortal
> And how much gratitude I owe for that
> my tongue, while I still live, must give report.'[2]

Hardy no longer had to reach for immortality; his argument in the *Apology* served more to illuminate than to reinforce his own claim. He located the value of his own work, and the work of all 'real mathematicians', in its permanence, in its seriousness, and in its beauty, these three qualities being interwoven. A millennium before Hardy, Abū Sahl al-Kūhī (*fl.* c. 970–c. 1000) had also linked the beauty and the permanence of mathematics;* perhaps one can consider as an exemplar of this connection the treatise in which he wrote of it.

But historical memory is fickle. This lesson is an old one. The text known as 'The Immortality of Writers', found on a Egyptian papyrus from the nineteenth dynasty (1295–1186 BCE), says that

> 'Man decays, his corpse is dust,
> All his kin have perished;
> But a book makes him remembered.'[3]

The anonymous author then named writers like Imhotep (*fl.* late 27th century BCE), Hardjedef (*fl.* 26th century BCE), and Ptahhotep (*fl.* 25th–24th century BCE),[4] and continued pointedly:

> 'Death made their names forgotten
> But books made them remembered!'[5]

Over four millennia after they died, they still live through their writings.[6] But if their names have not actually been forgotten, each is as a rule a hollow label that points to the *work*, not to the author.

And even the label may be lost. In his *Nuzhat al-Qulūb*, the historian and geographer Ḥamd Allāh Mustawfī (c. 1281–after 1339/40) quoted a poetic couplet:

> 'These are our works, and they declare us;
> Wherefore, after we are gone, look at our works.'[7]

Mustawfī chose, perhaps with a deliberate irony, to ascribe these lines only to 'a poet'.

Mathematics builds monuments equally exalting and effacing. A column so high that the statue on top is invisible, as Hardy's friend J. M. Lomas (1917–45) once suggested.[8] Mathematics, it seems, confers immortality by displacing the mathematician, and so mere names we hold.[1] T. L. Heath's (1861–1940) insistence on Euclid's (*fl.* 300 BCE) durability is revealing:

---

[1] Paraphrasing Horace, *Odes*, III.30, ll. 3–5.

[2] Dante, *Commedia*, *Inf.* XV, ll. 85–7.

* ▸ p. 207

[3] [Anonymous], 'The Immortality of Writers', ll. 38–40.

[4] *Ibid.*, ll. 46–53.

[5] *Ibid.*, ll. 62–3.

[6] *Ibid.*, n. 5.

[7] Quot. in Ḥamd-allāh Mustawfī, *Nuzhat-al-Qulūb*, p. 235.

[8] Hardy, *Apology*, Note.

> 'Euclid can never at any time be more than apparently in abeyance; he is immortal. Elementary geometry will also continue to form part of a complete education; and elementary geometry *is* Euclid'.[2]

'[E]lementary geometry *is* Euclid'. And, equivalently, Euclid *is* elementary geometry — *and not more*. Would he have objected? It is impossible to say: Euclid *the man* is a cipher.

[1] Cf. Bernard, 'The Scorn of the World', bk I, p. 99.

[2] Heath, intro. to Euclid, *Book I*, p. v.

So here is my motivation, my apology, and my manifesto:

I believe that there are certain intrinsic goods that should be cultivated, and the creation and contemplation of mathematical beauty is a nexus where there meet certain of these goods: aesthetic, intellectual, mathematical, moral, and philosophical.

I see the beauty of the mathematics — just as I see other kinds of intellectual beauty — as a place where aesthetic appreciation is completely free of the acquisitive desire. I see the mental endeavour and the aesthetic joy that are a sanctuary to those who come after me. I see and hope for the durability of my own small contribution.

Nothing remains, except for me to say that there is, I think, a nobility in what I have set out here: in the beauty of the mathematics; in the elation of discovery; in the pleasure of understanding; in the permanence of the ideas; and in the fading of the author.

Lector, si monumentum requiris, ecce liber.

OF HISTORY
ROMANS
CHINESE
SCYTHÆ
INDI
TARTARY
THE PARTHIAN EMPIRE
PERSIANS
SARACENS
TURKS
EASTERN EMPIRE
EMPERORS OF GERMANY
EASTERN OR GREEK EMPIRE
THE POPE'S TERRITORIES
FRANKS
FRENCH
SPANIARDS
NORMANS
ENGLISH
CALIDITAS
HUMIDITAS
AER
SUMMA
REMISSA
possibilis
impossibilis
Combinatio
PLATO
OVID METAMORPHOSES
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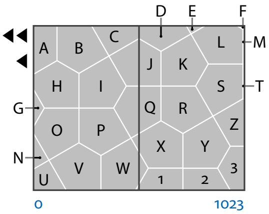


A Johan Heiberg's 1906 photograph of folios 16v and 17r from the Archimedes palimpsest, containing as undertext part of *On Floating Bodies*. **B** Periodic solutions of the three-body problem. **C** Intarsia geometric figure perhaps designed by Francesco di Giorgio, executed by Giuliano da Maiano Studiolo from the Ducal Palace, Gubbio (c. 1478–c. 1482). **D** Soma cube. **E** Jigsaw puzzle. **F** ? **G** Macrophotograph of a snowflake. **H** Prime spiral: primes coloured white; each other number n coloured according to $\Omega(n)$. **I** Lantern in the form of a small stellated dodecahedron. **J** Frontispiece of Leibniz's *Dissertatio de arte combinatoria* (1666). **K** Assortment of cubes with regular patterns of circular holes in their faces, by Jared Tarbell. **L** Joseph Priestley, *A New Chart of History*. **M** Detail from a tiled floor in the Winter Garden of the Ursulines, Onze-Lieve-Vrouw-Waver. **N** Japanese puzzle box. **O** Icosahedral die marked with Greek letters from Hellenistic/Roman Egypt. **P** Albrecht Dürer, *First Knot* (1506/7?). **Q** Mikael Hvidtfeldt Christensen, *Tetrahedron 4D*. **R** Domenico Remps, *Cabinet of Curiosities* (c. 1689). **S** Bookshelves. **T** Detail of the cover of G. H. Hardy, *An Annotated Mathematician's Apology*. **U** *Aloe polyphylla* exhibiting phyllotaxis. **V** Detail from *The Ideal City* of Urbino. **W** Endrass surface. **X** Sandstone engraved with a figure generated by a context-free grammar, by Jared Tarbell. **Y** Fountain pen and notebook. **Z** Detail of Johann Michael Bretschneider, *Blick in die Galerie eines Fürsten* (first half of 17th century). **1** Pocket watch. **2** Case of moveable type. **3** Card index.

A © 2008 The Walters Art Museum. [CC-BY-3.0] **E** © 2011 Pratap Sankar. [CC-BY-2.0] **G** © 2015 'Urmansky'. [CC-BY-4.0] **I** © 2012 'The Playful Geometer'. [CC-BY-2.0] **K** © 2010 Jared Tarbell. [CC-BY-2.0] **M** © 2019 Rob Oo. [CC-BY-2.0] **N** © 2007 Ross Berteig. [CC-BY-2.0] **Q** © 2011 Mikael Hvidtfeldt Christensen. [CC-BY-2.0] **S** © 2006 Stewart Butterfield. [CC-BY-2.0] **W** © 2019 Claudio Rocchini. [CC-BY-4.0] **X** © 2010 Jared Tarbell. [CC-BY-2.0] **Y** © 2017 Jacob Ehnmark. [CC-BY-2.0] **Z** © 2011 Städtische Museum Schloss Rheydt. [CC-BY-4.0] **2** © 2011 Miguel Mendez. [CC-BY-2.0] **3** © 2011 Marie-Lan Nguyen. [CC-BY-2.5] **C, L, O, P, R, U, V** Public domain.

Timeline

> ‘All our temporal language is contaminated with spatial imagery:
> we speak of “long” and “short” times,
> of “intervals” (literally, “spaces between”),
> of “before” and “after”
> — all implicit metaphors which depend upon
> a mental picture of time as a linear continuum.’
>
> — W. J. T. Mitchell, ‘Spatial Form in Literature’, p. 542.

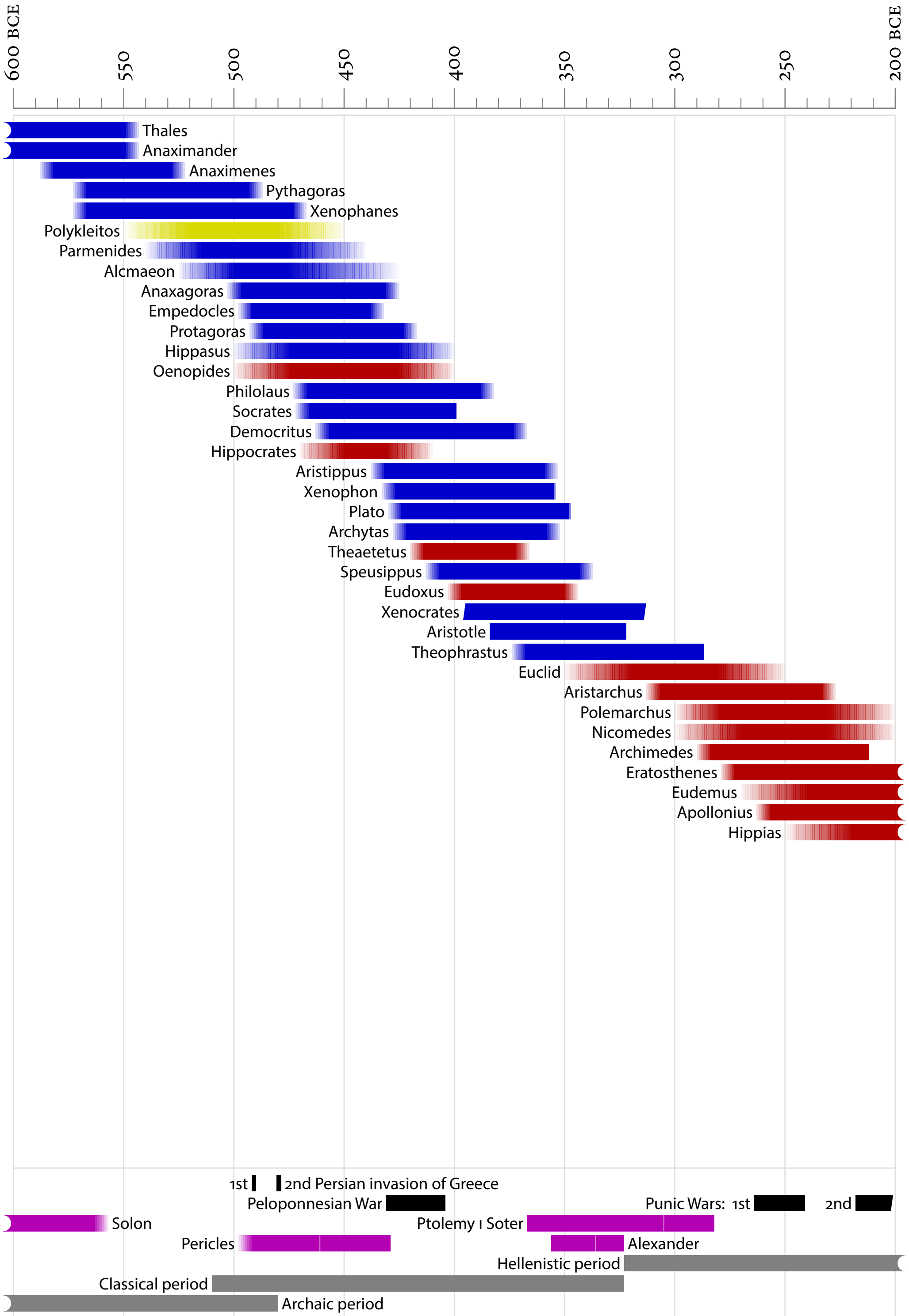
> ‘All my ambition in the BIOGRAPHICAL CHART
> I now preſent to the public is to be a humble ſecond
> to the great Hiſtorians, Chronologers, and Biographers’
>
> — Joseph Priestley, *A Description of a Chart of Biography*, p. 3.

The timeline in the following pages provides a unified chronological overview of the main figures discussed in this book, with some cultural and political context. The ranges marked in the timeline use the following conventions, with each person, event, or era classified according to their primary field of relevance:

🟥 Mathematics
🟧 Natural science
🟦 Philosophy
🟨 Art, architecture

■ Literature
🟩 Religion and theology
🟪 Political leaders
⬛ Political entities, events
▩ Other

▥ ▥ Uncertain dates
◢ ◤ Dates lying within a range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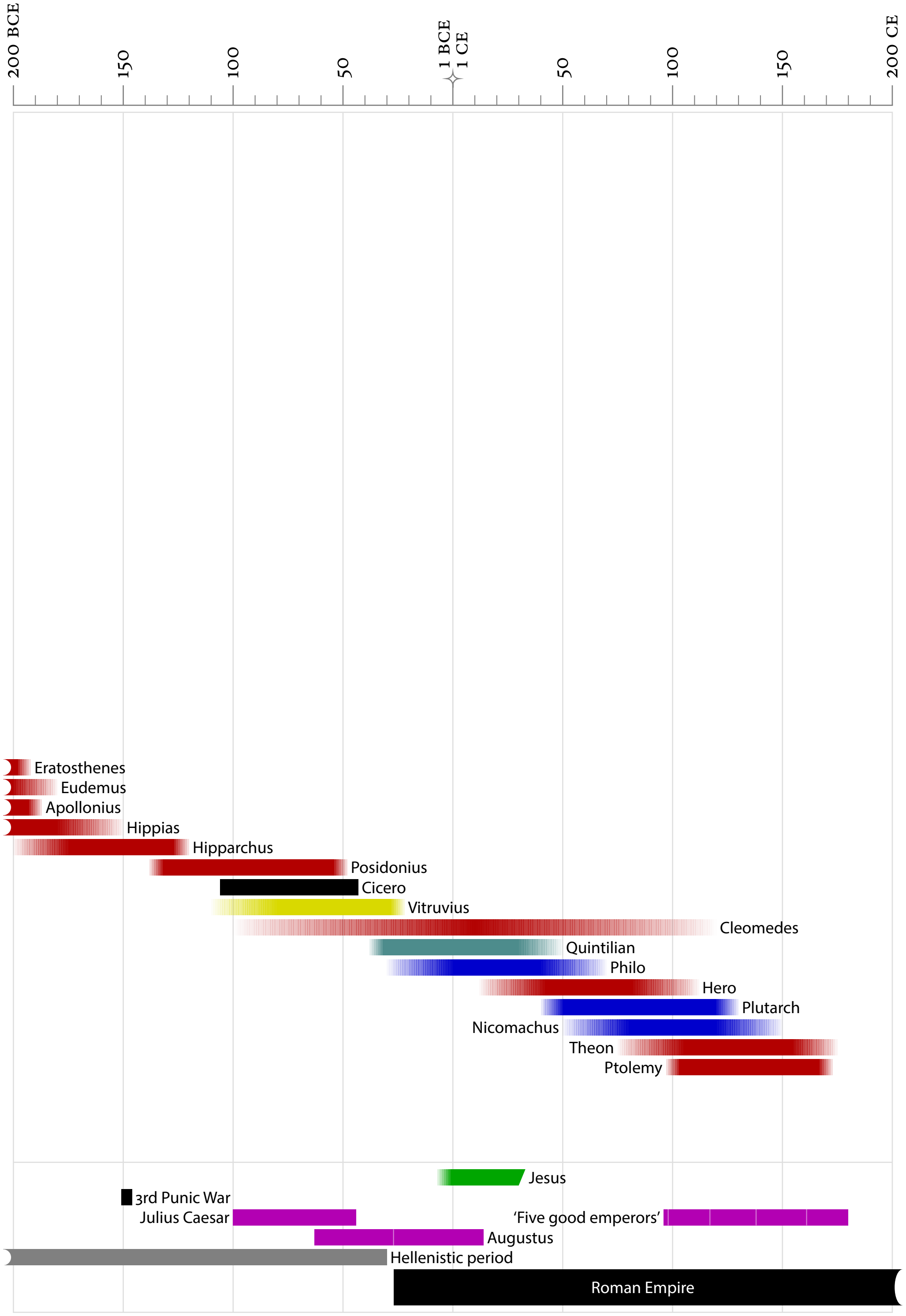
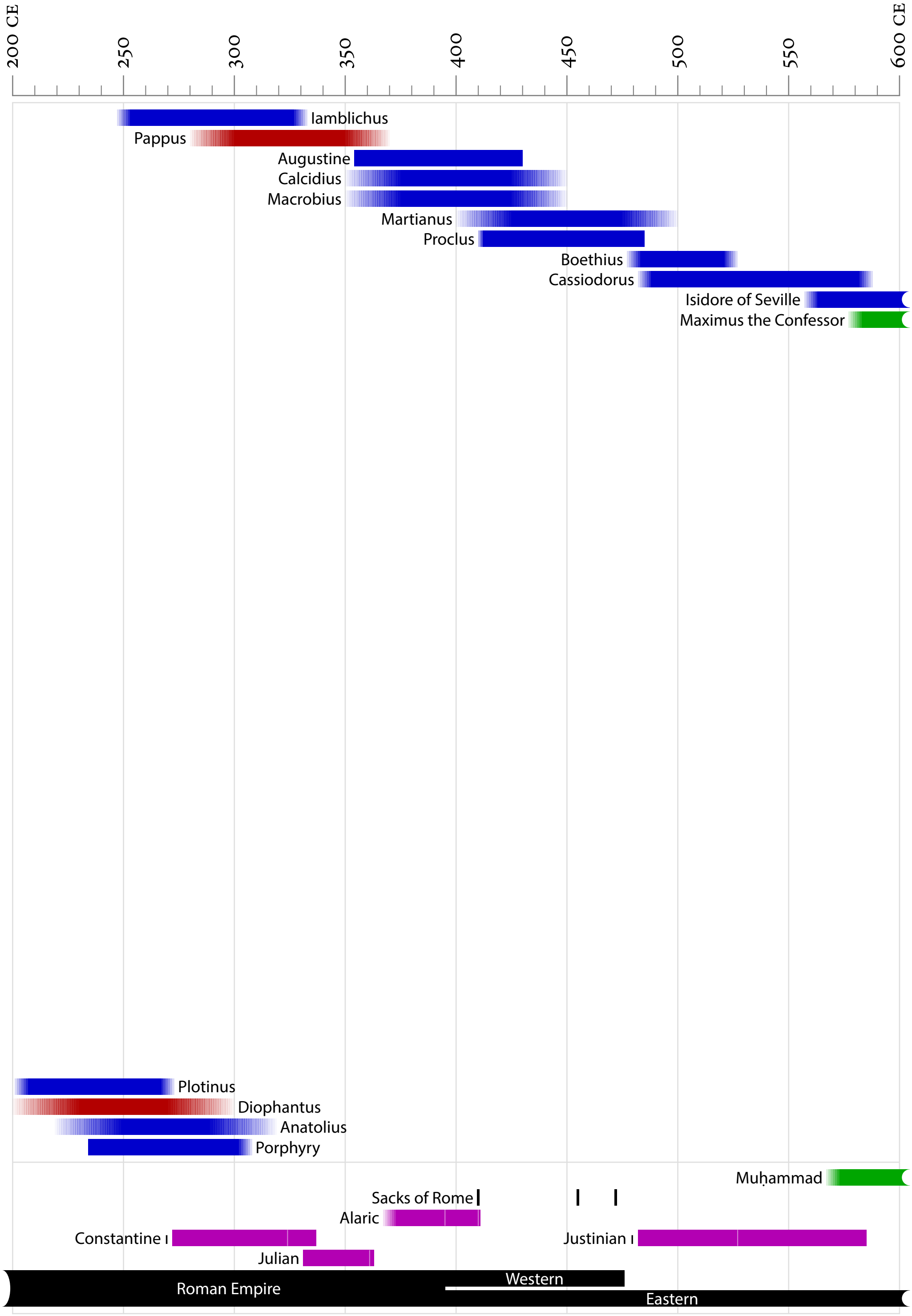
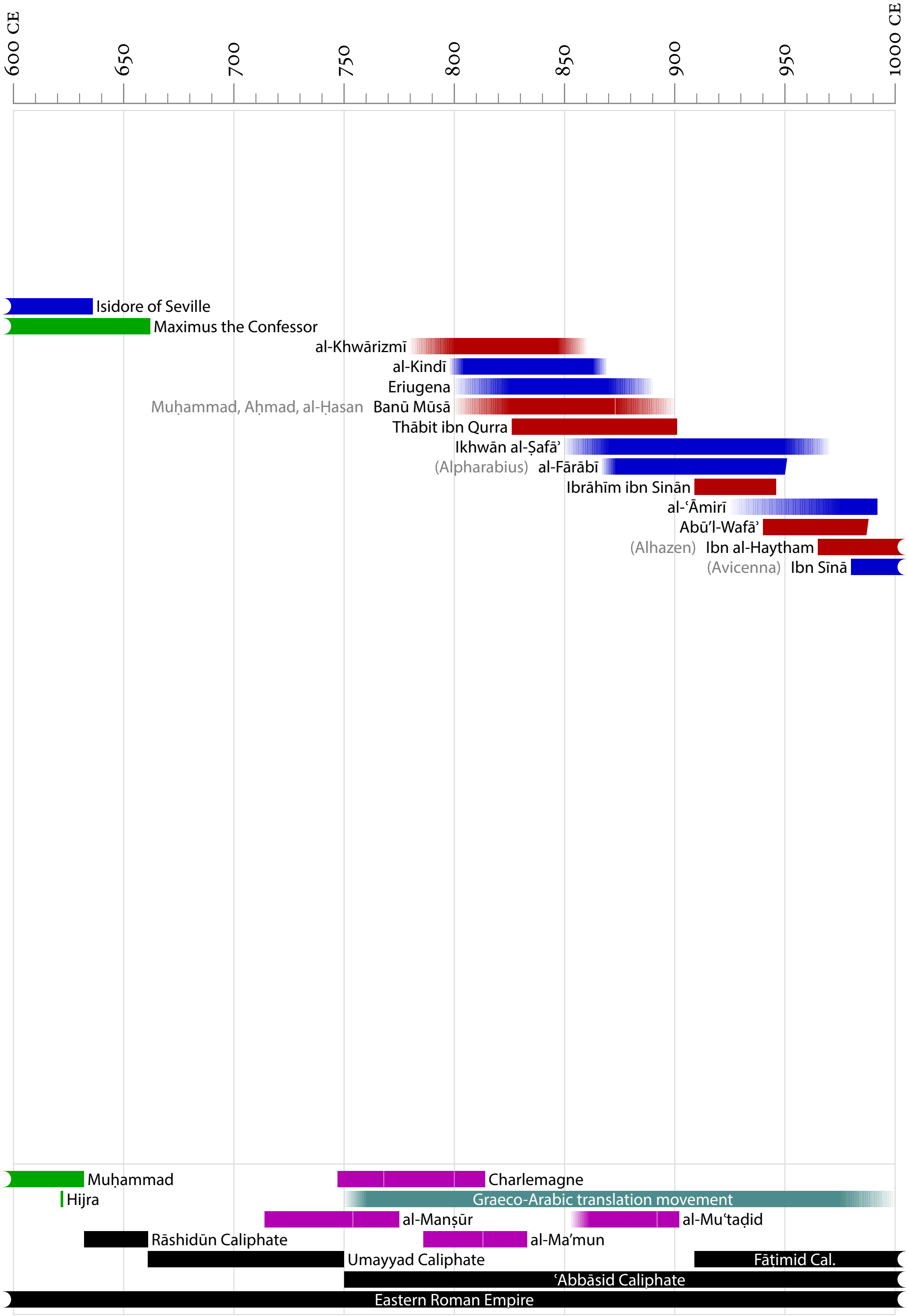
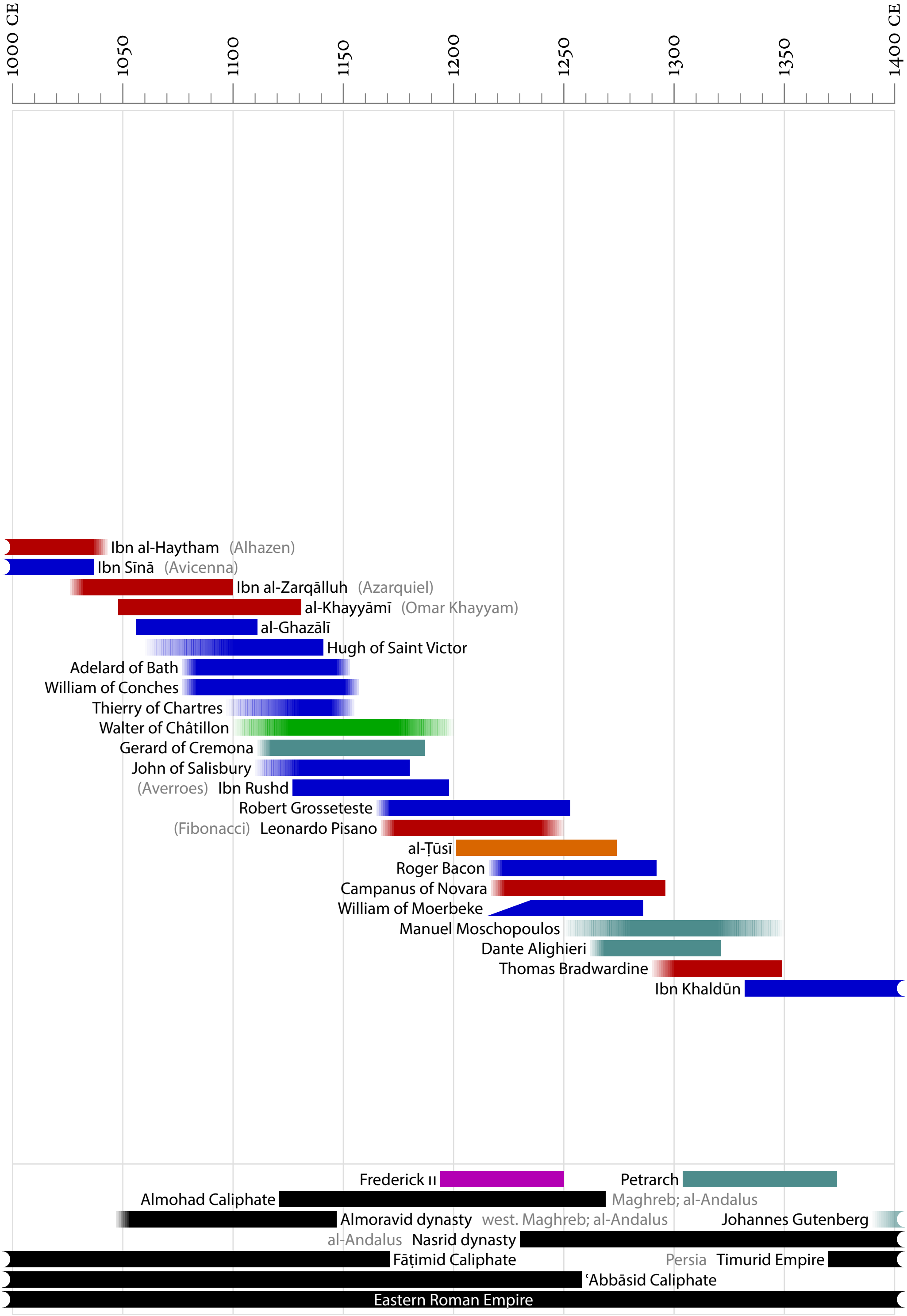
550
500
450
400
350
300
250
200 BCE
Thales
Anaximander
Anaximenes
Pythagoras
Xenophanes
Polykleitos
Parmenides
Alcmaeon
Anaxagoras
Empedocles
Protagoras
Hippasus
Oenopides
Philolaus
Socrates
Democritus
Hippocrates
Aristippus
Xenophon
Plato
Archytas
Theaetetus
Speusippus
Eudoxus
Xenocrates
Aristotle
Theophrastus
Euclid
Aristarchus
Polemarchus
Nicomedes
Archimedes
Eratosthenes
Eudemus
Apollonius
Hippias
1st
2nd Persian invasion of Greece
Peloponnesian War
Punic Wars: 1st
2nd
Solon
Ptolemy I Soter
Pericles
Alexander
Hellenistic period
Classical period
Archaic period

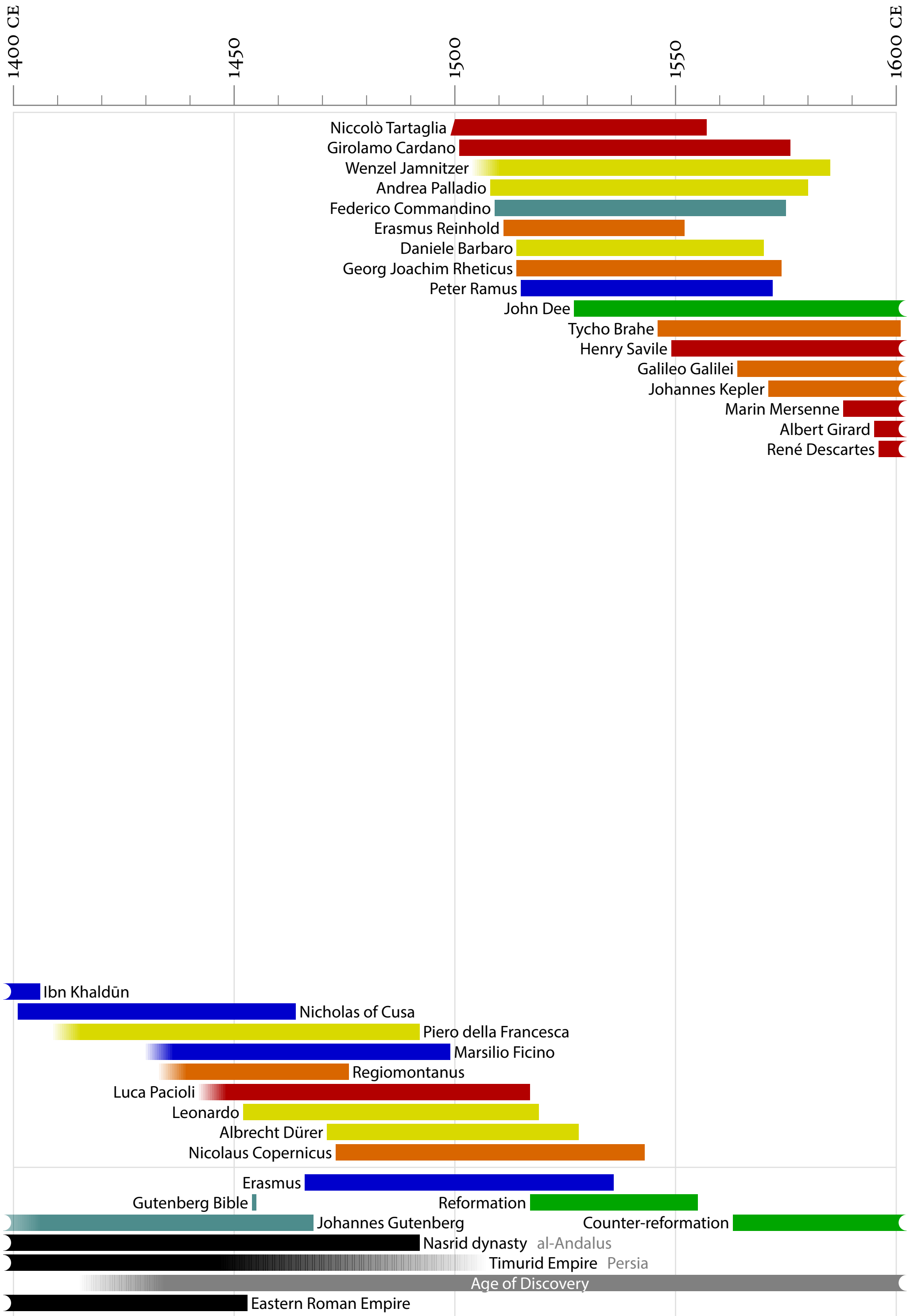

200 BCE
150
100
50
1 BCE
1 CE
50
100
150
200 CE
Eratosthenes
Eudemus
Apollonius
Hippias
Hipparchus
Posidonius
Cicero
Vitruvius
Cleomedes
Quintilian
Philo
Hero
Plutarch
Nicomachus
Theon
Ptolemy
Jesus
3rd Punic War
Julius Caesar
‘Five good emperors’
Augustus
Hellenistic period
Roman Empire

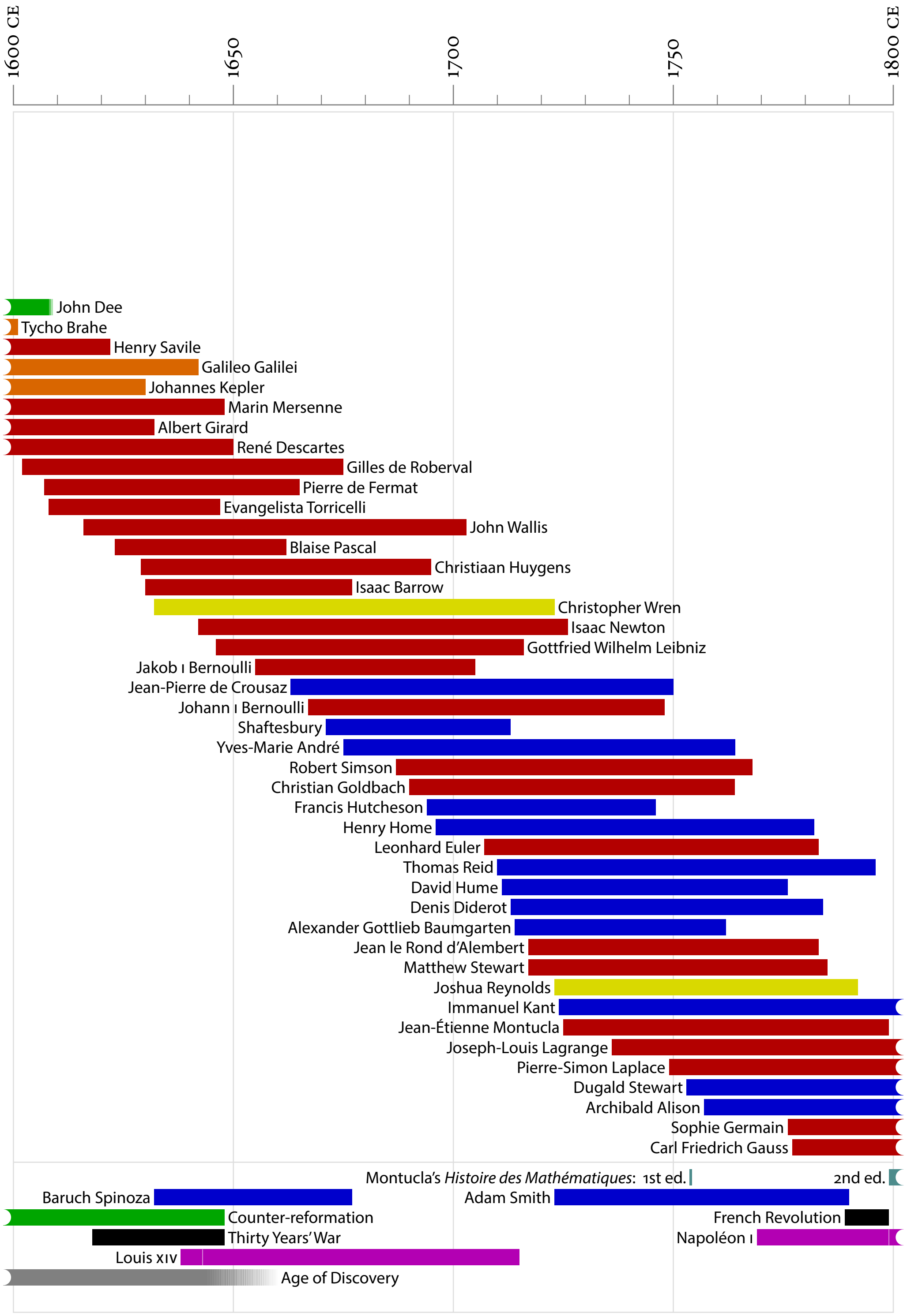


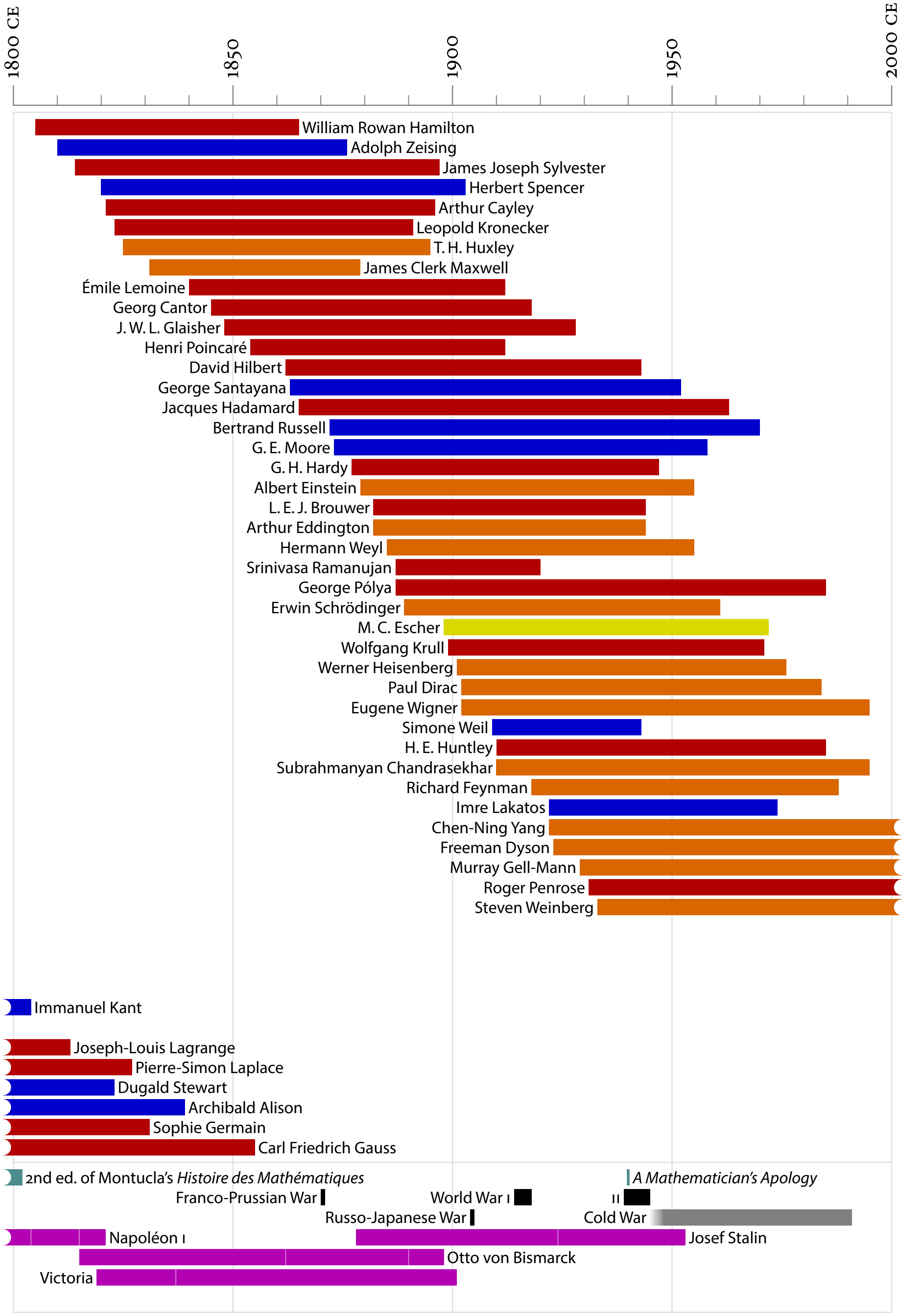
200 CE
250
300
350
400
450
500
550
600 CE
Iamblichus
Pappus
Augustine
Calcidius
Macrobius
Martianus
Proclus
Boethius
Cassiodorus
Isidore of Seville
Maximus the Confessor
Plotinus
Diophantus
Anatolius
Porphyry
Muḥammad
Sacks of Rome
Alaric
Constantine ı
Justinian ı
Julian
Roman Empire
Western
Eastern


600 CE
650
700
750
800
850
900
950
1000 CE
Isidore of Seville
Maximus the Confessor
al-Khwārizmī
al-Kindī
Eriugena
Muḥammad, Aḥmad, al-Ḥasan
Banū Mūsā
Thābit ibn Qurra
Ikhwān al-Ṣafā’
(Alpharabius)
al-Fārābī
Ibrāhīm ibn Sinān
al-ʿĀmirī
Abū’l-Wafā’
(Alhazen)
Ibn al-Haytham
(Avicenna)
Ibn Sīnā
Muḥammad
Hijra
Charlemagne
Graeco-Arabic translation movement
al-Manṣūr
al-Muʿtaḍid
Rāshidūn Caliphate
al-Ma’mun
Umayyad Caliphate
Fāṭimid Cal.
ʿAbbāsid Caliphate
Eastern Roman Empire


1000 CE
1050
1100
1150
1200
1250
1300
1350
1400 CE
Ibn al-Haytham (Alhazen)
Ibn Sīnā (Avicenna)
Ibn al-Zarqālluh (Azarquiel)
al-Khayyāmī (Omar Khayyam)
al-Ghazālī
Hugh of Saint Victor
Adelard of Bath
William of Conches
Thierry of Chartres
Walter of Châtillon
Gerard of Cremona
John of Salisbury
(Averroes) Ibn Rushd
Robert Grosseteste
(Fibonacci) Leonardo Pisano
al-Ṭūsī
Roger Bacon
Campanus of Novara
William of Moerbeke
Manuel Moschopoulos
Dante Alighieri
Thomas Bradwardine
Ibn Khaldūn
Frederick II
Petrarch
Almohad Caliphate
Maghreb; al-Andalus
Almoravid dynasty
west. Maghreb; al-Andalus
Johannes Gutenberg
al-Andalus
Nasrid dynasty
Fāṭimid Caliphate
Persia
Timurid Empire
ʻAbbāsid Caliphate
Eastern Roman Empire


1400 CE
1450
1500
1550
1600 CE
Niccolò Tartaglia
Girolamo Cardano
Wenzel Jamnitzer
Andrea Palladio
Federico Commandino
Erasmus Reinhold
Daniele Barbaro
Georg Joachim Rheticus
Peter Ramus
John Dee
Tycho Brahe
Henry Savile
Galileo Galilei
Johannes Kepler
Marin Mersenne
Albert Girard
René Descartes
Ibn Khaldūn
Nicholas of Cusa
Piero della Francesca
Marsilio Ficino
Regiomontanus
Luca Pacioli
Leonardo
Albrecht Dürer
Nicolaus Copernicus
Erasmus
Gutenberg Bible
Reformation
Johannes Gutenberg
Counter-reformation
Nasrid dynasty
al-Andalus
Timurid Empire
Persia
Age of Discovery
Eastern Roman Empire


1600 CE
1650
1700
1750
1800 CE
John Dee
Tycho Brahe
Henry Savile
Galileo Galilei
Johannes Kepler
Marin Mersenne
Albert Girard
René Descartes
Gilles de Roberval
Pierre de Fermat
Evangelista Torricelli
John Wallis
Blaise Pascal
Christiaan Huygens
Isaac Barrow
Christopher Wren
Isaac Newton
Gottfried Wilhelm Leibniz
Jakob ı Bernoulli
Jean-Pierre de Crousaz
Johann ı Bernoulli
Shaftesbury
Yves-Marie André
Robert Simson
Christian Goldbach
Francis Hutcheson
Henry Home
Leonhard Euler
Thomas Reid
David Hume
Denis Diderot
Alexander Gottlieb Baumgarten
Jean le Rond d'Alembert
Matthew Stewart
Joshua Reynolds
Immanuel Kant
Jean-Étienne Montucla
Joseph-Louis Lagrange
Pierre-Simon Laplace
Dugald Stewart
Archibald Alison
Sophie Germain
Carl Friedrich Gauss
Montucla's *Histoire des Mathématiques*: 1st ed.
2nd ed.
Baruch Spinoza
Adam Smith
Counter-reformation
French Revolution
Thirty Years' War
Napoléon ı
Louis xiv
Age of Discovery


1800 CE
1850
1900
1950
2000 CE
William Rowan Hamilton
Adolph Zeising
James Joseph Sylvester
Herbert Spencer
Arthur Cayley
Leopold Kronecker
T. H. Huxley
James Clerk Maxwell
Émile Lemoine
Georg Cantor
J. W. L. Glaisher
Henri Poincaré
David Hilbert
George Santayana
Jacques Hadamard
Bertrand Russell
G. E. Moore
G. H. Hardy
Albert Einstein
L. E. J. Brouwer
Arthur Eddington
Hermann Weyl
Srinivasa Ramanujan
George Pólya
Erwin Schrödinger
M. C. Escher
Wolfgang Krull
Werner Heisenberg
Paul Dirac
Eugene Wigner
Simone Weil
H. E. Huntley
Subrahmanyan Chandrasekhar
Richard Feynman
Imre Lakatos
Chen-Ning Yang
Freeman Dyson
Murray Gell-Mann
Roger Penrose
Steven Weinberg
Immanuel Kant
Joseph-Louis Lagrange
Pierre-Simon Laplace
Dugald Stewart
Archibald Alison
Sophie Germain
Carl Friedrich Gauss
2nd ed. of Montucla’s *Histoire des Mathématiques*
A Mathematician’s Apology
Franco-Prussian War
World War I
II
Russo-Japanese War
Cold War
Napoléon I
Josef Stalin
Otto von Bismarck
Victoria

Bibliography

> ‘ I drew up a bibliography of more than a hundred books,
> but in the end read only about thirty of them. ’
>
> — Umberto Eco, *Foucault’s Pendulum*, pt ‘Binah’, ch. 7
> (trans. William Weaver).

> ‘ I know that I was not able
> to find or read everything that has been written.
> And I do not claim that
> I expressed everything which was noteworthy
> even from what I was able to read ’
>
> — Vincent of Beauvais,
> *Speculum Naturale*, ‘Prologus’, ch. iv, col. 4
> (trans. Ann M. Blair, *Too Much to Know*, p. 43).

Abbreviations used in citations.

DK	DIELS, H. & KRANZ, W., eds. *Die Fragmente der Vorsokratiker*.
KRS	KIRK, G. S., RAVEN, J. E. & SCHOFIELD, M. *The Presocratic Philosophers*.
LSJ	LIDDELL, H. G. & SCOTT, R. *A Greek–English Lexicon*.
OEIS	SLOANE, N. J. A., ed. ‘The On-Line Encyclopedia of Integer Sequences’.

Standard pagination used in citations.

BEK	Bekker pagination	ARISTOTLE. [All works].
BOU	Ed. Bouyges	IBN RUSHD. Comm. Metaphysics Lām.
BRU	Ed. Bruns	ALEXANDER. *Quaestiones*.
DEF	Ed. de Falco & Klein	PSEUDO-IAMBLICHUS. *The Theology of Arithmetic*.
DHL	Ed. Diehl	PROCLUS. *Commentary on Plato’s* Timaeus.
DLS	Ed. Diels	SIMPLICIUS. *On Aristotle Physics*.
DÜR	Ed. Düring	PTOLEMY. *Harmonics*.
FES	Ed. Festa	IAMBLICHUS. *On the General Science of Mathematics*.
FLO	Ed. Floss	ERIUGENA. *Periphyseon* (*De Divisione Naturae*).
FRI	Ed. Friedlein	PROCLUS. *A Commentary on the First Book of Euclid’s Elements*.
HAY	Ed. Hayduck	ALEXANDER. *On Aristotle Metaphysics*.

HE6	6th edition	Home, H. *Elements of Criticism*.
HEI	Ed. Heiberg	Ptolemy. *Almagest*.
	Ed. Heiberg	Simplicius. *On Aristotle On the Heavens*.
HER	Ed. Hermann	Alcinous. *The Handbook of Platonism*.
KÜH	Kühn pagination	Galen. [All works].
KCP	1st/2nd editions (A/B)	Kant, I. *Critique of Pure Reason*.
KGS	*Gesammelte Schriften*, Akademie ed.	Kant, I. [All translated works except the *Critique of Pure Reason*].
MEI	Meibomius pagination	Aristoxenus. *Harmonics*; Diogenes Laertius. *Lives of Eminent Philosophers*.
STE	Stephanus pagination	Plato. [All works]; Plutarch. [All works]; Ps.-Plutarch. *Peri tōn Areskontōn tois Philosophois Physikōn Dogmatōn*; Timaeus Locrus. *De Natura Mundi et Animae*.
TOD	Ed. Todd	Cleomedes. *Lectures on Astronomy*.
WIL	Ed. Wilpert & Klibansky	Nicholas of Cusa. *On Learned Ignorance*.

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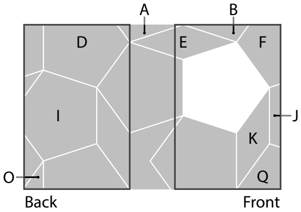

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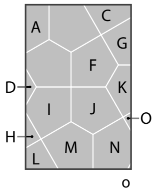

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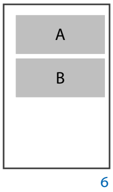

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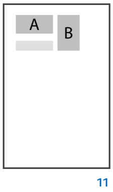

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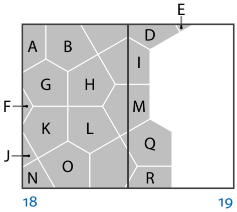
A Paolo Barbotti, *Cicero discovering the tomb of Archimedes*. URL: https://commons.wikimedia.org/wiki/File:Cicero_discovering_tomb_of_Archimedes_(_Paolo_Barbotti_).jpeg [Public domain] B William Blake, *The Ancient of Days*. URL: https://commons.wikimedia.org/wiki/File:Europe_a_Prophecy,_copy_D,_object_1_(Bentley_1,_Erdman_i,_Keynes_i)_British_Museum.jpg [Public domain] D Adapted from Goury, Jones & Gayangos, *Alhambra*, vol. I, pl. XXXVII. URL: https://archive.org/details/gri_33125012280661/page/n262/mode/1up [Public domain] E Rhind mathematical papyrus. URL: https://commons.wikimedia.org/wiki/File:Rhind_Mathematical_Papyrus.jpg [Public domain] F Adapted from 'prilfish' (Flickr user), https://flickr.com/photos/silkebaron/13805669085. © 2014 'prilfish'. [Licence: CC-BY-2.0] G 'Systema solare et planetarium ex hypothesi Copernicana secundum', from a design by Johann Gabriel Doppelmayr. Rijksmuseum. Object number: RP-P-AO-29-9. URL: http://hdl.handle.net/10934/RM0001.COLLECT.702461 [Public domain] H Leonardo da Vinci, 'Vitruvian Man'. URL: https://en.wikipedia.org/wiki/File:Da_Vinci_Vitruve_Luc_Viatour.jpg [Public domain] I Adapted from Kepler, *The Harmony of the World*, bk II, § XX, p. 111. URL: https://archive.org/details/ioanniskeppleri h00kepl/page/n80 [Public domain] J Adapted from Allen, *Celtic Art in Pagan and Christian Times*, p. 169. URL: https://archive.org/details/celticartinpagan00alle_0/page/169 [Public domain] K Adapted from Cellarius, *Harmonia Macrocosmica*, pl. between pp. 12, 13. URL: https://nla.gov.au/nla.obj-230714237 [Public domain] L Euclid, *Preclarissimus liber elementorum*, fol. 2r.



URL: https://commons.wikimedia.org/wiki/File:Thomas-Stanford_Plate01b.jpg [Public domain] M Stefano della Bella, 'Galileo and personifications of Astronomy, Perspective and Mathematics', frontispiece for *Opere di Galileo Galilei*. The Metropolitan Museum of Art. Accession number: 68.746. URL: https://www.metmuseum.org/art/collection/search/377877 [Public domain] N Detail from Raphael, *The School of Athens*. Vatican Museums. URL: https://commons.wikimedia.org/wiki/File:"The_School_of_Athens"_by_Raffaello_Sanzio_da_Urbino.jpg [Public domain] O Detail from Bernoulli, *Opera Omnia*, vol. 1, frontispiece. URL: https://www.e-rara.ch/zut/content/zoom/1051516 [Public domain] Q Adapted from Yale Babylonian Collection YPM BC 21354 (formerly YBC 7289), obverse. URL: https://collections.peabody.yale.edu/search/Record/YPM-BC-021354 [Public domain] R Adapted from Haüy, *Traité de Cristallographie*, Atlas, pl. 5. URL: https://www.e-rara.ch/zut/content/zoom/5805540 [Public domain]

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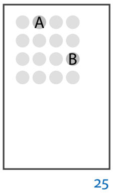

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Detail from Raphael, *The School of Athens*. Vatican Museums. URL: https://commons.wikimedia.org/wiki/File:"The_School_of_Athens"_by_Raffaello_Sanzio_da_Urbino.jpg [Public domain]

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Jacopo de' Barbari (attributed), *Portrait of Luca Pacioli*. National Museum of Capodimonte. Accession number: inv. Q 58. URL: https://commons.wikimedia.org/wiki/File:Pacioli.jpg [Public domain]

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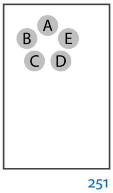
A Adapted from Pacioli, *Divina proportione*, pl. II. URL: https://archive.org/details/divinaproportion00paci/page/n187 [Public domain] B Adapted from Pacioli, *Divina proportione*, pl. VIII. URL: https://archive.org/details/divinaproportion00paci/page/n199 [Public domain] C Adapted from Pacioli, *Divina proportione*, pl. XXII. URL: https://archive.org/details/divinaproportion00paci/page/n231 [Public domain] D Adapted from Pacioli, *Divina proportione*, pl. XXVIII. URL: https://archive.org/details/divinaproportion00paci/page/n243 [Public domain] E Adapted from Pacioli, *Divina proportione*, pl. XVI. URL: https://archive.org/details/divinaproportion00paci/page/n219 [Public domain]



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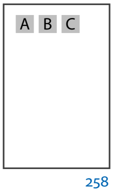
A Detail from Giovanni Jacopo Caraglio, engraving of Diogenes. The Metropolitan Museum of Art. Accession Number: 17.3.3416. URL: `https://www.metmuseum.org/art/collection/search/357255` [Public domain] B Detail from Nicolas Neufchâtel, *The Nuremberg Writing Master Johann Neudörfer mit einem Schüler*. URL: `https://commons.wikimedia.org/wiki/File:Der_Nürnberger_Schreibmeister_Johann_Neudörffer_mit_einem_Schüler_-_1561.jpg` [Public domain] C Detail from Lorenz Stoer, cover page from *Geometria et Perspectiva*. The Metropolitan Museum of Art. Accession Number: 1984.1085.1. URL: `https://www.metmuseum.org/art/collection/search/718394` [Public domain]



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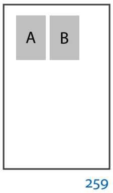

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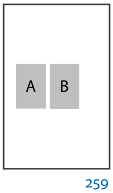

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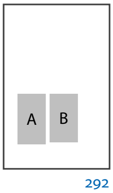

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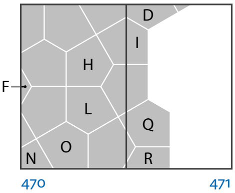

.biodiversitylibrary.org/page/34988432 [Public domain] I Adapted from https://www.rockini.name/math/images/rectified_icosahedral_honeycomb.png. © 2006 Claudio Rocchini. [Licence: CC-BY-4.0] L Adapted from photograph of Leonhard Fuchs, *De historia stirpivm commentarii insignes* (1542). Wellcome Collection EPB/D/2438, image L0051234. URL: https://wellcomecollection.org/works/kbf5pt3t [Public domain] N Adapted from Nelson & Segerman, {5, 4, 5} hyperbolic honeycomb. URL: http://www.hyperbolichoneycombs.org/ [Public domain] O https://www.flickr.com/photos/12131226. © 2012 Mikael Hvidtfeldt Christensen. [Licence: CC-BY-2.0] Q Background adapted from https://www.flickr.com/photos/jephy/14654979899/. © 2014 'Grempz' (Flickr user). [Licence: CC-BY-2.0] R Adapted from https://www.flickr.com/photos/fdecomite/29670663895/. © 2016 'fdecomite' (Flickr user). [Licence: CC-BY-2.0]

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Redrawn from Jeandel & Rao, 'An aperiodic set of 11 Wang tiles', Fig. 15. © 2015 Emmanuel Jeandel & Michael Rao. [Licence: CC-BY-4.0]

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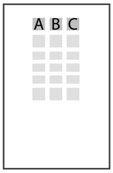

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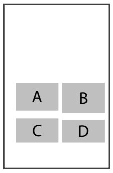
A Albert Bierstadt, *Looking Down Yosemite Valley, California*. Birmingham Museum of Art. Accession number: 1991.879. URL: https://commons.wikimedia.org/wiki/File:Looking_Down_Yosemite-Valley.jpg [Public domain] B Albert Bierstadt, *A Storm in the Rocky Mountains, Mt. Rosalie*. Brooklyn Museum. Accession number: 76.79. URL: https://commons.wikimedia.org/wiki/File:Albert_Bierstadt_-_A_Storm_in_the_Rocky_Mountains,_Mt._Rosalie_-_Google_Art_Project.jpg [Public domain] C John Constable, *The Hay Wain*. National Gallery. Accession number: NG1207. URL: https://commons.wikimedia.org/wiki/File:Joh



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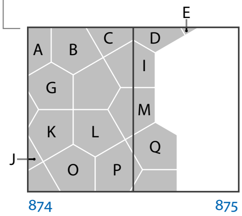

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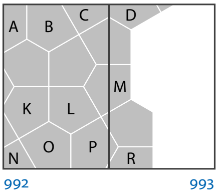

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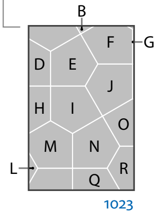

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> ‘ I shall read the Index with pleasure. ’
>
> — William Clarke,
> letter to William Bowyer dated 8 April 1767,
> quot. in John Nichols, *Literary Anecdotes*,
> vol. III, p. 46, n. ∗ cont. from p. 45.

> ‘ an intriguing asymmetry was introduced
> by the workings of the Index. ’
>
> — Elizabeth L. Eisenstein,
> *The Printing Press as an Agent of Change*, p. 145.

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> ‘I cannot promise such exactness in our Index,
> that no one name hath escaped our enquiry,
> some few, perchance, hardly slipping by,
> may tell tales against us.’
>
> — Thomas Fuller,
> *A Pisgah Sight of Palestine*, p. 669.

> ‘Begin to make order, and names arise.
> Names lead to more names —
> And to knowing when to stop.’
>
> — Lao-Tzu, *Dao De Jing*, § 32
> (trans. Stephen Addiss & Stanley Lombardo).

This index contains only names of individual persons, historical, pseudo-historical, mythical, or fictional. Locations, groups, and eponymous terms (‘Athens’, ‘Banū Mūsā’, ‘Cantor’s diagonal argument’) are in the subject index. The ordering of entries is strictly lexicographical, ignoring punctuation.

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> ‘ It is plain now that my life, for what it is worth, is finished, and that nothing I can do can perceptibly increase or diminish its value.
>
> […]
>
> I have never done anything “useful”. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world. […] I have just one chance of escaping a verdict of complete triviality, that I may be judged to have created something worth creating. ’

— G. H. Hardy, *A Mathematician’s Apology*, § 29.

> ‘ τῷ Ἀσκληπιῷ ὀφείλομεν ἀλεκτρυόνα·
> ἀλλὰ ἀπόδοτε καὶ μὴ ἀμελήσητε. ’

— Πλάτων, *Φαίδων*, STE 118a.

> ‘ Deyr fé, deyia frændr,
> deyr siálfr it sama;
> ek veit einn, at aldri deyr:
> dómr um dauðan hvern. ’

— *Hávamál*, v. 77 (ed. Neckel).

> ‘ 妻子王位財眷屬
> 死去無一來相親常隨業鬼繫縛我
> 苦受叫喚無邊際 ’

— 『平家物語』, 巻第六, 「慈心坊」.

COLOPHON

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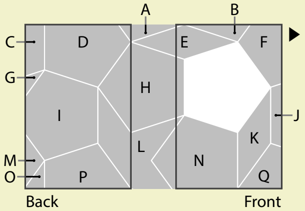

A Plot of a Lorenz attractor. **B** Diagram of the construction of a regular pentagon from Byrne, *The First Six Books of The Elements of Euclid*, p. 137. **C** Rotationally symmetric Venn diagram for seven sets. **D** The Higman–Sims graph and classes of edges in its construction. **E** Detail from Hans Holbein the Younger, *The Ambassadors*. **F** Crystals of iron pyrite. **G** Tessellation of the euclidean plane using diagrams of Napoleon's theorem. **H** Pentagonal Sierpiński gasket. **I** *Circogonia icosahedra*, a radiolarian with the form of a regular icosahedron. **J** Geometric design from a fifteen-century wall panel, thought to originate in Cairo. **K** Section through a nautilus shell. **L** Aperiodic tiling using the monotile discovered by Smith et al. **M** Three-dimensional tessellation using truncated octahedra. **N** Apollonian gasket. **O** Tiling from the Alhambra. **P** Illustration of the cyclic difference set $\{0, 1, 3, 8, 12, 18\}$. **Q** Rhombicuboctahedron drawn by Leonardo da Vinci, printed in Pacioli, *Divina proportione*, pl. xxxvi.

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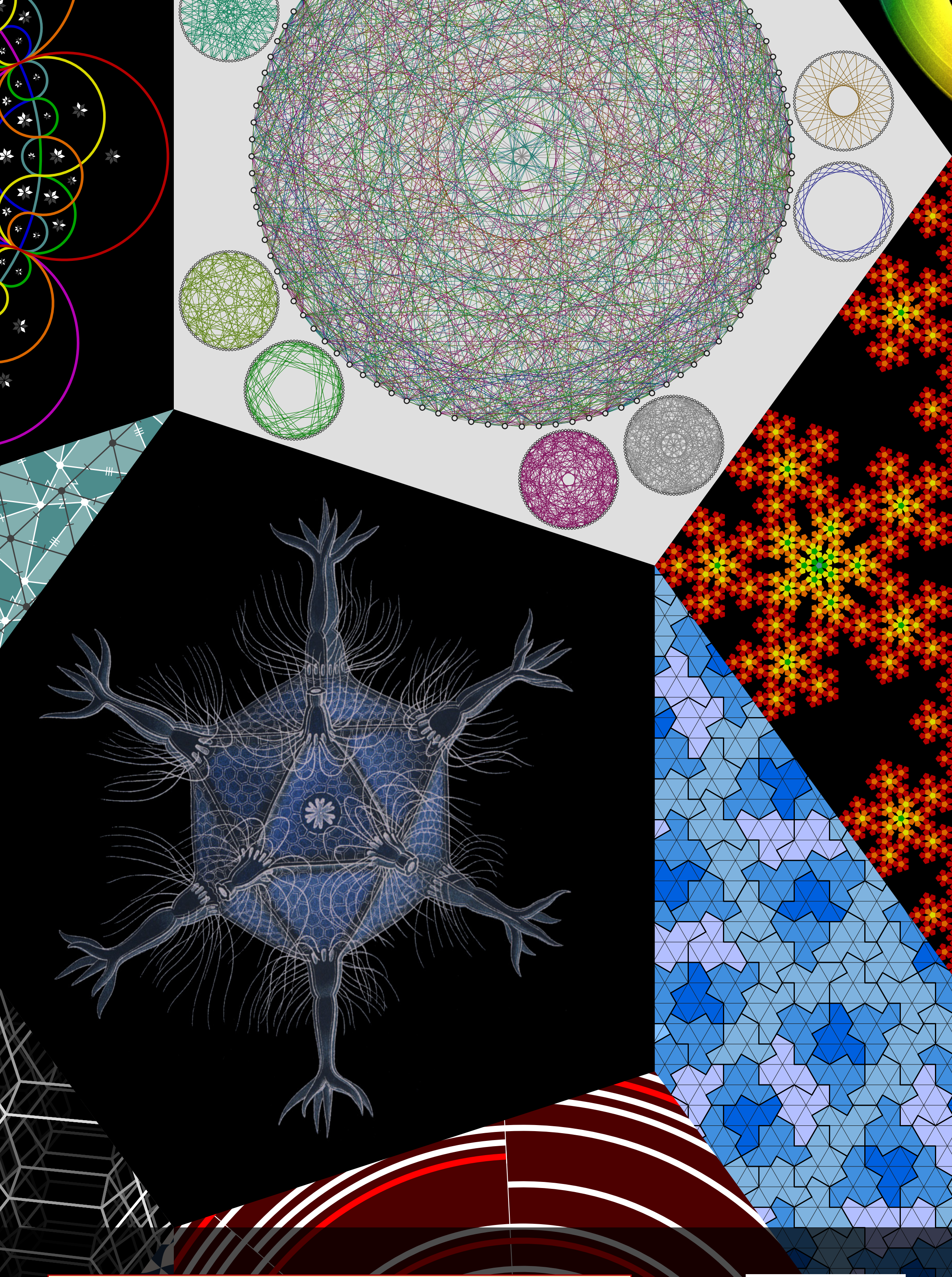

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